LINEAR QUADRATIC OPTIMAL CONTROL DESIGN USING CHEBYSHEV-BASED STATE PARAMETERIZATION

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by

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Abstract

A computationally attractive method for determining the optimal control of unconstrained linear dynamic systems with quadratic performance indices is presented. In the proposed method, the difference between each state variable and its initial condition is represented by a finite-term shifted Chebyshev series. The representation leads to a system of linear algebraic equations as the necessary condition of optimality. Simulation studies demonstrate computational advantages relative to a standard Riccati-based method, a transition matrix method, and a previous Fourier-based method.

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Introduction

Determining the optimal control of linear, lumped parameter models of dynamic systems is one of the principal "state space" design problems. The challenge is to find the optimal trajectories of the control and associated state which give the best tradeoff between performance and cost of control. Toward this end, variational methods can be used to cast the optimality condition as a two-point boundary-value problem (TPBVP). The most well-known solution is achieved via the Hamilton-Jacobi approach which converts the TPBVP to a terminal value problem involving a matrix differential Riccati equation. The Riccati equation provides the optimal solution in closed-loop form with natural advantages for physical implementation, although it is computationally intensive and sometimes difficult to employ in solving high order systems.

A preferred alternative for the optimal control solution of time-invariant problems is the open-loop transition matrix approach (Speyer, 1986). Typically, the transition matrix approach converts the TPBVP into an initial value problem. The transition matrix approach is also susceptible to numerical problems in determining the optimal control of high order systems (Yen and Nagurka, 1991). In particular, numerical instabilities are attributed principally to the errors associated with the computation of large dimension matrix exponentials (Golub and Van Loan, 1983).

In contrast to Riccati-based and transition matrix methods, approximate solution strategies, namely trajectory parameterization methods, have been investigated. In general, these approaches approximate the control, state, and/or co-state trajectories by finite-term series whose coefficient values are sought giving a near optimal solution. For example, approaches employing Walsh (Chen and Hsiao, 1975), block-pulse (Hsu and Cheng, 1981), Chebyshev (Paraskevopoulos, 1983; Vlassenbroeck and Van Dooren, 1988), Laguerre (Shih, *et al.*, 1986), and Fourier (Chung, 1987) series have been suggested. Like the state transition matrix approach, many of these approaches employ algorithms that convert the TPBVP into an initial value problem. By approximating the state and co-state vectors by truncated series, the initial value problem can be reduced to a static optimization problem represented by algebraic equations. However, the transition matrix (needed to convert the TPBVP to an initial value problem) must still be evaluated which, as mentioned above, can cause instability problems in high order systems.

State parameterization offers two important advantages for solving optimal control problems. First, the state initial condition can be satisfied directly. Second, the state equation can be treated as an algebraic equation in determining the corresponding control trajectory (since the state and hence state rate are known), assuming that no constraints on the control structure prevent an arbitrary representation of the state trajectory from being achieved.

This technical report extends previous work (Yen and Nagurka, 1991) for solving optimal control problems via Fourier-based state parameterization. The earlier work has shown computational advantages of a Fourier-based state approximation for solving linear quadratic (LQ) optimal control problems relative to standard methods. For systems with different numbers of state and control variables, artificial control variables were introduced to overcome the potential difficulty of trajectory inadmissibility.

The particular focus of this report is to explore a simplified parameterization approach employing a finite-term Chebyshev representation of the state trajectory. Chebyshev functions can nearly uniformly approximate a broad class of functions, making them computationally attractive (Vlassenbroeck and Van Dooren, 1988). Following the Fourier-based development, it is shown that the necessary condition of optimality can be derived as a system of linear algebraic equations from which an unknown state parameter vector can be solved. In contrast to the earlier work, a simplified state representation is adopted involving a constant term and shifted Chebyshev terms. This representation guarantees satisfaction of the state initial condition and enables the linear transformation of the unknown parameter vector in the solution procedure. The result is an accurate, robust, and computationally attractive method that is especially suited for high-order systems.

Chebyshev-based Approach

Problem Statement. The LQ optimal control problem involves finding the control $\mathbf{u}(t)$ and the corresponding state $\mathbf{x}(t)$ in the time interval [0,T] that minimizes the quadratic performance index L,

$$L = L_1 + L_2 \tag{1}$$

where

$$L_1 = \mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T) + \mathbf{h}^T\mathbf{x}(T) \tag{2}$$

$$L_2 = \int_0^T [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)+\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{u}(t)+\mathbf{q}^T(t)\mathbf{x}(t)+\mathbf{r}^T(t)\mathbf{u}(t)]dt \tag{3}$$

for the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4}$$

with known initial condition $\mathbf{x}(0)=\mathbf{x}_0$. The state vector $\mathbf{x}$ is N×1, the control vector $\mathbf{u}$ is M×1, the system matrix A is N×N, and the control influence matrix $\mathbf{B}$ is N×M. It is assumed that the columns of $\mathbf{B}$ are independent, weighting matrices **H**, **Q**, **R** and S and weighting vectors **h**, **q** and **r** have appropriate dimensions, and that **H**, **Q**, **R** and S are real and symmetric with **H** and **Q** being positive-semidefinite and **R** being positive definite.

State Parameterization. The LQ optimal control problem can be converted to an optimization problem by approximating each state variable by the summation of the initial condition and a K term series.

$$x_n(t) = x_{n0} + \sum_{k=1}^{K} c_k(t)\, y_{nk} \quad , \quad k=1,2,\ldots,K \text{ and } n=1,2,\ldots,N \tag{5}$$

where $x_{n0}=x_n(0)$ and y_{nk} is the k-th unknown coefficient of the basis function $c_k(t)$ for the n-th state variable. A variety of basis functions is available with the requirement that the summation vanishes at t=0 such that the initial condition x_{n0} is satisfied. Here, the proposed basis function is

$$c_k(t) = \psi_k(t) + (-1)^{k-1} \quad , \quad k=1,2,\ldots,K \tag{6}$$

where $\psi_k(t)$ is a shifted Chebyshev polynomial and the additional term $(-1)^{k-1}$ ensures $c_k(0)=0$.

In general, Chebyshev polynomials are orthogonal on the interval $\xi \in [-1,1]$ with the weighting function $(1-\xi^2)^{-1/2}$ and have the following analytical form:

$$\varphi_k(\xi) = \cos(k\cos^{-1}\xi) = \sum_{i=0}^{[k/2]} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{k-2i} \quad , \quad k=0,1,2,\ldots \tag{7}$$

where the notation [k/2] means the greatest integer smaller than k/2. In shifted Chebyshev polynomials the domain is transformed to values between 0 and T by introducing the change of variables $\xi = 2t/T-1$ giving

$$\psi_k(t) = \varphi_k(\xi) = \varphi_k(2\tau-1) \tag{8}$$

where nondimensional time $\tau = t/T$. From equations (7) and (8) the first few shifted

Chebyshev polynomials are

$$\psi_0(t) = 1 \;\; ; \;\; \psi_1(t) = 2\tau - 1 \;\; ; \;\; \psi_2(t) = 8\tau^2 - 8\tau + 1 \qquad \text{(9a-c)}$$

$$\psi_3(t) = 32\tau^3 - 48\tau^2 + 18\tau - 1 \;\; ; \;\; \psi_4(t) = 128\tau^4 - 256\tau^3 + 160\tau^2 - 32\tau + 1 \quad \text{(9d,e)}$$

The initial and terminal values of the shifted Chebyshev polynomial and their first time derivatives can be obtained as

$$\psi_k(0) = (-1)^k \;\; ; \;\; \dot{\psi}_k(0) = (-1)^{k+1}(2k^2/T) \;\; ; \;\; \psi_k(T) = 1 \;\; ; \;\; \dot{\psi}_k(T) = 2k^2/T \qquad \text{(10a-d)}$$

Equation (5) can be written alternatively as

$$x_n(t) = x_{n0} + \mathbf{c}^T(t)\,\mathbf{y}_n \qquad (11)$$

where

$$\mathbf{c}^T(t) = [c_1(t) \;\; c_2(t) \;\cdots\; c_K(t)] \;\; ; \;\; \mathbf{y}_n = [y_{n1} \;\; y_{n2} \;\cdots\; y_{nK}]^T \qquad (12),(13)$$

In equation (13) $\mathbf{y}_n$ is a state parameter vector (containing the unknown coefficients) for the n-th state variable.

The state vector containing the N state variables can be written in terms of a full state parameter vector $\mathbf{y}$, *i.e.*,

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{C}(t)\mathbf{y} \qquad (14)$$

where

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{c}^T(t) & & & \mathbf{O} \\ & \mathbf{c}^T(t) & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{c}^T(t) \end{bmatrix}_{N\times(N)(K)} \qquad (15)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} [y_{11} & y_{12} & \cdots & y_{1K}]^T \\ [y_{21} & y_{22} & \cdots & y_{2K}]^T \\ & \vdots & & \\ [y_{N1} & y_{N2} & \cdots & y_{NK}]^T \end{bmatrix}_{(N)(K)\times 1} \qquad (16)$$

In equations (15) and (16), the matrix dimensions are identified and the notation (N)(K) denotes N times K. From equation (14), the state rate vector can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{D}(t)\mathbf{y} \tag{17}$$

where

$$\mathbf{D}(t) = \dot{\mathbf{C}}(t) = \begin{bmatrix} \mathbf{d}^T(t) & & & \mathbf{O} \\ & \mathbf{d}^T(t) & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{d}^T(t) \end{bmatrix}_{N\times(N)(K)} \tag{18}$$

$$\mathbf{d}^T(t) = \left[\dot{c}_1(t)\ \ \dot{c}_2(t)\ \cdots\ \dot{c}_K(t)\right] \tag{19}$$

The control vector $\mathbf{u}(t)$ can also be expressed as a function of $\mathbf{y}$. From equations (4), (14), and (17),

$$\mathbf{u}(t) = \left[\mathbf{B}^{-1}(t)\mathbf{D}(t) - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{C}(t)\right]\mathbf{y} - \mathbf{B}^{-1}(t)\mathbf{A}(t)\mathbf{x}_0 \tag{20}$$

Equation (20) assumes that $\mathbf{B}^{-1}$ exists and implies that the lengths of the state and control vectors are the same (*i.e.*, M=N). This requirement is later relaxed (see subsection on General Linear Systems).

Approximation of Performance Index. The performance index can now be approximated as a function of the state parameter vector $\mathbf{y}$. First, equation (14) with t=T is substituted into equation (2) giving the cost L_1 as a quadratic function of y

$$L_1 = \mathbf{y}^T\left[\mathbf{H}\otimes\mathbf{c}(T)\mathbf{c}^T(T)\right]\mathbf{y} + \mathbf{y}^T\left[(2\mathbf{H}\mathbf{x}_0+\mathbf{h})\otimes\mathbf{c}^T(T)\right] + \mathbf{x}_0^T(\mathbf{H}\mathbf{x}_0+\mathbf{h}) \tag{21}$$

where $\otimes$ is a Kronecker product sign (Brewer, 1978), *e.g.*,

$$\mathbf{V}\otimes\mathbf{Z} = \begin{bmatrix} V_{11}\mathbf{Z} & \cdots & V_{1n}\mathbf{Z} \\ V_{21}\mathbf{Z} & & \vdots \\ \vdots & & \vdots \\ V_{n1}\mathbf{Z} & \cdots & V_{nn}\mathbf{Z} \end{bmatrix} \tag{22}$$

where $\mathbf{V}$ is an n×n matrix and $\mathbf{Z}$ is an arbitrary matrix. From equations (14) and (20) the integrand of equation (3) can be also expressed as a quadratic function of y, *i.e.*,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}+\mathbf{x}^T\mathbf{S}\mathbf{u}+\mathbf{q}^T\mathbf{x}+\mathbf{r}^T\mathbf{u} = \mathbf{y}^T\mathbf{P}\mathbf{y}+\mathbf{y}^T\mathbf{p}+\mathbf{x}_0^T\mathbf{p}_0 \tag{23}$$

where, for convenience, the time-dependent notation (t) has been dropped and

$$\mathbf{P} = \mathbf{F}_1\otimes\mathbf{c}\mathbf{c}^T+\mathbf{F}_2\otimes\mathbf{d}\mathbf{d}^T+\mathbf{F}_3\otimes\mathbf{d}\mathbf{c}^T \tag{24a}$$

$$\mathbf{p} = (2\mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1)\otimes\mathbf{c}+(\mathbf{F}_3\mathbf{x}_0+\mathbf{f}_2)\otimes\mathbf{d} \quad ; \quad \mathbf{p}_0 = \mathbf{F}_1\mathbf{x}_0+\mathbf{f}_1 \tag{24b,c}$$

In equations (24a–c) $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are N×N matrices and $\mathbf{f}_1$ and $\mathbf{f}_2$ are N×1 vectors given by

$$\mathbf{F}_1 = \mathbf{Q}+\mathbf{A}^T\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{A}-\mathbf{S}\mathbf{B}^{-1}\mathbf{A} \quad ; \quad \mathbf{F}_2 = \mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1} \tag{25a,b}$$

$$\mathbf{F}_3 = -2\mathbf{B}^{-T}\mathbf{R}\mathbf{B}^{-1}\mathbf{A}+\mathbf{B}^{-T}\mathbf{S} \quad ; \quad \mathbf{f}_1 = \mathbf{q}-\mathbf{A}^T\mathbf{B}^{-T}\mathbf{r} \quad ; \quad \mathbf{f}_2 = \mathbf{B}^{-T}\mathbf{r} \tag{25c–e}$$

and superscript $-T$ denotes inverse transpose. Hence, $\mathbf{P}$ is an (N)(K)×(N)(K) matrix, and $\mathbf{p}$ and $\mathbf{p}_0$ are (N)(K)×1 vectors. From equation (23), the integral part of the performance index can be expressed as

$$L_2 = \int_0^T \left(\mathbf{y}^T\mathbf{P}\mathbf{y}+\mathbf{y}^T\mathbf{p}+\mathbf{x}_0^T\mathbf{p}_0\right)dt = \mathbf{y}^T\mathbf{P}^*\mathbf{y}+\mathbf{y}^T\mathbf{p}^*+\mathbf{x}_0^T\mathbf{p}_0^* \tag{26}$$

where

$$\mathbf{P}^* = \int_0^T \mathbf{P}\,dt \quad ; \quad \mathbf{p}^* = \int_0^T \mathbf{p}\,dt \quad ; \quad \mathbf{p}_0^* = \int_0^T \mathbf{p}_0\,dt \tag{27a–c}$$

can be integrated numerically for time-varying problems. Combining equations (21) and (26) gives the performance index L as a quadratic function of $\mathbf{y}$, *i.e.*,

$$L = \mathbf{y}^T\mathbf{G}\mathbf{y} + \mathbf{y}^T\mathbf{g} + \mathbf{x}_0^T\left[\mathbf{H}\mathbf{x}_0+\mathbf{h}+\mathbf{p}_0^*\right] \tag{28}$$

where

$$\mathbf{G} = \mathbf{H}\otimes\mathbf{c}(T)\mathbf{c}(T)^T + \mathbf{P}^* \quad ; \quad \mathbf{g} = (2\mathbf{H}\mathbf{x}_0+\mathbf{h})\otimes\mathbf{c}^T(T) + \mathbf{p}^* \tag{29a,b}$$

For time-invariant problems, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are constants, and equations (27a–c) can be rewritten as

$$P^* = Fi\langle 8\rangle \quad (cc^{\prime})dt \; + F2\langle 8\rangle \quad (dd^{\prime})dt \; + F3® \quad (dc^{\prime})dt \tag{30a}$$

$$p^* = (2Fix_0+fi)\langle 8\rangle \int_{Jo} cdt + (F_3x_0+f2)® \int_{Jo} ddt \tag{30b}$$

$$_{r}; = T(FiXo+fi) \tag{30c}$$

The terms in the brackets can be evaluated numerically or derived in closed-form. For example, the first integral term of equation (30b) can be derived as

$$\int_{Jo} Cidt = \int_{Jo} (t)dt + (-ir^{1}T = l\, a_{0i} + (-ir^{1}T \quad , \quad i=l,...,K \tag{31}$$

where

$$aoi = 0 \quad \text{and} \quad \alpha oi = \frac{1 + (-i)^{y}}{1 - i^{2}}, \quad i * 1 \tag{32}$$

Closed-form relations for all bracketed terms in equations (30a,b) are presented in Appendix A.

Optimality Condition. The necessary condition of optimality can be obtained by differentiating equation (28) with respect to y. The resulting optimality condition is

$$(G + G^{\prime})y = -g \tag{33}$$

representing a system of linear algebraic equations from which the unknown vector **y** can be solved. Note that the state initial condition is embedded only in the right-hand side of equation (33). The coefficient matrix remains the same for problems with different initial conditions.

General Linear Systems. To apply the Chebyshev-based approach to systems with different numbers of state and control variables, a penalty function technique is proposed. The state-space model of equation (4) is modified to

$$x(t) = A(t)x(t) + B'(t)u'(t) \tag{34}$$

It is required that B^{1} be invertible and that the modified excitation $B'u^{\prime}$ be as close to the column space of B as possible. This can be done by choosing a well-conditioned B' and penalizing the orthogonal projection of BV onto the left-nullspace of B in a modified performance index,

$$L' = L + pE \tag{35}$$

with

$$E = \int_0 [(B'(t)u'(t) - B(t)u(t))^T(B'(t)u'(t) - B(t)u(t))]dt \tag{36}$$

Here, L is the original performance index of equation (1), p is a weighting constant chosen as a large positive number, and E is the integral of the orthogonal projection. E can be viewed as an error index indicating the proximity of the modified state equation to the original state equation. By equating Bu and B'u' and applying least squares approximation, the original u can be reconstructed as

$$u = Wu' \tag{37}$$

where

$$W = (B^T B)^{-1} B^T B' \tag{38}$$

With u from equation (37), the modified performance index L' can be rewritten as

$$L' = L_1 + \int_0 [x^T Q x + u'^T R' u' + x^T S' u' + q^T x + r'^T u']dt \tag{39}$$

where L_1 is from equation (2) and

$$R' = W^T R W + p(B' - BW)^T(B' - BW) \quad ; \quad S' = SW \quad ; \quad r' = r^T W \tag{40a-c}$$

Equations (34) and (39) represent a modified LQ problem solvable by the Chebyshev-based approach. Matrix B' can be chosen arbitrarily as long as it is invertible. A convenient choice is the identity matrix which minimizes function evaluations in equations (20) and (25a-e).

The procedure for solving a general time-invariant LQ optimal control problem is summarized below:

INPUT:	initial state $\mathbf{x}_0$; system matrix **A**; control influence matrix **B**; terminal time T; coefficient matrices **H**, **Q**, **R**, **S**; coefficient vectors **h**, **q**, **r**; number of Chebyshev-based polynomial terms, K.		
Step 1	*If*	$M \neq N$,	
	then	pick **B'**; compute **W** from equation (38); replace **B** by **B'**; replace **R**, **S**, and **r** by **R'**, **S'**, and **r'** from equations (40a–c).	
Step 2	*Compute*	$\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{f}_1$ and $\mathbf{f}_2$ from equations (25a–e).	
Step 3	*Compute*	$\mathbf{P}^*$, $\mathbf{p}^*$ and $\mathbf{p}_0^*$ from equations (30a–c).	
Step 4	*Compute*	**G** and **g** from equations (29a,b).	
Step 5	*Compute*	y from equation (33).	
Step 6	*Compute*	performance index L from equation (28).	
Step 7	*Evaluate*	state and state rate from equations (14) and (17).	
Step 8	*Evaluate*	control from equation (20).	
Step 9	*If*	$M \neq N$,	
	then	evaluate original control from equation (37).	

OUTPUT: performance index L; state trajectory $\mathbf{x}(t)$ and control trajectory $\mathbf{u}(t)$.

STOP

Simulation Studies

Example 1. Sage and White (1977) consider the one-dimensional diffusion equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial y^2} + u(y,t) \quad , \quad 0 \le t \le T \quad , \quad 0 \le y \le Y \tag{41}$$

with boundary conditions and initial condition

$$\frac{\partial x}{\partial y}(0,t) = \frac{\partial x}{\partial y}(Y,t) = 0 \quad ; \quad x(y,0) = 1 + y \qquad (42),(43)$$

The performance index to be minimized is

$$L = \frac{1}{2}\int_0^T \int_0^Y \left[x^2(y,t) + u^2(y,t)\right] dy\,dt \tag{44}$$

Using a finite difference approximation, this distributed parameter system can be approximated by the N-th order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{45}$$

where

$$\mathbf{A} = \frac{1}{(\Delta y)^2}\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \mathbf{O} \\ & & \ddots & & \\ \mathbf{O} & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix}_{N\times N} \quad ; \quad \mathbf{B} = \mathbf{I}_{N\times N} \quad ; \quad \Delta y = \frac{Y}{N-1} \tag{46a–c}$$

$$\mathbf{x} = [x_1 \; x_2 \; \ldots \; x_N]^T \quad , \quad x_n = x\big((n-1)\Delta y\big) \quad , \quad n = 1,2,\ldots,N \tag{46d}$$

$$\mathbf{u} = [u_1 \; u_2 \; \ldots \; u_N]^T \quad , \quad u_n = u\big((n-1)\Delta y\big) \quad , \quad n = 1,2,\ldots,N \tag{46e}$$

with initial conditions

$$x_n(0) = 1 + (n-1)\Delta y \quad , \quad n = 1,2,\ldots,N \tag{47}$$

The performance index can then be approximated by

$$L = \frac{\Delta y}{2}\int_0^T (\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{48}$$

where

$$\mathbf{Q} = \mathbf{R} = \text{diag}\left[\frac{1}{2}, \; 1, \ldots, \; 1, \; \frac{1}{2}\right]_{N\times N} \tag{49}$$

Simulation studies were conducted using a Macintosh IIcx. The optimal value of the performance index and the optimal trajectories of the state and control vectors at 101 equally-spaced points were solved by a Riccati equation solver (Speyer, 1986), a transition matrix approach (Speyer, 1986), a Fourier-based state parameterization approach (Yen and Nagurka, 1991), and the proposed Chebyshev-based approach.

The value of the performance index and the execution time (in seconds) for T=1, Y=4 and N=5, 8, 11, 14, 17, and 20 are summarized in Table 1. The Riccati equation solver provides accurate solutions in all cases, although it is time-consuming for the higher order systems. The transition matrix approach is accurate and computationally more efficient than the Riccati equation solver but it encounters numerical difficulties and fails to provide reasonable solutions for $N \geq 14$. In the Fourier-based and Chebyshev-based approaches the number of terms in the state approximation was selected to provide accurate solutions, defined (arbitrarily) as having a percent relative error of less than one percent. To achieve this high accuracy, two Fourier-type terms in addition to the fifth-order auxiliary polynomial terms are required in the Fourier-based approach and six shifted Chebyshev terms in addition to the constant (initial condition) term are needed in the Chebyshev-based approach. Both state parameterization approaches are computationally more efficient than the transition matrix approach for $N \geq 8$.

It is possible to interpret the results of the state parameterization methods in light of the number of equivalent linear algebraic equations. A K-term Chebyshev-based approach involves (N)(K) linear algebraic equations representing the conditions of optimality. In comparison, a K-term Fourier approach involves N(2K+3) linear algebraic equations (see Yen and Nagurka, 1991). The results suggest that the six-term Chebyshev-based approach is more accurate and computationally more efficient than the two-term Fourier-based approach in all cases. In particular, for N=20, the Chebyshev-based method (involving 120 equations) shows greater than 34 percent savings in execution time when compared with the two-term Fourier-based approach (with 140 equations). For N>17 in both approaches, the performance indices increase slightly as the system order grows, while the solutions from the Riccati equation solver indicate that the performance index should decrease. Adding terms to the series improves the accuracy of the solutions.

The time histories of the state variables for N=5 obtained via a transition matrix approach and a 6-term Chebyshev-based approach are plotted in Figure 1. The solutions from both approaches coincide indicating that convergence has been achieved. Similarly, for N=5, the control variable histories for the two approaches overlap for the scale shown in Figure 2.

Example 2. This example, adapted and modified from (Meirovitch, 1990, Example 6.3), considers a series arrangement of J masses and J springs. As shown in Figure 3, it represents a 2J order system with a single force input acting on the last mass, m_J. The displacement of mass m_j is denoted by q_j. The mass and stiffness matrices are

$$
\mathbf{M} = \begin{bmatrix} m_1 & & & \mathbf{O} \\ & m_2 & & \\ & & \ddots & \\ \mathbf{O} & & & m_J \end{bmatrix} \tag{50}
$$

$$
\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & \mathbf{O} & \\ & & & \ddots & & \\ & \mathbf{O} & & -k_{J-1} & k_{J-1}+k_J & -k_J \\ & & & & & k_J \end{bmatrix} \tag{51}
$$

The state equation of this system is given by equation (4) with

$$
\mathbf{x} = [x_1\ x_2 \cdots x_{2J}]^T = [q_1\ q_2 \cdots q_J\ \dot{q}_1\ \dot{q}_2 \cdots \dot{q}_J]^T \tag{52}
$$

$$
\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ & \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{O} \end{bmatrix} \quad ; \quad \mathbf{B} = [0\ 0\ \cdots\ 0\ \ 1/m_J]^T \tag{53),(54}
$$

The initial conditions are

$$
\mathbf{x}(0) = [x_1(0)\ \ x_2(0) \cdots x_{2J}(0)]^T \tag{55}
$$

where it is presumed

$$
x_J(0) = 1 \quad ; \quad x_j(0) = 0 \quad , \quad j = 1,2,\ldots,J-1,J+1,\ldots,2J \tag{56a,b}
$$

indicating that the last mass only has been displaced from rest.

The problem is to find the optimal control history, u(t), that minimizes the performance index

$$
L = \int_0^{10} \left\{ \mathbf{x}^T\mathbf{Q}\mathbf{x} + u^2 \right\} dt \quad ; \quad \mathbf{Q} = \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix} \tag{57),(58}
$$

The integrand term $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ represents the sum of kinetic and potential energies of the system. The inclusion of the integrand term u^2 reflects the desire to minimize the force (as well as the total energy).

Using the values $m_j=10\,kg$ and $k_j=1\,N/m$ ($j=1,2,\ldots,J$) for three different systems, J=3, 5, and 7, the optimal solutions were determined using a Riccati approach, a transition matrix approach, and the Chebyshev-based approach. In the latter approach an eight term series (*i.e.*, initial condition plus seven Chebyshev-type terms) was selected, the weighting constant $\rho=10^5$ was used, and **B'** was chosen as the identity matrix. The resulting values of the original performance index and the execution time are summarized in Table 2. For J=7 there is less than a 0.21 percent error in the value of the performance index and a time savings of greater than 52 percent relative to the transition matrix method (and over a 96 percent savings relative to the Riccati solver).

For J=3 the time histories of the state variables x_3 and x_6 (the displacement and velocity of the last mass, respectively) and the control variable u obtained using the Chebyshev-based approach are compared with the respective state and control variables of the transition matrix approach in Figures 4 and 5. To verify that the penalty function technique is successful, the error index, E, is evaluated by substituting back the state and control trajectories into equation (35). The results are $E=7.64\times10^{-7}$ for J=3, $E=8.09\times10^{-7}$ for J=5, and $E=8.09\times10^{-7}$ for J=7, indicating that the modified state equation closely approximates the original state equation.

Discussion

Selection of Terms of Chebyshev-based Series. The proposed approach provides near optimal solutions with the accuracy depending on the number of terms of the series. Increasing the number of terms improves the accuracy while sacrificing computation time. A recommended procedure for selecting the number of terms is to solve the problem using a K term series and a K+I term series, and to then check whether the relative error of the performance index is within a desired tolerance. When the difference I is arbitrarily large, the relative error essentially represents the error between the approximate and exact solutions. If the relative error is within the required tolerance, the K term series is acceptable.

Selection of Penalty Function Weighting Constant. An important factor affecting the solution accuracy of general systems is the weighting constant. To ensure that the modified excitation **B'u'** closely approximates the original excitation **Bu**, the weighting constant is chosen to be a large positive number. However, if the weighting constant is too large, the magnitude of the original performance index can become insignificant relative to the approximated performance index. On the other hand, if the weighting constant is too small, (**B'u'**–**Bu**) is not driven small enough to approximate the original system. When an exact solution is not available, it is useful to plot L vs. ρ and E vs. ρ to help determine the appropriate weighting constant. Figure 6 shows these relations for Example 2. It reveals that the performance index is least sensitive in the range $10^4<\rho<10^7$. Thus, $\rho=10^5$ is an appropriate weighting constant, and the corresponding E of less than 10^{-6} indicates a satisfactory approximation of the state equation.

Comparison of Available Approaches vs. Chebyshev-based Approach. Vlassenbroeck and Van Doreen (1988) proposed a Chebyshev-based state and control parameterization approach for solving nonlinear optimal control problems. To apply the Chebyshev polynomials, the system dynamics are transformed from time interval [0,T] to [–1,1] and then converted into equality constraints (with tedious analytical formulation). Since both state and control parameterization is employed, the optimal control problem is converted into an optimization problem with constraints. In contrast, the proposed Chebyshev-based approach employs shifted Chebyshev polynomials on time interval [0,T]. By applying state parameterization only, fewer unknown parameters are needed. The state equation is used to represent the control as a function of the state, circumventing equality constraints, and the LQ optimal control problem is then converted into an unconstrained optimization problem which may be cast as a system of linear algebraic equations. In summary, Vlassenbroeck and Van Doreen's approach is capable of solving nonlinear and constrained optimal control problems, while the Chebyshev-based approach of this paper provides a direct and fast solution procedure for linear optimal control problems.

Compared to the Fourier-based approach (Yen and Nagurka, 1991), the Chebyshev-based approach offers a simplified solution procedure with concomitant computational advantages. Although the Chebyshev-based approach is computationally more attractive, the Fourier-based approach is more flexible in that it can deal with a broader class of problems, namely those with general boundary conditions. It is capable of solving optimal control problems with known initial states, initial state rates, terminal states, and/or terminal state rates by isolating the known boundary conditions from the unknown parameters in the state parameter vector **y**.

Conclusion

This report has presented a robust and computationally efficient Chebyshev-based algorithm for solving LQ optimal control problems. A key reason underlying the computationally streamlined nature of the approach is that the necessary condition of optimality can be written as a set of linear algebraic equations. Another advantage of the approach, especially important for time-invariant problems, is the availability of closed-form formulas for the integrals of shifted Chebyshev polynomial terms needed in establishing the linear algebraic equations. Finally, a penalty function technique is promoted as a means to make the approach tractable for systems with different numbers of state and control variables. Simulation results demonstrate computational advantages of the proposed approach relative to a Riccati approach, a transition matrix approach, and a previous Fourier-based approach.

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Appendix A Integrals for Chebyshev-based Approach

A.1 Properties of Chebyshev Polynomials. Chebyshev polynomials are orthogonal on the interval $\xi \in [-1,1]$ with the weighting function $(1-\xi^2)^{-1/2}$ and have the following analytical form:

$$\varphi_k(\xi) = \cos(k\cos^{-1}\xi) \quad , \quad k=0,1,2,... \tag{A.1}$$

or

$$\varphi_k(\xi) = \sum_{i=0}^{[k/2]} (-1)^i \frac{k!}{(2i)!(k-2i)!}(1-\xi)^i \xi^{k-2i} \quad , \quad k=0,1,2,... \tag{A.2}$$

where the notation [k/2] means the greatest integer smaller than k/2. From equation (A.2), the first few Chebyshev polynomials can be extended as

$$\begin{aligned}
\varphi_0(\xi) &= 1 \\
\varphi_1(\xi) &= \xi \\
\varphi_2(\xi) &= 2\xi^2 - 1 \\
\varphi_3(\xi) &= 4\xi^3 - 3\xi \\
\varphi_4(\xi) &= 8\xi^4 - 8\xi^2 + 1 \\
\varphi_5(\xi) &= 16\xi^5 - 20\xi^3 + 5\xi
\end{aligned} \tag{A.3a-f}$$

The Chebyshev polynomials have several interesting properties, such as satisfying *(i)* the recurrence relations

$$\begin{aligned}
&\varphi_{k+1}(\xi) - 2\xi\varphi_k(\xi) + \varphi_{k-1}(\xi) = 0 \quad , \quad k=1,2,... \\
&(1-\xi^2)\dot{\varphi}_k(\xi) = -k\xi\varphi_k(\xi) + k\varphi_{k-1}(\xi) \quad , \quad k=1,2,...
\end{aligned} \tag{A.4a,b}$$

where the dot indicates differentiation with respect to time, *(ii)* the boundary and midpoint values

$\varphi_k(1) = 1$

$\varphi_k(-1) = (-1)^k$

$\varphi_{2k}(0) = (-1)^k$

$\varphi_{2k-1}(0)=0$ (A.5a-d)

and *(iii)* the product relations

<Pi($)<Pj($)= i[<Pi.j(^)+<PII^)] » (A.6a-b)

^) = yi+<P2k(^)]

These properties can be used to develop the recurrence formulas for the integrals in the next section.

A.2 Integrals of Chebyshev Polynomials. Applying the properties of the recurrence relation, product relation, and boundary values in Section A.1, the integrals of £<pj» ^'(pj, (pi(pj, <Hg and <j>i(pj can be derived by "integration by parts". Here, the integrals are denoted as follows.

(A.7)

(A.8)

<Pite)<Pjte)d4 (A.9)

•i (A. 10)

(A.11)

The recurrence formulas for these integrals are summarized below, where the subscripts i and j are non-negative integers.

((j + i x j - i - l) ! ' ' "' ' ' " ' U«<i_ixH-i>l . when j - i - 1 * 0

a;; = , otherwise (A.12)

p.,=1-(-1)'-i_ia.,_,), (A.13)

Yu = 2t'°<'J)-'olHl] (A.14)

Sij = { (A. 15)

0	,i=0
(3oi	, i*0, j=0
Pii	, i*0, j=1
2P,i-Poi	, i*0, j=2
4p3i-3pn	, i*0, j=3
\28i(j_2) - 8,,u) + 2i[Y(i-1)(j_2) - Y(i-i)G-2)]	. i*0. J^'

(A. 16)

0	J=0
Poi	.j-1
4Pu	.j=2
12p,,-3Poi	,j=3
12c,,_2) - e,,j-4) + 2j8i(j_i) + (8 - 2j)8,,j_,)	, j>4

where li-jl denotes the absolute value of (i-j).

A.3 Integrals of Shifted Chebyshev Polynomials. In shifted Chebyshev polynomials the domain is transformed to values between 0 and T by introducing the change of variables £ = 2t/T-l giving

Vk(0=<Pk(S) = <Pk(2T-l) (A.17)

where nondimensional time x = t/T. From equations (A.17) and (A.2) the first few shifted Chebyshev polynomials are

$$\begin{aligned}
\psi_0(t) &= 1 \\
\psi_1(t) &= 2\tau - 1 \\
\psi_2(t) &= 8\tau^2 - 8\tau + 1 \\
\psi_3(t) &= 32\tau^3 - 48\tau^2 + 18\tau - 1 \\
\psi_4(t) &= 128\tau^4 - 256\tau^3 + 160\tau^2 - 32\tau + 1
\end{aligned} \tag{A.18a-e}$$

The initial and terminal values of the shifted Chebyshev polynomial and their first time derivatives can be obtained as

$$\begin{aligned}
\psi_k(0) &= (-1)^k \\
\dot{\psi}_k(0) &= (-1)^{k+1}(2k^2/T) \\
\psi_k(T) &= 1 \\
\dot{\psi}_k(T) &= 2k^2/T
\end{aligned} \tag{A.19a–d}$$

In this section, the integrals of ψ_i, $\dot{\psi}_i$, $\psi_i\psi_j$, $\dot{\psi}_i\psi_j$ and $\dot{\psi}_i\dot{\psi}_j$ are developed. Equation (A.17) yields the following relations:

$$\begin{aligned}
\psi_k(t) &= \varphi_k(\xi) \\
\dot{\psi}_k(t) &= \frac{2}{T}\dot{\varphi}_k(\xi) \\
t &= \frac{T}{2}(1+\xi) \\
dt &= \frac{T}{2}d\xi
\end{aligned} \tag{A.20a–d}$$

Making use of equations (A.20a-d) with equations (A.7)-(A.11), the following integrals are obtained:

$$\int_0^T \psi_i(t)dt = \frac{T}{2}\alpha_{0i} \tag{A.21}$$

$$\int_0^T \dot{\psi}_i(t)dt = \beta_{0i} \tag{A.22}$$

$$\int_0^T \psi_i(t)\psi_j(t)\,dt = \frac{T}{2}\gamma_{ij} \tag{A.23}$$

$$\int_0^T \dot{\psi}_i(t)\psi_j(t)\,dt = \delta_{ij} \tag{A.24}$$

$$\int_0^T \dot{\psi}_i(t)\dot{\psi}_j(t)\,dt = \frac{2}{T}\varepsilon_{ij} \tag{A.25}$$

where i and j are non-negative integers.

A.4 Closed-form Integrals for Chebyshev-based Approach. The basis function of the Chebyshev-based approach is repeated here as

$$c_k(t) = \psi_k(t) + (-1)^{k-1} \quad , \quad k=1,2,\ldots,K \tag{A.26}$$

The closed-form integrals of the basis function can be derived from equations (A.21)-(A.26) as

$$\int_0^T c_i\,dt = \frac{T}{2}\alpha_{0i} + (-1)^{i-1}T \tag{A.27}$$

$$\int_0^T d_i\,dt = \beta_{0i} \tag{A.28}$$

$$\int_0^T (c_i c_j)\,dt = \frac{T}{2}\left[\gamma_{ij} + (-1)^{j-1}\alpha_{i0} + (-1)^{i-1}\alpha_{0j} + 2(-1)^{i+j-2}T\right] \tag{A.29}$$

$$\int_0^T (d_i c_j)\,dt = \delta_{ij} + (-1)^{j-1}\beta_{0i} \tag{A.30}$$

$$\int_0^T (d_i d_j)\,dt = \frac{T}{2}\varepsilon_{ij} \tag{A.31}$$

where α, β, γ, δ, and ε are defined in equations (A.12)-(A.16). Equations (A.27)-(A.31) show the closed-form relations of the integral parts required for the Chebyshev-based approach in solving time-invariant optimal control problems.

Table 1. Simulation Results for Example 1

N	Riccati *		Transition Matrix		Fourier-based		Chebyshev-based	
	Perf. index	Time (s)	Perf. index	Time (s)	Perf. index	Time (s)	Perf. index	Time (s)
5	15.180	12.22	15.180	1.83	15.180	2.05	15.180	1.63
8	15.056	64.18	15.056	6.60	15.056	5.23	15.056	3.77
11	15.027	212.97	15.027	16.78	15.031	10.77	15.030	7.40
14	15.016	520.32	15.440	33.42	15.030	19.20	15.029	12.73
17	15.011	1100.50	unstable	–	15.042	30.62	15.042	20.28
20	15.008	4797.15	unstable	–	15.061	46.73	15.061	30.68

* For N=5 to N=17, the Riccati equation is integrated backward using a fourth-order Runge-Kutta routine with a time step of 0.01 second. For N=20, the time step is reduced to 0.005 second to ensure a numerically stable solution.

Table 2. Simulation Results for Example 2

METHOD	J=3		J=5		J=7	
	Perf. index	Time (s)	Perf. index	Time (s)	Perf. index	Time (s)
Riccati*	7.6205	22.9	7.6204	145	7.6204	511
Transition Matrix	7.6205	3.75	7.6204	13.2	7.6204	34.7
Chebyshev-based	7.6055	2.33	7.6049	7.70	7.6049	16.6

* The Riccati equation is integrated backward using a fourth-order Runge-Kutta routine with a time step of 0.1 second.

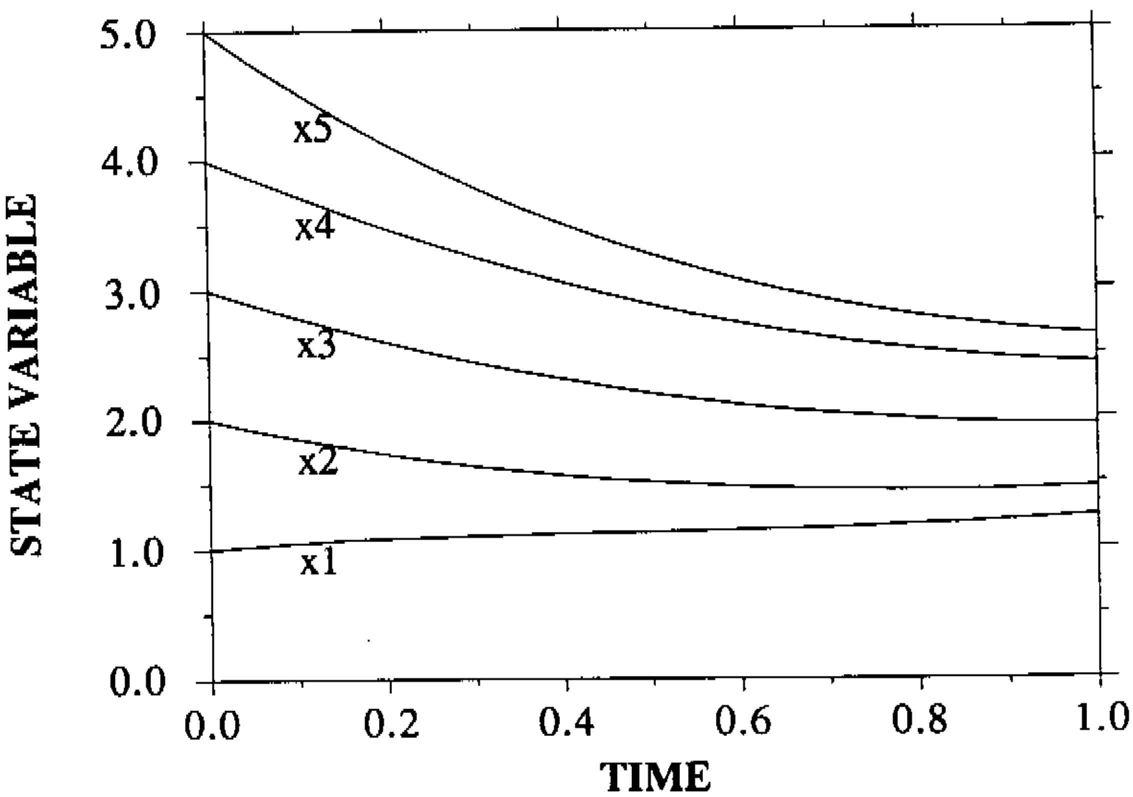

Figure 1. State Variable History of Example 1

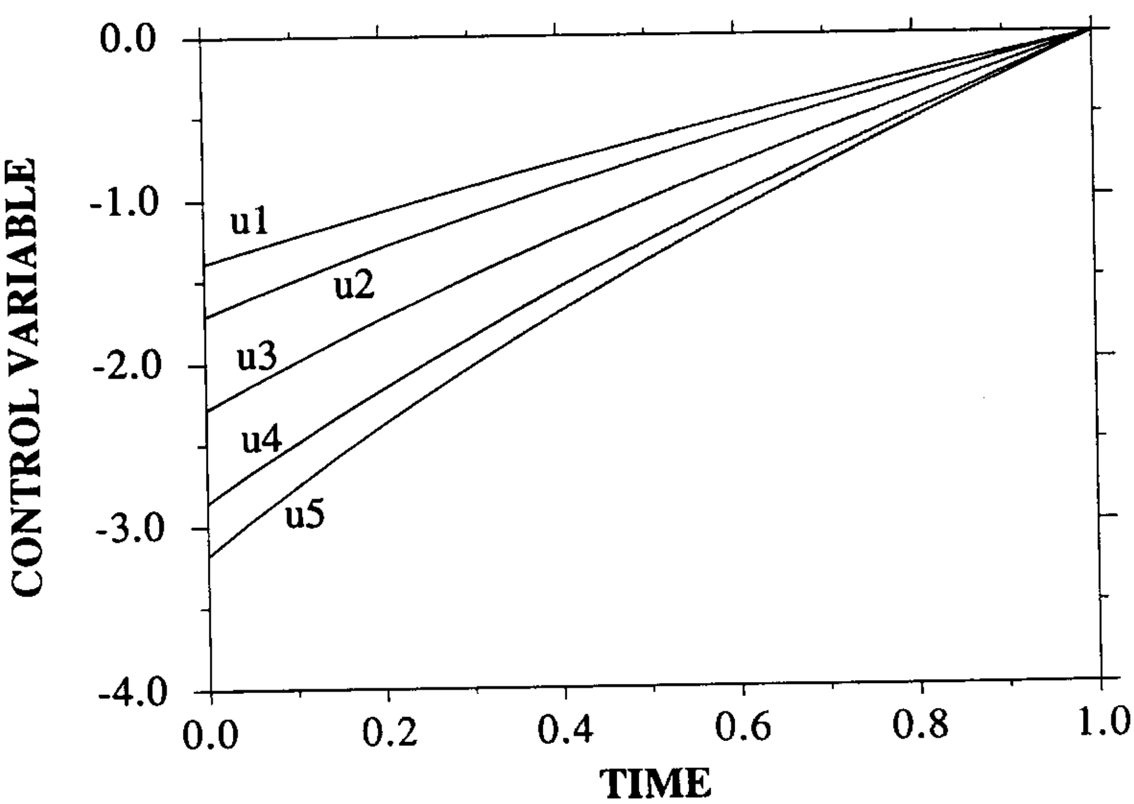

Figure 2. Control Variable History of Example 1

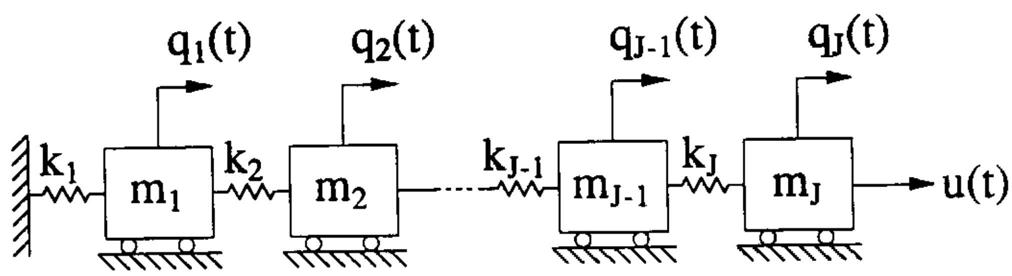

Figure 3. 2J Order System of Example 2

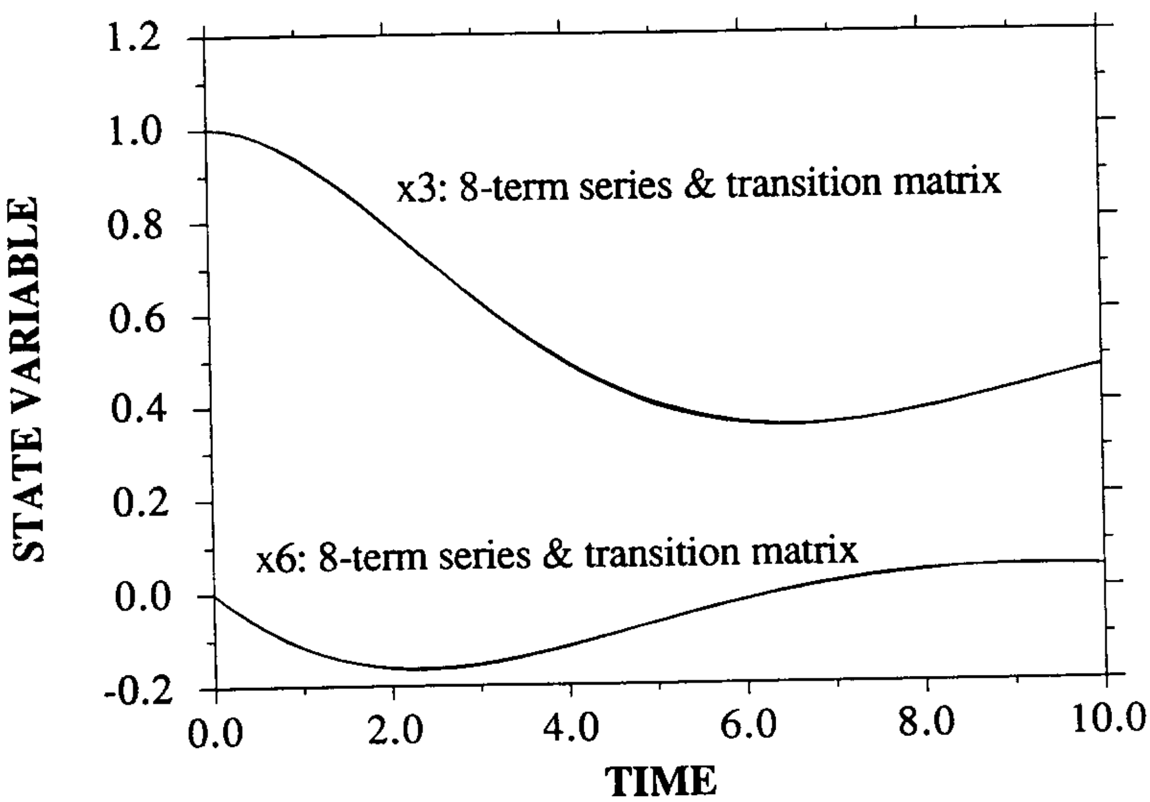

Figure 4. State Variable History of Example 2

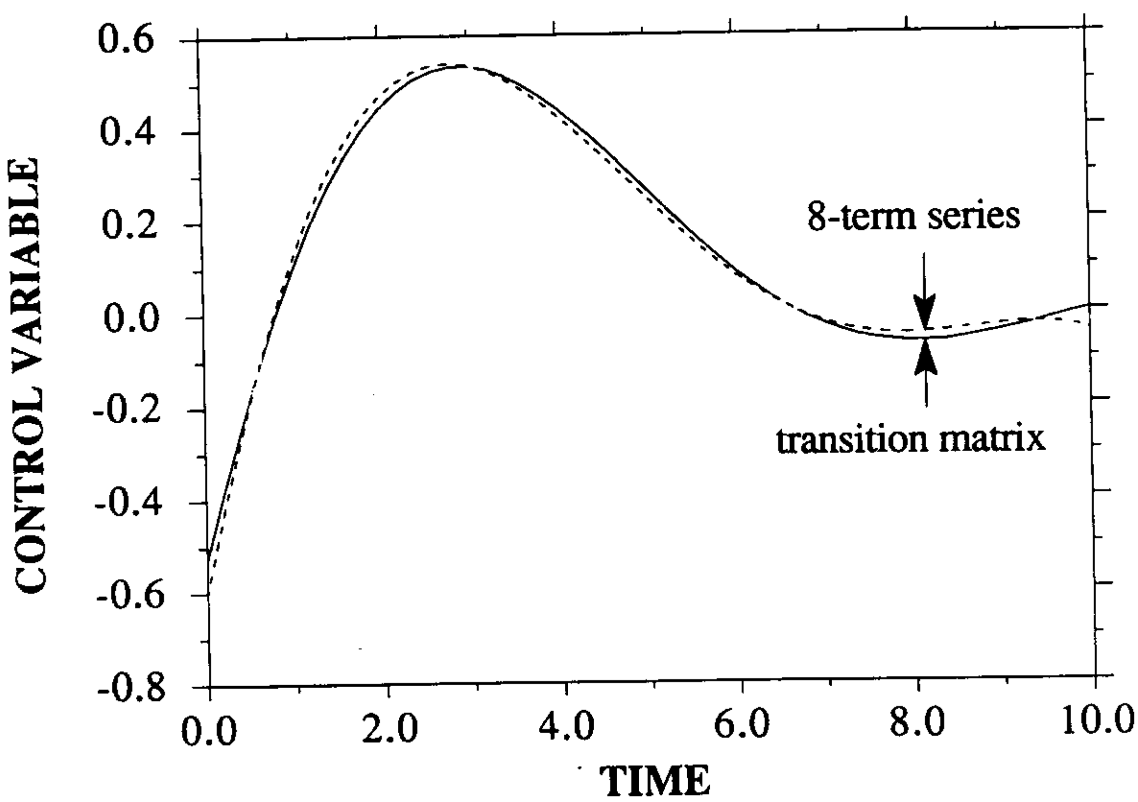

Figure 5. Control Variable History of Example 2

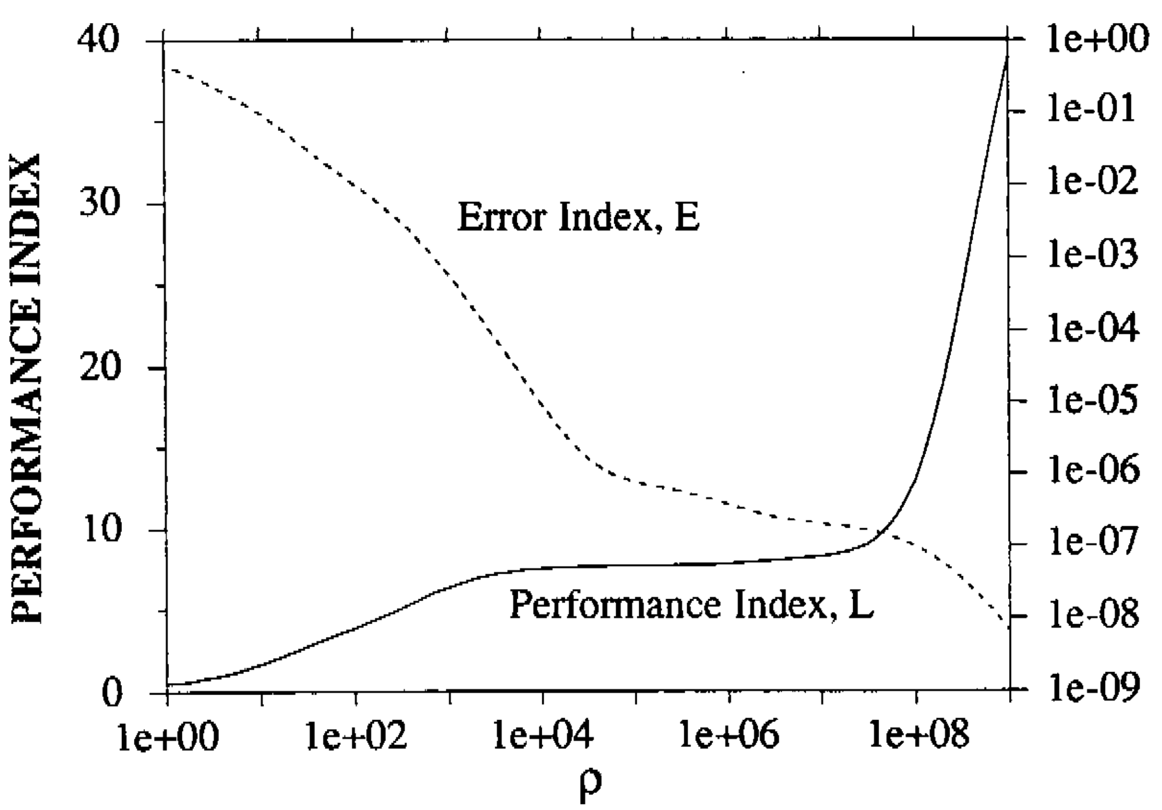

Figure 6. Performance and Error Indices vs. Weighting for Example 2