QUASI-LINEAR ELLIPTIC
BOUNDARY VALUE PROBLEMS

Martin H. Schultz

1971

1. INTRODUCTION

Let Ω be a region in R^n and $\partial\Omega$ denote the boundary of Ω. We consider quasi-linear elliptic boundary value problems of the form

$$(1)\quad L[u] \equiv \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u) = f(x,u,\ldots)\ ,\ x \in \Omega,\ a_{\alpha\beta}(x) = a_{\beta\alpha}(x), |\alpha|, |\beta| \le m,$$

$$(2)\quad L[u] \equiv \lambda f(x,u,\ldots)\ ,\ x \in \Omega,$$

subject to the boundary conditions

(3) $D^j u(x) = 0$, $x \in \partial\Omega$, $0 \le j \le m-1$, where we have freely used multi-index notation, cf. [1], [2], or [3], and $f(x,u,\ldots)$ denotes a function of x, u, and possibly all derivatives $D^{\alpha}u$ with $|\alpha| \le m$.

This class of problems has been studied in [1], [2], and [3] under the restrictions that the coefficients $a_{\alpha\beta}(x)$ are measurable and uniformly bounded in Ω, that there exists a positive constant C such that

$$(4)\quad (L[w],w)_{L^2(\Omega)} \ge C\ ||w||^2_{W^{m,2}(\Omega)} \equiv C(\sum_{|\alpha|\le m} \int_{\Omega} |D^{\alpha}w(x)|^2 dx) \text{ for all}$$

$w \in W_0^{m,2}(\Omega)$, i.e, for all w in the closure of $C_0^{\infty}(\Omega)$ with respect to $||\cdot||_{W^{m,2}(\Omega)}$, and that f depends on x and $D^{\alpha}u$ with $|\alpha| \le m-1$, but <u>not</u> $D^{\alpha}u$ with $|\alpha| = m$.

In this paper, we extend the results of [1] to problems in which the differential operator L satisfies a weaker "positive definite" hypothesis than (4) and f depends on x and $D^{\alpha}u$ with $|\alpha| \le m$. The price of this extension is a slightly stronger hypothesis on the smoothness of the coefficients $a_{\alpha\beta}(x)$.

2. MAIN RESULTS

Throughout this paper the coefficients $a_{\alpha\beta}(x)$ are assumed (I) to be bounded, measurable functions such that the domain of L, $\mathcal{D}(L)$, in $L^2(\Omega)$ can be taken to be those C^2 functions satisfying the boundary conditions (3) and (II) to be such that there exists a positive constant C such that

(5) $(L[u],u)_{L^2(\Omega)} \geq C \, ||u||^2_{L^2(\Omega)}$ for all $u \in \mathcal{D}(L)$. We remark that for assumption (I), it suffices to assume that $D^j a_{\alpha\beta}(x) \in C^0(\Omega)$ for all $|j| \leq |\alpha|$.

Let H denote the Hilbert space which is the completion of $\mathcal{D}(L)$ with respect to the norm

(6) $||u||_H \equiv (L[u],u)^{1/2}_{L^2(\Omega)}$. It follows from Theorem 2, pg. 323 of [5] that $H \subset L^2(\Omega)$.

<u>Theorem 1</u>. If L satisfies assumptions (I) and (II) and is such that any set which is bounded in H is precompact in $L^2(\Omega)$, then L^{-1} is defined as a compact mapping from $L^2(\Omega)$ to H.

<u>Proof</u>. It follows from Theorem 3, pg. 222 of [5] that, under these hypotheses, L has a discrete spectrum and hence, from Theorem 2, pg. 461 of [5] that L^{-1} is defined from $L^2(\Omega)$ to $H \subset L^2(\Omega)$ and is compact, when viewed as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$. To show that L^{-1} is a compact mapping from $L^2(\Omega)$ to H, let S be any bounded set in $L^2(\Omega)$. Since $L^{-1}(S)$ is precompact in $L^2(\Omega)$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset S$ such that $\{L^{-1}u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$, i.e., $\lim_{n,k\to\infty} ||L^{-1}u_n - L^{-1}u_k||_{L^2(\Omega)} = 0$. But, since $||L^{-1}u_n - L^{-1}u_k||^2_H = (L[L^{-1}u_n - L^{-1}u_k], L^{-1}u_n - L^{-1}u_k)_{L^2(\Omega)}$

$\leq ||u_n - u_k||_{L^2(\Omega)} \; ||L^{-1}u_n - L^{-1}u_k||_{L^2(\Omega)} \to 0$ as $n,k \to \infty$,

$\lim_{n,k\to\infty} ||L^{-1}u_n - L^{-1}u_k||_H = 0$ and $L^{-1}(S)$ is precompact in H. QED.

Combining Theorem 1 with the Leray-Schander Fixed Point Theorem, we have

Theorem 2. Let L satisfy the hypotheses of Theorem 1 and $f(x,u,\ldots)$ be such that the mapping $F: u \rightsquigarrow f(x,u,\ldots)$ is defined from H to $L^2(\Omega)$ and such that F is a bounded mapping, i.e., F maps bounded subsets of H into bounded subsets of $L^2(\Omega)$. Then $T \equiv L^{-1}F$ is a compact mapping from H to H and if T satisfies the Leray-Schander condition on $S_R(H) \equiv \{x \in H \mid ||x|| = R\}$, i.e., no solutions of the equation $\lambda Tx = x$, $\lambda \in (0,1]$, lie on the sphere $S_R(H)$, then T has a fixed point in $B_R(H) \equiv \{x \in H \mid ||x|| < R\}$.

We remark that if there exists a continuous, non-negative function $g(r)$, for $0 \leq r < \infty$, with $||F(u)||_{L^2(\Omega)} \leq g(||u||_H)$ for all $u \in H$, then F is a bounded mapping from H to $L^2(\Omega)$.

The fixed point of T given by Theorem 2 is called a generalized solution of (1), (3). It may be shown as in [1] that if the coefficients $a_{\alpha\beta}(x)$, the function $f(x,u,\ldots)$, and the domain Ω are sufficiently smooth, then the above generalized solution is a classical solution.

Corollary 1. If the mapping $F: H \to L^2(\Omega)$ is uniformly bounded, i.e., there exists a positive constant K such that $||F(u)||_{L^2(\Omega)} \leq K$ for all $u \in H$, then T has a fixed point and (1), (3) has a generalized solution.

The following two results for the eigenvalue problem (2), (3) are analogues of Theorem 3 of [1].

Theorem 3. Let the hypotheses of Theorem 2 hold. If $F(0) \neq 0$, then for any $R > 0$, there exists a $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, there exists a nontrivial solution of $u = \lambda L^{-1}F(u)$.

Theorem 4. Let the hypotheses of Theorem 2 hold. Given $\lambda_0 > 0$, either there exists $u \in H$ such that $u = \lambda_0 L^{-1}F(u)$ or for any $R > 0$ there exists $u \in S_R(H)$ such that $u = \lambda L^{-1}F(u)$ for some $\lambda < \lambda_0$.

The λ and u given by either Theorem 3 or Theorem 4 are respectively called a generalized eigenvalue and eigenfunction of (2), (3).

By using the Sobolev Imbedding Theorem, cf. Theorem I.4.1 of [2], we can give some general conditions under which the hypotheses of Theorem 2 hold.

Theorem 5. Let L satisfy assumption (I) and assume that there exists a positive constant C such that

(7) $(L[u],u)_{L^2(\Omega)} \geq C\, ||u||^2_{W^{j,2}(\Omega)}$ for all $u \in \mathcal{D}(L)$, for some $0 \leq j \leq m$.

Then $H \subset W_0^{j,2}(\Omega)$ and L^{-1} is a compact mapping from $L^2(\Omega)$ to H. If $f(x,u,\ldots)$ depends on x and $D^\alpha u$ with $|\alpha| \leq j$ and there exists a continuous, non-negative function h on $\prod_{|\beta|<j-\frac{n}{2}} R^+_\beta$ and a positive real number t such that

(8) $$|f(x,u,\ldots)| \leq h\,(\ldots, |D^\beta u|, \ldots)\,\{1+ \sum_{|\beta|=j-\frac{n}{2}} |D^\beta u|^t + \sum_{j-\frac{n}{2}<|\beta|\leq j-1} |D^\beta u|^\phi + \sum_{|\beta|=j} |D^\beta u|\,\},$$ for all $x \in \Omega$ where $\phi = n(n-2j+2|\beta|)^{-1}$,

then $F: u \rightsquigarrow f(x,u,\ldots)$ is a bounded mapping from H to $L^2(\Omega)$.

Theorem 6. Let L satisfy assumption (I) and assume that there exists a positive constant C such that

(9) $$(L[w],w)_{L^2(\Omega)} \geq C\, ||w||^2_{C^j(\Omega)} \equiv C \max_{x\in\Omega} \sum_{|\alpha|\leq j} |D^\alpha w(x)| \text{ for all } w \in \mathcal{D}(L)$$

and some $0 \leq j \leq m-1$. Then, $H \subset C^j(\Omega)$ and L^{-1} is a compact mapping from

$L^2(\Omega)$ to H. If $f(x,u,\ldots)$ depends on x and $D^{\alpha}u$ with $|\alpha| \leq j$ and there exists a continuous, non-negative function h on $\prod_{|\beta|\leq j} R^+_{\beta}$ such that

(10) $|f(x,u,\ldots)| \leq h(\ldots, |D^{\beta}u|,\ldots)$ for all $x \in \Omega$, then F; $u \dashrightarrow f(x,u,\ldots)$ is a bounded mapping from H to $L^2(\Omega)$.

3. EXAMPLES

As our first example, we consider the nonlinear two-point boundary value problem

(11) $L[u] \equiv \sum_{j=0}^{m} (-1)^{j+1} D^j[p_j(x) D^j u(x)] = f(x,u(x),\ldots,D^m u(x)),\ 0 < x < 1,$

subject to the boundary conditions

(12) $D^j u(0) = D^j u(1) = 0\ ,\ 0 \le j \le m-1.$

We assume that $p_j(x) \in C^j(0,1)$, $f(x,u,\ldots,D^m u)$ is continuous with respect to $x,u,\ldots,D^m u$, there exists a positive constant C such that

(13) $\int_0^1 \sum_{j=0}^{m} p_j(x)\ (D^j w(x))^2\, dx \ge C\|w\|^2_{W^{m,2}[0,1]}$ for all $w(x) \in C^{2m}(0,1)$

and satisfying the boundary conditions (12), and there exists a continuous, non-negative function h on R^m such that

(14) $|f(x,u,\ldots,D^m u)| \le h(u,\ldots,D^{m-1}u)\ \{1 + |D^m u|\}$

for all $x \in [0,1]$ and all $u,Du,\ldots,D^m u \in R.$ Then it follows from Theorem 5 that $H \equiv W_0^{m,2}[0,1]$ and $L^{-1}F$ is compact in H. If, in addition $f(x,u,\ldots,D^m u)$ is uniformly bounded, it follows from the Corollary of Theorem 2, that $L^{-1}F$ has a fixed point and hence (11), (12) has a generalized solution. We remark that under these hypotheses this generalized solution is a classical solution.

As our second example we consider the second order, nonlinear, two-point boundary value problem

(15) $L[u] \equiv -\ D[p_1(x)Du] + p_0(x)u(x) = f(x,u)$, $0 < x < 1$ subject to the boundary conditions

(16) $u(0) = u(1) = 0$.

We assume that $p_1(x) \in C^1(0,1)$, $p_0(x) \in C^0(0,1)$, $p_1(x) \geq 0$ for all $x \in (0,1)$, $p_0(x) \geq 0$ for all $x \in (0,1)$, and $A \equiv \int_0^1 \frac{dx}{p_1(x)} < \infty$. For example, taking $p_1(x) \equiv x^\sigma$, $0 < \sigma < 1$, and $p_0(x) \equiv 0$, we obtain the singular differential operator considered in [4].

Since $u^2(x) = (\int_0^x Du(x)dx)^2 = (\int_0^x \frac{1}{\sqrt{p_1(x)}} \sqrt{p_1(x)}\, Du(x)dx)^2$

$\leq \int_0^1 \frac{dx}{p_1(x)} \int_0^1 p_1(x) \, (Du(x))^2 \, dx \leq A \int_0^1 p(x))Du(x))^2 \, dx$, for all $x \in [0,1]$,

we have $||u||_{C^0[0,1]} \leq \sqrt{A}\, ||u||_H$ for all $u \in \mathcal{D}(L)$ and hence $H \subset C^0[0,1]$. Thus, by Theorem 6, if we assume that $f(x,u) \in C^0([0,1] \times R)$, then $F: u \to f(x,u)$ is a bounded mapping of H into $L^2[0,1]$. Moreover, it follows from the discussion on pg. 246 of [5] that every bounded subset of H is precompact in $L^2[0,1]$ and hence L^{-1} is a compact mapping from $L^2[0,1]$ to H. Thus, $L^{-1}F$ is a compact mapping in H. If in addition $f(x,u)$ is uniformly bounded, then it follows from the Corollary of Theorem 2 that $L^{-1}F$ has a fixed point and hence (15), (16) has a generalized solution. Moreover, under these hypotheses, this generalized solution is a classical solution.

References

[1] Adams, R. A., "A quasi-linear elliptic boundary value problem", Canad. J. Math. 18 (1966), 1105-1112.

[2] Berger, M.S., "An eigenvalue problem for nonlinear elliptic partial differential equations", Trans. Amer. Math.Soc. 120 (1965), 145-184.

[3] Browder, F. E., "Variational methods for nonlinear elliptic eigenvalue problems", Bull. Amer. Math.Soc., Vol. 71, No. 1 (1965), 176-183.

[4] Jamet, P., "Numerical methods for singular linear boundary value problems", Doctoral thesis, University of Wisconsin, Madison, Wisconsin, 1967.

[5] Mikhlin, S. G., "Variational methods in mathematical physics" (584 pp), The Macmillan Co., New York (1964).