FORMATION OF SINGULARITIES FOR VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN HIGHER DIMENSIONS

by

G. T. Kossioris
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Abstract: In this work we study the generation of singularities (shock waves) of the solution of the Cauchy problem for Hamilton-Jacobi equations in several space variables, under no assumption on convexity or concavity of the hamiltonian. We study the problem in the class of viscosity solutions, which are the correct class of weak solutions. We first examine the way the characteristics cross by identifying the set of critical points of the characteristic manifold with the caustic set of the related lagrangian mapping. We construct the viscosity solution by selecting a single-valued branch of the multi-valued function given as a solution by the method of characteristics. We finally discuss how the shocks propagate and undergo catastrophe in the case of two space variables.

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1 Introduction

This paper considers the formation of singularities of the solution of the Cauchy problem for the Hamilton-Jacobi equation

$$u_t + f(u_x) = 0, \ \ (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad t > 0, \qquad (1.1)$$

$$(P)$$

$$u(x,0) = \phi(x),$$

where $\phi, f \in C^\infty(\mathbb{R}^n)$. The initial value problem (P) is well-posed in the sense of viscosity solutions introduced by Crandall and Lions [11]. For small time t the viscosity solution is determined by using the classical method of characteristics. There is, in general, a critical time at which the characteristics start crossing and we can assign different values at the same point as a solution. Thus, the function given as the solution of (P) via the method of characteristics is multi-valued. Therefore, we expect that the globally defined viscosity solution is not C^∞. Indeed, singularities appear, i.e. discontinuities in the derivatives of the solution accross a set of surfaces (shock waves).

In the present work we continue the program started in [16] and we investigate how the shocks are generated in higher dimensions. The Hamiltonian function f may be neither convex nor concave. We first study the way the characteristics cross. We present two alternative approaches. The first one is based on the theory of caustics developed by Arnold [1] (see also [5], [7], [3], [6], [4]); the second uses techniques of the theory of critical points of smooth mappings presented in [19]. We then study the multi-valuedness of the function v given as the solution of (P) by means of the classical method of characteristics. We finally choose the proper single-valued branch as solution and prove that this is actually the viscosity solution. The theory of caustics describes the different ways the caustics may bifurcate in lower dimensions. The information can be used to study how the shocks propagate further and undergo catastrophe.

In the case where f is convex, Fleming [13] has proved that, except for a set of lower dimension, the set of singularities lies on the union of n-dimensional manifolds. Assuming that the convex function f is only continuous, Cannarsa and Soner [8] have proved some regularity properties of the Lipschitz continuous viscosity solution. The complete structure of the shock curves in the case $n = 1$ has been investigated in [16] (see also references cited there). The method of constructing the solution by selecting a single-valued branch of the function given by the method of characteristics is originated in Tsuji [21], who has investigated the case of a semi-concave solution corresponding to a convex Hamiltonian f. In particular, [21] examines the two-dimensional problem using the theory of critical points of smooth mappings from a plane to a plane obtained by Whitney [22]. A rigorous proof for the case of a conservation law in higher dimensions has been obtained by Nakane [19]. The author uses the characterization of critical points of smooth mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by Morin [18].

In this work we study the propagation of shock waves for the correct class of generalized solutions of (P), that is, the viscosity solutions. We also establish the proper setting for studying the way the characteristics cross, which is along the lines of the theory of caustics developed by Arnold [1].

The paper is organized as follows: In Section 2 we review the basic facts about the method of characteristics and viscosity solutions. We show that the viscosity criterion across a shock surface holds provided that the graph of the Hamiltonian f lies below or above the line segment defined by the jump across the shock. In Section 3 we study the way the characteristics cross. In Section 4 we study the multi-valuedness of the function v. The viscosity solution is constructed in Section 5. Finally, in Section 6, we discuss how the shock waves propagate and interact, for $n = 2$.

By the time this work was completed, the author has received Nakane [20], which uses the techniques presented in [19] to study the formation of shocks for the semi-concave solution of (P) corresponding to a convex Hamiltonian f.

2 Preliminaries

2.1 Method of characteristics

The classical method of characteristics for the Cauchy problem (P) reduces the solution of the partial differential equation to solving the following ordinary-differential initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f'(p), \dfrac{dp}{dt} = 0, \dfrac{dv}{dt} = -f(p) + p^T \cdot f'(p), \ \ x, p \in I\!R^n, \\ \\ x(0) = y, p(0) = \phi'(y), v(0) = \phi(y), \ \ y \in I\!R^n. \end{cases} \tag{2.1}$$

In the sequel a vector will be considered as a $(n, 1)$ matrix and its transpose will be denoted by a superscript $(.^T)$. The operation of multiplication is taken in the sense of matrices. The equations (2.1) are the *characteristic equations* related to (P) and their solutions are given explicitly by the equations

$$x(y,t) = y + tf'(\phi'(y)), \tag{2.2}$$

$$p(y,t) = \phi'(y), \ \ v(y,t) = t\{-f(\phi'(y)) + (\phi'(y))^T \cdot f'(\phi'(y))\} + \phi(y). \tag{2.3}$$

Suppose that we want to find the value of the solution u of (P) at the point (x, t) using the classical method of characteristics. Then for a fixed time t, t sufficiently small, there exists a unique characteristic line (2.2) passing through the point (x, t) originating from the point $(y, 0)$. Indeed consider the mapping

$$x_t : y \rightarrow x = x_t(y) \equiv x(y,t).$$

Then for t near zero,

$$\left| \frac{\partial x_t(y)}{\partial y} \right| \neq 0,$$

where $|\cdot|$ denotes the determinant of the matrix $\dfrac{\partial x_t}{\partial y}$ and x_t is a C^∞-diffeomorphism. If

$$x_t^{-1} : x \to y = x_t^{-1}(x) \quad , \quad x \in I\!R^n,$$

is the inverse of x_t, then the solution of (P) is given by

$$u(x,t) = v(x_t^{-1}(x), t).$$

In general, there is a critical time t_0 beyond which

$$\left|\frac{\partial x_t(y)}{\partial y}\right| = 0, \;\; y \in \Sigma_t^y \subset I\!R^n,$$

where $\Sigma_t^y \neq \emptyset$ is the set of critical points of x_t. We denote by Σ_t^x the image of Σ_t^y under the mapping x_t. For $t > t_0$ the characteristic lines (2.2) start crossing and x_t^{-1} becomes multi-valued. Therefore, the map

$$x \to v(x_t^{-1}(x), t)$$

is also multi-valued. The folds of its graph, however, have a certain form due to the Hamiltonian structure of the characteristic equations.

2.2 The notion of viscosity solution

Since the characteristics cross, it is not possible, in general, to have a globally defined smooth solution. This difficulty has been overcome with the introduction of the correct class of generalized solutions, called *viscosity solutions* ([11]; see also [10]) for which (P) is globally well-posed. In the present work, the viscosity solutions are taken to be continuous. Beyond the critical time they are no longer C^∞, because discontinuities appear in the derivatives of the solution.

Let $\Gamma(x,t)$ be a smooth n-dimensional surface in $\mathbb{R}^n_x \times \mathbb{R}_t$. Let $\mathcal{O}$ be a bounded domain separated by $\Gamma(x,t)$ into two domains $\mathcal{O}^+$ and $\mathcal{O}^-$ such that $\mathcal{O} = \mathcal{O}^+ \cup \mathcal{O}^- \cup \Gamma(x,t)$. Then we have the following.

Theorem 2.1. *Let $u \in C(\mathcal{O})$ and $u = u^+$ in $\mathcal{O}^+ \cup \Gamma(x,t), u = u^-$ in $\mathcal{O}^- \cup \Gamma(x,t)$ where $u^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma(x,t))$. Then u is a viscosity solution of* (1.1) *in $\mathcal{O}$ if and only if the following conditions hold true:*

(a) *u^+ and u^- are classical solutions of* (1.1) *in $\mathcal{O}^+$ and $\mathcal{O}^-$, respectively.*

(b) *If the vector $\tilde{\eta} = (-(u_x^+ - u_x^-), f(u_x^+) - f(u_x^-))$ points into $\mathcal{O}^+$, then*

$$f((1-\lambda)u_x^+ + \lambda u_x^-) \leq (1-\lambda)f(u_x^+) + \lambda f(u_x^-);$$

if $\tilde{\eta}$ points into $\mathcal{O}^-$, then

$$f((1-\lambda)u_x^- + \lambda u_x^+) \geq (1-\lambda)f(u_x^-) + \lambda f(u_x^+),$$

where $\lambda \in [0,1]$. In particular, the graph of f lies, respectively, below or above the line segment joining the points $(u_x^+, f(u_x^+))$ and $(u_x^-, f(u_x^-))$.

The proof of the theorem is given in [16]. The condition (b) will be referred in the sequel as the *viscosity criterion*. The surface $\Gamma(x,t)$ in the neighborhood of which u has the properties specified in the above theorem is called a *shock surface*.

3 The caustic structure of the critical points of the characteristic manifold

As we have seen in the previous section, there is in general a critical time t_0 beyond which x_t^{-1} becomes multi-valued. Due to the special form of the characteristic equations, the critical points of x_t are generically of specific type. Indeed, the characteristic equations are given in Hamilton's canonical form, i.e.

$$\frac{dx_i}{dt} = \frac{\partial f(p)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial f(p)}{\partial x_i} = 0, \quad i = 1, \cdots, n. \tag{3.1}$$

If we consider the mapping

$$g_t : (x_0, p_0) \in \mathbb{R}^{2n}_{(x,p)} \to (x, p) = (x_0 + tf'(p_0), p_0) \in \mathbb{R}^{2n}_{(x,p)},$$

then, since

$$\left| \frac{\partial(x,p)}{\partial(x_0, p_0)} \right| = 1,$$

g_t defines an 1-parameter group of diffeomorphisms, i.e. a *Hamiltonian phase flow* with Hamiltonian function f (see [2], p.204). In the sequel we shall use the notation $\mathbb{R}^{2n}_{(x,p)}$ to denote that $\mathbb{R}^{2n}$ is considered as the range set of the variable (x, p).

For the sake of completeness, we next recall some elementary facts from symplectic geometry. The space $\mathbb{R}^{2n}_{(x,p)}$ equipped with the standard symplectic structure

$$w^2 = x_1 \wedge p_1 + \cdots + x_n \wedge p_n \tag{3.2}$$

is a symplectic space; see [2, p.219].

Definition 3.1. *An n-dimensional submanifold of the symplectic space $I\!R^{2n}_{(x,p)}$ on which the symplectic form w^2 induces the null form is called a* Lagrangian submanifold.

Consider the manifold

$$W_0 = \Big((x_0, p_0) \in I\!R^{2n}_{(x,p)}, x_0 = y, p_0 = \phi'(y), y \in I\!R^n_y\Big).$$

The following proposition was established in [2], p. 440.

Lemma 3.1. *For any smooth function ϕ the graph W_0 of its gradient is a Lagrangian submanifold of $I\!R^{2n}_{(x,p)}$.*

We also have the following proposition (see [5], p.289).

Proposition 3.1. *The Hamiltonian phase flow preserves the symplectic structure.*

Lemma 3.1 and the Proposition 3.1 yield

Lemma 3.2. *The image $W_t = g_t(W_0)$ of the manifold W_0 under the Hamiltonian phase flow g_t is a Lagrangian submanifold of $I\!R^{2n}_{(x,p)}$.*

Let

$$\sqcap : I\!R^{2n}_{(x,p)} \to I\!R^n_x, \ \sqcap(x,p) = x,$$

be the projection mapping and

$$\sqcap_t : W_t \subset \mathbb{R}^{2n}_{(x,p)} \to \mathbb{R}^n_x, \ \sqcap_t(W_t) \equiv \sqcap(W_t)$$

its restriction on W_t.

Definition 3.2. *The mapping* $\sqcap_t : W_t \to \mathbb{R}^n_x$ *is called a Lagrangian mapping. The set of critical values of the Lagrangian mapping is said to be a caustic.* By representing the manifold W_t in y-coordinates we easily obtain

Proposition 3.2. *The caustic of the Lagrangian mapping* $\sqcap_t$ *is the image* Σ_t^x *of* Σ_t^y *under the mapping* x_t.

Our first goal is to study the multi-valuedness of the mapping x_t^{-1}. This is equivalent to studying the structure of the surface Σ_t^x; in particular, the form of the generated caustic and the way caustics undergo metamorphosis as t evolves. By metamorphosis we mean how caustics interact and change from one type to another and the way that different branches of caustics cross. The question of the generation and bifurcation of caustics has been answered along the lines of the critical point theory of smooth mappings; (see e.g. [5], Section 22.3). The obtained results are *generic*, that is they hold for a residual subset of the set of Lagrangian mappings. A subset is called residual, if it is the intersection of a countable number of sets, each of which is both open and dense in the set of Lagrangian mappings. For $n \leq 4$ there are only finitely many inequivalent types of metamorphoses of caustics (see [5], [1], [7], [4]). The exact pictures for $n = 2$ are given in [7], [4 Fig. 36], [6] and for $n = 3$ are given in [5, Fig. 64, 65], [4, Fig. 40,41], [7]; see also [3]. For $n > 4$ the caustics may be unstable and may bifurcate in infinitely many ways. The propagation of shocks, however, is not affected by the crossing of the different branches of caustics. The generated shock remains attached to the branches by which it has been initially determined and does not follow the evolution of the others.

In this work we shall investigate the way the shocks are generated, except in Section 6 where we discuss how the shocks propagate further in the case $n = 2$. To accomplish the goal above, we need to describe the multivaluedness of the mapping x_t^{-1} just after the first time t_0 for which $\Sigma_t^y \neq \emptyset$. This can be done without the use of the results of the theory of caustics, by expressing the mapping x_t in canonical coordinates, following the lines of [19]. This task is undertaken in the rest of this section.

Let $\lambda_i(y), y \in I\!R^n$ be the eigenvalues of the matrix $f''(\phi'(y)) \cdot \phi''(y)$. We make the following generic assumptions (see [1], [6]):

(A1) λ_n has a negative nondegenerate local minimum for $y = y_0$ and $\Sigma_t^y = \emptyset$ for $t < t_0, \Sigma_{t_0}^y = \{y_0\}$.

(A2) $\lambda_n(y) < \lambda_{n-1}(y) \leq \cdots \leq \lambda_1(y)$ in a neighborhood of the point y_0.

Then it is obvious that for $t > t_0, t$ near t_0,

$$\Sigma_t^y = \{y \in I\!R_y^n : 1 + t\lambda_n(y) = 0\}.$$

Moreover, $t_0 = -1/\lambda_n(y_0)$ and Σ_t^y is an n-dimensional manifold in $\mathbb{R}_y^n$. We have the following:

Lemma 3.3. *At least one of the principal minors of order $(n-1)$ of the matrix $\frac{\partial x_{t_0}(y_0)}{\partial y}$ is different from zero.*

Proof: Let $\tilde{\lambda}_i = 1, \cdots, n$, be the eigenvalues of the matrix $\frac{\partial x_t(y_0)}{\partial y}$ and $\sigma(\tilde{\lambda})$ its characteristic polynomial. Since $\tilde{\lambda}_i = 1 + t_0\lambda_i(y_0)$, using (A2) we get

$$\frac{d\sigma(\tilde{\lambda}_n)}{d\tilde{\lambda}} = \prod_{\lambda=1}^{n-1}\left(\tilde{\lambda}_n - \tilde{\lambda}_i\right) = \triangle_{n-1} \neq 0,$$

where $\triangle_{n-1}$ is the sum of the principal minors of order $(n-1)$. $\square$

We next study the multi-valuedness of the mapping x_t^{-1}. Consider the mappings

$$\tilde{Y}_t : \mathbb{R}_y^n \to \mathbb{R}_Y^n, \ \ \tilde{X}_t : \mathbb{R}_Y^n \to \mathbb{R}_x^n,$$

defined by

$$\left.\begin{aligned} Y_i &= y_i + tf_{p_i}(\phi'(y)), \ \ i = 1, \cdots, n-1 \\ Y_n &= y_n \end{aligned}\right\} (\tilde{Y}_t)$$

and

$$\left.\begin{aligned} x_i &= Y_i, \ \ i = 1, \cdots, n-1 \\ x_n &= Y_n + tf_{p_n}\left(\phi'(\tilde{Y}_t^{-1}(Y))\right) \end{aligned}\right\} (\tilde{X}_t)$$

In view of Lemma 3.3, without loss of generality, we can assume that $\tilde{Y}_t$, $t > t_0, t$ near t_0, is a C^∞-diffeomorphism. Moreover,

$$\Sigma_t^Y = \tilde{Y}_t(\Sigma_t^y) = \{Y \in \mathbb{R}_Y^n : \frac{\partial x_n}{\partial Y_n} = 0\}.$$

Next, we define the sets

$$\Sigma_t^{Y,1} = \{Y \in \Sigma_t^Y : \frac{\partial^2 x_n}{\partial Y_n^2} = 0\},$$

$$\Sigma^Y = \{(Y,t) : Y \in \Sigma_t^Y\}, \qquad \Sigma^{Y,1} = \{(Y,t) : Y \in \Sigma_t^{Y,1}\},$$

$$\Sigma^x = \{(x,t) : x \in \tilde{X}_t(\Sigma_t^Y)\}, \ \Sigma^{x,1} = \{(x,t) : x \in \tilde{X}_t(\Sigma_t^{Y,1})\}.$$

We, finally, denote by Ω_t^Y the domain bounded by Σ_t^y and by Ω_t^x the domain $\tilde{X}_t^{-1}(\Omega_t^Y)$.

We now state the following definition.

Definition 3.3. *A critical point Y_0 of the mapping $\tilde{X}_t : I\!R_Y^n \to I\!R_x^n$ is said to be a fold (respectively cusp), if in a neighborhood of Y_0, $\tilde{X}_t$* is differentiably equivalent (*see* [5, page 8]) *to the mapping*

$$x_1 = Y_1, \cdots, x_{n-1} = Y_{n-1}, x_n = Y_n^2,$$

$$(\textit{respectively}, \ \ x_1 = Y_1, \cdots, x_{n-1} = Y_{n-1}, x_n = Y_n^3 + Y_{n-1}Y_n).$$

According to the characterization of cusp points given in [18] (see also [19]) a point of $\Sigma^{Y,1}$ is a cusp if and only if

$$\frac{\partial^3 x_n}{\partial Y_n^3} \neq 0, \ \frac{\partial^2 x_n}{\partial t \partial Y_n} \neq 0. \tag{3.3}$$

We use the above relations to obtain

Proposition 3.3. *For t near t_0 the following hold:*

i) *The set $\Sigma^{x,1}$ consists of cusps and the set $\Sigma^x \setminus \Sigma^{x,1}$ consists of folds.*

ii) *The set* $\Sigma^{Y,1}$ *is a* C^∞*-submanifold of* $I\!R^{n+1}_{(Y,t)}$ *of codimension 2, parametrized by* $(Y_1,\cdots,Y_{n-1})$.

iii) *The set* $\Sigma^{x,1}$ *is a* C^∞*-submanifold of* $I\!R^{n+1}_{(x,t)}$ *of codimension 2, parametrized by* $(x_1,\cdots,x_{n-1})$.

Proof: It is easy to see that

$$\frac{\partial x_n}{\partial Y_n}=\left|\frac{\partial \tilde{X}_t}{\partial Y}\right|=\left|\frac{\partial x}{\partial y}\right|\cdot\left|\frac{\partial \tilde{Y}_t}{\partial y}\right|^{-1}=\Big(1+t\lambda_n(y)\Big)\cdot\left[\prod_{i=1}^{n-1}(1+t\lambda_i(y))\right]\cdot\left|\frac{\partial \tilde{Y}_t}{\partial y}\right|^{-1}.$$

Therefore

$$\Sigma_t^Y=\{Y\in I\!R^n_Y:1+t\lambda_n(\tilde{Y}_t^{-1}(Y))=0\},\Sigma_t^{Y,1}=\{Y\in\Sigma_t^Y:\frac{\partial}{\partial Y_n}\lambda_n(\tilde{Y}_t^{-1}(Y))=0$$

and for $Y\in\sum_t^{Y,1}$,

$$\frac{\partial^3 x_n}{\partial Y_n^3}=t\frac{\partial^2}{\partial Y_n^2}\lambda_n(\tilde{Y}_t^{-1}(Y))\cdot\left[\prod_{i=1}^{n-1}(1+t\lambda_i(\tilde{Y}_t^{-1}(Y))\right]\cdot\left|\frac{\partial \tilde{Y}_t}{\partial y}\right|^{-1},$$

$$\frac{\partial^2 x_n}{\partial t\partial Y_n}=\lambda_n(\tilde{Y}_t^{-1}(Y))\cdot\left[\prod_{i=1}^{n-1}(1+t\lambda_i(\tilde{Y}_t^{-1}(Y)))\right]\cdot\left|\frac{\partial \tilde{Y}_t}{\partial y}\right|^{-1}\neq 0.$$

Therefore

$$\frac{\partial^2\lambda_n(y_0)}{\partial Y_n^2}=\Big(\frac{\partial\tilde{Y}_t^{-1}(Y_0)}{\partial Y_n}\Big)^T\cdot\frac{\partial^2\lambda_n(y_0)}{\partial y^2}\cdot\frac{\partial\tilde{Y}_t^{-1}(Y_0)}{\partial Y_n}>0,\ \ Y_0=\tilde{Y}_t(y_0),$$

and hence

$$\frac{\partial^2\lambda_n(\tilde{Y}_t^{-1}(Y))}{\partial Y_n^2}\neq 0,\ \ Y\in\Sigma_t^{Y,1}.$$

According to (3.3), $\Sigma^{Y,1}$ is the set of cusp points. Since, for $t=t_0, y=y_0$,

$$\left|\frac{\partial(1+t\lambda_n(y),\partial\lambda_n/\partial Y_n)}{\partial(t,Y_n)}\right|=\lambda_1(y_0)\cdot\frac{\partial^2\lambda_1(y_0)}{\partial Y_n^2}\neq 0,$$

(ii) is easily deduced from the implicit function theorem. $\square$

The geometry of Σ_t^x described by Proposition 3.3 and the theory of caustics is the same. According to the theory of caustics, Σ_t^x, t near t_0, has the form of a "pancake"; see Figure 1, where $x' = (x_1, \ldots, x_{n-1}), Y' = (Y_1, \ldots, Y_{n-1})$. The set $\Sigma_t^{x,1}$ of the cusps consists of the "points" $X_A, X_B, (Y_A = \tilde{X}_t^{-1}(X_A), Y_B = \tilde{X}_t^{-1}(X_B))$, which represent a manifold of codimension 2 in $I\!R_x^n$. See e.g. Figure 2, for the case $n = 3$, where the caustic looks like a "flying saucer" whose circular edge is the set $\Sigma_t^{x,1}$. A mapping is three-valued around a cusp point; see e.g. [22]. Therefore, the image of a level curve $\ell : I\!R \to I\!R_Y^n, \ell(s) = (Y' = \text{const.}, Y_n = s)$, in the domain Ω_t^x is triply-folded. See Figure 1, where the three folds have been drawn separated to show the change of the orientation of $\tilde{X}_t^{-1}(\ell(s))$ as s increases. Hence, the inverse mapping $\tilde{X}_t^{-1}$ is three-valued on Ω_t^x.

We conclude this section by introducing several notations which will be used in the sequel. In particular, we denote by $Y = \Phi^i(x;t), \Phi^i = (\Phi_1^i, \ldots, \Phi_n^i) \in I\!R_Y^n$, the three branches of $\tilde{X}_t^{-1}$ defined on Ω_t^x. Let also $\Omega_{t,i}^Y$ be the image of Ω_t^x under the mapping Φ^i. We name $\Omega_{t,i}^Y$ in such a way that $\Omega_{t,1}^Y$ is met first by a point moving on ℓ, along the positive direction, and $\Omega_{t,2}^Y = \Omega_t^Y$. The sets Σ^Y and Σ^x described by Proposition 3.3 are depicted in Figure 3. The level surfaces t=const. are the previously described caustics Σ_t^Y and Σ_t^x.

4 Formation of shocks

Here, we study the way the graph of $v(x_t^{-1}(x), t)$ folds for $t > t_0$. We accomplish that by examining the relative position of the different branches of v on a properly chosen level surface, that is Y' =const., in the $I\!R^{2n+1}_{(Y,v)}$ space. Throughout this section we shall assume that t is sufficiently close to t_0. We shall also introduce the notation A^{τ} to denote the matrix $(A^T)^{-1}$, where A is a nonsingular matrix. We have the following

Lemma 4.1. *Let $s \to y(s)$, $s \in (a, \beta) \subset I\!R$, be a smooth simple curve in the $y-$space, then*

$$\frac{d}{ds}v(y(s), t) = (\phi'(y))^T \frac{d}{ds}x_t(y(s)).$$

Proof: We have

$$\frac{d}{ds}v(y(s), t) = \left(\frac{\partial v}{\partial y}\right)^T \cdot \frac{dy}{ds} = \left[\left(\frac{\partial}{\partial y}(\tilde{X}_t \circ \tilde{Y}_t)(y)\right)^T \cdot \phi'(y(s))\right]^T \cdot \frac{dy}{ds}$$
$$= (\phi'(y(s)))^T \left[\left(\frac{\partial x}{\partial y}\right) \cdot \frac{dy}{ds}\right] = (\phi'(y(s)))^T \cdot \frac{dx_t(y(s))}{ds}. \qquad \square$$

Lemma 4.2. *Consider the mapping $\tilde{Y}_t' : R^{n-1}_{y'} \to R^{n-1}_{Y'}$, where $(Y', Y_n) =$ $\tilde{Y}_t(y', y_n)$ and $y' = (y_1, \ldots, y_{n-1})$. The following identities hold:*

$$\frac{\partial}{\partial Y'} = \left(\frac{\partial \tilde{Y}_t'}{\partial y'}\right)^{\tau} \cdot \frac{\partial}{\partial y'}, \tag{4.1}$$

$$\frac{\partial}{\partial Y_n} = -\left(\frac{\partial \tilde{Y}_t'}{\partial y_n}\right)^T \cdot \left[\left(\frac{\partial \tilde{Y}_t'}{\partial y'}\right)^{\tau} \cdot \frac{\partial}{\partial y'}\right] + \frac{\partial}{\partial y_n}, \tag{4.2}$$

$$\frac{\partial}{\partial x_n} = \left|\frac{\partial \tilde{X}_t}{\partial \tilde{Y}_t}\right|^{-1} \cdot \frac{\partial}{\partial Y_n}, \tag{4.3}$$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial Y_i} - t\left|\frac{\partial \tilde{X}_t}{\partial \tilde{Y}_t}\right|^{-1} \frac{\partial}{\partial Y_i}(f_{p_n}(\phi_y'(\tilde{Y}_t^{-1}(Y)))\frac{\partial}{\partial Y_n},\ i \neq n, \tag{4.4}$$

provided that $\left|\frac{\partial \tilde{X}_t}{\partial \tilde{Y}_t}\right| \neq 0.$

Proof: The chain rule yields

$$\frac{\partial}{\partial y} = \left(\frac{\partial \tilde{Y}_t}{\partial y}\right)^T \frac{\partial}{\partial Y}.$$

Thereby,

$$\frac{\partial}{\partial y'} = \left(\frac{\partial \tilde{Y}_t'}{\partial y'}\right)^T \frac{\partial}{\partial Y'},\ \frac{\partial}{\partial y_n} = \left(\frac{\partial \tilde{Y}_t'}{\partial y_n}\right)^T \frac{\partial}{\partial Y'} + \frac{\partial}{\partial Y_n}.$$

Using these equalities, (4.2) is easily obtained.

Equalities (4.3) and (4.4) are obtained from the relation

$$\frac{\partial}{\partial Y} = \left(\frac{\partial \tilde{X}_t}{\partial \tilde{Y}_t}\right)^T \frac{\partial}{\partial x}. \qquad \square$$

We can study the geometry of the graph of v more easily if the level curves Y' =const. are chosen in such a way that the matrix $f''(\phi'(y_0))$ is diagonal. The following lemma is helpful in this direction.

Lemma 4.3. *By an affine transformation of coordinates we may assume that the matrix $f''(\phi'(y_0))$ is diagonal. The corresponding mapping x_t of the transformed problem has the same eigenvalues.*

Proof: Let $B = (b_{ij}),\ i,j = 1,\ldots,n$, be a nonsingular matrix. If

$$\tilde{x}_j = y_{0j} + \sum_{i=1}^{n} b_{ji}(x_i - y_{0i}), \tag{4.5}$$

then

$$u_x = B^T \cdot \tilde{u}_{\tilde{x}}$$

where

$$\tilde{u}(\tilde{x},t) = u(y_0 + B^{-1}(\tilde{x} - y_0), t).$$

Let

$$\tilde{\phi}(\tilde{x}) = \tilde{u}(\tilde{x},0) = \phi(y_0 + B^{-1}(\tilde{x} - y_0)).$$

The problem (P) is transformed to the equivalent

$$\begin{cases} \tilde{u}_t + \tilde{f}(\tilde{u}_{\tilde{x}}) = 0 \\ \tilde{u}(\tilde{x},0) = \tilde{\phi}(\tilde{x}) \end{cases} \qquad (\tilde{P})$$

where $\tilde{f}(\tilde{p}) = f(B^T \cdot \tilde{p})$. Moreover,

$$\tilde{f}''(\tilde{p}) = B \cdot f''(P) \cdot B^T.$$

We choose as B the orthonormal matrix which diagonalizes $f''(\phi'(y_0))$. It is also easily seen that

$$\frac{\partial \tilde{x}_t(\tilde{y})}{\partial \tilde{y}} = B \cdot (I + tf''(\phi'(y))) \cdot \phi''(y)) \cdot B^{-1}. \tag{4.6}$$

where

$$\tilde{y}_j = y_{0j} + \sum_{i=1}^{n} b_{ji}(y_i - y_{0i}).$$

Therefore $\dfrac{\partial \tilde{x}_t}{\partial \tilde{y}}$ and $\dfrac{\partial x_t}{\partial y}$ have the same eigenvalues. $\square$

From now on we shall assume that the matrix $f''(\phi'(y_0))$ is diagonal.

Lemma 4.4. *The following inequality holds:*

$$\frac{\partial}{\partial Y_n}\left(\frac{\partial \phi}{\partial y_n}\right) \neq 0\ ,\ Y = \tilde{Y}_{t_0}(y_0).$$

Proof: Since $\left|\dfrac{\partial \tilde{X}_t}{\partial \tilde{Y}_t}\right| = 0$ for $Y = \tilde{Y}_{t_0}(y_0)$,

$$1 + t_0\, \frac{\partial}{\partial Y_n}(f_{p_n}(\phi'(y))) = 0,\ \ y = y_0.$$

Using the fact that $f''(\phi'(y_0))$ is diagonal, we get

$$1 + t_0 f_{p_n p_n}(\phi'(y))\frac{\partial}{\partial Y_n}\left(\frac{\partial \phi(y)}{\partial y_n}\right) = 0,\ \ y = y_0. \qquad \square$$

Let $u^i(x,t) = v(\tilde{Y}_t^{-1}(\Phi_i(x;t)),t)$, $i = 1,2,3$, be the three branches of $v(x_t^{-1}(x),t), x \in \Omega_t^x$.

Proposition 4.1. *For any $x \in \Omega_t^x, t > t_0$,*

$$(u^2(x,t) - u^1(x,t))(u^2(x,t) - u^3(x,t)) > 0.$$

Proof: Let $s \to x(s), s \in I\!R$, be the level curve in the $I\!R_x^n$ space, given by

$$x'(s) = x_0',\ x_n = s,\ x_0' \in I\!R_{x'}^{n-1},$$

and assume that it intersects Ω_t^x. If $y^i(s) = \tilde{Y}_t^{-1}(\Phi^i(x(s);t))$, then Lemma 4.1 yields

$$\frac{\partial u^1}{\partial x_n}(x(s),t) - \frac{\partial u^2}{\partial x_n}(x(s),t) = \frac{\partial \phi}{\partial y_n}(y^1(s)) - \frac{\partial \phi}{\partial y_n}(y^2(s)) \qquad (4.7)$$

$$\frac{\partial u^3}{\partial x_n}(x(s),t) - \frac{\partial u^2}{\partial x_n}(x(s),t) = \frac{\partial \phi}{\partial y_n}(y^3(s)) - \frac{\partial \phi}{\partial y_n}(y^2(s)), \tag{4.8}$$

for x in Ω_t^x. In view of (4.2)

$$\frac{\partial x_n}{\partial Y_n} = -t\left(\frac{\partial \tilde{Y}_t'}{\partial y_n}\right)^T \cdot \left[\left(\frac{\partial \tilde{Y}'}{\partial y'}\right)^T \cdot \frac{\partial}{\partial y'}(f_{p_n}(\phi'(y)))\right]$$

$$+1 + t\frac{\partial}{\partial y_n}(f_{p_n}(\phi'(y))),\ y = y_0,\ t = t_0. \tag{4.9}$$

The right-hand side of the above equation is the Schur complement of the matrix $\left(\frac{\partial \tilde{Y}_{t_0}'}{\partial y'}\right)^T$ in $\left(\frac{\partial x_{t_0}(y_0)}{\partial y}\right)^T$ (see [15] p. 22),
therefore

$$\frac{\partial x_n(y_0)}{\partial Y_n} = \left|\frac{\partial x_{t_0}(y_0)}{\partial y}\right| \left|\frac{\partial \tilde{Y}_{t_0}'}{\partial y'}\right|^{-1}.$$

In view of Lemma 3.3, we may assume that

$$\left|\frac{\partial \tilde{Y}_{t_0}'}{\partial y'}\right| > 0, \quad y = y_0. \tag{4.10}$$

Since $|\frac{\partial x_t(y)}{\partial y}|$ is less than zero in Ω_t^y, $t > t_0$, and it changes sign outside Ω_t^y, we have that

$$\frac{\partial x_n(y^i(s))}{\partial Y_n} > 0,\ i = 1,3, \quad \frac{\partial x_n(y^2(s))}{\partial Y_n} < 0,$$

away from the caustic Σ_t^y. Therefore

$$Y_n(y^1(s);t) < Y_n(y^2(s);t) < Y_n(y^3(s);t). \tag{4.11}$$

In view of Lemma 4.4, we may also assume that

$$\frac{\partial}{\partial Y_n}\left(\frac{\partial \phi}{\partial y_n}\right) < 0, \quad Y = \tilde{Y}_{t_0}(y_0). \tag{4.12}$$

By means of (4.7), (4.8), (4.11), (4.12) we conclude that

$$\begin{aligned} &\frac{\partial}{\partial x_n}\left[u^1(x(s),t) - u^2(x(s),t)\right] > 0, \\ &\frac{\partial}{\partial x_n}\left[u^3(x(s),t) \ - \ u^2(x(s),t)\right] < 0, \end{aligned} \tag{4.13}$$

for $x(s) \in \Omega_t^x, t > t_0$. Finally, if s_1, s_2 correspond to the points $x(s_1), x(s_2) \in \Sigma_t^x$ and $x_n(s_2) > x_n(s_1)$, then

$$u^1(x(s_2),t) = u^2(x(s_2),t), \ u^3(x(s_1),t) = \ u^2(x(s_1),t).$$

Therefore, in view of (4.13), we obtain

$$u^3(x(s),t) < u^2(x(s),t), \quad u^1(x(s),t) < u^2(x(s),t), \quad x \in \Omega_t^x. \quad \square \tag{4.14}$$

The graph of the function $u(x_t^{-1}(x),t)$ with respect to x is depicted in Figure 4. If the opposite inequality holds in (4.12), inequalities (4.14) are reversed. The cusp points of Σ_t^x correspond to the swallowtail singularities of the graph of $v(x_t^{-1}(x),t)$. This is in accordance with the results obtained for the one-dimensional problem (see [16]).

5 Construction of the shock surface

In this section we shall construct the viscosity solution of the problem (P) up to the time $t_0 + \varepsilon, \varepsilon$ sufficiently small. This is accomplished by selecting a single-valued branch of $v(x_t^{-1}(x), t)$ and proving that it is a viscosity solution. In the case of the one-dimensional problem, i.e. $n = 1$, we have that

$$f''(\phi'(y_0))\phi''(y_0)) = -\frac{1}{t_0},$$

where (y_0, t_0) corresponds to the point of generation of the shock. In particular, the function f is convex or concave in a neighborhood of the point $\phi'(y_0)$ and the viscosity criterion is automatically satisfied (see [16]). For $n \neq 1$ we expect an analogous situation, i. e. a property related to convexity or concavity must hold but only in a specific direction. Along this direction the viscosity criterion must be satisfied. This direction is along the level curves $Y' =$const. of the space $\mathbb{R}^n_Y$.

In order that we simplify the calculations we have to make a proper transformation of coordinates. Thus, we study the solution $\tilde{u}(\tilde{x}, t)$ of the transformed problem $(\tilde{P})$, where $\tilde{B}$ substitutes for B in (4.5). The matrix $\tilde{B}$ will be chosen properly in the sequel. The viscosity solution of (P) is obtained by virtue of

$$u(x, t) = \tilde{u}(y_0 + \tilde{B} \cdot (x - y_0), t),$$

see e.g. [11]. The solution $\tilde{u}$ is classically defined outside $\Omega_t^{\tilde{x}}$. We construct the solution in $\Omega_t^{\tilde{x}}$ by choosing a single-valued branch of the corresponding v

function, which we denote by $\tilde{v}$. In order to prove that the so chosen function is the viscosity solution, we shall make the following assumption

$$(A2') \qquad \lambda_n = \lambda_n(y_0) < \lambda_{n-1} = \lambda_{n-1}(y_0) < \ldots < \lambda_1 = \lambda_1(y_0),$$

which is stronger than (A2). According to (4.6), let $\tilde{B}$ be the matrix which diagonalizes $H = (f'' \cdot \phi'')(y_0)$ with diagonal elements $H_{ii} = \lambda_i(y_0), i = 1, \ldots, n$. Since an affine transformation does not alter the type of multi-valuedness of a function, the graph of $\tilde{v}$ folds in the way described in the previous section. In the sequel, for convenience, we omit writing the tildes. Let $u^i(x,t)$ be the branches of $v(x_t^{-1}(x), t)$ defined along the lines of the previous section. Let $\Gamma(x,t)$ be the smooth surface along which the two branches $u^1(x,t)$ and $u^3(x,t)$ intersect and $\Gamma_t(x)$ be its projection in the x-space (see Figure 4). We denote by $\Omega_{t,1}^x$ and $\Omega_{t,2}^x$ the two parts of Ω_t^x into which Ω_t^x is separated by $\Gamma(x,t)$ and we assume that $\Omega_{t,1}^x$ is the domain which is met first by a point moving along the positive direction of the level curve Y'=const. of the space $\mathbb{R}_Y^n$. Let u be the function defined by

$$u(x,t) = \begin{cases} v(x_t^{-1}(x), t), & x \not\in \Omega_t^x \\ u^1(x,t), & x \in \Omega_{t,1}^x \\ u^3(x,t), & x \in \Omega_{t,2}^x. \end{cases} \tag{5.1}$$

We want to prove that u is the viscosity solution of $(\tilde{P})$. Since u is a classical solution away from $\Gamma(x,t)$, the viscosity criterion must be satisfied across

$\Gamma(x,t)$. That is, the following conditions must hold across the surface $\Gamma_t(x)$:

$$\begin{cases} \text{if } \dfrac{\partial u^1}{\partial x_n} > \dfrac{\partial u^3}{\partial x_n}, \text{ then} \\ f\Big((1-\lambda)\dfrac{\partial u^3}{\partial x} + \lambda\dfrac{\partial u^1}{\partial x}\Big) \le (1-\lambda) f\Big(\dfrac{\partial u^3}{\partial x}\Big) + \lambda f\Big(\dfrac{\partial u^1}{\partial x}\Big), \end{cases}$$

and

$$\begin{cases} \text{if } \dfrac{\partial u^1}{\partial x_n} < \dfrac{\partial u^3}{\partial x_n}, \text{ then} \\ f\Big((1-\lambda)\dfrac{\partial u^1}{\partial x} + \lambda\dfrac{\partial u^3}{\partial x}\Big) \ge (1-\lambda) f\Big(\dfrac{\partial u^1}{\partial x}\Big) + \lambda f\Big(\dfrac{\partial u^3}{\partial x}\Big), \lambda \in [0,1]. \end{cases}$$

But for t near t_0,

$$\frac{\partial u^3}{\partial x} - \frac{\partial u^1}{\partial x} = [\Phi_n^3(x,t) - \Phi_n^1(x,t)]\frac{\partial}{\partial Y_n}\Big[\frac{\partial \phi}{\partial y}(y)\Big], y = y_0 + \theta(t, y_0), \qquad (5.2)$$

where $\theta(t, y_0) \to 0$ uniformly as $t \to t_0$. In view of (4.2),

$$\frac{\partial}{\partial Y_n}\Big(\frac{\partial \phi(y)}{\partial y}\Big) = \frac{\partial}{\partial y_n}\Big(\frac{\partial \phi}{\partial y}\Big),\ y = y_0.$$

It is easily seen that

$$\Big[\frac{\partial}{\partial y}\Big(\frac{\partial \phi(y)}{\partial y_n}\Big)\Big]^T \cdot f''(\phi'(y)) \cdot \frac{\partial}{\partial y}\Big(\frac{\partial \phi(y)}{\partial y_n}\Big) =$$

$$= \frac{\partial^2 \phi(y)}{\partial y_n^2}\Big(\sum_{i=1}^{n} \frac{\partial^2 f(\phi'(y))}{\partial p_i \partial p_n} \frac{\partial^2 \phi(y)}{\partial y_i \partial y_n}\Big) = -\frac{1}{t_0}\frac{\partial^2 \phi(y)}{\partial y_n^2}, y = y_0. \qquad (5.3)$$

In view of (5.2) and (5.3), the viscosity criterion is satisfied provided that

$$\frac{\partial^2 \phi(y_0)}{\partial y_n^2} \neq 0.$$

In the sequel we shall prove that $\dfrac{\partial^2 \phi(y_0)}{\partial y_n^2} = 0$ contradicts the assumption $(A1)$. To this end, we write

$$f_{ij} = \frac{\partial^2 f(\phi'(y_0))}{\partial p_i \partial p_j}, \ \ \phi_{ij} = \frac{\partial^2 \phi(y_0)}{\partial y_i \partial y_j}, \ \ 1 \le i, j \le n,$$

and we assume that $\phi_{nn} = 0$. Then

$$\sum_{j=1}^{n-1} f_{ij}\phi_{jn} = 0, \ \ i = 1, \ldots, n-1.$$

Let $M(n-1, n-1)$ be the matrix which is obtained from $f''(\phi'(y_0))$ by deleting the n^{th} row and column and let $|M|(n-1, n-1)$ denote its determinant. Then

$$|M|(n-1, n-1) = 0,$$

otherwise $\phi_{in} = 0, i = 1, \cdots, n-1$ and $\lambda_n = 0$. We next assume that the rank of $M(n-1, n-1)$ is k, $1 \le k \le n-2$ and that $M(k,k)$ is the matrix resulting from $f''(\phi'(y_0))$ by deleting the rows and the columns of order greater than k. For convenience we assume that $|M|(k,k) \neq 0$. Consider the system

$$\sum_{j=1}^{k} f_{ij}\phi_{jn} = -\sum_{j=k+1}^{n-1} f_{ij}\phi_{jn}, \ \ i = 1, \ldots, k.$$

Solving the above system we obtain

$$\phi_{jn} = -\frac{|D|_j}{|M|(k,k)}, \ \ j = 1, \cdots, k, \tag{5.4}$$

where D_j is obtained by replacing the j^{th} column of $M(k,k)$ with the vector

$$\Big(\sum_{j=k+1}^{n-1} f_{1j}\phi_{jn}, \cdots, \sum_{j=k+1}^{n-1} f_{kj}\phi_{jn}\Big)^T.$$

We note that

$$|D|_j = \sum_{i=k+1}^{n-1} |D|_{ji}\phi_{in}, \tag{5.5}$$

where D_{ji} is obtained by replacing the j^{th} column of $M(k,k)$ with the vector $(f_{1i},\ldots,f_{ki})^T$. Using (5.4), (5.5) we obtain

$$\lambda_n = \sum_{i=k+1}^{n-1} \Big\{ -\Big(\sum_{j=1}^{k} \frac{f_{nj}|D|_{ji}}{|M|(k,k)}\Big) + f_{ni}\Big\}\phi_{in}. \tag{5.6}$$

We now define the quantity

$$A_i = -\sum_{j=1}^{k} \frac{f_{nj}|D|_{ji}}{|M|(k,k)} + f_{ni}, \;\; k+1 \le i \le n-1.$$

Since

$$|M|(k,k)\cdot A_i = \begin{vmatrix} & f_{1i} \\ M(k,k) & \vdots \\ & f_{ki} \\ f_{n1}\cdots f_{nk} & f_{ni} \end{vmatrix} = 0,$$

the right hand-side of (5.6) equals to zero, which contradicts $(A1)$. Therefore we have the following

Lemma 5.1. *Under the assumptions* $(A1)$, $(A2')$, *we have that*

$$\frac{\partial^2\phi(y_0)}{\partial y_n^2} \neq 0.$$

The above lemma and (5.3) yield

Theorem 5.1. *Under the assumptions* $(A1)$, $(A2')$ *the function* u *given by* (5.1) *is the viscosity solution of* $(\tilde{P})$, *up to a time* t *near* t_0.

6 Discussion

The multivaluedness of $v(x_t^{-1}(x), t)$, under the assumption $(A2)$, is depicted in Figure 4. Under the stronger assumption $(A2')$, we have shown that u defined by (5.1) is the desired viscosity solution. Since only $\lambda_n(y)$ participates in the generation of Σ_t^y, the formation of the shock should not depend on the presence of multiple eigenvalues $\lambda_i(y), i \neq n$. Thus the assumption $(A2')$ is more technical than essential.

As we have seen in [16], in order to continue the solution further we have to solve all the possible local Riemann problems at a time level $t_\alpha > t_0$. We may accomplish that in two steps. The first one is to study how the characteristics are crossing and the initial shock surface is transformed for $t > t_\alpha$. This is related to the way the caustics undergo metamorphosis and it can be understood using the theory of caustics. Using this information, we need to study the multivaluedness of $v(x_t^{-1}(x), t)$. The second step is to construct the solution by selecting a single-valued branch of v. Thus, we have to prove that the proper branches of the graph of v, corresponding to the different domains into which the initial plane is divided by the initial shock surface, intersect. We then have to show that the viscosity criterion is satisfied across the intersection. Intersection usually occurs at least in the one-dimensional case, as a consequence of convexity or concavity of f in the direction of the jump. Thus the viscosity criterion is automatically satisfied. For an arbitrary f the change of the sign of curvature (the analogue of inflection points for the one-dimensional problem) can force the branches of the graph of v to be detached. In this case we have to fill up the gap with a rarefaction-wave type solution such that the viscosity criterion is satisfied. Since locally the viscosity criterion is actually one-dimensional, we can follow the construction presented in [16]. To accomplish that, we have to study the geometry of the domain covered by the constructed rarefaction waves and the way they fit with the existing characteristics. Valuable insight in this direction can be obtained by the related work for conservation laws. See [9] and references cited there.

As $t \to \infty$, the surface Σ_t^y tends to the level surfaces $\{y \in \mathbb{R}_y^n : \lambda_i(y) = 0, i = 1, \ldots, n\}$. We can use this fact to study the asymptotic profile of the shock surfaces and the asymptotic behavior of the viscosity solution $u(x,t)$ as $t \to \infty$. See [14] for a result in the two-dimensional conservation law (cf. [12], [17], for the one-dimensional problem).

In the sequel we discuss how we can use the results on the bifurcation of caustics to get insight on how the shocks propagate further and undergo catastrophe. We only consider the case where $n = 2$ and f is convex or concave, so that the viscosity criterion is automatically satisfied. We primarily use the results presented in [6]. The five types of bifurcation of causics are presented in Figures 5a-5e. The "birth of a pancake" (type A_3, "lips") has been discussed in Section 3 and corresponds to the case where $\lambda_2(y)$ has a negative local minimum. Figure 5b represents the type A_3, "beak-to-beak", which occurs when $\lambda_2(y)$ has a saddle point. There are two versions of this type. The geometric conditions under which the two versions take place are given in [6]. In the first one, the two corresponding pancakes $\Sigma^x_{t,a}, \Sigma^x_{t,b}$ meet at their edges for $t = t_\alpha$. The different pictures of Σ^x_t for $t < t_\alpha, t = t_\alpha$ and $t > t_\alpha$ are given in Figures 6a-6c. The corresponding shocks $\Gamma_{t,a}, \quad \Gamma_{t,b}$ also meet at their ends and they are depicted by a full line. In the second version the junction takes place through the boundary points of the pancakes, see Figures 7a-7d. The shocks initially meet at a middle point and then undergo catastrophe. They interact giving one branched shock (cf. [21]). The Figure 5c represents the type A_4, "swallowtail", where x_t is locally equivalent to the swallowtail

$$\begin{cases} x_1 = y_1^4 - y_2 y_1 - (t - t_\alpha) y_1^2 \\ \\ x_2 = y_2. \end{cases}$$

In Figures 8a-8d we see how the appearance of the swallowtail changes the shape of the pancake. In Figure 8b the swallowtail touches the shock for the first time. The shock undergoes catastrophe, it bends and another branch springs out (Figure 8c). This branch can move out of the region of the pancake and interact with another shock (Figure 8d). Figures 5d, 5e depict the types $D_4^{\pm}$ which correspond to the case where $\lambda_1(y_\alpha) = \lambda_2(y_\alpha)$. The two types are the "purse" D_4^+ and the "pyramid" D_4^-. In a neighborhood of y_α the two surfaces $\lambda_1(y), \lambda_2(y)$ look like a cone. The local graph of the eigenvalues corresponding to the two cases is depicted in the Figures 9a and 9b respectively. In the D_4^+ case, the mapping x_t is locally equivalent to the hyperbolic umbilic

$$x_1 = y_2^3 + 3y_1^2 + 2(t - t_\alpha)y_2$$

$$x_2 = y_1^3 + 3y_2^2 + 2(t - t_\alpha)y_1.$$

The Figures 10a-10c depict the caustics corresponding to the two eigenvalues for $t < t_\alpha, t = t_\alpha$ and $t > t_\alpha$. The metamorphosis of the graph of $v(x_t^{-1}(x), t)$ is given in Figures 11a - 11c, respectively. The type D_4^-, "pyramid", involves the interaction of three shocks and we are not going to discuss it further. It is worth-while to notice that the metamorphosis of a caustic does not always cause the catastrophe of a shock, unless it affects directly the branches defining the shock surface.

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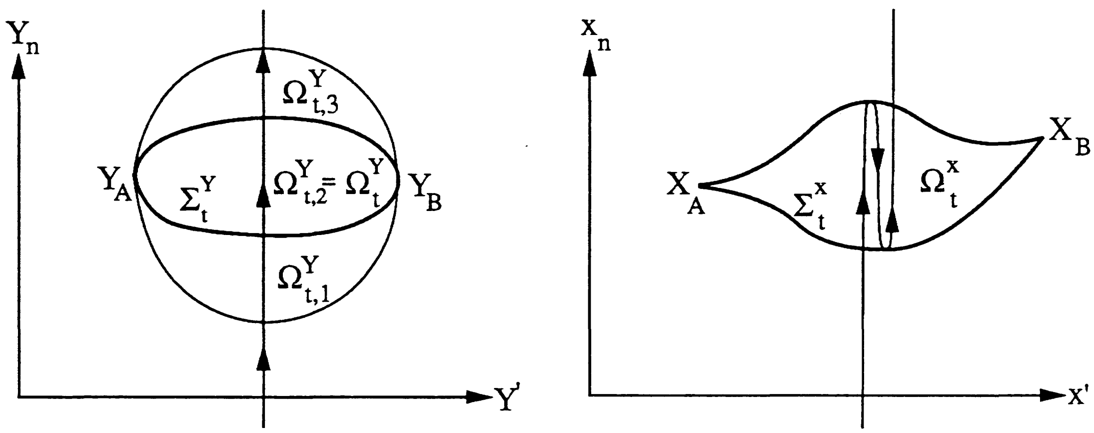


Figure 1

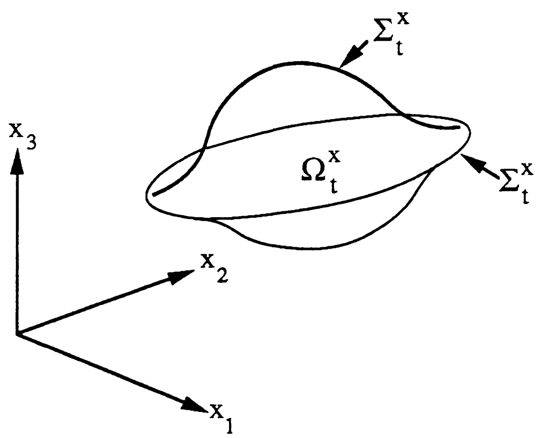


Figure 2

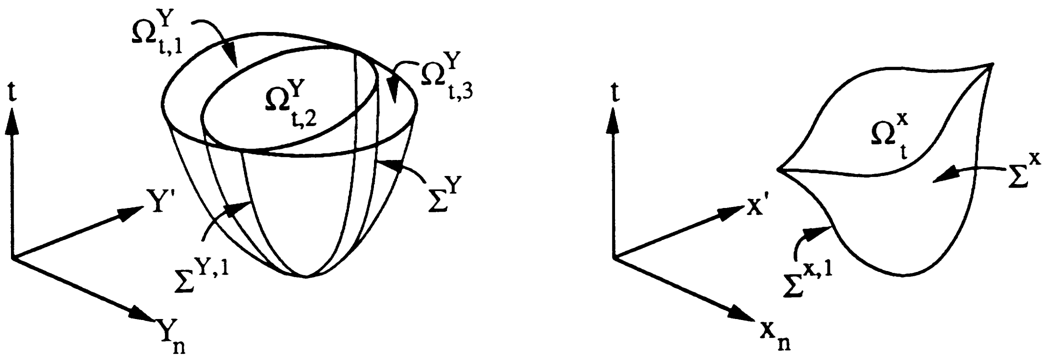


Figure3

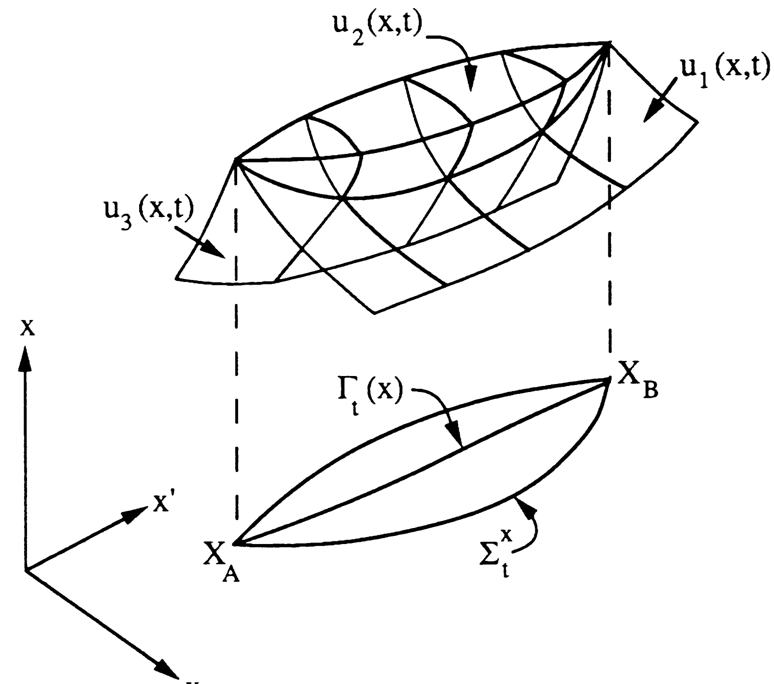
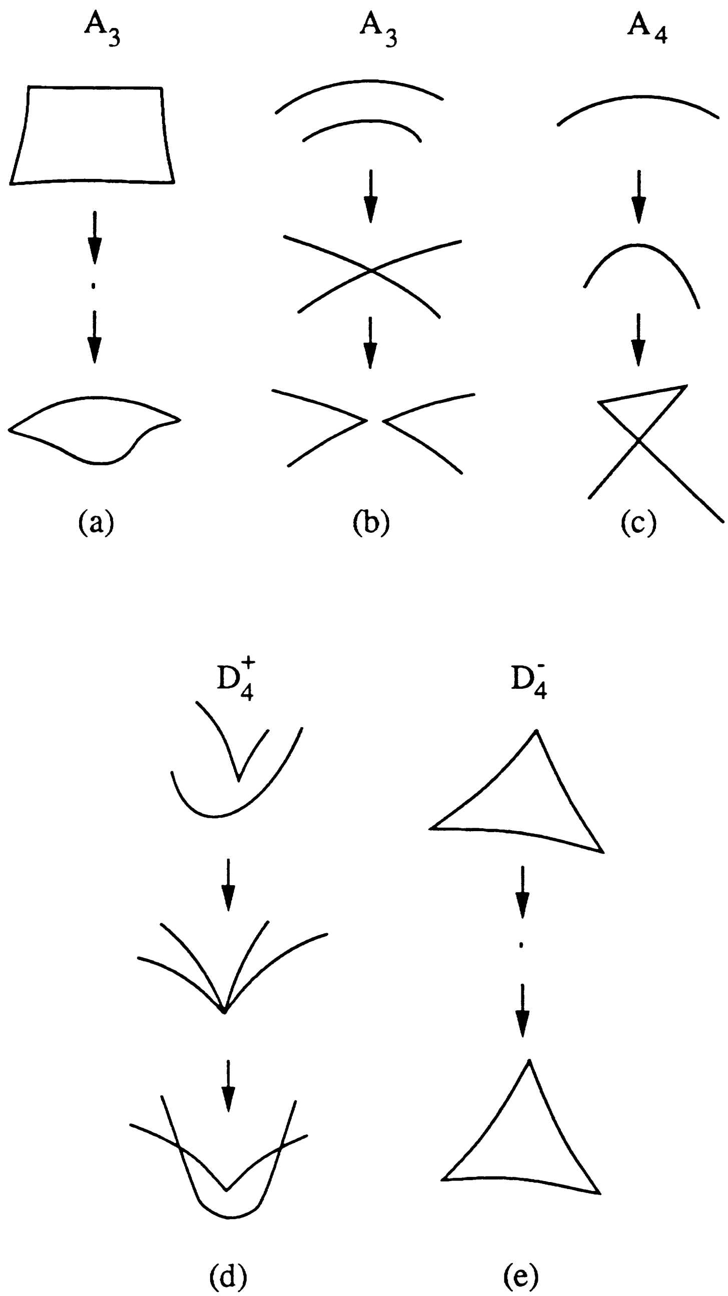
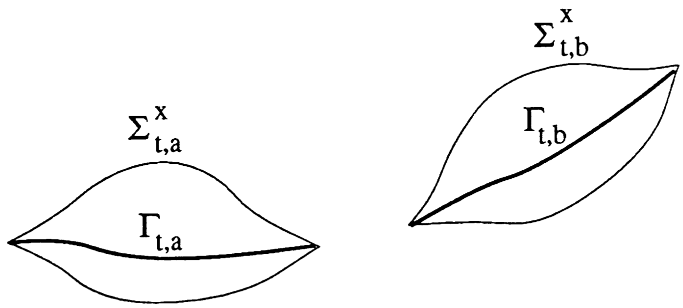


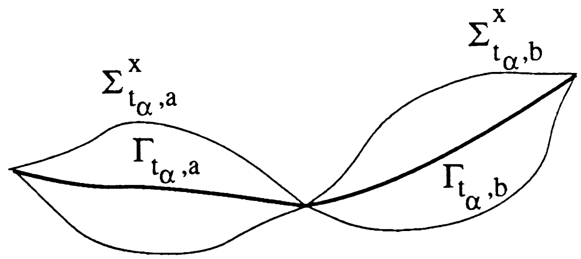
Figure 4



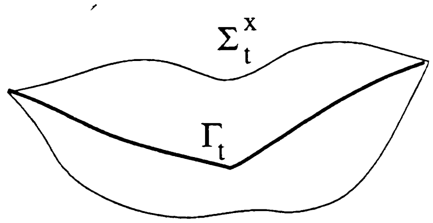
Figures 5a - 5e



(a) , $t < t_\alpha$

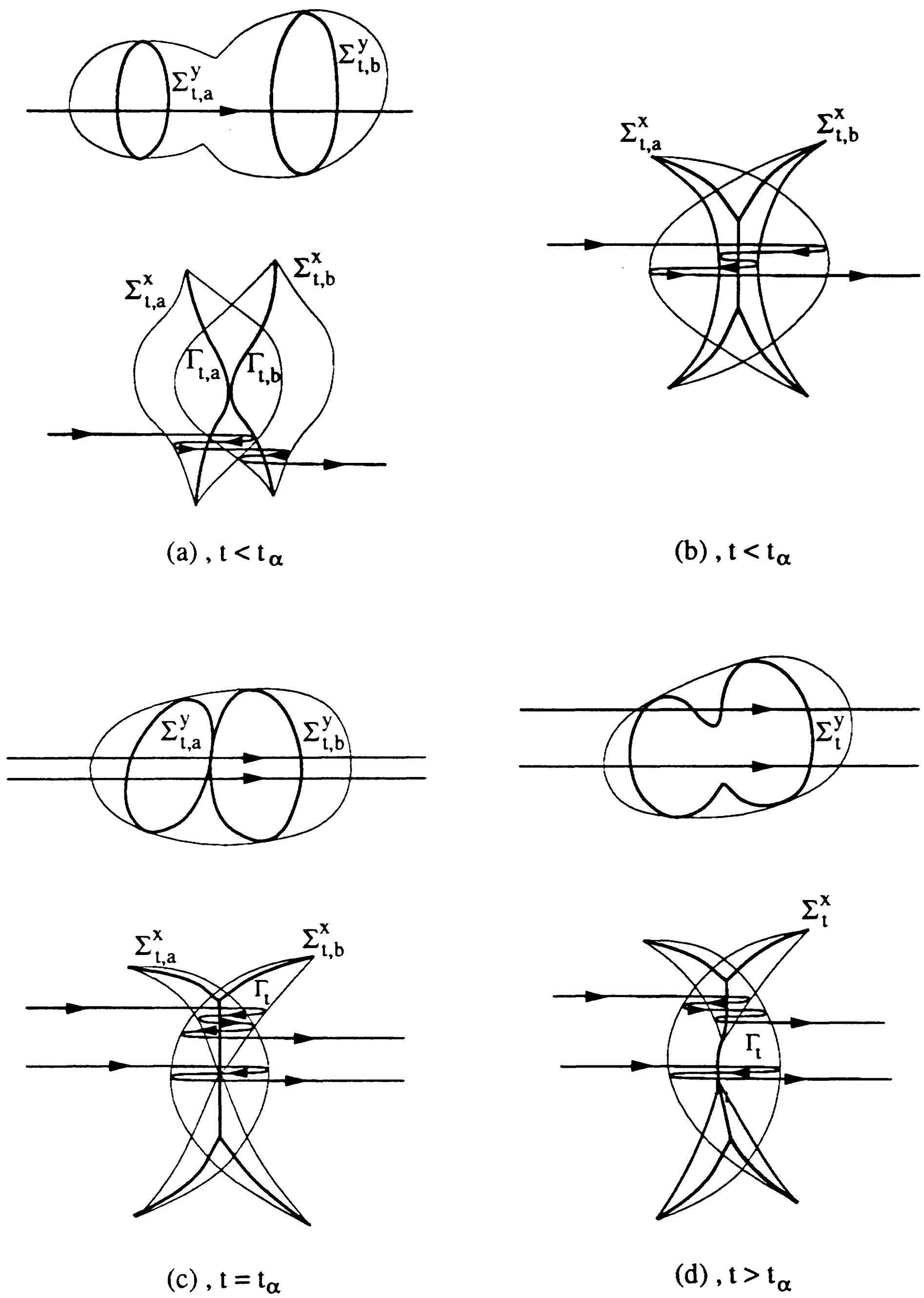


(b) , $t = t_\alpha$

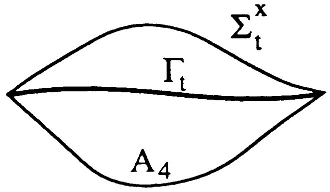


(c) , $t > t_\alpha$

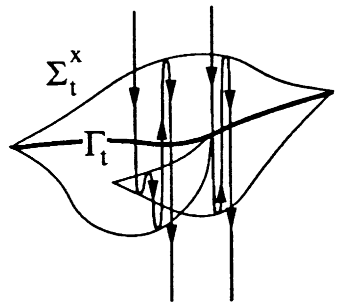
Figures 6a - 6c



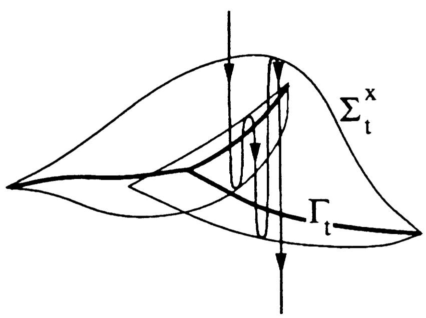
Figures 7a - 7d



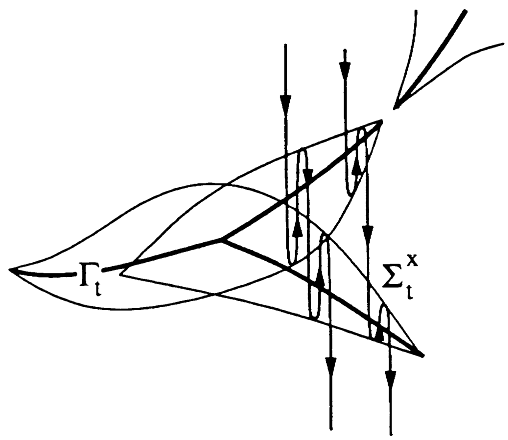
(a) , $t = t_\alpha$



(b) , $t > t_\alpha$

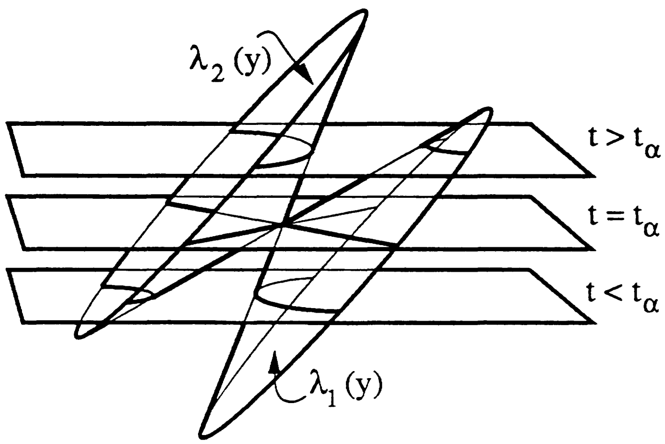


(c) , $t > t_\alpha$

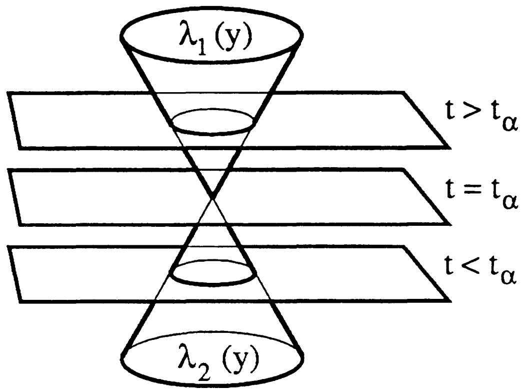


(d) , $t > t_\alpha$

Figures 8a - 8d

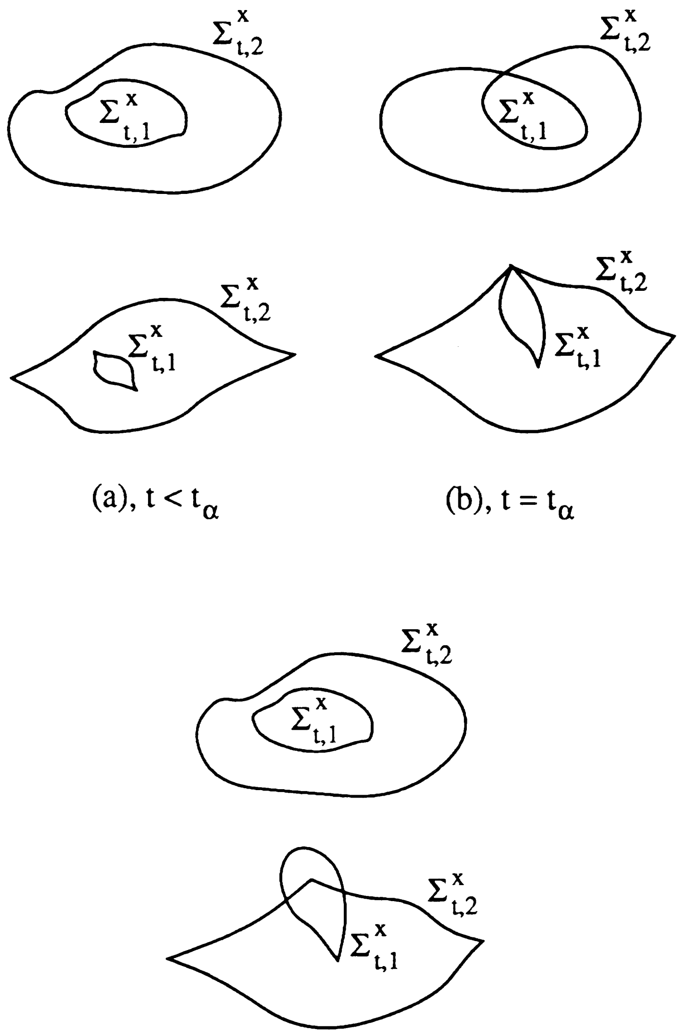


(a)



(b)

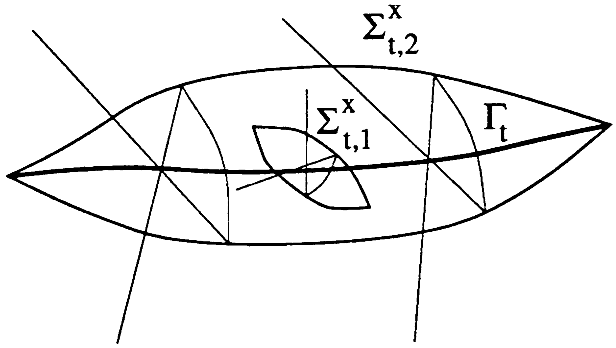
Figures 9a-9b



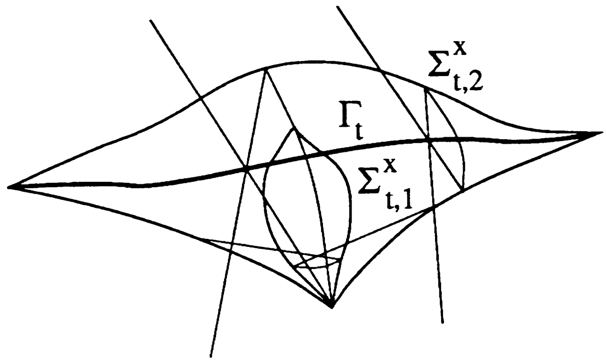
(a), $t < t_\alpha$ (b), $t = t_\alpha$

(c), $t > t_\alpha$

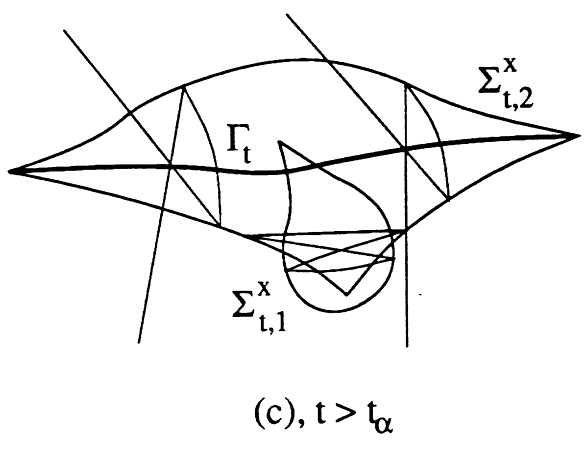
Figures 10a -10c



(a), $t < t_\alpha$



(b), $t = t_\alpha$



(c), $t > t_\alpha$

Figures 11a - 11c

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G.T. Kossioris

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13. ABSTRACT (*Maximum 200 words*)

In this work we study the generation of singularities (shock waves) of the solution of the Cauchy problem for Hamilton-Jacobi equations in several space variables, under no assumption on convexity or concavity of the hamiltonian. We study the problem in the class of viscosity solutions, which are the correct class of weak solutions. We first examine the way the characteristics cross by identifying the set of critical points of the characteristic manifold with the caustic set of the related lagrangian mapping. We construct the viscosity solution by selecting a single-valued branch of the multi-valued function given as a solution by the method of characteristics. We finally discuss how the shocks propagate the undergo catastrophe in the case of two space variables.

14. SUBJECT TERMS

shock waves, cingularities, Hamilton-Jacobi equations, viscosity solutions, caustics

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