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COMPUTATION OF THE ZEROS OF THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP

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ABSTRACT

We describe a computation which shows that the Riemann zeta function $\zeta(s)$ has exactly 70,000,000 zeros $\sigma + it$ in the region $0 < t < 30,549,654$. Moreover, all these zeros are simple and lie on the line $\sigma = \frac{1}{2}$. (A similar result for the first 3,500,000 zeros was established by Rosser, Yohe and Schoenfeld.) Counts of the number of Gram blocks of various types and the number of failures of "Rosser's rule" up to Gram number 70,000,000 are given.

AMS (MOS) Subject Classifications (1970)

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Key Words and Phrases

Gram blocks, Riemann hypothesis, Riemann zeta function, Riemann-Siegel formula, Rosser's rule, Turing's theorem, zeta functions.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ is the analytic function of $s = \sigma + it$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\sigma > 1$, and by analytic continuation for $\sigma \leqslant 1$, $s \neq 1$. Apart from "trivial" zeros at the negative even integers, all zeros of $\zeta(s)$ lie in the critical strip $0 < \sigma < 1$. The Riemann hypothesis is the conjecture [22] that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. For the number-theoretic significance of the Riemann hypothesis see, for example, Edwards [6], Ingham [10] and Lehman [12].

Since $\zeta(\bar{s}) = \bar{\zeta}(s)$, we need only consider zeros $\rho_j = \sigma_j + it_j$ with $t_j > 0$. We assume that the zeros ρ_j are counted according to their multiplicities and ordered so that $0 < t_j \leq t_{j+1}$ (and $\sigma_j \leq \sigma_{j+1}$ if $t_j = t_{j+1}$) for $j \geqslant 1$. By "the first n zeros of $\zeta(s)$" we mean $\rho_1, \ldots, \rho_n$. For brevity we let $H(n)$ denote the statement that the first n zeros of $\zeta(s)$ are simple and lie on the critical line. Thus, $H(n)$ holds for arbitrarily large n if and only if the Riemann hypothesis is true and all zeros of $\zeta(s)$ are simple.

In the era of hand computation, Gram [7], Backlund [2], Hutchinson [9], and Titchmarsh and Comrie [26] established H(10), H(79), H(138) and H(1,041) respectively. For a description of these computations see Edwards [6].

D. H. Lehmer [13,14] performed the first extensive computation of zeros of $\zeta(s)$ on a digital computer, and established H(25,000). Using similar methods, Meller [16], Lehman [11] , and Rosser, Yohe and Schoenfeld

[23] established H(35,337), H(250,000), and H(3,500,000) respectively.

Using essentially the method introduced by Lehmer, we have established H(70,000,001). Moreover, there are precisely 70,000,000 zeros with $0 < t_j < 30{,}549{,}654$. The computational method is outlined in Section 4, and additional details are given in Section 5. In Section 6 the results are summarized and various statistics regarding the distribution of the first 70,000,000 zeros are tabulated. Preliminary results are given in Sections 2 and 3.

2. PROPERTIES OF $\zeta(s)$

In this section we summarize some well-known properties of $\zeta(s)$ which form the basis for the computational method described in Section 4.

2.1 The Functional Equation for $\zeta(s)$

$\zeta(s)$ satisfies a functional equation which may be written in the form

$$\xi(s) = \xi(1-s) ,$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) .$$

It follows that, if

$$(2.1) \qquad \theta(t) = \arg\left[\pi^{-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\right] = \mathfrak{I}\left[\ln\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\right] - \tfrac{1}{2}t\ln\pi ,$$

then

$$(2.2) \qquad Z(t) = \exp\left[i\theta(t)\right]\zeta(\tfrac{1}{2}+it)$$

is real for real t. Thus, simple zeros of $\zeta(s)$ on the critical line can be located by finding changes of sign of $Z(t)$.

(The first few zeros of $Z(t)$ are $t_1 = 14.1347$, $t_2 = 21.0220$, $t_3 = 25.0109$, ... : see Haselgrove and Miller [8].)

2.2 The Asymptotic Expansion for $\theta(t)$

From (2.1) and Stirling's formula for $\ln\Gamma(s/2)$, we obtain the following asymptotic expansion for the phase $\theta(t)$:

$$\theta(t) = \tfrac{1}{2}t \ln\left(\frac{t}{2\pi}\right) - \tfrac{1}{2}t - \frac{\pi}{8} + \sum_{k=1}^{n} \frac{B_{2k}(1-2^{1-2k})}{4k(2k-1)} t^{1-2k} + r_n(t) ,$$

where $B_2 = 1/6$, $B_4 = -1/30$, ... are Bernoulli numbers, and

$$|r_n(t)| < \frac{(2n)!}{(2\pi)^{2n+2} t^{2n+1}} + \exp(-\pi t)$$

for all $t > 0$ and $n \geq 0$.

$\theta(t)$ has a minimum of approximately -3.53 near $t = 2\pi$, and is monotonic increasing for $t \geq 7$. For $m \geq -1$, we define the m-th Gram point g_m to be the unique solution in $[7,\infty)$ of

$$\theta(g_m) = m\pi \quad . \tag{2.3}$$

Thus $g_{-1} = 9.6669$, $g_0 = 17.8456$, $g_1 = 23.1703$, ...

For future reference we note the inequality

$$g_{m+1} - g_m > \frac{2\pi}{\ln[g_{m+1}/(2\pi)]} > g_{m+2} - g_{m+1} , \tag{2.4}$$

which holds for all $m \geq -1$.

2.3 The Euler-Maclaurin Formula for $\zeta(s)$

$\zeta(s)$ may be calculated to any desired accuracy by taking m and n large enough in the Euler-Maclaurin formula

$$(2.5) \qquad \zeta(s) = \sum_{j=1}^{n-1} j^{-s} + \tfrac{1}{2}n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{m} T_{k,n}(s) + E_{m,n}(s) ,$$

where

$$T_{k,n}(s) = \frac{B_{2k}}{(2k)!} n^{1-s-2k} \prod_{j=0}^{2k-2} (s+j)$$

and

$$|E_{m,n}(s)| < |T_{m+1,n}(s)(s+2m+1)/(\sigma+2m+1)|$$

for all $m \geq 0$, $n \geq 1$, and $\sigma = \mathcal{R}(s) > -(2m+1)$.

If (2.5) is used to obtain $\zeta(\tfrac{1}{2} + it)$ to within a specified absolute tolerance, then it is necessary to take $n \gtrsim t/(2\pi)$. It is also sufficient to take $n = O(t)$ and $m = O(t)$. Thus, the computational work required is roughly proportional to t.

2.4 The Riemann-Siegel Formula for $Z(t)$

The Riemann-Siegel formula [5,6,25] is an asymptotic expansion for $Z(t)$ (defined by (2.2)). The Rieman-Siegel formula is an improvement over the Euler-Maclaurin expansion for computing $Z(t)$ if t is large, because the work required is $O(t^{\frac{1}{2}})$ instead of $O(t)$.

Let $\tau = t/(2\pi)$, $m = \lfloor \tau^{\frac{1}{2}} \rfloor$, and $z = 2(\tau^{\frac{1}{2}} - m) - 1$. Then the Riemann-Siegel formula with $n+1$ terms in the asymptotic expansion is

$$(2.6) \qquad Z(t) = \sum_{k=1}^{m} 2k^{-\frac{1}{2}}\cos[t.\ln(k) - \theta(t)] + (-1)^{m+1}\,\tau^{-\frac{1}{4}} \sum_{j=0}^{n} \Phi_j(z)(-1)^j\tau^{-j/2} + R_n(\tau) \quad ,$$

where

$$R_n(\tau) = 0(\tau^{-(2n+3)/4})$$

for $n \geq -1$ and $\tau > 0$. Here the $\Phi_j(z)$ are certain entire functions which may be expressed in terms of the derivatives of

$$\Phi_0(z) \equiv \Phi(z) = \cos[\pi(4z^2+3)/8]/\cos(\pi z) \quad .$$

Expressions for $\Phi_1, \ldots \Phi_{19}$ are given in the review of [5]. For our purposes it is sufficient to note that

$$\Phi_1(z) = \Phi^{(3)}(z)/(12\pi^2)$$

and

$$\Phi_2(z) = \Phi^{(2)}(z)/(16\pi^2) + \Phi^{(6)}(z)/(288\pi^4) \quad .$$

To establish changes of sign of $Z(t)$ we need rigorous bounds on the error $R_n(\tau)$. Titchmarsh [27 , pg. 331] showed that

$$|R_0(\tau)| < \frac{3}{2}\tau^{-3/4} \quad \text{for} \quad \tau > 125 \quad ,$$

and Rosser et al [23] used the bound

$$(2.7) \qquad |R_2(\tau)| < 2.28\tau^{-7/4} \quad \text{for} \quad \tau > 2000\,.^{\dagger}$$

† The number "2.88" appearing in [23] should have been "2.28".

This bound is extremely conservative : computation of $\max_{z \in [-1,1]} |\Phi_j(z)|$ for $j = 3,4, \ldots$ (and computation of $R_2(\tau)$ for small τ) indicates that the constants 2.28 and 2000 in (2.7) may be replaced by 0.006 and 10 respectively. In the computation described below we took $n = 2$ in (2.6) and used only the weak bound

(2.8) $$|R_2(\tau)| < 3\tau^{-7/4} \quad \text{for} \quad \tau > 2000.$$

The effect of rounding errors in accumulating the first sum in (2.6) was more of a problem than the inherent error (2.8) : see Section 5.

3. GRAM BLOCKS AND THE LITTLEWOOD-TURING THEOREM

"Gram's law" is the observation [7] that $Z(t)$ usually changes sign in each "Gram interval" $G_j = [g_j, g_{j+1})$, $j \geq -1$. A plausible explanation for this is that the leading (k=1) term in (2.6) at $t = g_j$ is $2(-1)^j$. Following Rosser et al [23] , we call a Gram point g_j *good* if $(-1)^j Z(g_j) > 0$, and *bad* otherwise. (The first bad Gram point is g_{126}.) A "Gram block of length k" is an interval $B_j = [g_j, g_{j+k})$ such that g_j and g_{j+k} are good Gram points, $g_{j+1}, \ldots , g_{j+k-1}$ are bad Gram points, and $k \geq 1$. We say that B_j satisfies "Rosser's rule" if $Z(t)$ has at least k zeros in B_j . Rosser's rule fails infinitely often [6] , but it is still an extremely useful heuristic. The first exception is $B_{13,999,525}$ (see Table 3).

Let $N(T)$ denote the number of zeros (counted according to their multiplicities) of $\zeta(s)$ in the region $0 < \Im(s) \leq T$, and

(3.1) $$S(t) = N(t) - 1 - \theta(t)/\pi \quad .$$

It is each to show that Gram's law holds in regions where $|S(t)| < 1$, and Rosser's rule holds in regions where $|S(t)| < 2$. Thus , the success of these heuristics is closely related to the distribution of values of $S(t)$.

Turing [28] showed that the following theorem, based on an idea of Littlewood [15], could be used to bound $N(t)$ for certain values of t.

Theorem 3.1

If $A = 3.1$, $B = 4.8$, $C = 100$, and $C < u < v$, then

$$\left| \int_u^v S(t)\,dt \right| < A.\ln(v) + B .$$

Proof

Turing's proof contains some errors, so one can not rely on his values of A, B and C (which are smaller than those stated above). A correct proof was given by Lehman [11].

Since our program works with Gram blocks (see Section 4), it is convenient to deduce the following result from Theorem 3.1 .

Theorem 3.2

If K consecutive Gram blocks with union $[g_n, g_p)$ satisfy Rosser's rule, where

$$K \geq \tfrac{1}{2}[\ln(g_p)]^2 , \tag{3.2}$$

then

$$N(g_n) \leq n + 1 \tag{3.3}$$

and

$$N(g_p) \geq p + 1 . \tag{3.4}$$

Before proving Theorem 3.2 we need some lemmas.

Lemma 3.1

Let $B_j = [g_j, g_k)$ be a Gram block which satisfies Rosser's rule. Then

(3.5) $$S(g_j) \leq S(g_k) ,$$

(3.6) $$S(g_j) - 2 \leq S(t) \leq S(g_k) + 2 \quad \text{for } t \in B_j ,$$

(3.7) $$S(g_j) - 1 \leq S(t) \quad \text{for } t \in G_j ,$$

and

(3.8) $$S(t) \leq S(g_k) + 1 \quad \text{for } t \in G_{k-1} .$$

Proof

From (2.3) we have $\theta(g_k) - \theta(g_j) = \pi(k-j)$, and as B_j satisfies Rosser's rule we have $N(g_k) - N(g_j) \geq k-j$. Thus, (3.5) follows from (3.1). The remaining inequalities follow similarly from the assumption that B_j satisfies Rosser's rule.

Lemma 3.2

Under the conditions of Lemma 3.1 ,

(3.9) $$\int_{g_j}^{g_k} [S(t) - S(g_j) + 2] \; dt \geq g_{j+1} - g_j$$

and

(3.10) $$\int_{g_j}^{g_k} [S(g_k) - S(t) + 2] \; dt \geq g_k - g_{k-1} .$$

Proof

(3.9) follows by integrating (3.7) over $[g_j, g_{j+1})$ and (3.6) over $[g_{j+1}, g_k)$. (3.10) follows similarly from (3.6) and (3.8).

Lemma 3.3

Under the conditions of Theorem 3.2 ,

$$\int_{g_n}^{g_p} [S(t) - S(g_n) + 2] \ dt \ \geq \ K(g_p - g_{p-1})$$

and

$$\int_{g_n}^{g_p} [S(g_p) - S(t) + 2] \ dt \ \geq \ K(g_p - g_{p-1}) \ .$$

Proof

The inequalities follow from (2.4), (3.5) and Lemma 3.2.

Lemma 3.4

Under the conditions of Theorem 3.2, if

$$K(g_p - g_{p-1}) \ \geq \ A.\ln(g_p) + B$$

(where A and B are as in Theorem 3.1), then

(3.11) $$S(g_n) \leq 0 \leq S(g_p) \ .$$

Proof

Since g_n and g_p are good Gram points, $S(g_n)$ and $S(g_p)$ must be even integers. Thus, the result follows from Theorem 3.1 and Lemma 3.3.

Proof of Theorem 3.2

If $n \leq 125$ then $N(g_n) = n+1$ and, from (3.5), (3.4) also holds. Hence, we assume that $n > 125$, and thus $g_n > C$.

It is easy to show that

$$\tfrac{1}{2}x^2 > \frac{x - \ln(2\pi)}{2\pi} (Ax + B)$$

holds for all real x. Taking $x = \ln(g_p)$ and using (2.4) and (3.2), we see from Lemma 3.4 that (3.11) holds. Thus, the result follows from (2.3) and (3.1).

4. THE COMPUTATIONAL METHOD

The first (and most expensive) part of the computational verification of $H(n+1)$ is the location of $n+1$ sign changes of $Z(t)$ in (g_{-1}, g_n) . Our program works in the following way. Suppose that $j+1$ sign changes have been found in (g_{-1}, g_j) , where g_j is a good Gram point. Then $Z(g_{j+1})$, $Z(g_{j+2})$, ... are evaluated until the next good Gram point g_{j+k} is found. The program then evaluates $Z(t)$ for various $t \in B_j = [g_j, g_{j+k})$, until either

(a) k sign changes are found in B_j, when j is replaced by j+k and the process continues; or

(b) after a large number of evaluations of $Z(t)$ the program gives up and calls for help.

Case (b) could arise because of a pair of very close zeros of $Z(t)$ in B_j (or a zero of even multiplicity), or because B_j does not satisfy Rosser's rule. In fact, during the computation to n = 70,000,000, case (b) occurred only 15 times. In each case B_j contained $k - 2$ zeros of $Z(t)$, and the preceding or following Gram block of length k' contained $k' + 2$ zeros of $Z(t)$: see Table 3.

In this way we found the required $n+1$ sign changes, establishing that $N(g_n) \geq n+1$. By running the computation a little further we also showed that there are $K = 165$ Gram blocks in $[g_n, g_{n+200})$, and all of them satisfy Rosser's rule. Applying Theorem 3.2 gives $N(g_n) \leq n+1$. Thus, $N(g_n) = n+1$, and $H(n+1)$ holds. By locating the n-th and (n+1)-th zeros, it may be shown that $N(30,549,654) = n = 70,000,000$, as claimed in the abstract.

5. COMPUTATIONAL DETAILS

In Section 4 we glossed over an essential point: how can the sign of $Z(t)$ be determined with certainty? If $Z(t)$ is evaluated numerically from the Riemann-Siegel formula (2.6) , the effect of rounding errors must be considered as well as the inherent error $R_n(\tau)$.

5.1 Methods for Evaluating $Z(t)$

It is desirable to have at least two methods for evaluating $Z(t)$: a fast method which usually determines the sign of $Z(t)$ unambiguously, and a slower but more accurate method which may be used if the fast method fails. We used the Euler-Maclaurin formula (2.5) both for small t and for checking purposes, but for brevity we shall only analyse the use of the Riemann-Siegel formula (2.6). We shall also assume that $n = 2$ in (2.6), and that $t > 20,000\pi$. Our program uses the following two methods to evaluate the Riemann-Siegel sum

$$s(t) = \sum_{k=1}^{m} 2k^{-\frac{1}{2}} \cos[t.\ln(k) - \theta(t)] \quad . \tag{5.1}$$

Method A : The constants $\ln(k)$, $k = 1, 2, \ldots$ are precomputed (using double-precision) and stored in a table. For each value of k, $f = \text{frac}\{\frac{1}{2\pi}[t.\ln(k) - \theta(t)]\}$ is computed using double-precision, then rounded to single-precision. (Here frac(x) denotes the fractional part of x.) Then $\cos(2\pi f) = \cos[t.\ln(k) - \theta(t)]$ is approximated by a precomputed piecewise linear approximation, the result multiplied by the precomputed single-precision constant $2k^{-\frac{1}{2}}$, and the sum (5.1) accumulated in double precision.

Method B : As for method A except that all computations are done using double-precision arithmetic, and $\cos(2\pi f)$ is evaluated as accurately as possible.

All computations were performed on a Univac 1100/42 computer, which has a 36-bit word and hardware single and double-precision floating-point arithmetic (using 27 and 60-bit binary fractions, respectively).

5.2 Rounding Error Analysis of Methods A and B

The analysis is similar to that of Lehman [11] and Rosser et al [23] so we shall omit detailed (and tedious) proofs of the following results. Recall that $m = \lfloor \tau^{\frac{1}{2}} \rfloor \geq 100$. Lemmas 5.1 and 5.2 are elementary, and Lemma 5.3 follows easily from them.

Lemma 5.1

$$\sum_{k=1}^{m} k^{-\frac{1}{2}} \leq 2m^{\frac{1}{2}} \leq 2\tau^{\frac{1}{4}}$$

and

$$\sum_{k=1}^{m} k^{-\frac{1}{2}} \ln(k) \leq 2m^{\frac{1}{2}} \ln(m) \leq \tau^{\frac{1}{4}} \ln(\tau) \quad .$$

Lemma 5.2

$$\theta(t) \; < \; \pi\tau\ln(\tau) \; .$$

Lemma 5.3

Suppose that

$$|L(k) - \ln(k)| \; \leqslant \; \delta_1 \ln(k) \qquad \text{for} \quad k = 1, 2, \ldots, m \; ,$$

$$|\tilde{\theta}(t) - \theta(t)| \; < \; \delta_2 \theta(t) \; ,$$

$$|\tilde{c}(x) - \cos(x)| \; \lesssim \; \delta_3 \qquad \text{for} \quad 0 \leqslant x < 2\pi \; ,$$

and

$$\tilde{s}(t) \; = \; \sum_{k=1}^{m} 2k^{-\frac{1}{2}} \tilde{c}[t.L(k) - \tilde{\theta}(t)] \; .$$

Then

$$|\tilde{s}(t) - s(t)| \; \leqslant \; 4\pi\tau^{5/4} \ln(\tau)(\delta_1 + \delta_2) + 4\tau^{\frac{1}{4}}\delta_3 \; .$$

Lemma 5.3 accounts for the error in the computed value of $s(t)$, given bounds on the relative errors in the evaluation of $\ln(k)$ and $\theta(t)$ and on the absolute error in the evaluation of $\cos(x)$. By the techniques of backward error analysis [29] , we can account for errors caused by the computation of $t.L(k) - \tilde{\theta}(t)$, the computation of $2k^{-\frac{1}{2}}$ and multiplication by $\tilde{c}(x)$, and the final summation, by increasing $\delta_1 + \delta_2$ slightly. Since the required change in $\delta_1 + \delta_2$ is small, we shall omit details of the analysis.

For both methods A and B, analysis of the algorithm used to compute double-precision logarithms and $\theta(t)$ gives the (conservative) bounds $\delta_1 \lesssim 2^{-59}$ and $\delta_2 \lesssim 3\text{x}2^{-59}$. (We assume here that τ is exactly representable as a floating-point number. This is true in our program, where τ is used rather than t in the critical computations.)

For method A we approximate $\cos(2\pi x)$ for $0 \le x < 1$ using piecewise linear approximations on the intervals $[ih, (i+1)h)$ for $i = 0, \ldots, 1023$ and $h = 2^{-10}$. It is easy to show that, with exact arithmetic, the approximation error is bounded by $2^{-22}\pi^2 < 2.36\text{x}10^{-6}$. Allowing for rounding errors in evaluating the linear approximations $a + bx$ (with $|a| \le 3\pi/2$, $|b| \le 2\pi$, $0 \le x < 1$) increases this bound by $(13 + 2\pi)2^{-27}$, so we have $\delta_3 \le 2.497\text{x}10^{-6}$.

For method B it turns out that δ_3 is negligible, because the errors in the cosine and logarithm evaluation are the same order of magnitude, but the error in the evaluation of $\ln(k)$ contributes much more to the bound on the error in $\tilde{s}(t)$ because it is amplified by the factor $t \ge 20{,}000\pi$.

It is possible to allow for errors in evaluating $\tau^{\frac{1}{2}}$ (and hence m) and the $\Phi_j(z)$ in (2.6), but as these contribute little to the final error bound we shall omit the details. Collecting the results, and including the inherent error (2.8), we have the following bounds for the error in the computed value $\tilde{Z}(t)$ (rounded to single-precision) of $Z(t)$:

$$(5.2) \qquad |\tilde{Z}(t) - Z(t)| \le \begin{cases} (2\text{x}10^{-5} + 5\text{x}10^{-16}\tau\ln(\tau) + 3\tau^{-2})\tau^{\frac{1}{4}} & \text{for method A}, \\ (5\text{x}10^{-16}\tau\ln(\tau) + 3\tau^{-2})\tau^{\frac{1}{4}} + 8\text{x}10^{-9}|\tilde{Z}(t)| & \text{for method B}. \end{cases}$$

These are the bounds actually used in the program, and are weaker than could be justified by the analysis sketched above.

5.3 Efficiency Considerations

When evaluating $Z(t)$ our program always tries method A first. If the computed $|\tilde{Z}(t)|$ is smaller than the bound (5.2), the sign of $Z(t)$ can not be guaranteed, so method B is used. (Method B is also used once in 1000 evaluations to give a dynamic check on the consistency of the error bounds (5.2).) Occasionally method B is unable to guarantee the sign of $Z(t)$. If we are searching for sign changes in a Gram block and t is not a Gram point, we simply discard t and try another nearby point. If t is a Gram point the sign of $Z(t)$ must be determined to ensure the accuracy of Tables 1-4 below. Thus, we occasionally use a multiple-precision arithmetic package [4] to evaluate $Z(t)$ accurately at Gram points. (Actually, method B always gives the correct sign of $Z(g_n)$ for $n \leq 70{,}000{,}000$, even though the bound (5.2) is too weak to guarantee this.)

Nearly all the computation time is spent in the inner loop of method A, so not much would be gained by speeding up method B or increasing the accuracy of method A. It also seems unlikely that the inner loop could be speeded up much without using a faster machine, as the loop compiles into only 19 machine instructions which execute in about 22 μsec. (The double-precision evaluation of $\cos(2\pi x)$ using the standard library routine [1] takes about 79 μsec, and the inner loop of method B takes about 150 μsec.)

To separate the first 70,000,000 zeros our program evaluated $Z(t)$ at about 99,000,000 points. Thus, the heuristic of using Rosser's rule is very efficient - the number of evaluations of $Z(t)$ could not be reduced by more than 29 percent.

Our program requires about $35[n/\ln(n)]^{1/2}$ μsec of CPU time per Gram point near g_n, $n \leq 10^8$. Thus, the time required to verify $H(n)$ is about $6.5 \times 10^{-9} n[n/\ln(n)]^{1/2}$ hours. Our program is about 3.6 times faster than the CDC 3600 program of Rosser et al [23], and about 11 times faster than the IBM 7090 program of Lehman [11]. This is roughly what one would expect, given the relative speeds of the different machines. (The times given for our program are approximate because of the variability of factors such as the ratio of primary to extended memory references, system load, etc.)

6. SUMMARY OF THE COMPUTATIONAL RESULTS

During the course of the verification of H(70,000,001) we accumulated various statistics which are summarized in Tables 1 to 4. Table 1 gives the number $J(k,n)$ of Gram blocks B_j of length $k \leq 7$ with $0 \leq j < n$ and various $n \leq 70{,}000{,}000$. (Note that $B_{-1} = [g_{-1}, g_0)$ and the zero $t_1 \in B_{-1}$ are excluded from the statistics given in Tables 1 to 4.) No Gram blocks of length greater than 7 were found.† The average block length up to $n = 70{,}000{,}000$ is 1.1873, and appears to increase slowly with n. (If the $Z(g_j)$ had random independently distributed signs then the average block length would be 2.)

In Table 2 we give the number of Gram intervals $G_j = [g_j, g_{j+1})$, $0 \leq j < n$, which contain exactly m zeros of $Z(t)$, $0 \leq m \leq 4$. About 74 percent of the Gram intervals up to $n = 70{,}000{,}000$ contain precisely one zero, and this percentage decreases slowly with n. We found only one Gram interval ($G_{61,331,768}$) which contains more than three zeros.

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† Blocks of length 8, e.g. $B_{1,801,894,493}$, have been found by a different method (mentioned later).

TABLE 1 : Number of Gram Blocks of Given Length

n	J(1,n)	J(2,n)	J(3,n)	J(4,n)	J(5,n)	J(6,n)	J(7,n)
100	100						
200	194	3					
500	474	13					
1,000	916	42					
2,000	1,766	117					
5,000	4,283	348	7				
10,000	8,374	780	22				
20,000	16,404	1,680	76	2			
50,000	39,911	4,545	325	6			
100,000	78,694	9,445	779	19	1		
200,000	155,327	19,338	1,928	52	1		
500,000	382,162	49,374	6,040	230	10		
1,000,000	755,132	100,203	13,822	709	32		
2,000,000	1,493,597	202,964	30,659	2,018	84	1	
5,000,000	3,683,812	513,502	85,804	7,559	294	11	
10,000,000	7,297,808	1,034,545	184,107	19,115	821	36	
20,000,000	14,468,638	2,079,342	390,564	46,989	2,422	151	2
30,000,000	21,596,795	3,126,675	604,103	78,370	4,491	264	4
40,000,000	28,697,661	4,176,596	821,276	112,050	6,951	387	6
50,000,000	35,780,082	5,227,670	1,041,204	147,419	9,623	514	13
60,000,000	42,844,351	6,280,945	1,263,391	184,290	12,450	668	24
70,000,000	49,898,904	7,333,132	1,487,914	222,034	15,530	849	30

TABLE 2 : Number of Gram Intervals Containing Exactly m Zeros

n	m = 0	m = 1	m = 2	m = 3	m = 4
100	0	100			
200	3	194	3		
500	13	474	13		
1,000	42	916	42		
2,000	117	1,766	117		
5,000	358	4,287	352	3	
10,000	808	8,390	796	6	
20,000	1,770	16,472	1,746	12	
50,000	4,915	40,209	4,837	39	
100,000	10,330	79,427	10,157	86	
200,000	21,528	157,153	21,110	209	
500,000	56,236	388,110	55,072	582	
1,000,000	116,055	769,179	113,477	1,289	
2,000,000	238,441	1,525,833	233,011	2,715	
5,000,000	614,253	3,778,577	600,087	7,083	
10,000,000	1,253,556	7,507,820	1,223,692	14,932	
20,000,000	2,550,785	14,929,745	2,488,155	31,315	
30,000,000	3,861,692	22,324,402	3,766,121	47,785	
40,000,000	5,181,785	29,700,949	5,052,747	64,519	
50,000,000	6,507,746	37,065,811	6,345,140	81,303	
60,000,000	7,839,959	44,418,273	7,643,577	98,191	
70,000,000	9,174,803	51,765,709	8,944,174	115,313	1

In Table 3 we list the 15 exceptions to Rosser's rule up to $B_{70,000,000}$. Each exception is associated with a small region where $|S(t)|$ exceeds 2, and the table gives the local extreme values of $|S(t)|$. Selberg [24] has shown that

$$S(t) = \Omega_{\pm}[(\ln t)^{1/3}(\ln \ln t)^{-7/3}] ,$$

and, assuming the Riemann Hypothesis, Montgomery [19] has shown that

$$S(t) = \Omega_{\pm}[(\ln t)^{\frac{1}{2}}(\ln \ln t)^{-\frac{1}{2}}] .$$

Unfortunately, it appears that the "interesting" region where $|S(t)|$ greatly exceeds 2 is well outside the range of feasible computation by the Riemann-Siegel formula.

TABLE 3 : Exceptions to Rosser's Rule

n	Type	Extreme $S(t)$
13,999,525	1	-2.004138
30,783,329	1	-2.002594
30,930,927	2	+2.050625
37,592,215	1	-2.076426
40,870,156	1	-2.003797
43,628,107	1	-2.024243
46,082,042	1	-2.031132
46,875,667	1	-2.004600
49,624,541	2	+2.001841
50,799,238	1	-2.028778
55,221,454	2	+2.024216
56,948,780	2	+2.017714
60,515,663	1	-2.008143
61,331,766	3	-2.054298
69,784,844	2	+2.063683

Type 1 is block B_n of length 2 with no zeros, immediately followed by block B_{n+2} of length 1 with 3 zeros.

Type 2 is block B_n of length 2 with no zeros, immediately preceded by block B_{n-1} of length 1 with 3 zeros.

Type 3 is block B_n of length 2 with no zeros, immediately followed by block B_{n+2} of length 2 with 4 zeros.

All exceptions to Rosser's rule up to $B_{70,000,000}$ are included.

Let $B_m = [g_m, g_{m+j})$ be a Gram block which satisfies Rosser's rule and has length $j \geq 2$. We say that B_m is of type (j, k) if $1 \leq k \leq j$ and $[g_{m+k-1}, g_{m+k})$ contains at least two zeros of $Z(t)$. This is neither an unambiguous nor a complete classification, but it is sufficient to deal with all nontrivial Gram blocks up to $B_{70,000,000}$, except for those noted in Table 3. The first occurrences of Gram blocks of various types are noted in Table 4. No blocks of type (7, 1) or (7, 7) occur up to $B_{70,000,000}$.

TABLE 4 : First Occurrences of Gram Blocks of Various Types

j	k	n
2	1	133
2	2	125
3	1	3,356
3	2	2,144
3	3	4,921
4	1	83,701
4	2	39,889
4	3	18,243
4	4	67,433
5	1	1,833,652
5	2	243,021
5	3	601,944
5	4	68,084
5	5	455,256
6	1	20,046,223
6	2	2,656,216
6	3	4,718,714
6	4	1,181,229
6	5	2,842,089
6	6	19,986,469
7	2	13,869,654
7	3	17,121,221
7	4	37,091,042
7	5	20,641,464
7	6	52,266,282

B_n is the first Gram block of type (j,k)

Our program did not explicitly search for pairs of close zeros of $Z(t)$, but we did detect some such pairs when the program had difficulty in finding the expected number of sign changes in the Gram block containing them. For example,

$$t_{n+1} - t_n < 0.00053 \quad \text{and} \quad \max_{t\in(t_n, t_{n+1})} |Z(t)| < 0.00000248$$

for $n = 41,820,581$. This is a more extreme example of the phenomenon first observed by Lehmer [13,14]. See also Montgomery [17,18,20,21].

Our program regularly printed out the largest value of $|Z(g_j)|$ found so far. For example, $Z(g_{65,379,394}) > 75.6$, and the first 42 terms in the Riemann-Siegel sum (5.1) are positive at this point!

In all cases where an exception to Rosser's rule was observed, there was a large local maximum of $|Z(t)|$ nearby. This suggests that "interesting" regions might be predicted by finding values of t such that the first few terms in the Riemann-Siegel sum reinforce each other. Preliminary computations suggest that this is a promising approach. To verify the feasibility of such computations for Gram numbers near 10^{10} we ran our program (slightly modified) from g_{n-400} to g_{n+6400}, where $n = 10^{10}$. All Gram blocks in this region satisfy Rosser's rule and, using Theorem 3.2, we can show that $\rho_n, \rho_{n+1}, \ldots, \rho_{n+6000}$ are simple and lie on the critical line.

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REFERENCES

1. ANONYMOUS, *Sperry Univac 1100 Series Fortran V Library Programmer Reference Manual (UP 7876, rev. 4)*, Sperry Rand Corp., 1974.

2. R. BACKLUND, "Sur les zéros de la fonction $\zeta(s)$ de Riemann", *C. R. Acad. Sci. Paris*, v.158, 1914, pp. 1979-1982.

3. R. P. BRENT, "The first 40,000,000 zeros of the Riemann zeta function lie on the critical line", *AMS Notices*, v. 24, 1977, p. A-417.

4. R. P. BRENT, "A Fortran multiple precision arithmetic package", *ACM Trans. Math. Software*, v. 4, 1978, pp. 57-70.

5. F. D. CRARY & J. B. ROSSER, *High Precision Coefficients Related to the Zeta Function*, MRC Technical Summary Report #1344, Univ. of Wisconsin, Madison, May 1975, 171 pp.; RMT 11, *Math. Comp.*, v. 31, 1977, pp. 803-804.

6. H. M. EDWARDS, *Riemann's Zeta Function*, Academic Press, New York, 1974.

7. J. GRAM, "Sur les zéros de la fonction $\zeta(s)$ de Riemann" *Acta Math.*, v. 27, 1903, pp. 289-304.

8. C. B. HASELGROVE in collaboration with J.C.P. MILLER, *Tables of the Riemann Zeta Function*, Roy. Soc. Math. Tables No. 6, Cambridge Univ. Press, New York, 1960; RMT 6, *Math. Comp.*, v. 15, 1961, pp. 84-86. MR 22 #8679.

9. J. I. HUTCHINSON, "On the roots of the Riemann zeta-function", *Trans. Amer. Math. Soc.*, v. 27, 1925, pp. 49-60.

10. A. E. INGHAM, *The Distribution of Prime Numbers*, Cambridge Tracts in Math. and Math. Phys., No. 30, Cambridge Univ. Press, London and New York, 1932. (Republished by Stechert-Hafner, New York and London, 1964.)

11. R. S. LEHMAN, "Separation of zeros of the Riemann zeta-function", *Math. Comp.*, v. 20, 1966, pp. 523-541.

12. R. S. LEHMAN, "On the difference $\pi(x) - \mathrm{li}(x)$", *Acta Arithmetica*, v.11, 1966, pp. 397-410.

13. D. H. LEHMER, "Extended computation of the Riemann zeta-function", *Mathematika*, v. 3, 1956, pp. 102-108; RMT 108, *MTAC*, v. 11, 1957, p. 273. MR 19, p. 121.

14. D. H. LEHMER, "On the roots of the Riemann zeta-function", *Acta Math.*, v. 95, 1956, pp. 291-298; RMT 52, *MTAC*, v.11, 1957, pp. 107-108. MR 19, p. 121.

15. J. E. LITTLEWOOD, "On the zeros of the Riemann zeta-function", *Proc. Cambridge Phil. Soc.*, v. 22, 1924, pp. 295-318.

16. N. A. MELLER, "Computations connected with the check of Riemann's hypothesis", *Dokl. Akad. Nauk SSSR*, v. 123, 1958, pp. 246-248 (in Russian). MR 20#6396.

17. H. L. MONTGOMERY, "The pair correlation of zeros of the zeta function", *Proc. Sympos. Pure Math.*, v.24, Amer. Math. Soc., Providence, Rhode Island, 1973, pp. 181-193.

18. H. L. MONTGOMERY, "Distribution of the zeros of the Riemann zeta function", *Proc. International Congress of Mathematicians*, Vancouver, 1974, pp. 379-381.

19. H. L. MONTGOMERY, "Extreme values of the Riemann zeta function", *Comment. Math. Helvetici*, v.52, 1977, pp. 511-518.

20. H. L. MONTGOMERY, "Problems concerning prime numbers", *Proc. Sympos. Pure Math.*, v. 28, Amer. Math. Soc., Providence, Rhode Island, 1976, pp. 307-310.

21. H. L. MONTGOMERY & P. J. WEINBERGER, "Notes on small class numbers", *Acta Arith.*, v. 24, 1974, pp. 529-542.

22. B. RIEMANN, *Gesammelte Werke*, Teubner, Leipzig, 1892. (Reprinted by Dover Books, New York, 1953.)

23. J. B. ROSSER, J. M. YOHE & L. SCHOENFELD, "Rigorous computation and the zeros of the Riemann zeta-function", *Information Processing 68 (Proc. IFIP Congress, Edinburgh, 1968), v. 1*, North-Holland, Amsterdam, 1969, pp. 70-76. MR 41 #2892. Errata: *Math. Comp.*, v.29, 1975, p.243.

24. A. SELBERG, "Contributions to the theory of the Riemann zeta-function", *Arch. Math. Naturvid.*, v. 48, 1946, pp. 99-155.

25. C. L. SIEGEL, "Über Riemanns Nachlass zur analytischen Zahlentheorie", *Quellen Studien zur Geschichte der Math. Astron. und Phys. Abt. B: Studien 2*, 1932, pp. 45-48. (Also in *Gesammelte Abhandlungen*, v. 1, Springer-Verlag, New York, 1966.)

26. E. C. TITCHMARSH, "The zeros of the Riemann zeta-function", *Proc. Roy. Soc. London*, v. 151, 1935, pp. 234-255; also *ibid*, v. 157, 1936, pp. 261-263.

27. E. C. TITCHMARSH, *The Theory of the Riemann Zeta-Function,* Oxford Univ. Press, New York, 1951. MR 13, 741.

28. A. M. TURING, "Some calculations of the Riemann zeta-function", *Proc. London Math. Soc.,* ser. 3, v. 3, 1953, pp. 99-117.

29. J. H. WILKINSON, *Rounding Errors in Algebraic Processes,* Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

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