# A BOUND ON THE MULTIPLICATION EFFICIENCY OF ITERATION

by

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# ABSTRACT

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$, define the multiplication efficiency measure E to be $\frac{\log_2 p}{M}$, where p is the order of convergence, and M is the number of multiplications or divisions needed to compute $\varphi$. Then, if $\varphi$ is any multivariate rational function, $E \le 1$. Since $E = 1$ for the sequence $\{x_i\}$ generated by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-\frac{1}{2}$, the bound on E is sharp.

Let $P_M$ denote the maximal order for a sequence generated by an iteration with M multiplications. Then $P_M \le 2^M$ for all positive integer M. Moreover this bound is sharp.

# I. INTRODUCTION

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$, define the multiplication efficiency measure E to be $\frac{\log_2 p}{M}$, where p is the order of convergence, and M is the number of multiplications or divisions needed to compute $\varphi$. In [1] Paterson showed that if

(i) $\varphi$ is a rational function,

(ii) $d = 1$,

(iii) $\lim_{i\to\infty} x_i$ is an algebraic number, and

(iv) $\varphi$ has rational coefficients,

then $E \le 1$. In this note we show $E \le 1$ removing all these restrictions except (i). Since condition (i) is not a restriction for a computer algorithm, this is a very general result. In particular, we shall show that $E = 1$ for the sequence $\{x_i\}$ defined by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-\frac{1}{2}$. Hence our bound on E is sharp.

Let $P_M$ denote the maximal order for a sequence generated by an iteration with M multiplications. Since $E \le 1$, it follows that $P_M \le 2^M$ for all positive integer M. Moreover, we shall show that this bound is sharp.

Paterson used results from approximation by rational numbers to obtain his result, while we use a completely different approach here. With the technique we use here, the case $d = 1$ would be very easy to prove. We show that a rational iteration function which generates a $p^{th}$ order convergent sequence must have degree (degree will be defined below) $\ge p$, and therefore must employ at least $\lceil \log_2 p \rceil$ multiplications or divisions (except by constants). Hence, $E = \frac{\log_2 p}{M} \le 1$.

The result belongs to analytic computational complexity which deals with optimality theory of analytic processes [2].

## II. NOTATION

We work over the field of real numbers or the field of complex numbers. Let $\{x_i\}$ be any convergent sequence with limit $\alpha$, and $x_i \neq \alpha$ for all i. Denote $e_i = |x_i - \alpha|$ for all i.

<u>Definition 1</u>: (<u>Order</u>) <u>The sequence $\{x_i\}$ has an order $p > 1$ (or $\{x_i\}$ is a $p^{th}$ order sequence) iff $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^{p-\varepsilon}} = 0$ and $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^{p+\varepsilon}} \neq 0$ for any $\varepsilon > 0$.</u>

From the above definition, it is easy to see that if $\{x_i\}$ has order p, then

(2.1) $p = \sup\{r \mid \lim_{i\to\infty} \frac{e_{i+1}}{e_i^r} = 0\}$, and

(2.2) for any fixed positive integer n, $\{x_{in}\}_{i=0}^{\infty}$ has order $p^n$.

It should be noted that in our proofs the only properties of order needed are (2.1) and (2.2), although (2.1) has been used as a definition of order by many people. Definition 1 is the weakest definition on order we have found which enjoys both properties (2.1) and (2.2).

For each number $\alpha$, we define a class $F(\alpha)$ of convergent sequences with the same limit $\alpha$ as follows: $\{x_i\} \in F(\alpha)$ iff

(i) $x_i \neq \alpha$ for all but finitely many i

(ii) $\{x_i\}$ has an order $p > 1$

(iii) $x_{i+1} = \alpha(x_i, x_{i-1}, \ldots, x_{i-d+1})$ for all i, for some multivariate rational expression $\alpha(y_1, y_2, \ldots, y_d)$ of d variables, say, $\varphi(y_1,\ldots,y_d) = \dfrac{\varphi_1(y_1,y_2,\ldots,y_d)}{\varphi_2(y_1,y_2,\ldots,y_d)}$, where $\varphi_1(y_1,y_2,\ldots,y_d)$ and $\varphi_2(y_1,y_2,\ldots,y_d)$ are two relatively prime multivariate polynomials of d variables $y_1,y_2,\ldots,y_d$. We say that $\{x_i\}$ is generated by the rational iteration $\varphi$. For examples of these $\varphi$'s, see [3].

Consider a sequence in $F(\alpha)$ generated by $\varphi$. For the purpose of this note, we assume the cost in generating the sequence to be the number of multiplications or divisions needed to compute $\varphi$ at each stage. Then it is natural to give the following definition about the measure of efficiency.

Definition 2: (Multiplication Efficiency) The multiplication efficiency E of a sequence in $F(\alpha)$ generated by $\varphi$ is defined to be $\frac{\log_2 p}{M}$ where p is the order of the sequence and M is the number of multiplications or divisions needed to compute $\varphi$, after doing any preconditioning of coefficients (i.e., preconditioning is not counted).

Definition 3: (Optimality) A sequence in $F(\alpha)$ is called optimal if it has the largest multiplication efficiency among all sequences in $F(\alpha)$.

From (2.2) we can check that a very desirable property holds, namely, for any fixed positive integer n, $\{x_i\}$ and $\{x_{in}\}_{i=0}^{\infty}$ have the same multiplication efficiency. In fact, this invariance under composition property implies that any efficiency measure must be a strictly monotonic function of E [4]. Therefore, as far as optimality is concerned, it makes no difference if E or any other possible efficiency measure is used. For instance, the efficiency measure $p^{\frac{1}{M}}$ will give the same answer in optimality problems as E will, since it is a strictly monotonic function of E.

*Definition 4*: (*Degree*) *Let* $\varphi(y_1,y_2,\ldots,y_d) = \dfrac{\varphi_1(y_1,y_2,\ldots,y_d)}{\varphi_2(y_1,y_2,\ldots,y_d)}$ *be a multivariate rational expression, where* $\varphi_1(y_1,y_2,\ldots,y_d)$ *and* $\varphi_2(y_1,y_2,\ldots,y_d)$ *are two relatively prime multivariate polynomials. If* $D(\varphi_i)$ *is the degree of* $\varphi_i(y_1,y_2,\ldots,y_d)$ *for* $i = 1,2$, *then the degree* $D(\varphi)$ *of* $\varphi(y_1,y_2,\ldots,y_d)$ *is defined to be* $\max(D(\varphi_1), D(\varphi_2))$.

## III. PRELIMINARY LEMMA

For each positive integer d, we define an order (>) on the set $I_d = \{(j_1,j_2,\dots,j_d) | j_i$ is a non-negative integer for $i = 1,2,\dots,d\}$ as follows: for $(j_1,j_2,\dots,j_d)$, $(\ell_1,\ell_2,\dots,\ell_d) \in I_d$, $(j_1,j_2,\dots,j_d) > (\ell_1,\ell_2,\dots,\ell_d)$ iff there exists $k \in \{1,2,\dots,d\}$ such that $j_k > \ell_k$ and $j_i = \ell_i$ for $i < k$.

<u>Lemma 1</u>: For any number $\alpha$, let $\{x_i\}$ be any $p^{th}$ order sequence in $F(\alpha)$ generated by $\varphi$, and let $e_i = |x_i - \alpha|$ for all i. Suppose that $\varphi$ has d variables. Then we have the following:

(i) if $(j_1,j_2,\dots,j_d) \in I_d$ with $\sum_{i=1}^{d} j_i < p$,

then $\lim_{i\to\infty} \frac{e_i^{p-\epsilon}}{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}} = 0$, for $\epsilon > 0$ and

sufficiently small, and

(ii) if $(j_1,j_2,\dots,j_d)$, $(\ell_1,\ell_2,\dots,\ell_d) \in I_d$

with $(j_1,j_2,\dots,j_d) > (\ell_1,\ell_2,\dots,\ell_d)$

and $\sum_{i=1}^{d} \ell_i < p$, then

$$\lim_{i\to\infty} \frac{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}}{e_i^{\ell_1} e_{i-1}^{\ell_2} \cdots e_{i-d+1}^{\ell_d}} = 0.$$

<u>Proof</u>:

(i) Choose $\epsilon$ such that $0 < \epsilon < p - \sum_{i=1}^{d} j_i$ and $0 < \epsilon < p - 1$. Then lim e. , lim - ± - p-e-1 'i-1 0, and then

lim 0̲ = lim *i-2 = 0.
i-^oo'i-2

In general, lim e. — - 0 for any positive integer k. Hence,
i-.»'i-k

d
£ 1

0 * lim e?-" } * lim ef lim( /e.xj, ) \ . . M /e. x j ") - 0,
i-KO ^1 'd i-K» 'l 'd i-»«\'i V'i-d+1/
'i • "V d + 1 'i " 'i-d+i . . • '• ,
'1 '2 'd
d e. e. . . . . e.

(ii) Choose € such that 0 < e<p- S Z.. Let Q. - —~ *f
i=i h h V
'i 'i-1***'i-d+1

Suppose that j > and j = for i < k. Then when i is so large that e^ < 1, we have

•Wl 'd
'k " \ 'i-k •••'i-d+1
'i * V k . 1 * V1 \
'i-k '-^i-d+1

'k+1 'd
'i-k 'i-d+1
'i-k+1 * V1 \
'i-k "^i-d+1

'k+2 Jd
e.
p-e i-k \ *d
'i-k 'i-k-1-"'i-d+1

Case 1, $p - \epsilon + j_{k+i} - \ell_{k+i} \geq 1$ for $k+i = k+1,\dots,d$. Repeating the above procedure, we get

$$Q_i \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}} \cdot e_{i-k} \cdot \frac{e_{i-k-1}^{j_{k+2}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-1}^{\ell_{k+2}} \cdots e_{i-d+1}^{\ell_d}}$$

$$= \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}} \cdot \frac{e_{i-k}}{e_{i-k-1}^{p-\epsilon}} \cdot e_{i-k-1}^{(p-\epsilon+j_{k+2}-\ell_{k+2})}$$

$$\cdot \frac{e_{i-k-2}^{j_{k+3}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-2}^{\ell_{k+3}} \cdots e_{i-d+1}^{\ell_d}}$$

$$\leq \dots \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}} \cdot \frac{e_{i-k}}{e_{i-k-1}^{p-\epsilon}} \cdot \dots \cdot \frac{e_{i-d+2}}{e_{i-d+1}^{p-\epsilon}} .$$

Case 2, $p-\epsilon+j_{k+n}-\ell_{k+n} < 1$ and $p-\epsilon+j_{k+i}-\ell_{k+i} \geq 1$ for $k + i = k+1,\dots,k+n-1$ for some $n$ with $k + n - 1 < d$. Since $p-\epsilon-\ell_{k+n} > 0$, $j_{k+n} < p-\epsilon+j_{k+n}-\ell_{k+n} < 1$. Hence we must have $j_{k+n} = 0$. Consequently, $1 > p-\epsilon-\ell_{k+n} > \sum_{i=1}^{d} \ell_i - \ell_{k+n}$. This implies that $\ell_i = 0$ for all $i$ except $i = k+n$. Then

$$Q_i \leq \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}} \cdot \dots \cdot \frac{e_{i-k-n+2}}{e_{i-k-n+1}^{p-\epsilon}} \cdot e_{i-k-n+1}^{p-\epsilon+j_{k+n}-\ell_{k+n}} \cdot e_{i-k-n}^{j_{k+n+1}} \cdot \dots \cdot e_{i-d+1}^{j_d} .$$

Note that $p-\epsilon+j_{k+n} - \ell_{k+n} > 0$. Therefore, in both cases, $\lim_{i\to\infty} Q_i = 0$. ∎

## IV. MAIN RESULT

<u>Theorem 1: For any number $\alpha$, let $\{x_i\}$ be any $p^{th}$ order sequence generated by $\varphi$. Then $D(\varphi) \geq p$.</u>

<u>Proof</u>: Write

$$(4.1) \quad \varphi_1(y_1,y_2,\ldots,y_d) - \alpha\varphi_2(y_1,y_2,\ldots,y_d) = \sum_{(j_1,\ldots,j_d)\in I_d} C(j_1,\ldots,j_d)(y_1-\alpha)^{j_1}\ldots(y_d-\alpha)^{j_d}$$

for constants $C(j_1,\ldots,j_d)$. Suppose that $D(\varphi) < p$. Then $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d) \in I_d$ with $\sum_{i=1}^{d} j_i \geq p$: Moreover, we shall use induction to show that $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d)$ with $\sum_{i=1}^{d} j_i < p$. Note that for $\epsilon > 0$,

$$0 = \lim_{i\to\infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^{p-\epsilon}} = \lim_{i\to\infty} \frac{|\varphi(x_i,x_{i-1},\ldots,x_{i-d+1})-\alpha|}{|x_i - \alpha|^{p-\epsilon}}$$

Then, by (4.1), we have

$$(4.2) \quad \lim_{i\to\infty} \frac{\left|\sum_{j_1+j_2+\ldots+j_d<p} C(j_1,\ldots,j_d)(x_i-\alpha)^{j_1}\ldots(x_{i-d+1}-\alpha)^{j_d}\right|}{e_i^{p-\epsilon}} = 0 .$$

Since $\lim_{i\to\infty} e_k = 0$ for $k=i,\ldots,i-d+1$, from (4.2) it follows that $C(0,\ldots,0) = 0$. Suppose that $C(j_1,\ldots,j_d) = 0$ whenever $(j_1,\ldots,j_d) < (\ell_1,\ldots,\ell_d)$ for some $(\ell_1,\ldots,\ell_d) \in I_d$ with $\sum_{i=1}^{d} \ell_i < p$. (4.2) may be written as

$$\lim_{i\to\infty} \frac{\left| \sum_{(j_1,\ldots,j_d) \geq (\ell_1,\ldots,\ell_d)} C(j_1,\ldots,j_d) \frac{(x_i-\alpha)^{j_1}\cdots(x_{i-d+1}-\alpha)^{j_d}}{e_i^{\ell_1}\cdots e_{i-d+1}^{\ell_d}} \right|}{\dfrac{e_i^{p-\varepsilon}}{e_i^{\ell_1}\cdots e_{i-d+1}^{\ell_d}}} = 0.$$

Using Lemma 1 for sufficiently small $\varepsilon$, we must have $C(\ell_1,\ldots,\ell_d) = 0$. This completes the induction proof.

Hence $C(j_1,\ldots,j_d) = 0$ for all $(j_1,\ldots,j_d) \in I_d$.

From (4.1), $\varphi_1(y_1,\ldots,y_d) - \alpha\varphi_2(y_1,\ldots,y_d) \equiv 0$.

Hence $\varphi(y_1,\ldots,y_d) \equiv \alpha$. This is a contradiction.

Hence, $D(\varphi) \geq p$. ■

Theorem 2: If $\varphi(y_1,\ldots,y_d)$ is a multivariate rational expression and $\bar{M}$ is the number of multiplications or divisions (except by constants) needed to compute $\varphi(y_1,\ldots,y_d)$, then $\bar{M} \geq \log_2 D(\varphi)$.

Proof: Observe that we compute $\varphi(y_1,\ldots,y_d)$ through a sequence of arithmetic operations. Let $R_i(y_1,\ldots,y_d)$ be the result immediately following the $i^{th}$ multiplication or division (except by constants) for $i=1,2,\ldots,\bar{M}$. Let $R_0(y_1,\ldots,y_d)$ be one of $y_1,\ldots,y_d$. Observe that we have either

$$(4.3)\quad R_{n+1}(y_1,\ldots,y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,\ldots,y_d) + A_{n+1}\right)$$

$$\times \left(\sum_{i=1}^{n} N_{i,n+1}R_i(y_1,\ldots,y_d) + B_{n+1}\right), \text{or}$$

$$(4.4)\quad R_{n+1}(y_1,\ldots,y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,\ldots,y_d) + A_{n+1}\right)$$

$$\div \left( \sum_{i=1}^{n} N_{i,n+1} R_i(y_1,\dots,y_d) + B_{n+1} \right)$$

where $M_{i,n+1}$, $N_{i,n+1}$, $A_{n+1}$, $B_{n+1}$ are many numbers, for $n=0,1,\dots,\bar{M}-1$.

We claim that, for $n=1,2,\dots,\bar{M}$, the following is true. For any numbers $k_0,\dots,k_n$, C, we have

$$(4.5) \quad \sum_{i=0}^{n} k_i R_i(y_1,\dots,y_d) + C = \frac{P_n(y_1,\dots,y_d;\ k_0,\dots,k_n,\ C)}{Q_n(y_1,\dots,y_d)}$$

where $P_n(y_1,\dots,y_d;\ k_0,\dots,k_n,C)$ is a multivariate polynomial depending on $k_0,k_1,\dots,k_n$, C and $Q_n(y_1,y_2,\dots,y_d)$ is a multivariate polynomial independent of $k_0,k_1,\dots,k_n$, C; moreover, both polynomials have degrees $\le 2^n$. We prove it by induction. It is clear that (4.5) is true for $n = 1$. Suppose that (4.5) is true for all $n \le N$ for some $N < \bar{M}$. Suppose that (4.3) is true for $n = N$. Then by (4.5) for $n = N$, we have

$$\sum_{i=0}^{N+1} k_i R_i(y_1,\dots,y_d) + C = k_{N+1} R_{N+1}(y_1,\dots,y_d) + \sum_{i=0}^{N} k_i R_i(y_1,\dots,y_d) + C$$

$$= k_{N+1} \left( \sum_{i=0}^{N} M_{i,N+1} R_i(y_1,\dots,y_d) + A_{N+1} \right) \times \left( \sum_{i=1}^{N} N_{i,N+1} R_i(y_1,\dots,y_d) + B_{N+1} \right)$$

$$+ \sum_{i=0}^{N} k_i R_i(y_1,\dots,y_d) + C = \frac{P_{N+1}(y_1,\dots,y_d;\ k_0,\dots,k_N,\ C)}{Q_{N+1}(y_1,\dots,y_d)}$$

where $P_{N+1}(y_1,\dots,y_d;\ k_0,\dots,k_N,\ C) = k_{N+1}\ P_N(y_1,\dots,y_d;\ M_{0,N+1},\dots,M_{N,N+1},\ A_{N+1})$

$\cdot\ P_N(y_1,\dots,y_d;\ N_{0,N+1},\dots,N_{N,N+1},\ B_{N+1}) + P_N(y_1,\dots,y_d;\ k_0,\dots,k_N,C)\ Q_N(y_1,\dots,y_d)$,

and $Q_{N+1}(y_1,\dots,y_d) = Q_N(y_1,\dots,y_d)^2$. Then by the induction hypothesis, we have that $\sum_{i=0}^{N+1} k_i R_i(y_1,\dots,y_d) + C$ has degree $\le 2^{N+1}$.

Similarly, from (4.4) we also have that $\sum_{i=0}^{N+1} k_i R_i(y_1,\ldots,y_d) + C$ has the form $\frac{P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,C)}{Q_{N+1}(y_1,\ldots,y_d)}$ with degree $\le 2^{N+1}$ for some $P_{N+1}(y_1,\ldots,y_d;\ k_0,\ldots,k_N,C)$ and $Q_{N+1}(y_1,\ldots,y_d)$.

Hence, both cases imply that (4.5) is true for $n = N+1$. This completes the induction. Therefore, for any numbers $k_0,\ldots,k_n,C$, the degree of $\sum_{i=0}^{n} k_i R_i + C$ will not reach $D(\varphi)$ until $n \ge \log_2 D(\varphi)$. This implies that $\bar{M} \ge \log_2 D(\varphi)$. This completes the proof. ■

Note that $M \ge \bar{M}$, since preconditioning is only performed on constant coefficients. Thus, by Theorem 1, $M \ge \bar{M} \ge \log_2 D(\varphi) \ge \log_2 p$. Therefore, we have the following

<u>MAIN RESULT</u>: $E = \frac{\log_2 p}{M} \le 1.$

Now consider the sequence generated by $\psi(x) = x^2 + x - \frac{1}{4}$ with the limit $-1/2$. Since $\psi'(-1/2) = 0$ and $\psi''(-1/2) \ne 0$, we can easily show that this sequence has order 2. Obviously $M=1$ for this sequence. Thus $E = \frac{\log_2 2}{1} = 1$. Similarly, $E=1$ for the second order sequence generated by $\Gamma(x) = \frac{1}{x} + x - 1$ with the limit 1. Either example shows that our bound on E is sharp. Moreover, we have the following interesting result.

Let $P_M$ denote the maximal order for a sequence generated by an iteration with M multiplications. From our main result, we have the following

<u>Corollary</u>: <u>$P_M \le 2^M$ for all positive integer M. Moreover this bound is sharp.</u>

<u>Proof</u>: Let $\psi_M$ be the composition of $\psi$ with itself M times where $\psi(x) = x^2 + x - \frac{1}{4}$ as before. Then the sequence generated by $\psi_M$ has order $2^M$ and $\psi_M$ employs M

multiplications. Hence for each M the maximal order is achieved by the sequence generated by $\psi_M$. ■

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13. ABSTRACT

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Let $P_M$ denote the maximal order for a sequence generated by an iteration with M multiplications. Then $P_M \le 2^M$ for all positive integer M. Moreover this bound is sharp.

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