BOX SKELETONS OF DISCRETE OBJECTS

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Abstract

In a previous report, it was suggested that the Medial Axis Transform (MAT) would be an extremely useful design tool, if only algorithms could be defined for transforming a conventional geometric representation to one using MAT's. Certain specific properties of the MAT were shown to be the useful ones. It was also pointed out that the MAT can be regarded as a special case of a class of objects called *skeletons*, based upon the euclidean metric. We then proposed that a different set, defined using the *box metric*, can be used in place of the MAT for at least some engineering applications.

In that report, a procedure was defined which, we claimed, identified the box skeleton. The purpose of this report is to prove that claim.

This report describes the procedure, and then proves that the procedure yields the box skeleton.

[1] An NSF Engineering Research Center at Carnegie Mellon University

1 Introduction and Terminology

An object to be represented in any geometric model is assumed to be sufficiently *well-behaved*. Our purpose here is to exclude arbitrarily convoluted boundaries, such as of the function $\sin(\frac{1}{x})$, since an object bounded by such a function has non-intuitive topological properties. It is probably sufficient to say that an object in our modeling (flat) space is a *regular, semi-analytic* set (after [4]). A *discrete* object is defined to be (the closure of) the union of a finite number of pixels/voxels. We make no assumptions about any "continuous" object underlying the discrete one: all our work will concentrate on the derivation of a skeletal set abstracting the shape of the given discrete object.

The following terms are taken to be well-understood; the reader is referred to standard texts on geometry, topology and homotopy theory, as also our previous reports, [2] [3] [1]:
flat space, open set, closed set, closure, interior, neighborhood, norm, L_1 (diamond) norm, L_2 (euclidean) norm, L_∞ (box) norm, flat span (affine hull), pixel, voxel.

R is the set of real numbers

$P = \{x \in R \mid x \geq 0\}, P^\times = P \setminus \{0\}$.

$\mathcal{E}$ is the modeling (flat) space, one of R^2 or R^3. We shall sometimes refer to R^n, but it should be understood in such cases that $n \in \{2,3\}$.

If $f : A \rightarrow B$ denotes a mapping, and if $A_1 \subset A$ and $B_1 \subset B$, then $f_>(A_1)$ denotes the *image* of A_1 under f, and $f^<(B_1)$ denotes the *pre-image* or *inverse image* of B_1 under f.

The symbol **Clo** is used to denote the closure, **Int** to denote the interior, and **Bdy** to denote the boundary of a set.

An *object* is the closure of some open set in $\mathcal{E}$, with a restriction on the convolution of the boundary as mentioned above. We usually denote objects by O, perhaps with a subscript. Thus, $O = \mathbf{Clo}(\mathbf{Int}(O))$. The term *discrete object* is used to refer to objects which are (the closure of) the union of some finite number of pixels or voxels in space.

Given a norm $d : \mathcal{E} \times \mathcal{E} \rightarrow P^\times$, a *norming cell* based on that norm is the set $\{x \mid d(0,x) \leq 1\}$.

A *cell* is defined as the set $C(d,c,r) = \{x \mid d(x,c) \leq r\}$, where d is a given norm, c is the center of the cell, and $r \in P^\times$, generically called the *radius*, is specified somehow.

A cell *contained* in some object O is a cell which is a subset of O. A *maximal cell* contained in O is any cell, contained in O, which is not a subset of any other cell also contained in O.

Since, for the most part, we shall use the box norm, we define $\mathbf{Box}(x,r)$ to refer to a cell, based on the box norm, with center x and radius r.

Definition 1: Given an object and a norm, the closure of the set of centers of maximal cells contained in the object is defined to be the *skeleton* of the object, denoted by S. Associated with this skeleton is a function $\mathbf{r} : S \rightarrow P$, called the *radius function*, which gives the radius of the maximal cell at each point in the skeleton.

We shall concentrate on the *euclidean* skeleton (the MAT), defined using the euclidean norm, and the *box* skeleton, defined using the box norm.

2 Graphs, Dual Graphs

In order to present a procedure for obtaining the box skeleton of a discrete object, the following terms and notation are introduced. [2]

A ***unit box*** is a pixel if £ is 2D, and a voxel if it is 3D.

A ***space graph*** G^* is a partition of the space, £. The subsets under this partition, called the elements of G^*, are all connected components, and furthermore are restricted to being open boxes in their flat spans in £. These elements are required to be mutually exclusive and collectively exhaustive.

The dimension of an element is understood to be the dimension of its ***flat span.*** Then, the OD subsets are called ***vertices,*** the ID subsets are called ***edges,*** 2D ones are ***faces,*** and 3D ones are ***solids.*** Thus, ***£*** = \\$J_{A\epsilon G}$* A^*

Edges of a space graph are parallel to one of the principal axes, and faces are parallel to one of the principal planes. Two elements of a space graph are said to be adjacent if their closures have points in common. Two distinct, adjacent edges have exactly one vertex in common. Two distinct, adjacent faces have exacdy one edge and two vertices, or exactly one vertex in common, and so forth. Thus, a space graph in 3D space consists of points, line segments of unit length, unit squares, and unit cubes.

V(A) is the set of all subsets of *A.*

VG(£) is the set of all space graphs of £.

The ***dual mapping*** V^* : $G^* \longrightarrow G^{**}$, where G^{**} *e VG(£),* is an adjacency- preserving mapping. The image of G^* under this mapping, 2>*>(G^*), is also a graph. For the case where £ is 2D, the mapping, and hence G^{**}, is specified as follows (see figure 1):

1. The image of every face is a dual vertex at the centroid of the face.

2. The image of every edge is a dual edge crossing the given edge. The end-points of the dual edge are the dual images of the two faces adjacent to the given edge.

3. The image of every vertex is a dual face in which the vertex lies. The corners of this dual face are the dual images of faces adjacent the given vertex.

This can be extended to 3D as below:

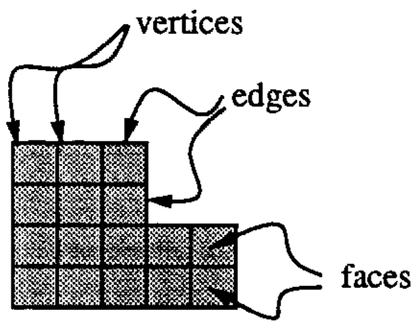
1. The image of every solid is a dual vertex at the centroid the of solid.

2. The image of every face is a dual edge through the face. The ends of this dual are the images of the two solids on either side of the face.

3. The image of every edge is a dual face through which it passes. The corners of this dual face are the images of the solids adjacent to the edge.

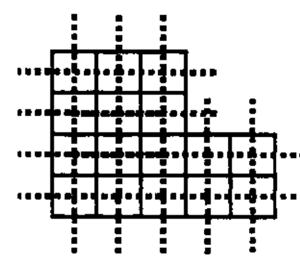
4. The image of every vertex is a dual solid, bounded by the images of the edges adjacent to the vertex.

[2] This terminology is largely consistent with [1], except that certain problems with the presentation there have been addressed.



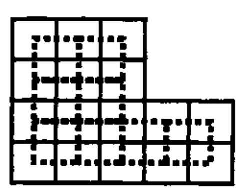
A 2D discrete object as a graph

(a)



Part of graph (bold) and part of its dual (dotted)

(b)



The graph (bold) and the restricted dual (dotted)

(c)

Figure 1: Graphs and duals.

That $\mathcal{D}^*$ is adjacency preserving can be seen by constructing an *adjacency graph*, as follows. The elements of the adjacency graph are termed *nodes* and *links*, to avoid confusion. Each element of G^* has a corresponding (unique) node in the adjacency graph. The links of the adjacency graph join two nodes whose corresponding elements in G^* are adjacent. Then it is easy to see that the adjacency graph of any G^* and that of $\mathcal{D}^*{}_{>}(G^*)$ are isomorphic. It is thus apparent that the dual graph of any G^* is an abstract representation of the adjacencies of elements in G^*.

Given any object O, the *restriction* of any $G^* \in \mathcal{P}_G(\mathcal{E})$ to the object O is defined to be the subgraph of G^* whose vertices are in O, and denote it by $G^* \mid_O$. We use the notation $\mathcal{P}_G \mid_O$ to denote the set of all such restrictions.

If a restriction to O, $G = G^* \mid_O$, is such that $O = \bigcup_{A \in G} A$, then G^* is said to be *compatible* with O, and the restriction G is called the *object graph* of O.

We use the notation $\bigcup G$ to represent the union of all elements of G. Thus, if O is an object, G^* is a space graph compatible with O, and $G = G^* \mid_O$, then $O \triangleq \bigcup G$.

Let O be an object, let $G^* \in \mathcal{P}_G(\mathcal{E})$, and $G = G^* \mid_O$. Then it is of interest to examine the restriction of the dual to O, $\mathcal{D}^*(G^*) \mid_O$. Since this will be of great use later, a new mapping is defined from $\mathcal{P}_G \mid_O$ to itself,

$$\mathcal{D} : \mathcal{P}_G \mid_O \rightarrow \mathcal{P}_G \mid_O$$

which associates each element $G \in \mathcal{P}_G \mid_O$ with an image, also in $\mathcal{P}_G \mid_O$, such that this image is the restriction of $\mathcal{D}^*(G^*)$ to O.

It is not difficult to see that given any space graph G^*, the dual $\mathcal{D}^*(G^*)$ is also a graph. Furthermore, the second dual of a graph is the graph itself. That is, $\mathcal{D}^*(\mathcal{D}^*(G^*)) = G^*$.

Given a discrete object, it is immediately obvious that the set of pixels/voxels and their boundary elements gives the discrete object the structure of an object graph. We can then speak of the space graph of which this object graph is a restriction.

3 Dual-based Thinning Procedure $\mathcal{T}$

Based on the definition of the dual mapping above, we define a thinning procedure, called procedure $\mathcal{T}$, as follows: the procedure starts with some object O, and an associated object graph G. Let G^* denote the space graph of which G is a restriction. We define $O_0 = O$, $G_0 = G$, $S_0 = \phi$, and i = 0. Then the following steps are applied:

1. If $O_i \neq \phi$, identify $\mathcal{D}(G_i)$.

2. Let $O_{i+1} = \mathbf{Clo}(\mathbf{Int}(\bigcup \mathcal{D}(G_i)))$,

3. $S_{i+i} = \bigcup \mathcal{D}(G_i) \setminus O_{i+1}$.

4. If i is odd, $G_{i+1} = G^* \mid_{O_{i+1}}$. If i is even, $G_{i+1} = \mathcal{D}^*(G^*) \mid_{O_{i+1}}$.

5. If $O_{i+1} \neq \phi$, increment i by 1, and repeat from step 1.

Proposition 1: *Procedure $\mathcal{T}$ terminates for every discrete object O.*

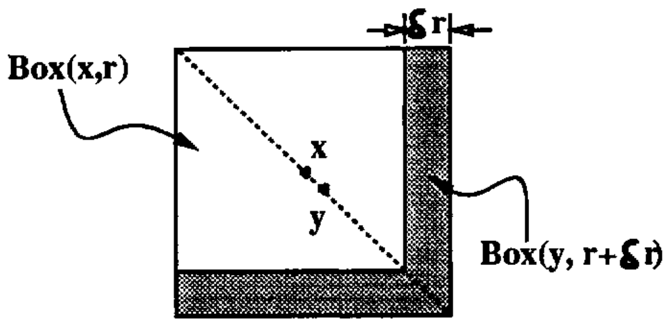


Figure 2: Growing boxes.

The term *facet* is used to refer to the $n - 1$ dimensional flat components of the boundary of a box. Three-dimensional boxes have six facets (which are the faces of the boundary), and 2D boxes have four facets (which are the edges of its boundary).

A facet f is said to be *constrained* in O if $f \cap \mathbf{Bdy}(O) \neq \phi$.

In three dimensions, a *cycle* of facets at a vertex is defined to be the set which includes a vertex, the edges adjacent to the vertex, and the faces adjacent to the vertex. In two dimensions, this reduces to a vertex and the two edges adjacent to it.

A cycle C of facets of some box in O is said to be constrained if for some element $A \in C$, $A \cap \mathbf{Bdy}(O) \neq \phi$.

Given some box, it is possible to construct another box which contains the given box as follows: keeping one vertex (say v) fixed, expand the given box such that the vertex at the other end of the body diagonal at v moves along this diagonal by some distance, say δr[3]. If the given box was $\mathbf{Box}(x, r)$, then the new box is $\mathbf{Box}(y, r + \delta r)$, where y is at distance $\frac{r+\delta r}{2}$ from x along the body diagonal at x (see figure 2).

The important thing to note is that every box is a *scaled* version of every other box. Hence it is clear that if two boxes of unequal radius are aligned at a vertex (as in the figure), then the larger one can be obtained by "pulling" the diagonally opposite vertex along the diagonal, while preserving the shape of the box.

This suggests an interesting property of *maximal boxes* in discrete objects, and motivates the following proposition:

Proposition 2: *A box* $\mathbf{Box}(x, r) \subset O$ *is maximal in* O if and only if *the cycle of facets at every vertex of the box is constrained.*

Corollary 2.1: *A box* $\mathbf{Box}(x, r) \subset O$ *is maximal in* O *if at least one pair of non-adjacent facets is constrained.*

Corollary 2.2: *A box* $\mathbf{Box}(x, r) \subset O$ *is maximal in* O *if all the vertices of the box are on the boundary of* O.

Proposition 3: *Let* O_i, $i \in N$ *be an object at some stage* i *of procedure* $\mathcal{T}$. *Then* $S_{i+1} \subset S^B(O_i)$.

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[3]note that all distances are measured using the box norm.

Proof: By definition, no point of $Si+\$ is adjacent to a solid (face in 2D) in ***OM.*** We use this to show that for each x 6 S;+i, Box(x, \) is maximal in <9,, and hence that C $S'(Qi)$.

For any x **G** S,+i, exactly one of the following is true:

1. x is on a vertex of $V(Gi)$.
2. x is on an edge of 2>(G,).
3. (3D only) x is on a face of $V(Gi)$.

We consider each in turn.

Case **1:** If x is on a vertex of X>(G;), say v, then

(2T)-(v) = Int(Box(;t, 1))

i.e., v is the dual of a solid (face in 2D), centered at v, and of radius ~. Now for any vertex v* of Box(v, £), 2>*(v*) is a solid adjacent to v. Since v 6 ***SM,*** "© conclude that Z>*(v*) £ X>(G,), and hence every vertex of **BOX(JC**, 1) is in Bdy(0/). Then, by Corollary 2.2 above, Box(x, 1) is maximal in G,.

Case **2:** Let x be on an edge of $T>(Gi)$, say $e*$. We break this into two situations, 2D and 3D.

2 D : $e*$ is the dual of some edge in G,, say e. The end points of e are vertices of G,. Call these vertices Vi and V2. Clearly, the images of these vertices under $V*$ cannot be in X>(G,), since otherwise $e*$ would be adjacent to some face of $V(Gi)$.

Hence we conclude that Vi, V2 6 Bdy(0,). It is evident that Vi, V2 € Bdy(Box(jc, £)). Furthermore, since the definition of an edge excludes the endpoints, it is also evident that vi, V2 are on *opposite* facets of **BOX(JC**, |). By Corollary 2.1 we conclude that **BOX(JC**, £) is maximal.

3 D : $e*$ is the dual of some face of G,-, say /. Once again, we can see that the corners of/ must be in Bdy(Oi), and hence we conclude that every cycle of facets of BOX(JC, \) is constrained, and the box is therefore maximal.

Case **3:** (3D only) Let x be on a face of say / *. / * is the dual of some edge of G,-, say e. Once again, it is clear that the ends of e must be on Bdy(G,), and by Corollary 2.1 we conclude that BOX(JC, \) is maximal in 0;.

Hence we conclude that

S_{i} C $S'(OdM$

Proposition 4: *Let i e N | 1 be some stage in procedure T. Then*

1. 0,-COM.
2. *The minimum distance from any point on the boundary of O, to the boundary of O^i is \.*

Proof: Let G* and G** be the space graphs compatible with and G,, respectively. Then it is clear that all the elements of G** with all adjacent vertices in 0,-1 are themselves also in Since O, is (by definition) the union of all such elements, it follows that Oi **C** $O,-\$.

Now consider any point x € Bdy(<9;). Then exactly one of the following is true:

1. x is on a vertex of G_i.

2. x is on an edge of G_i.

3. x is on a face of G_i. Note that this is possible only in 3D.

We consider each in turn.

Case 1: x is on a vertex (say v) of G_i. Consider the solid (face in 2D) of G_{i-1} of which this vertex is a dual. Since v is on $\mathbf{Bdy}(O_i)$, it is clear that *atleast* one of the vertices of $(D^*)^-(v)$ must be on the boundary of O_{i-1}. Since this is a discrete object, we conclude that minimum distance from v to the boundary of O_{i-1} is $\frac{1}{2}$.

Case 2: x is on an edge of G_i.

We first present our arguments for the 2D case. Let e be the edge of G_i on which x lies. Then e is the dual of some edge in G_{i-1}, say e'. Then, since $e \in \mathbf{Bdy}(O_i)$, it is clear that exactly one of the endpoints of e' is in $\mathbf{Bdy}(O_{i-1})$. Under the box norm, the distance from any point on e to each of the endpoints of e' is precisely $\frac{1}{2}$. Since all the O's are discrete, it is not hard to see that this is also the nearest point on the boundary of O_{i-1}.

In 3D, the edge (say e) is the dual of some face of G_{i-1}. Once again, it is clear that atleast one of the vertices of the face must be in $\mathbf{Bdy}(O_{i-1})$. Also, these vertices are the nearest points on $\mathbf{Bdy}(O_{i-1})$ from any point on e, with distance $\frac{1}{2}$.

Case 3: (3D only) x is on a face of G_i. This face (say f) is the dual of some edge in G_{i-1}. Once again, precisely one of the endpoints of this edge must be on the boundary of O_{i-1}, and (since O_{i-1} is discrete) this is also the nearest point of $\mathbf{Bdy}(O_{i-1})$ from any point of f, and the distance is $\frac{1}{2}$. ■

Proposition 5: *Let $i \in N+1$ be some stage in procedure $\mathcal{T}$. Then*

$$S^B(O_i) \subset S^B(O_{i-1}).$$

Proof: We shall show that

$$\forall x \notin S^B(O_{i-1}),\ x \notin S^B(O_i),$$

and hence conclude that $S^B(O_i) \subset S^B(O_{i-1})$.

To show this, let x be some point in *the interior* of O_i such that

$$x \notin S^B(O_{i-1}).$$

Then, by definition, there is no box centered at x which is maximal in O_{i-1}. Let $\mathbf{Box}(x,r)$be the largest box, centered at x, which can be fit into O_{i-1}.

Since $x \in \mathbf{Int}(O_i)$, it is clear (since the object O is discrete) that r must be strictly greater than $\frac{1}{2}$.

Since $\mathbf{Box}(x,r)$is *not* maximal, then by Proposition 2 there must be one vertex of $\mathbf{Box}(x,r)$such that the cycle of facets at that vertex is unconstrained. Let v be that vertex, and let C be the corresponding cycle of facets.

Since C is an unconstrained cycle at v, it is clear that each point in each member of C has a strictly positive distance to the boundary of O_{i-1}.

Now consider $\mathbf{Box}(x, r - \frac{1}{2})$. This is possible, since r is strictly greater than $\frac{1}{2}$. Since this box has the same center as $\mathbf{Box}(x, r)$, there is a natural homeomorphism between the two boxes, and for any point of one box we can unambiguously refer to the "corresponding" point on the other box. Then it is clear that the boundary of O_i, which consists of points in O_{i-1} with distance $\frac{1}{2}$to the boundary of O_{i-1}, has strictly positive distance to each point in each member of the cycle of $\mathbf{Box}(x, r - \frac{1}{2})$ corresponding to C. This means that the vertex of $\mathbf{Box}(x, r - \frac{1}{2})$ corresponding to v has an unconstrained cycle, and hence $\mathbf{Box}(x, r - \frac{1}{2})$ is not maximal in O_i.

Furthermore, it is clear that $S^B(O_i) \cap \mathbf{Bdy}(O_i) = \phi$. Hence we conclude that $S^B(O_i) \subset S^B(O_{i-1})$. ■

Proposition 6: *Let $O_i, i \in N$, be a non-empty object at some stage i of procedure $\mathcal{T}$. If $\mathbf{Box}(x, \frac{1}{2}) \subset O_i$ is maximal in O_i, then $x \in S_{i+1}$.*

Proof: Since the diameter of the box is *unity*, it immediately follows that the solids (faces in 2D) with which $\mathbf{Box}(x, \frac{1}{2})$ has a non-empty intersection can be in only a few configurations. We explore these configurations, and show that in each case $x \in S_{i+1}$.

2D: The only possible configurations of pixels which overlap with a *maximal* box of radius unity are shown in figure 3.

Consider the first case (figure 3(a)), in which the box precisely overlaps the pixel. Since x is now the center of the pixel, $x \in \bigcup \mathcal{D}(G_i)$. To show that x is in S_i, we need to show that x is not adjacent to any face of O_{i+1}. We do this by contradiction. Assume, then, that there is a face of O_{i+1} adjacent to x. There are four possible configurations; without loss of generality, we can choose any one of these as representative. Figure 3(c) shows the point x, the maximal box (shown shaded), the dual face which we assume to be adjacent to x (shown with vertical hatching), and the faces which must be present in G_i (shown with horizontal hatching) for the dual face assumed to exist. It is then clear that the southeast vertex of $\mathbf{Box}(x, \frac{1}{2})$ does not have a constrained cycle of facets, and by proposition 2 we conclude that $\mathbf{Box}(x, \frac{1}{2})$ is not maximal.

Now consider the second case, shown in figure 3(b). It is immediately apparent that x is on the dual edge joining the centers of the two pixels, and is thus in some element of $\mathcal{D}(G_i)$. Once again, we assume that this edge of $\mathcal{D}(G_i)$ is, in fact, adjacent to some face of O_{i+1}. Without loss of generality, we consider the case when this face of O_{i+1} is *below* the edge in figure 3(d), shown with vertical hatching. Then, for this dual face to exist, several other faces of G_i *must* exist, shown with horizontal hatching in the figure. Once again, it is clear that the southeast vertex of $\mathbf{Box}(x, \frac{1}{2})$ is seen to have an unconstrained facet cycle, and hence cannot be maximal.

In both the 2D cases, we thus conclude that $x \in \bigcup \mathcal{D}(G_i)$, and also that x is not adjacent to any face of O_i. Then by definition $x \in S_{i+1}$.

3D: In 3D, there are three possible configurations, shown in figure 4 (a), (b) and (c). In each case, it is not difficult to see that the point $x \in \mathcal{D}(G_i)$. The question is, is this point adjacent to some solid of O_{i+1}?

In each case, the assumption that the center of the $\mathbf{Box}(x, \frac{1}{2})$ (the box is shown shaded in figure 4(a), (b), (c)) is adjacent to some solid of $\mathcal{D}(G_i)$ immediately leads to the conclusion that some corner of the box of must have at least one vertex with an unconstrained cycle, which violates the assumption of maximality of $\mathbf{Box}(x, \frac{1}{2})$.

This concludes the proof.■

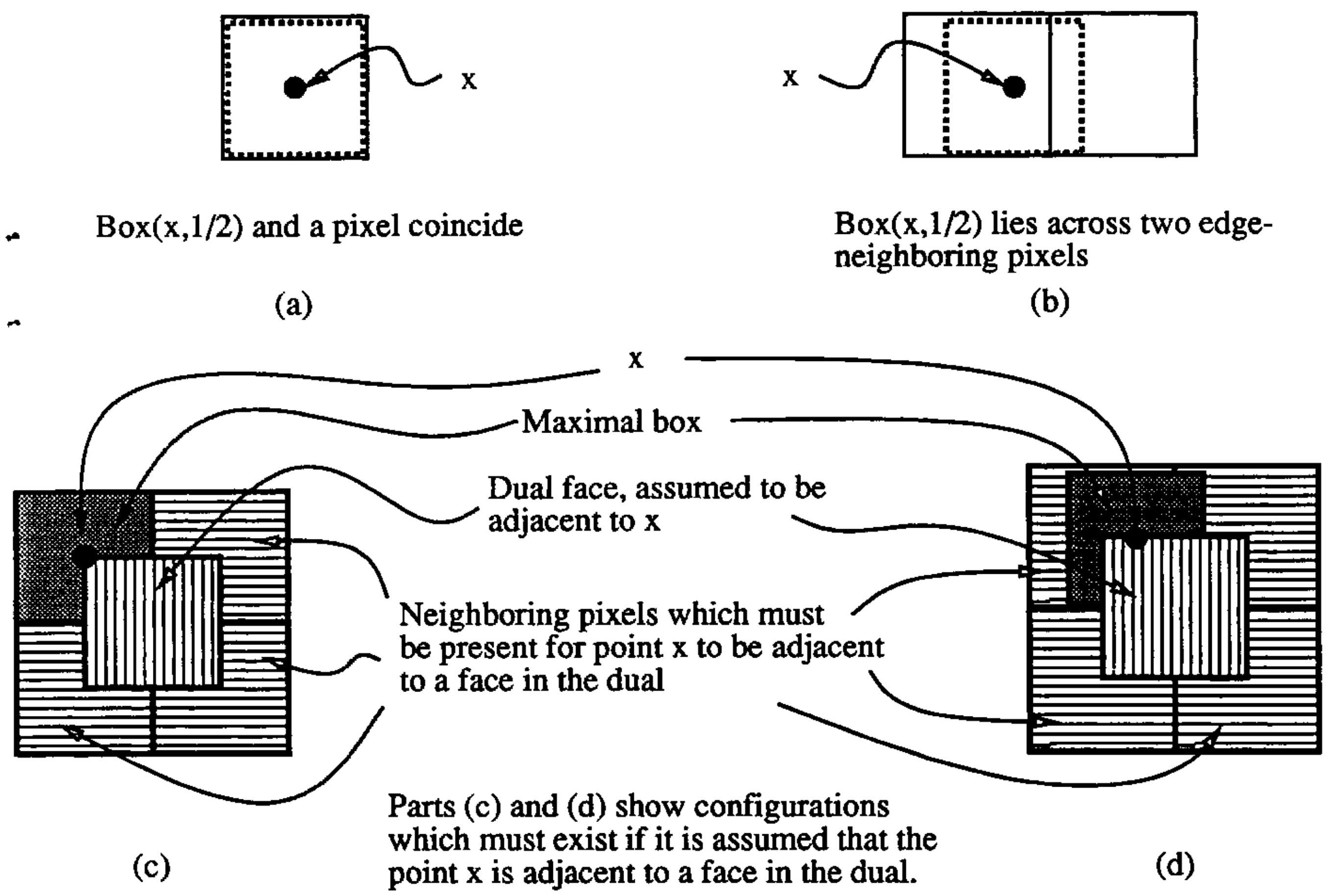

Figure 3: Figure for 2D case of Proposition 6.

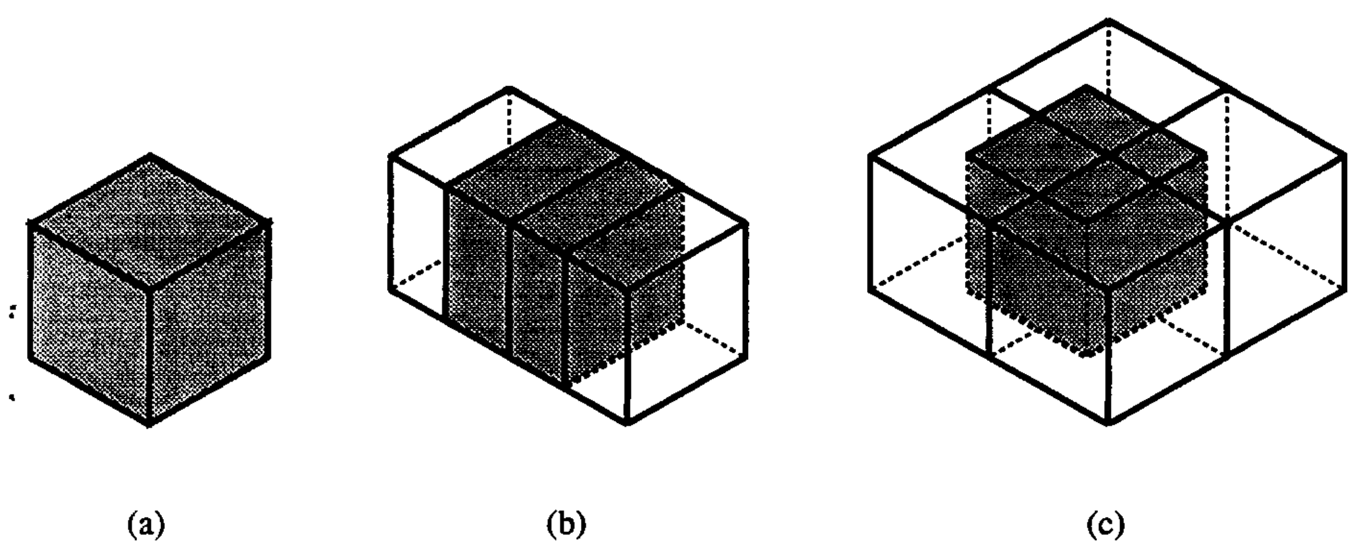

Figure 4: Figure for 3D case of Proposition 6.

Proposition 7: *Let $O_i and O_{i+1}$, $i \in N$, be non-empty objects at stages i and $i+1$ of procedure $\mathcal{T}$. Then the following are equivalent statements:*

1. $\mathbf{Box}(x, r+\frac{1}{2}) \subset O_i$ *is maximal in* O_i.
2. $\mathbf{Box}(x, r) \subset O_{i+1}$ *is maximal in* O_{i+1}.

Proof: Consider the two boxes, $\mathbf{Box}(x, r+\frac{1}{2})$ and $\mathbf{Box}(x, r)$. It is clear that there is a natural homeomorphism between these, associating each point in one with a unique point in the other.

Now assume that one of these boxes is maximal, corresponding to one of the statements of the proposition. Proposition 2 gives necessary and sufficient conditions for the common boundary between the given box and its object. Then from proposition 4 it is not hard to show that for each point in this common boundary, the corresponding point on the boundary of the other box is also on the boundary of its containing object, and hence that the other box is also maximal.■

Definition: $S^{\mathcal{T}} \triangleq \mathrm{Clo}(\bigcup_{i \in N} S_i)$.

From propositions 1-7 above and the principle of mathematical induction, the theorem below immediately follows:

Theorem: *Let O be a discrete object. If procedure $\mathcal{T}$ is applied starting with $O_0 = O$, then*

$$S^{\mathcal{T}}(O) = S^{B}(O).$$

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