REGULARITY OF STOCHASTIC DELAY EQUATIONS UNDER Ptk ORDER DEGENERACY

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The purpose of this note is to present an extension of a theorem proved by the authors in [1]. Let ft denote the space of all continuous paths $u : R^+ \to R^n$, o^O) = 0, with the topology of uniform convergence on compact subsets of R^+. Suppose (ft^jP) is the complete probability space with & = Borel ft and P Wiener measure on ft.

Theorem 1

Suppose that $g : R^d \to R^{d \times n}$ ***denotes a*** C^m ***map from*** R^d ***into the space*** $R^{d\times n}$ **o / d a** ***matrices, with bounded derivatives of all orders. Assume that there is a positive integer*** **p** ***and positive constants X and 6 such that***

$$g(v)g(v)^* \geq A\min(|v|^{2}P,£)I \qquad \text{(C)}$$

for all **v G** R^d ***, where*** **| |, I,** ***and*** * ***denote respectively the Euclidean norm on*** R^d***, the*** **d * d** ***identity matrix, and matrix transposition. Let*** **x** ***denote the solution of the follwoing stochastic differential delay equation:***

$$\left.\begin{aligned} dx(t) &= g(x(t-r))dW(t), \quad t > 0 \ , \\ x(t) &= \eta(t) \quad , \quad -r \leq t \leq 0 \end{aligned}\right\} \qquad \text{(I)}$$

where **W** ***is normalized n—dimensional Brownian motion on*** **(ft^P)** ***and i is a strictly positive time delay. Suppose the initial path*** ***rj*** **6** $L^{\infty}([-r,0],R^d)$ ***and is such that***

$\int_{-r}^{0} |r/(s)|^2 ds > 0$. *Define* s_Q e [-r,0] *by*

$$s_Q := \sup\{s : s \in [-1,0], \int_{-r}^{s} | rfa) 1^2 du = 0\}.$$

Then for each $t > s_Q + r$, ***the random variable*** $x(t)$ ***has a distribution which is absolutely continuous and has a*** C^{∞} ***density with respect to Lebesgue measure on*** R^d.

Remark:

We proved the theorem in [1] in the special case $p = 1$ using the methodology of the Malliavin calculus. The significance of the result lies in the fact that in view of the time delay r the solution of equation (I) is a non-Markov R^d—valued process and is therefore not amenable to analysis via classical PDE techniques. Indeed prior to [1] the only existing regularity result for non—Markov diffusions was a theorem of Kusuoka and Stroock [2], which requires the assumption that the diffusion coefficient g be *bounded away from* 0.

The proof of Theorem 1 relies on the following result, which we give in its most general form as we feel it may also be of interest in its own right.

Theorem 2

Suppose that y ***is a process in*** R^d ***defined by an Itb integral of the type***

$$y(t) = z + \int^{t} A(s)dW(s), \qquad t \geq 0$$

where $z \in R^d$ ***and*** $A : [0,\infty) \times \Omega \to R^{d \times n}$ ***is a bounded measurable process, adapted to the filtration of*** W. ***Let*** $h : R^d \to R$ ***be a measurable function satisfying the condition***

$$|h(v)| \geq A\min\{|v|^p, 6) \qquad v \in R^d. \tag{D}$$

Let $0 < a < b$ ***and suppose that***

$$P\left[\int_a^b \|y(s)\|^2 ds < \epsilon\right] = o(\epsilon^k) \quad \textit{as} \quad \epsilon \to o+, \quad \textit{for all} \quad k \geq 1. \tag{E}$$

Then

$$P\left[\int_a^b (h(y(s))^2\, ds < \epsilon\right] = o(\epsilon^k) \quad \textit{as} \quad \epsilon \to o+, \quad \textit{for all} \quad k \geq 1\,.$$

N.B. $\|\cdot\|$ *denotes any norm on the space* $\mathbb{R}^{d\times n}$ *of* $d \times n$ *matrices.*

Proof:

Note that we proved this theorem in [1, Lemma 3] for the case $p = 1$. Define $f(v) = |h(v)|^{1/p}$, $v \in \mathbb{R}^d$. Then f satisfies the condition

$$|f(v)| \geq \lambda^{1/p} \min(|v|, \delta^{1/p}), \qquad v \in \mathbb{R}^d.$$

Condition (E) together with Jensen's inequality and Lemma 3 of [1] now imply

$$P\left[\int_a^b h(y(s))^2\, ds < \epsilon^p\right] \leq P\left[\int_a^b f(y(s))^2\, ds < \epsilon\right]$$

$$= o(\epsilon^k) \quad \text{for all} \quad k \geq 1$$

from which the result clearly follows. □

Following [1] we define

$$h(v) \equiv \inf\{|g(v)^*(e)| : e \in S^{d-1}\}\,, \qquad v \in \mathbb{R}^d.$$

Then (C) implies that h satisfies (D); thus h satisfies the conclusion of Theorem 2. We

observe that this step is the only part of the argument in [1] in which the lower bound condition on g is used. Hence the argument in [1] suffices to complete the proof of Theorem 1.

In conclusion we remark that Theorem 1 is a significant extension of the main result in [1]. For example in the one dimensional situation vanishing to some finite order at 0 occurs for any analytic function not identically zero in some neighborhood of 0, whereas in the previous version of the theorem the hypotheses imply that g has essentially linear behavior at 0.

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