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FORMATION OF SINGULARITIES FOR VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN ONE SPACE VARIABLE

by

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a single-valued branch of the multi-valued function given as a solution by the method of characteristics. The singularities correspond to the intersection of the different branches. Such singularities are called genuine shocks. Due to the presence of the inflection points this intersection may be empty. In this case we join the two parts by a rarefaction-wave type solution. The two junctions correspond respectively to a weak wave and a contact discontinuity shock curve.

In the case where f is convex, Fleming [7] proved that, except for a set of lower dimension, the set of singularities lies on the union of n-dimensional smooth manifolds. Assuming that the convex function f is only continuous, Cannarsa and Soner [3] have proved certain regularity properties of Lipschitz continuous viscosity solutions. Jensen and Souganidis [11] studied the stationary problem with nonconvex f, when $n = 1$. Using a blow-up argument, they proved that the singular set is countable.

The structure of the singular points obtained in this work is analogous to that of the weak solution of the related one-dimensional conservation law; see Dafermos [6]. The latter is obtained using the method of generalized characteristics. This method is qualitative; it describes the different forms of the curves of the singular points and their relative position as trajectory solutions of a differential equation with discontinuous right-hand side. It is, however, restricted to the one-dimensional case.

The present approach relates naturally the hamiltonian flow corresponding to the Hamilton–Jacobi equation with the unique viscosity solution. We give the exact geometric construction of the path of a generated singularity in terms of the initial data. We state explicit criteria which force a type of singularity to change into another and we explain the way they interact. We, moreover, obtain additional geometric features, not given in [6], which complete the geometric picture of a singularity. Finally, the method can be extended to higher dimensions using the theory of critical points of smooth mappings (see e.g. Arnold et al [1]). In the present work, however, we have systematically avoided the language of critical point theory in order to present clearly the phenomena related to the singularities themselves. The correct setting of the problem in higher dimensions and some first results how the singularities propagate will be presented in a forthcoming paper [12] (see also [13], [14], [16]).

The paper is organized as follows: In Subsection 2.1 we explain the method of characteristics. In Subsection 2.2 we give the necessary background from the theory of viscosity solutions. The assumptions and the main results are given in Subsection 2.3. In Section 3 we investigate how the shocks are generated. In Section 4 we make the constructions of all the possible local Riemann problems. The above constructions are used in Section 5 to study the global structure of the shock waves.

2 Preliminaries and statement of the main results

2.1 Method of characteristics

The classical method of characteristics for the Cauchy problem (P) reduces the solution of the partial differential equation to solving a system of ordinary differential equations given by

$$\frac{dx}{dt} = f'(p), \ \frac{dp}{dt} = 0, \ \frac{dv}{dt} = -f(p) + pf'(p)$$

with initial conditions $x(0) = y$, $p(0) = \phi'(y)$, $v(0) = \phi(y)$, $y \in I\!R$. The solution of the above system is given explicitly by

$$x(y,t) = f'(\phi'(y))t + y, \tag{2.1}$$

$$p(y,t) = \phi'(y), \quad v(y,t) = \{-f(\phi'(y)) + \phi'(y)f'(\phi'(y))\}t + \phi(y). \tag{2.2}$$

The value of the solution u at (x,t) is found as follows: If we assume $\dfrac{\partial x}{\partial y} \neq 0$, then there exists a unique characteristic line (2.1) passing through the point (x,t) originating at the point $(y,0)$. In particular, $\dfrac{\partial x}{\partial y} \neq 0$ yields that the map

$$x : y \to x = x(y;t) \equiv x(y,t)$$

is a C^∞-diffeomorphism. If we denote by

$$x^{-1} : x \to y = x^{-1}(x;t)$$

its inverse, then

$$u(x,t) = v(x^{-1}(x;t),t).$$

In general, there is a critical time beyond which the Jacobian $\partial x/\partial y$ may vanish. The characteristic lines (2.1) start crossing and x^{-1} becomes multi-valued. As a consequence, the map $x \rightarrow v(x^{-1}(x;t),t)$ is also multi-valued. That is, the graph of v starts folding in a specific way, generating a picture which looks like a swallowtail (see Fig. 4).

2.2 The notion of viscosity solution

Since the characteristics may cross, the value of $v(x^{-1}(x;t),t)$ at a certain point (x,t) is determined by more than one points $(y,0)$. Therefore, we can assign different values of v as the solution $u(x,t)$ from the left or the right of the point (x,t) along the $x-$direction. That is, we have to look for a solution which is not in the class of C^∞ functions. This difficulty has been overcome with the introduction by Crandall and Lions [5] of the correct class of generalized solutions, called *viscosity solutions*, for which (P) is globally well-posed. Such a solution need not be differentiable anywhere, as the only regularity required in its definition is continuity. Even the requirement of continuity has been relaxed; see e.g. Ishii [9] where f, ϕ are not assumed continuous. Since in our case f and ϕ are C^∞ we shall only deal with continuous viscosity solutions. Beyond the critical time, the (globally defined continuous) viscosity solution is not C^∞ and discontinuities in the derivatives of the solution appear. We next state the definition of a viscosity solution.

Definition 2.2.1 *The function $u \in C(\mathcal{O})$ is a viscosity solution of (1.1) in the open domain $\mathcal{O} \subset I\!R \times I\!R^+$ provided*

$$\psi_t(x,t) + f(\psi_x(x,t)) \leq 0, \tag{2.3}$$

respectively

$$\psi_t(x,t) + f(\psi_x(x,t)) \geq 0. \tag{2.4}$$

for any $\psi \in C^1(\mathcal{O})$ *for which* $u - \psi$ *attains a local maximum, respectively local minimum, at the point* $(x,t) \in \mathcal{O}$.

Definition 2.2.2. *The function* $u \in C(I\!R \times [0,\infty))$ *is a viscosity solution of the Cauchy problem* (P) *if and only if it is a viscosity solution of (1.1) in the domain* $I\!R \times (0,\infty)$ *and satisfies the initial condition*

$$\lim_{t \to 0+} u(x,t) = \phi(x).$$

The inequalities (2.3) and (2.4) will be referred in the sequel as the *viscosity criterion* at the point (x,t).

We next state the viscosity criterion in a form which turns out to be more useful for the construction of the solution. To this end, assume that $\mathcal{O} \subset I\!R \times (0,\infty)$ is open and that there is a smooth curve $t \to x = \chi(t),\ t \in (t_1,t_2) \subset I\!R$, with negative slope, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-$, $\Gamma = \{(x,t) :\ x = \chi(t)\}$, where $\mathcal{O}^+$ lies on the right of Γ. Finally, let $\eta(x,t)$ be the unit normal on $\Gamma(x,t)$ pointing into $\mathcal{O}^+$. Then

Theorem 2.1. ([4]) *Let* $u \in C(\mathcal{O})$ *and* $u = u^+$ *in* $\mathcal{O}^+ \cup \Gamma, u = u^-$ *in* $\mathcal{O}^- \cup \Gamma$ *where* $u^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma)$. *Then* u *is a viscosity solution of (1.1) in* $\mathcal{O}$ *if and only if the following conditions hold:*

a) u^+ *and* u^- *are classical solutions of (1.1) in* $\mathcal{O}^+$ *and* $\mathcal{O}^-$, *respectively.*

b) Let $T(x,t) = \{\tau \in I\!R^2 : \eta(x,t) \cdot \tau = 0\}$ *be the tangent space to* Γ *at* (x,t) *and* P_T *the orthogonal projection of* $I\!R^2$ *onto* $T(x,t)$*. If*

$$a = Du^+(x,t) \cdot \eta(x,t) \leq Du^-(x,t) \cdot \eta(x,t) = b,$$

then

$$H(P_T Du^\pm(x,t) + \xi \cdot \eta(x,t)) \leq 0, \qquad \text{for} \qquad a \leq \xi \leq b,$$

while, if

$$a = Du^+(x,t) \cdot \eta(x,t) \geq Du^-(x,t) \cdot \eta(x,t) = b,$$

then

$$H(P_T Du^\pm(x,t) + \xi \cdot \eta(x,t)) \geq 0, \qquad \text{for} \qquad b \leq \xi \leq a,$$

where $H(q) = q_2 + f(q_1),\ q = (q_1, q_2) \in R^2$ *and* $D = (\frac{\partial}{\partial x}, \frac{\partial}{dt})$.

The curve $\chi(t)$ in the neighborhood of which u has the properties specified in the theorem above is called a *shock curve.*

We now state the viscosity criterion in a standard form which has a clear geometric representation.

Theorem 2.2 *Let the function u be defined as in Theorem 2.1. Then u is a viscosity solution of (1.1) in* $\mathcal{O}$ *if and only if the following conditions hold:*

a) u^+ and u^- are classical solutions of (1.1) in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively,

b) $f\left((1-\lambda)u_x^+ + \lambda u_x^-\right) \leq (1-\lambda)f(u^+) + \lambda f(u_x^-)$ for $u_x^- > u_x^+$

across the shock curve, while

$f((1-\lambda)u_x^- + \lambda u_x^+) \geq (1-\lambda)f(u_x^-) + \lambda f(u_x^+)$ for $u_x^- < u_x^+$,

where $\lambda \in [0,1]$. That is, the graph of f lies respectively below or above the line segment joining the points $(u_x^+, f(u_x^+))$ and $(u_x^-, f(u_x^-))$.

Proof. The vector $\tilde{\eta} = (-(u_x^+ - u_x^-), f(u_x^+) - f(u_x^-))$ is normal to $T(x,t)$. Moreover,

$$Du^+ \cdot \frac{\tilde{\eta}}{\|\tilde{\eta}\|} \leq Du^- \cdot \frac{\tilde{\eta}}{\|\tilde{\eta}\|}.$$

Therefore $b \geq a$ (resp. $b \leq a$) if and only if $\tilde{\eta}$ (resp. $-\tilde{\eta}$) points into $\mathcal{O}^+$. This happens if and only if $u_x^+ < u_x^-$ (resp. $u_x^+ > u_x^-$). The inequalities

$$Du^+ \cdot \tilde{\eta} \leq \xi\|\tilde{\eta}\| \leq Du^- \cdot \tilde{\eta} \quad \text{or} \quad -Du^+ \cdot \tilde{\eta} \geq \xi\|\tilde{\eta}\| \geq -Du^- \cdot \tilde{\eta}$$

are equivalent to

$$0 \leq \lambda = \frac{\xi\|\tilde{\eta}\| \mp Du^\pm \cdot \tilde{\eta}}{\|\tilde{\eta}\|^2} \leq 1.$$

Moreover,

$$\xi\|\tilde{\eta}\| = \lambda\|\tilde{\eta}\|^2 \pm Du^\pm \cdot \tilde{\eta}, \; P_T Du^\pm = Du^\pm - \frac{(Du^\pm \cdot \tilde{\eta})}{\|\tilde{\eta}\|^2}\tilde{\eta}.$$

Therefore,

$$P_T Du^\pm \pm \xi \frac{\tilde{\eta}}{\|\tilde{\eta}\|} = (u_x^\pm, -f(u_x^\pm)) + (\mp\lambda(u_x^+ - u_x^-), \pm\lambda(f(u_x^+) - f(u_x^-)).$$

Substituting the above expression in H yields the result. □

It is an immediate consequence of Theorem 2.2 that the viscosity criterion is analogous to the Oleinik E- condition for the corresponding conservation law. (See e.g. [11]).

2.3 Assumptions and main results

Throughout this paper we make the following assumptions:

A1) The function $h(y) = f''(\phi'(y))\phi''(y), y \in I\!R$, has finitely many isolated critical points and ϕ has compact support.

A2) The function $f(p), p \in I\!R$, has just one inflection point normalized so that

$$f(0) = f'(0) = f''(0) = 0, \ pf''(p) < 0, p \neq 0.$$

Although this approach could work without assuming (A1), this condition is imposed to guarantee that the solution is initially C^∞ and only finitely many shocks are generated. Assumption (A2) is somewhat special, but it captures the effect of inflection points of the hamiltonian f and it illuminates completely how the method works. The reader who wishes to get a feeling of the interaction of several inflection points may consult the related papers in conservation laws [2], [8], [10].

In order to present the main results of the present work, we need to give some additional definitions. The shock curves are either *genuine shocks* or *left contact discontinuities* (see Figures 1, 2 respectively). The jump of the derivatives of the viscosity solution across a genuine shock is generated by two incoming characteristics (incoming waves). The shock is initially constructed by choosing a continuous single-valued branch of $v(x^{-1}(x;t), t)$. We continue the solution further by constructing the solution of all the possible local Riemann problems. Let $v^-(x^{-1}(x;t),t)$ and $v^+(x^{-1}(x;t),t)$ be the solutions defined by the characteristics coming from the left and the right of the initial discontinuity point. If the two graphs intersect, then their intersection determines a genuine shock curve.

Due to the presence of the inflection point of f, the two graphs may, however, have no common point. In that case we construct the solution by joining v^- and v^+ with a *rarefaction- wave* type solution (see Section 4). The junction of the graph of the rarefaction wave with the graph of v^+ determines the contact discontinuity curve. The jump across the shock is determined by the right incoming waves. The shock is a convex curve emitting tangentially from the left outgoing waves called *rarefaction waves.* The rarefaction waves are carried by characteristic lines originating from the contact discontinuity and not the initial axe. They are present because of the inflection point of f. Characteristic lines across which all the derivatives of the solution of order less or equal m are continuous while that of order $(m+1)$ has a finite jump are called *weak waves of order* m.

The main results of the present work are summarized in the following theorem.

Theorem 2.3 *The viscosity solution u of (P) is a C^∞ function on $(-\infty,+\infty)\times[0,\infty)\backslash(\mathcal{S}\cup\mathcal{W})$ where*

1) *$\mathcal{S}$ is either empty or is the finite union of genuine shocks and left contact discontinuities.*

2) *$\mathcal{W}$ is either empty or is the finite union of weak waves of order m.*

3) *A genuine shock is either generated at the point where the characteristics start crossing for the first time or springs from a weak wave of order 1. In the latter case, at the point of birth of the weak wave the shock splits into two forward shocks. The one on the right is a left contact discontinuity, the one on the left is a genuine shock determined from the right by the incoming rarefaction waves.*

4) *A genuine shock can turn into a left contact discontinuity. A left contact discontinuity can change back into a genuine shock. In that case the shock is defined from the left by the outgoing rarefaction waves.*

5) *Weak waves of order 1 are triggered tangentially at the points where a genuine shock turns into a contact discontinuity. Weak waves of order $m\geq 2$ are emitted tangentially on the left of the contact discontinuity due to the collision from the right with a weak wave of order $m-1$. All weak waves terminate upon colliding with a shock.*

6) The collision of two or more shocks produces a single outgoing genuine shock.

The generation of a shock is caused by the presence of a negative local minimum of the function $h(y)$. Contact discontinuities are always the result of deformation of genuine shocks. The number of existing shocks at any time is less or equal to the number of the negative local minima of $h(y)$. The conditions under which a genuine shock turns into a contact discontinuity are given in Theorem 5.1. Such conditions cannot be obtained with the techniques used in [6]. Moreover, properties 3) and 4) of Theorem 5.1 do not appear in the related literature of conservation laws.

In Section 3 we show how the shocks are generated. In Section 4 we make all the local constructions needed to continue the shock further. The results obtained in these sections are combined in Section 5 to get the global structure of the shocks.

3 Generation of shocks

In order to understand the mechanism of generation of shocks let us assume that the function $h(y) = \frac{d}{dy}(f'(\phi'(y)))$ achieves a negative global minimum at the point $y = y_\alpha$. As long as the Jacobian $\frac{\partial x}{\partial y} = 1 + th(y)$ is positive, the solution is classically defined. The first time t_α that the Jacobian is 0 is given by

$$t_\alpha = -\frac{1}{h(y_\alpha)}.$$

In view of (A1), for $t \in (t_\alpha, t_\alpha + \varepsilon), \varepsilon > 0$ small enough, there exists a neighborhood $(y_\alpha - \delta, y_\alpha + \delta), \delta > 0$, and points, $\bar{y}_i = \bar{y}_i(t) \in (y_\alpha - \delta, y_\alpha + \delta)$, $i = 1, \dots, 4$, such that $\bar{y}_i < \bar{y}_{i+1}$ and

$$\frac{\partial x}{\partial y} > 0, y \in (y_1, y_2) \cup (y_3, y_4), \quad \frac{\partial x}{\partial y} < 0, y \in (y_2, y_3).$$

Moreover $\frac{\partial x(\bar{y}_2; t)}{\partial y} = \frac{\partial x(\bar{y}_3; t)}{\partial y} = 0$. For t fixed, the graph of the function $x(y; t)$ with respect to the variable y is depicted in Figure 3. The inverse mapping $x^{-1} : x \to y = x^{-1}(x; t)$, $t\epsilon(t_\alpha, t_\alpha + \varepsilon)$, is clearly three-valued on the interval (x_1, x_2), where $\bar{y}_{i+2} = \bar{y}_i = x^{-1}(x_i; t), i = 1, 2$. We denote by $y_i = y_i(x; t), i = 1, 2, 3$, the three branches of x^{-1}, mapping (x_1, x_2) onto (y_i, y_{i+1}).

Lemma 3.1 *If $f''(\phi'(y_\alpha)) > 0$ (resp. $f''(\phi'(y_\alpha) < 0)$, then for $t\epsilon(t_\alpha, t_\alpha +$*

ε)

$$\phi'(y_3(x;t)) < \phi'(y_2(x;t)) < \phi'(y_1(x;t))$$

$$(\text{resp. } \phi'(y_3(x;t)) > \phi'(y_2(x;t)) > \phi'(y_1(x;t))).$$

Proof: Since $f''(\phi'(y_\alpha))\phi''(y_\alpha) = -1/t_\alpha$, $f''(\phi'(y))\phi''(y) < 0$ on a neighborhood of y_α and $\phi'(y)$ is monotone around y_α. □

Theorem 3.1 *Let* $u_i(x,t) = v(y_i(x;t),t), i = 1,2,3,\ t\epsilon(t_\alpha, t_\alpha + \varepsilon)$, *where* v *is defined by (2.2) and* $F(x,t) = u_3(x,t) - u_1(x,t),\ x\epsilon[x_1, x_2]$. *Then, if* $f''(\phi'(y_\alpha)) > 0$ (resp. $f''(\phi'(y_\alpha)) < 0$),

a) $\dfrac{\partial}{\partial x}F(x,t) < 0$, (resp. $\dfrac{\partial}{\partial x}F(x,t) > 0$), $x\epsilon(x_1,x_2)$,

b) $F(x_1,t) > 0$ *and* $F(x_2,t) < 0$ (resp. $F(x_1,t) < 0$ *and* $F(x_2,t) > 0$)

c) there exists a unique point $\chi(t)\epsilon(x_1,x_2)$ *such that* $F(\chi(t),t) = 0$. *Moreover, the curve* $\chi(t)$ *is smooth on* $(t_\alpha, t_\alpha + \varepsilon)$.

Proof: Let us assume $\phi''(y) < 0, y\epsilon(y_\alpha - \delta, y_\alpha + \delta)$. In this case Lemma 3.1 yields

$$\frac{\partial F}{\partial x}(x,t) = \frac{\partial}{\partial x}(u_3(x,t) - u_2(x,t)) + \frac{\partial}{\partial x}(u_2(x,t) - u_1(x,t)) < 0.$$

Since

$$\frac{\partial}{\partial x_3}(u_3(x,t) - u_2(x,t)) < 0,\ x\epsilon(x_1,x_2),\quad u_3(x_1,t) = u_2(x_1,t),$$

we have that

$$u_3(x,t) - u_2(x,t) < 0, \; x \epsilon (x_1, x_2].$$

Similarly we conclude

$$u_2(x,t) > u_1(x,t), \; x \epsilon [x_1, x_2).$$

Moreover,

$$F(x_1,t) = (u_3(x_1,t) - u_2(x_1,t)) + (u_2(x_1,t) - u_1(x_1,t)) > 0,$$

$$F(x_2,t) = (u_3(x_2,t) - u_2(x_2,t)) + (u_2(x_2,t) - u_1(x_2,t)) < 0.$$

Conclusion (c) follows by the implicit function theorem. In this case the graph of $v(x^{-1}(x;t),t)$ is depicted in Figure 4. In the case where $\phi''(y) > 0$, the graph of u_2 lies below the graphs of u_1, u_3. □

Theorem 3.2 *The function $u(x,t), (x,t) \epsilon (x_1,x_2) \times (t_\alpha, t_\alpha + \varepsilon)$, defined by*

$$u(x,t) = \begin{cases} u_1(x,t), & x \leq \chi(t) \\ u_3(x,t), & x \geq \chi(t), \end{cases}$$

is a local viscosity solution of (1.1).

Proof: Let $u_x(\chi(t)\pm, t)$ denote the x-derivatives of u from the right and the left at the point $(\chi(t), t)$. Since the values $u_x(\chi(t)\pm, t)$ correspond to the points where the function f is convex or concave, the viscosity criterion is satisfied. □

This construction is originated in [15]. In the case of a conservation law the graph of $v(x^{-1}(x,t),t)$ folds in a different way, see [8, p. 112], and the shock curve is constructed using stable manifold theory. We denote by x_s^{-1} the single-valued branch of x^{-1} defined on the interval (x_1, x_2), for $t\epsilon(t_\alpha, t_\alpha+\varepsilon)$, given by

$$x_s^{-1}(x;t) = \begin{cases} y_1(x;t), & x < \chi(t) \\ \\ y_3(x;t), & x > \chi(t), \end{cases} \tag{3.1}$$

while $x_s^{-1} = x^{-1}$ for $t \in (0, t_\alpha], x \in I\!R$. It is obvious from the previous analysis that any negative local minimum of the function $h(y)$ may give rise to the generation of a shock following the above construction. In view of (A1) there are only finitely many such points $y_{\alpha_i}, i = 1, \ldots, m$, where $y_{\alpha_1} = y_\alpha$. Let χ_i be the corresponding shocks and assume that ε is sufficiently small such that the χ_i's do not interact. Then following (3.1), we can choose a single-valued branch x_s^{-1} of x^{-1} defined for every $x\epsilon I\!R$.

Theorem 3.3. *There exists a single-valued branch of x^{-1} denoted by $x_s^{-1} = x_s(x;t), x\epsilon I\!R, t\epsilon(t_\alpha, t_\alpha+\varepsilon), \varepsilon > 0$ small enough, such that the function*

$$u(x,t) = v(x_s^{-1}(x;t),t)$$

is a viscosity solution of (P) up to the time level $t_\alpha + \varepsilon$.

Proof: By construction $u(x,t)$ is C^∞ away from the points $(\chi_i(t), t)$ and satisfies (1.1). Since the values $u_x(\chi_i(t)\pm, t)$ correspond to the points where f is convex or concave, the viscosity criterion is satisfied. □

4 Construction of the solution of local Riemann problems

In view of the construction of Section 3, we have extended the solution beyond the critical time t_α. In order to further construct the solution, we have to solve a new Cauchy problem with initial data $\tilde{\phi}(x) = \lim_{t \uparrow t_\alpha} u(x,t)$ at the time level $\tilde{t}_\alpha = t_\alpha + \varepsilon$. Here ε is the supremum of all the ε's such that Theorem 3.3 holds. In the sequel to simplify notation, we write t_α instead of $\tilde{t}_\alpha$. The function $\tilde{\phi}$ is C^∞, except at finitely many points where it is only continuous and $\tilde{\phi}'$ undergoes a finite jump. The viscosity criterion is satisfied across the jump, i.e. the inequalities of Theorem 2.2 hold for $u_x^+ = \tilde{\phi}'(x_\alpha+)$ and $u_x^- = \tilde{\phi}'(x_\alpha-)$. The points of discontinuity correspond to the trace of the shocks at the time t_α or to the points where a new shock is generated.

To construct the solution of the Cauchy problem we have to solve all the possible local Riemann problems:

$$u_t + f(u_x) = 0,\ (x,t)\epsilon(x_\alpha - \tilde{\delta}, x_\alpha + \tilde{\delta}) \times (t_\alpha, t_\alpha + \varepsilon),\ \varepsilon, \tilde{\delta} > 0 \tag{P_α}$$

$$u(x, t_\alpha) = \tilde{\phi}(x),$$

where $\tilde{\phi}\epsilon C^\infty((x_\alpha - \tilde{\delta}, x_\alpha + \tilde{\delta})\backslash\{x_\alpha\}) \cap C((x_\alpha - \tilde{\delta}, x_\alpha + \tilde{\delta}))$, for $\varepsilon, \tilde{\delta}$ sufficiently small.

Here $\tilde{\phi}'$ is assumed discontinuous only at the point x_α across which the viscosity criterion is satisfied. We shall also assume that x^{-1}, $t = t_\alpha$, is single-valued on $(x_\alpha - \tilde{\delta}, x_\alpha) \cup (x_\alpha, x_\alpha + \tilde{\delta})$ with range the disjoint union $(x_\alpha^- - \delta, x_\alpha^-) \cup (x_\alpha^+, x_\alpha^+ + \delta)$, where $x_\alpha = x(x_\alpha^-; t_\alpha) = x(x_\alpha^+; t_\alpha)$, $x_\alpha^- < x_\alpha^+$, and

$$1 + t_\alpha \frac{d}{dy}(f'(\phi'(y))) > 0, y \in (x_\alpha^- - \delta, x_\alpha^-) \cup (x_\alpha^+, x_\alpha^+ + \delta). \tag{4.1}$$

The task of the construction of the solutions of the local Riemann problems is undertaken in this section. There are only four constructions in addition to that presented in Section 3. Construction I corresponds to the case where the incoming shock for $t < t_\alpha$ meets the characteristic coming from the left at the point (x_α, t_α) at an oblique angle. In the case where the characteristic meets the shock tangentially, we have the constructions II and III. The different subcases depend on the sign of $\phi''(y), y \neq x_\alpha^\pm$, and whether $1 + t_\alpha f'(\phi'(x_\alpha^-))\phi''(x_\alpha^-)$ may or may not equal to zero. The last construction corresponds to the case where $\tilde{\phi}'(x_\alpha -) = \tilde{\phi}'(x_\alpha +)$, while the higher order derivatives are discontinuous. It is related to the generation of a new shock and is caused by the interaction of regular waves with rarefaction waves. On the other hand, the shock constructed in Section 3 was the result of the breaking of regular waves. In the following section we are going to use these constructions to trace the paths of the shocks, as t evolves, and the way they interact.

Before we present any of these constructions, we need to study how the characteristics cross. To this end, we consider the sets

$$Q^+ = \{(x,t) : x = y + tf'(\phi'(y)), x_\alpha^+ < y < x_\alpha^+ + \delta, t_\alpha < t < t_\alpha + \varepsilon\}$$

$$Q^- = \{(x,t) : x = y + tf'(\phi'(y)), x_\alpha^- - \delta < y < x_\alpha^-, t_\alpha < t < t_\alpha + \varepsilon\}.$$

The domain Q^+ (resp. Q^-) is spanned by the characteristics originating from $(x_\alpha^+, x_\alpha^+ + \delta)$ (resp. $(x_\alpha^- - \delta, x_\alpha^-)$). Subsequently we shall assume that ε, δ are sufficiently small so the assumptions made in each case hold for any (x,t) in the domain of definition of P_α.

Lemma 4.1. *If*

$$1 + t_\alpha \frac{d}{dx}(f'(\phi'(x_\alpha^+))) = 0 \quad (\text{resp. } 1 + t_\alpha \frac{d}{dx}(f'(\phi'(x_\alpha^-))) = 0)$$

then the left (resp. right) boundary of Q^+ (resp. Q^-) is a smooth concave (resp. convex) curve $\sigma(t)$, which is the envelope of the characteristic lines. Otherwise the boundary is the corresponding characteristic line originating from (x_α, t_α).

Proof: Let us assume that the first equality holds. In view of (A1) and (4.1), we have that $h'(y) > 0, y > x_\alpha^+$. Applying the implicit function theorem, we conclude that, for any $t > t_\alpha$ there exists a unique $\sigma^+(t) > x_\alpha^+$ such that

$$1 + tf''(\phi'(\sigma^+(t))\phi''(\sigma^+(t)) = 0.$$

Since $\sigma(t) = \sigma^+(t) + tf'(\phi'(\sigma^+(t))$, we have

$$\frac{d\sigma}{dt} = f'(\phi'(\sigma^+(t)).$$

Differentiating the equation defining $\sigma^+(t)$, yields $\frac{d\sigma^+}{dt} > 0$, therefore

$$\frac{d^2\sigma}{dt^2} = f''(\phi'(\sigma^+(t))\phi''(\sigma^+(t))\frac{d\sigma^+}{dt} < 0. \tag{4.2}$$

The other case is proved similarly. □

We shall now study how we can define $v(x^{-1}(x;t),t)$ to be single-valued in Q^+ and Q^-. We assume that

$$1 + t_\alpha \frac{d}{dy}(f'(\phi'(x_\alpha^-)) = 0$$

and consider the map

$$x : y \to x = x(y;t) = f'(\phi'(y))t + y,\ x_\alpha^- - \delta < y < x_\alpha^-,\ \ t\epsilon(t_\alpha, t_\alpha + \varepsilon).$$

For a fixed t, x^{-1} is two-valued on the interval $(x(x_\alpha^-;t), \sigma(t))$. Let $y_1 = y_1(x;t)$ be the continuous single-valued branch of x^{-1} mapping
$(x(x_\alpha^- - \delta;t), \sigma(t))$ onto $(x_\alpha^- - \delta, \sigma^-(t))$, where

$$\sigma(t) = \sigma^-(t) + tf'(\phi'(\sigma^-(t))),\quad \sigma^-(t) < x_\alpha^-.$$

We define $u^-(x,t) = v(y_1(x,t),t),\ (x,t) \in Q^-$, where

$$v(y,t) = \{-f(\phi'(y)) + \phi'(y)f'(\phi'(y))\}t + \phi(y).$$

The value of the function u^- at the point $(x,t) \in Q^-$, near the boundary curve $\sigma(t)$, is defined through the unique characteristic originating from the initial axis, which is tangent to the curve σ (see Fig. 5). In the same way we define u^+ on Q^+. In the case where

$$1 + t_\alpha \frac{d}{dy}(f'(\phi'(x_\alpha^-)) \neq 0,$$

the above analysis is superfluous since x^{-1} is single -valued on Q^-. Since the viscosity criterion is satisfied across the point (x_α, t_α), the domains Q^+ and Q^- overlap. (See Figure 5, where for definiteness it is assumed that $1 + t_\alpha \frac{d}{dy}(f'(\phi'(x_\alpha^+)) \neq 0)$. Outside the overlapping region $Q^+ \cap Q^-$, the solution can be defined as u^+ or u^-. We have to construct the solution on $Q^+ \cap Q^-$. This is done by first examining the graphs of u^+ and u^- on $Q^+ \cap Q^-$. If they intersect, the viscosity criterion is satisfied across the intersection and the solution is obtained by neglecting the unnecessary parts of the graphs. Otherwise a contact discontinuity appears. In that case we explicitly construct the contact discontinuity curve and the outgoing rarefaction waves. Therefore, we have the following constructions depending on the sign of $\phi''(y)$ on $(x_\alpha^- - \delta, x_\alpha^-)$ and $(x_\alpha^+, x_\alpha^+ + \delta)$ and on whether or not $1 + t_\alpha \frac{d}{dy}(f'(\phi'(x_\alpha^\pm))$ equals to zero.

Construction I We assume that if $\phi'(x_\alpha^-) > \phi'(x_\alpha^+)$, then

$$f'(\phi'(x_\alpha^-)) > \frac{f(\phi'(x_\alpha^+)) - f(\phi'(x_\alpha^-))}{\phi'(x_\alpha^+) - \phi'(x_\alpha^-)}, \tag{4.3}$$

while if $\phi'(x_\alpha^-) < \phi'(x_\alpha^+)$, then the opposite inequality holds. In this case we have a single genuine shock. Define $s(x_\alpha, t_\alpha) = \phi'(x_\alpha^-) - \phi'(x_\alpha^+)$.

Lemma 4.2. *If $s(x_\alpha, t_\alpha)) > 0$ (resp. < 0), then*

$$\frac{d}{dt}[u^\mp(x(x_\alpha^\pm;t),t) - u^\pm(x(x_\alpha^\pm;t),t)] < 0 \quad (resp. \ > 0),$$

where $x(x_\alpha^\pm;t)$ stands for the characteristics originating from the points $(x_\alpha^\pm, 0)$.

Proof: We only present here the proof of the case $s(x_\alpha;t_\alpha) > 0$. The other case follows similarly. A simple calculation yields

$$\frac{d}{dt}[u^-(x(x_\alpha^+;t),t) - u^+(x(x_\alpha^+;t),t)] =$$

$$= f'(\phi'(x_\alpha^+))[u_x^-(x(x_\alpha^+;t),t) - \phi'(x_\alpha^+)]$$

$$-[f(u_x^-(x(x_\alpha^+;t),t)) - f(\phi'(x_\alpha^+)].$$

On the other hand, since $u_x^-(x(x_\alpha^+;t),t)$ and $\phi'(x_\alpha^+)$ lie in the convex region of f,

$$f'(\phi'(x_\alpha^+)) < \frac{f(u_x^-(x(x_\alpha^+;t),t) - f(\phi'(x_\alpha^+))}{u_x^-(x(x_\alpha^+;t),t) - \phi'(x_\alpha^+)}.$$

Combining the above relations we obtain the first inequality.

The second inequality is proved using

$$\frac{d}{dt}[u^+(x(x_\alpha^-;t),t) - u^-(x(x_\alpha^-;t),t)] =$$

$$= f'(\phi'(x_\alpha^-))[u_x^+(x(x_\alpha^-;t),t) - \phi'(x_\alpha^-)]$$

$$-[f(u_x^+(x(x_\alpha^-;t),t) - \phi'(x_\alpha^-)]$$

and

$$f'(\phi'(x_\alpha^-)) > \frac{f(u_x^+(x(x_\alpha^-;t),t)) - f(\phi'(x_\alpha^-))}{u_x^+(x(x_\alpha^-;t),t) - \phi'(x_\alpha^-)}.$$

The inequality above holds for $t = t_\alpha$ by means of (4.3). In view of continuity it holds also for t near t_α $\square$.

For a fixed time t, $\frac{\partial}{\partial x}[u^+(x,t) - u^-(x,t)] < 0$ (resp. > 0) in $Q^+ \cap Q^-$, for $s(x_\alpha, t_\alpha) < 0$ (resp. > 0). Thereby, in view of the Lemma 4.2 and implicit function theorem, there exists a unique $\chi(t)$, $x(x_\alpha^+;t) < \chi(t) < x(x_\alpha^-;t)$, such that $u^+(\chi(t),t) = u^-(\chi(t),t)$ and $\chi(t)$ is a smooth curve (see Figure 6). Therefore we have the following theorem.

Theorem 4.1 *There exists a unique smooth curve* $\chi : (t_\alpha, t_\alpha + \varepsilon) \to \mathbb{R}$, *such that*

$$u^+(\chi(t),t) = u^-(\chi(t),t), \;\; x(x_\alpha^+;t) < \chi(t) < x(x_\alpha^-;t).$$

Moreover, the function

$$u(x,t) = \begin{cases} u^-(x,t), & x \le \chi(t) \\ u^+(x,t), & x \ge \chi(t) \end{cases} \tag{4.4}$$

is a local viscosity solution of (P_α).

Proof: In view of (4.3) the viscosity criterion is satisfied across χ. □

Let us now assume that instead of (4.3) we have

$$f'(\phi'(x_\alpha^-)) = \frac{f(\phi'(x_\alpha^+)) - f(\phi'(x_\alpha^-))}{\phi'(x_\alpha+) - \phi'(x_\alpha-)} \tag{4.5}$$

and $\phi''(y) < 0, y > x_\alpha^+$, for $s(x_\alpha, t_\alpha) > 0$ (resp. $\phi''(y) > 0$ for $s(x_\alpha, t_\alpha) < 0$). We follow the previous construction to construct a curve $\chi, x(x_\alpha^+; t) < \chi(t) < x(x_\alpha^-; t)$, such that

$$\frac{d\chi}{dt} = \frac{f(u_x^+(\chi(t), t)) - f(u_x^-(\chi(t), t))}{u_x^+(\chi(t), t) - u_x^-(\chi(t), t)}$$

and we define u using (4.4).

If $\phi''(y) > 0, y < x_\alpha^-$, for $s(x_\alpha, t_\alpha) > 0$ (*resp.* $\phi''(y) < 0$ for $s(x_\alpha, t_\alpha) < 0$), then the geometric properties of f and the monotonicity of ϕ', for $y \neq x_\alpha^\pm$, yield that the viscosity criterion is satisfied across χ. If the opposite inequality holds and the viscosity criterion is not satisfied for $t > t_\alpha$ then, in view of $(A1)$, $\frac{d\chi}{dt} > f'(\phi'(\chi^-(t))$ in a half-neighborhood of t_α, where

$$\chi(t) = \chi^-(t) + tf'(\phi'(\chi(t))), \quad \chi^-(t) \leq x_\alpha^-.$$

Differentiating the last equality yields

$$\frac{d\chi}{dt} - f'(\phi(\chi^-(t)) = [1 + tf''(\phi'(\chi^-(t))\phi''(\chi^-(t))]\frac{d\chi^-}{dt}.$$

Therefore $\frac{d\chi^-}{dt} > 0, t > t_\alpha$ and hence $\chi^-(t) > \chi^-(t_\alpha) = x_\alpha^-$, which is a contradiction. Therefore we have the following theorem.

Theorem 4.2 *Assume that (4.5) holds and that* $\phi''(y) < 0, y \neq x_\alpha^\pm$, *for* $s(x_\alpha, t_\alpha) > 0$, *while* $\phi''(y) > 0$ *for* $s(x_\alpha, t_\alpha) < 0$. *Then defining* u *by (4.4), the viscosity criterion is satisfied across* χ *and* u *is the local viscosity solution of* (P_α)

<u>*Construction II*</u> Here we assume that

$$f'(\phi'(x_\alpha^-)) = \frac{f(\phi'(x_\alpha^+)) - f(\phi'(x_\alpha^-))}{\phi'(x_\alpha^+) - \phi'(x_\alpha^-)}$$

and

$$1 + t_\alpha f''(\phi'(x_\alpha^-))\phi''(x_\alpha^-) > 0,$$

$\phi''(y) > 0, y > x_\alpha^+$, if $s(x_\alpha, t_\alpha) > 0$, while $\phi''(y) < 0$, if $s(x_\alpha, t_\alpha) < 0$.
Under these assumptions, it is easy to check that the graphs of u^+ and u^- do not intersect for $t > t_\alpha$. A new way to build the solution is required. This is the case where a contact discontinuity appears. We first construct the contact discontinuity shock curve and then define the solution in the

different regions. Let us consider the functions $(\cdot)^*$, $(\cdot)_*$, defined by

$$(\cdot)^* : \eta \ni I\!R^- \to \eta^* \in I\!R^+, \ (\cdot)_* : \eta \ni I\!R^+ \to \eta_* \in I\!R^-,$$

$$\eta^* = \sup_q \{ q > \eta : \frac{f(\eta) - f(q)}{\eta - q} < \frac{f(\eta) - f(v)}{\eta - v}, v \in (\eta, q) \}$$

$$\eta_* = \inf_q \{ q < \eta : \frac{f(\eta) - f(q)}{\eta - q} > \frac{f(\eta) - f(v)}{\eta - v}, v \in (q, \eta) \}.$$

Let us assume that $s(x_\alpha, t_\alpha) > 0$. We will first construct the contact discontinuity as solution of the following initial value problem

$$\begin{cases} \dfrac{d\chi}{dt} = f'((u_x^+(\chi(t), t))^*), \quad t > t_\alpha, \\ \\ \chi(t_\alpha) = x_\alpha. \end{cases} \tag{4.6}$$

Since

$$\frac{d\chi}{dt}(t_\alpha) = f'((\phi'(x_\alpha^+))^*) = f'(\phi'(x_\alpha^-)) > f'(\phi'(x_\alpha^+)),$$

there exists a unique local solution $\chi(t)$. By using the identities

$$\frac{d^2\chi}{dt^2} = f''((u_x^+(\chi(t), t))^*) \frac{d(u_x^+)^*}{du_x^+} \frac{d}{dt}(u_x^+(\chi(t), t)).$$

and

$$\frac{d}{dt} u_x^+(\chi(t), t) = u_{xx}^+(\chi(t), t)[f'(((u_x^+(\chi(t), t))^*) - f'(u_x^+(\chi(t), t))]$$

we obtain

$$\frac{d^2\chi}{dt^2} = f''((u_x^+(\chi(t), t)^*) \frac{d(u_x^+)^*}{du_x^+} \frac{\phi''(\chi^+(t))}{1 + tf''(\phi'(\chi^+(t)))\phi''(\chi^+(t))} \cdot \tag{4.7}$$

$$[f'((u_x^+(\chi(t), t))^*) - f'(u_x^+(\chi(t), t))],$$

where $\chi(t) = \chi^+(t) + tf'(\phi'(\chi^+(t))$. Therefore, $\frac{d^2\chi}{dt^2} > 0$ and the following Lemma holds (see [2]), which describes the construction of the rarefaction waves.

Lemma 4.3 *Let $\chi \in C^\infty((t_\alpha, t_\alpha + \varepsilon))$, be such that $\chi(t_\alpha) = x_\alpha$ and $\frac{d^2\chi}{dt^2} > 0$. For each $t_1 \in (t_\alpha, t_\alpha + \varepsilon)$ let $L(t_1)$ be the half line*

$$L(t_1) = \frac{d\chi(t_1)}{dt}(t - t_1) + \chi(t_1), \quad t \geq t_1,$$

and $\tilde{Q} = \{(x,t) : t_\alpha < t < t_\alpha + \varepsilon, \chi(t) > x > x(x_\alpha^-; t)$. Then every point $(x,t) \in \tilde{Q}$ lies on exactly one half line $L(t_1)$. Furthermore, if $g = g(t) \in C^\infty((t_\alpha, t_\alpha + \varepsilon))$, then the function w defined on $\tilde{Q}$ by $w(x,t) = g(t_1), (x,t) \in L(t_1)$, satisfies $w(x,t) \in C^\infty(\tilde{Q})$.

We next define u to be $u^-(x,t)$, on the left of $x(x_\alpha^-; t)$ (see Fig. 7) and $u^+(x,t)$ on the right of $\chi(t)$. We need to define the solution on $\tilde{Q}$. The curve $\chi(t)$ has been constructed to have the right slope $f'(u_x^+(\chi(t), t))^*)$. The previous lemma yields that the domain $\tilde{Q}$ is spanned by the half lines $L(t_1)$. Since on the right of the curve $\chi(t)$ at $t = t_1$, the gradient of the solution has the value $u_x^+(\chi(t_1), t_1)$, we assign $(u_x^+(\chi(t_1), t_1))^*$ as its value from the left and keep it constant along $L(t_1)$. That is the solution $u(x,t)$ on $\tilde{Q}$ is carried by rarefaction waves corresponding to the half lines $L(t_1)$.

Applying the previous lemma with $g(t_1) = (u_x^+(x(t_1), t_1))^*$ we obtain

Theorem 4.3 *The function u defined by*

$$u(x,t) = u^-(x,t), \quad (x,t) \in Q^-,$$

$$u(x,t) = \tilde{u}(x,t) = \int_{x(x_\alpha^-;t)}^{x} w(y,t)dy + u^-(x(x_\alpha^-;t),t), \quad (x,t) \in \tilde{Q},$$

$$u(x,t) = u^+(x,t), \quad (x,t) \in Q^+\backslash(\tilde{Q} \cup Q^-),$$

is a local viscosity solution of (P_α).

Proof: Let $(x,t) \in \tilde{Q}$, then making the change of variables $x = \chi'(t_1)(t-t_1) + \chi(t_1)$, we get

$$w_t(x,t) = -\frac{g'(t_1)(\chi'(t_1))^2}{(x-\chi(t_1))\chi''(t_1)}, \quad w_x(x,t) = \frac{g'(t_1)\chi'(t_1)}{(x-\chi(t_1))\chi''(t_1)}, \quad (x,t) \in \tilde{Q}.$$

In view of the previous identities and the definition of $\tilde{u}$, we have

$$(\tilde{u}_t(x,t) + f(\tilde{u}_x(x,t)))_x = w_t(x,t) + f'(w(x,t))w_x(x,t) =$$

$$= w_t(x,t) + f'((\tilde{u}_x^+(\chi(t_1),t_1))^*)w_x(x,t) =$$

$$= w_t(x,t) + x'(t_1)w_x(x,t) = 0.$$

Therefore $\tilde{u}_t(x,t) + f(\tilde{u}_x(x,t)) =$ constant, for $(x,t) \in \tilde{Q}, t$ fixed. On the other hand, $\lim_{x \to x(x_\alpha^-;t)+} \tilde{u}_x(x,t) = \phi'(x_\alpha^-)$ and $\lim_{x \to x(x_\alpha^-;t)+} \tilde{u}_t(x,t) = -f(\phi'(x_\alpha^-))$. Thereby,

$$\lim_{x \to x(x_\alpha^-;t)+} (\tilde{u}_t + (f(\tilde{u}_x)) = 0,$$

hence

$$\tilde{u}_t + f(\tilde{u}_x) = 0, \quad (x,t) \in \tilde{Q}.$$

By definition, however, $\tilde{u}_x(\chi(t)-,t) = (u_x^+(\chi(t),t))^*$, therefore

$$\tilde{u}_t(\chi(t)-,t) = -f((u_x^+(\chi(t),t))^*).$$

This yields that $\tilde{u}$ has a C^1 extension up to the boundary of $\tilde{Q}$. To conclude, we need to prove that u is continuous across $\chi(t)$, i.e.

$$I(t) = \tilde{u}(\chi(t),t) - u^+(\chi(t),t) = 0, \quad t \geq t_\alpha.$$

This is easily checked using that $I(t_\alpha) = 0$ and the relation

$$\frac{dI}{dt} = \chi'(t)(u_x^+(\chi(t),t))^* + \tilde{u}_t(\chi(t),t)$$

$$-u_x^+(\chi(t),t) \cdot \chi'(t) - u_t^+(\chi(t),t) =$$

$$= [(u_x^+(\chi(t),t))^* - u_x^+(\chi(t),t)]\chi'(t)$$

$$-[f((u_x^+(\chi(t),t)^*) - f(u_x^+(\chi(t),t))] = 0. \quad \square$$

In the case $s(x_\alpha,t_\alpha) < 0$, we repeat the same construction except that we substitute the upper star $(\cdot)^*$ function with the lower star $(\cdot)_*$. This construction is originated in [2], where the contact discontinuity curve and the rarefaction waves for a conservation law are obtained.

Construction III Here we assume that the conditions of the case II hold except that $1 + t_\alpha f''(\phi(x_\alpha^-))\phi''(x_\alpha^-) = 0$. Moreover, we assume $s(x_\alpha, t_\alpha) > 0$; the case of the opposite inequality can be treated similarly. The assumptions imply that $\phi''(y) > 0$ for $y < x_\alpha^-$. The geometry of f and the monotonicity of ϕ' yield

$$\frac{d}{dt}[u^-(x(x_\alpha^-;t),t) - u^+(x(x_\alpha^-;t),t)] < 0. \tag{4.8}$$

Let $\sigma(t)$ be the right boundary of Q^- and

$$A(t) = u^+(\sigma(t),t) - u^-(\sigma(t),t), \ t \geq t_\alpha.$$

Then

$$\begin{aligned} \frac{dA}{dt} &= f'(\phi'(\sigma^-(t)))[\phi'(\sigma^+(t)) - \phi'(\sigma^-(t))] \\ &\quad -[f(\phi'(\sigma^+(t))) - f(\phi'(\sigma^-(t)))], \end{aligned} \tag{4.9}$$

where

$$\sigma(t) = \sigma^-(t) + tf'(\phi'(\sigma^-(t))) = \sigma^+(t) + tf'(\phi'(\sigma^+(t))), \tag{4.10}$$

and $\sigma^+(t) \geq x_\alpha^+$, $\sigma^-(t) \leq x_\alpha^-$. Differentiating (4.9), we get

$$\begin{aligned} \frac{d^2A}{dt^2} &= [f'(\phi'(\sigma^-(t))) - f'(\phi'(\sigma^+(t)))]\phi''(\sigma^+(t))\frac{d\sigma^+}{dt} \\ &+\frac{d}{dy}(f'(\phi'(\sigma^-(t))))\frac{d\sigma^-}{dt}[\phi'(\sigma^+(t)) - \phi'(\sigma^-(t)))], \quad t > t_\alpha. \end{aligned} \tag{4.11}$$

Differentiating (4.10) and using that $\sigma(t)$ is the right boundary of Q^-, we get

$$\frac{d\sigma^+}{dt} = \frac{f'(\phi'(\sigma^-(t)) - f'(\phi'(\sigma^+(t))}{1 + t\frac{d}{dy}f'(\phi'(\sigma^+(t))}.$$

Moreover (4.2), with σ^- in place of σ^+, the previous identity and equation (4.11) yield

$$\frac{d^2A}{dt^2} = \frac{d\sigma^2}{dt^2}[\phi'(x_\alpha^+) - \phi'(x_\alpha^-)]$$

$$+\phi''(x_\alpha^+)[f'(\phi'(x_\alpha^+) - f'(\phi'(x_\alpha^-)]^2(1 + t_\alpha \frac{d}{dy} f'(\phi'(x_\alpha^+)))^{-1}, \tag{4.12}$$

for $t = t_\alpha$. Let us now denote by $\tilde{\chi}$ the curve defined by (4.6). In view of (4.7) and

$$\frac{du^*}{du} = \frac{f'(u)}{f''(u^*)(u - u^*)}$$

for $t = t_\alpha$ we obtain

$$\frac{d^2\tilde{\chi}}{dt^2} \frac{\phi'(x_\alpha^+) - \phi'(x_\alpha^-)}{f'(\phi'(x_\alpha^+))}$$

$$= \frac{\phi''(x_\alpha^+)}{1 + t_\alpha \frac{d}{dy} f'(\phi'(x_\alpha^+))}[f'(\phi'(x_\alpha^-)) - f'(\phi'(x_\alpha^+))].$$

By means of the above equality (4.12) is written in the form.

$$\frac{d^2A(t_\alpha)}{dt^2} = \left[\frac{d\sigma^2(t_\alpha)}{dt^2} + \left(\frac{f'(\phi'(x_\alpha^-))}{f'(\phi'(x_\alpha^+))} - 1\right)\frac{d^2\tilde{\chi}(t_\alpha)}{dt^2}\right] \cdot$$

$$[\phi'(x_\alpha^+) - \phi'(x_\alpha^-)]. \tag{4.13}$$

We need to examine the following subcases.

<u>Case IIIa</u> Assume that $\sigma(t) \geq \tilde{\chi}(t), \quad t \geq t_\alpha$. The construction results in a single genuine shock. Since both $\tilde{\chi}$ and σ are convex near t_α, we have $\frac{d^2\sigma(t_\alpha)}{dt^2} \geq \frac{d^2\tilde{\chi}(t_\alpha)}{dt^2}$ and (4.13) yields that $\frac{d^2A(t_\alpha)}{dt^2} < 0$. Therefore in view of the definition of $A(t)$ and (4.8) we have the following

Lemma 4.4. *Under the assumptions of case IIIa*

$$\frac{d}{dt}[u^-(x(x_\alpha^-;t),t) - u^+(x(x_\alpha^-;t),t)] < 0,\ t > t_\alpha,$$

and

$$\frac{d}{dt}[u^+(\sigma(t),t) - u^-(\sigma(t),t)] < 0.$$

Following the lines of Theorem 4.1, there exists a unique smooth curve χ with $x(x_\alpha^-;t) < \chi(t) < \sigma(t)$, such that $u^+(\chi(t),t) = u^-(\chi(t),t)$. The solution is given by (4.4).

Case IIIb Here we assume that $\sigma(t) < \tilde{\chi}(t), t > t_\alpha$. In this subcase two shocks spring from the point (x_α, t_α). The right one is a left contact discontinuity. The left one is a genuine shock defined from the right by the outgoint rarefaction waves. If $\tilde{Q}, \tilde{u}$ are defined as in construction II, we write $\tilde{Q} = \tilde{Q}_1 \cup \tilde{Q}$ where $\tilde{Q}_1$ is the domain bounded by the curves $x(x_\alpha^-;t)$ and $\sigma(t)$, $\tilde{Q}_2$ the domain bounded by the curves $\sigma(t)$ and $\tilde{\chi}(t)$ (see Figure 8).

Lemma 4.5 *The following inequalities hold:*

$$\frac{d}{dt}[\tilde{u}(x(x_\alpha^-;t),t) - u^-(x(x_\alpha^-;t),t)] < 0$$

$$\frac{d}{dt}[\tilde{u}(\sigma(t),t) - u^-(\sigma(t),t)] > 0.$$

Proof: Since ϕ' is increasing for $y < x_\alpha^-$, then

$$f'(\phi'(x_\alpha^-)) < \frac{f(\phi'(x_\alpha^-)) - f(u_x^-(x(x_\alpha^-;t),t))}{\phi'(x_\alpha^-) - u_x^-(x(x_\alpha^-;t),t)},$$

which results in the first inequality of the Lemma. Moreover,

$$\frac{d}{dt}[\tilde{u}(\sigma(t),t) - u^-(\sigma(t),t)] =$$

$$f'(\phi'(\sigma^-(t)))[\tilde{u}_x(\sigma(t),t) - \phi'(\sigma^-(t))]$$

$$-[f(\tilde{u}_x(\sigma(t),t)) - f(\phi'(\sigma^-(t)))].$$

Since $f'(\phi'(\sigma^-(t)), f'(\tilde{u}_x(\sigma(t),t))$ are the slopes of the incoming characteristics at $(\sigma(t),t)$ we have $f'(\phi'(\sigma^-(t))) > f'(\tilde{u}_x(\sigma(t),t))$. Since both $\phi'(\sigma^-(t))$ and $\tilde{u}_x(\sigma(t),t)$ belong to the concave region of f, $\phi'(\sigma^-(t)) < \tilde{u}_x(\sigma(t),t)$. Therefore,

$$f'(\phi'(\sigma^-(t))) > \frac{f(\tilde{u}_x(\sigma(t),t)) - f(\phi'(x_\alpha^-))}{\tilde{u}_x(\sigma(t),t) - \phi'(x_\alpha^-)},$$

hence the result. □

In view of the Lemma there exists a unique χ with $x(x_\alpha^-;t) < \chi(t) < \sigma(t)$, such that

$$\tilde{u}(\chi(t),t) = u^-(\chi(t),t).$$

Theorem 4.4 *The function u defined by*

$$u(x,t) = \begin{cases} u^-(x,t), & x \leq \chi(t) \\ \tilde{u}(x,t), & \chi(t) \leq x \leq \tilde{\chi}(t) \\ u^+(x,t), & \tilde{\chi}(t) \leq x \end{cases}$$

is a local viscosity solution of (P_α).

Proof: The viscosity criterion is satisfied by construction across $\tilde{\chi}(t)$ (see construction II). Since the derivatives accross χ correspond to the concave area of f, the viscosity criterion is also satisfied across χ. □

<u>*Construction IV*</u> Assume that $\tilde{\phi}'(x_\alpha -) = \tilde{\phi}'(x_\alpha +) = \tilde{\phi}'(x_\alpha) \neq 0$,

$$1 + t_\alpha f''(\phi'(x_\alpha^-))\phi''(x_\alpha^-) = 0,$$

and $\tilde{\phi}''(x) < 0,\ x > x_\alpha$, if $\tilde{\phi}'(x_\alpha) > 0$ or $\tilde{\phi}''(x) > 0$ if $\tilde{\phi}'(x_\alpha) < 0$. We only present here the case $\tilde{\phi}'(x_\alpha) > 0$. If $\tilde{\phi}'(x_\alpha) < 0$ we argue similarly. Here the construction results in a single genuine shock.

Lemma 4.6 *The following inequalities hold:*
$\frac{d}{dt}[u^+(x(x_\alpha^-;t),t) - u^-(x(x_\alpha^-;t),t)] < 0,\quad \frac{d}{dt}[u^+(\sigma(t),t) - u^-(\sigma(t),t)] > 0.$

The proof can be given along the lines that of Lemma 4.5. Therefore, there exists a unique point $x(x_\alpha^-;t) < \chi(t) < \sigma(t)$ such that $u^+(\chi(t),t) = u^-(\chi(t),t)$ and $\chi(t)$ is a smooth curve. The solution is constructed as in Theorem 4.1.

5 Propagation of a single shock and interaction of shocks

In this section we shall use the constructions undertaken in Sections 3 and 4 to investigate the global structure of the shock curves. In view of Theorem 3.3, we have extended the solution up to the time $\tilde{t}_\alpha$ (see the beginning of Section 4). If two or more shocks intersect at the same point, the viscosity criterion is satisfied across the resulting discontinuity. Moreover, the corresponding chord is not tangent to the graph of f. Hence, we follow construction I to extend locally the solution. The outgoing shock is a genuine one. We use the constructions presented in Section 4 to continue locally the shock curves for $t > \tilde{t}_\alpha$. Working along the lines of the proof of the Theorem 3.3, we can find a single-valued branch x_s^{-1} of x^{-1} defined in the region not covered by rarefaction waves. The solution of (P) is given by $v(x_s^{-1}(x;t))$. In the region covered by the rarefaction waves, the solution is defined according to the Theorem 4.3. In view of (A1), the construction of the shock can change only finitely many times. Therefore, we can construct the solution for any time t.

Let $\mathcal{S}$ be the union of genuine shocks and contact discontinuities and $\mathcal{W}$ the union of weak waves. We have the following theorem.

Theorem 5.1. *The complement of the union $\mathcal{S} \cup \mathcal{W}$ of the shock* *curves and weak waves consists of the two domains $\mathcal{G}$ and $\mathcal{R}$.*

1. *$\mathcal{G}$ is a simply-connected domain covered by regular waves originating from the initial axis. A single-valued branch x_s^{-1} of x^{-1} can be defined on $\mathcal{G}$ such that the solution u of (P) on $\mathcal{G}$ is given by*

$$u(x,t) = v(x_s^{-1}(x;t)).$$

2. *The set $\mathcal{R}$ is the union of simply connected domains and it is covered by rarefaction waves originating from contact discontinuities. The solution u on $\mathcal{R}$ is obtained according to Theorem 4.3.*

To understand better the geometric structure of the generated shocks, we need to study how a shock propagates further and how it changes from one type to another

To this end, let us assume that a negative local minimum of the function $h(y) = \frac{d}{dy}(f'(\phi'(y))$ at the point y_α gives rise to a genuine shock. The shock is generated at the point $(x(y_\alpha; t_\alpha), t_\alpha)$; see Figure 9. We follow construction I to continue the shock further. Let us assume that $\phi''(y_\alpha) < 0$. Then, we always have $u_x(\chi(t)+, t) < u_x(\chi(t)-, t)$. Otherwise, there would be a time t' for which $u_x(\chi(t')+, t') = u_x(\chi(t')-, t')$ and the shock would be degenerated in a characteristic line. For a time t near the time t_α of the generation of the shock, $u_x(\chi(t)-, t) < 0$. If, for a later time $t, u_x(\chi(t)-, t) < (u_x(\chi(t)+, t))^*$ or $u_x(\chi(t)-, t) = u_x(\chi(t)+, t))^*$ and $u_{xx}(x,t) < 0, x > \chi(t)$, we follow construction I to continue the solution further. If the equality holds but $u_{xx}(\chi(t),t) > 0, x > \chi(t)$ and $1 + t\ h(\chi^-(t)) \neq 0$, the characteristic coming from the left stimulates the shock which changes to a contact discontinuity and starts emitting tangentially from the left rarefaction waves. Let us assume that this happens for $t = t_\beta$ (see Fig. 9.). In the case where $1 + t_\beta h(\chi^-(t_\beta)) = 0$ the genuine shock either continues as a genuine one or splits into a left contact discontinuity and a genuine shock lying on its left (see Fig. 10). More precisely, we have the following Theorem.

Theorem 5.2 *Let u be the viscosity solution of (P) constructed up to the time level t_β. We assume that the genuine shock χ generated at a previous time passes through the point $(\chi(t_\beta), t_\beta)$,*

$$\frac{d\chi}{dt}(t_\beta) = f'(u_x(\chi(t_\beta)-, t_\beta)),$$

and either

$$i)\ if \quad u_x(\chi(t_\beta -, t_\beta) > u_x(\chi(t_\beta)+, t_\beta) \quad and$$

$$u_{xx}(x, t_\beta) > 0 \quad for \quad \chi(t_\beta) < x < \chi(t_\beta) + \delta, \quad \delta > 0 \quad small\ enough$$

or

$$ii)\ u_x(\chi(t_\beta)-, t_\beta) < u_x(\chi(t_\beta)+, t_\beta) \quad and$$

$$u_{xx}(x, t_\beta) < 0 \quad for \quad \chi(t_\beta) < x < \chi(t_\beta) + \delta.$$

Then we have the following cases: If

$$1 + th(\chi^{-}(t_{\beta})) \neq 0,$$

the genuine shock turns into a left contact discontinuity following construction II. If

$$1 + th(\chi^{-}(t_{\beta})) = 0,$$

then either

a) the conditions of Construction IIIa hold and the shock continues as a genuine one;

or

b) the conditions of Contstruction IIIb hold and the shock splits into a fleft contact discontinuity and a genuine one lying on the left. The genuine shock is defined from the right by the outgoing rarefaction waves.

Next, we assume that only a contact discontinuity springs at the time t_{β}. The condition for this contact discontinuity to change into a genuine shock at a later time $t' > t_{\beta}$, is that $u_x(\chi(t'), t') = 0$ and $u_{xx}(x, t') < 0$ (resp. $> 0), x > \chi(t')$, for $s(\chi(t'), t') > 0$ (resp. < 0). On the other hand, the structure of the rarefaction waves yields that $u_{xx}(x, t') < 0$ (resp. > 0) for $x < \chi(t')$. If we solve the new Riemann problem around the point $(\chi(t'), t')$, the graphs of the corresponding functions u^{+}, u^{-} will intersect along a smooth curve across which the viscosity criterion must be satisfied and the contact discontinuity changes into a genuine shock according to Theorem 4.2.

In the case where $1+t\ h(\chi^-(t_\beta)) \neq 0$ the characteristics on the left of the weak wave $x(x(t_\beta)-;t)$ may start crossing at the time level $t_\gamma > t_\beta$. The weak wave is terminated at this point and a genuine shock $\chi_1(t)$ is born following construction IV. The genuine shock enters the rarefaction wave region (see Fig. 11).

We denote by $\chi_1^-(t_\gamma)$ and $\chi^-(t_\alpha)$ the traces on the initial axis of the left characteristics passing through the points $(\chi_1(t_\gamma), t_\gamma), (\chi(t_\alpha), t_\alpha)$ respectively. Since there exists a point $y \in (\chi_1^-(t_\gamma), \chi^-(t_\alpha))$ such that $\phi''(y) = 0$, the function $h(y)$ has at least a negative local minimum on the interval $[\chi_1^-(t_\gamma), \chi^-(t_\alpha))$. Moreover, if more than two shocks intersect, we have a unique genuine outgoing shock and the number of existing shocks is less or equal the number of the negative local minima of $h(y)$. Due to the construction of the contact discontinuity, if the shock is hitted from the right by a weak wave of order m, it continues as a contact discontinuity and the triggered weak wave from the left is of order $m+1$.

All the results obtained in this section are summarized in Section 2.3.

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References

[1] V. I. ARNOLD, S.M. GUSEIN–ZADE & A. N. VARCHENKO, Singularities of Differentiable Maps, Volume 1, MonographS in Mathematics Vol. 82, Birkhäuser, 1985.

[2] D. P. BALLOU, Solutions to nonlinear hyperbolic Cauchy problems without convexity conditions, in Trans. Amer. Math. Soc. 152 (1970), 441-460.

[3] P. CANNARSA & H. M. SONER, On the singularities of the viscusity solutions to Hamilton–Jacobi–Bellman equations, Indiana U. Math. J., 36 (3) (1987), 501–523.

[4] M. G. CRANDALL, L.C. EVANS & P.-L. LIONS, Some properties of viscosity solutions of Hamilton-Jacobi equations, in Trans. Amer. Math. Soc. 282 (1984), 487-502.

[5] M. G. CRANDALL & P.-L. LIONS, Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc. 277 (1983), 1-42.

[6] C. M. DAFERMOS, Regularity and large time behaviour of solutions of a conservation law without convexity, Proc. Royal Soc. Edinb. 99A (1985), 201-239.

[7] W. H. FLEMING, The Cauchy problem for a nonlinear first order partial differential equation, J. Diff. Eqs., 5 (1969), 515–530.

[8] J. GUCKENHEIMER, Solving a single conservation law, Lecture Notes in Mathematics 468, Springer Verlag, New York 1975, 108-134.

[9] H. ISHII, Perron's Method for Hamilton–Jacobi Equations, Duke Math. J., 55 (2) (1987), 369–384.

[10] G. JENNINGS, Piecewise smooth solutions of a single conservation law exist, Adv. in Math., 33 (1979), 192-205.

[11] R. JENSEN & P. E. SOUGANIDIS, A regularity result for viscosity solutions of Hamilton–Jacobi equations in one space dimension, Trans. Amer. Math. Soc. 301 (1987), 137–147.

[12] G. T. KOSSIORIS, Formation of singularities for viscosity solutions of Hamilton-Jacobi equations in higher dimensions, Department of Mathematics, Carnegie Mellon University, Research Report No. 91-120, March 1991.

[13] S. NAKANE, Formation of shocks for a single conservation law, SIAM J. Math. Anal. 19 (1988), 1391-1408.

[14] S. NAKANE, Formation of singularities for Hamilton–Jacobi equation in several space dimensions, 1989, to appear.

[15] M. TSUJI, Solution globale et propagation des singularités pour l'équation de Hamilton–Jacobi, C.R. Acad. Sci. Paris, 289 (1979), 394–400.

[16] M. TSUJI, Formation of singularities for Hamilton-Jacobi equation II, J. Math. Kyoto Univ. 26 (1986) 299-308.

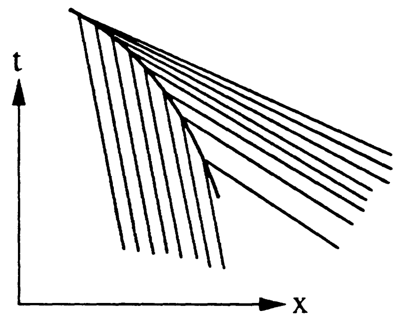
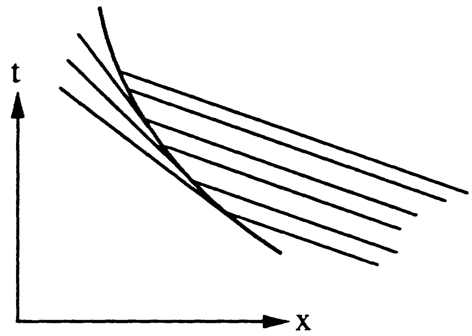


Figure 1



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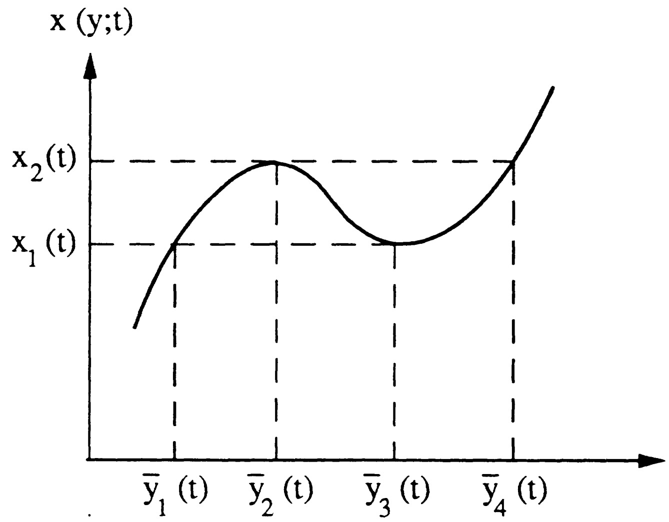

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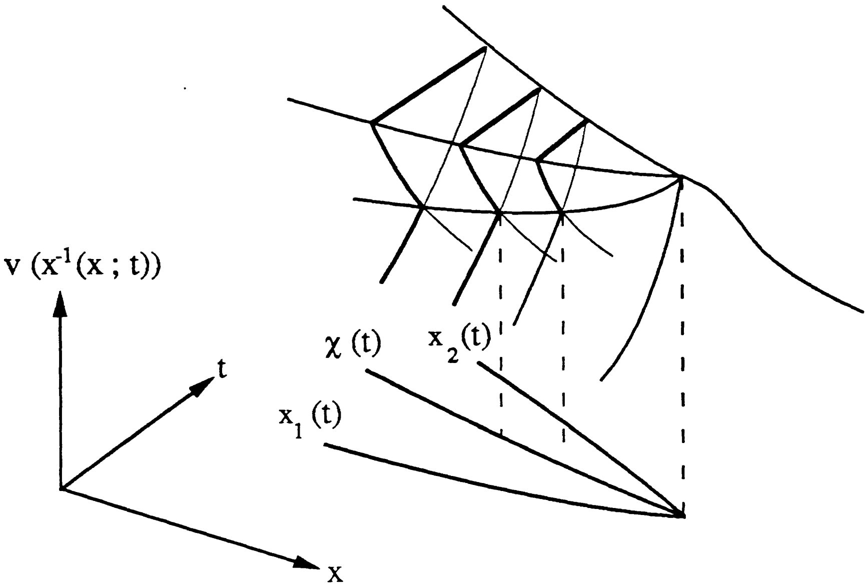

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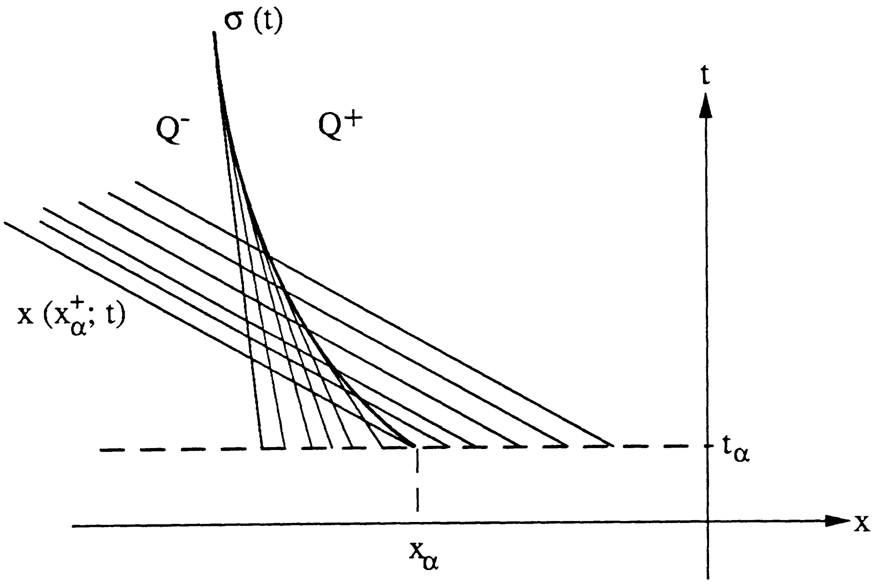

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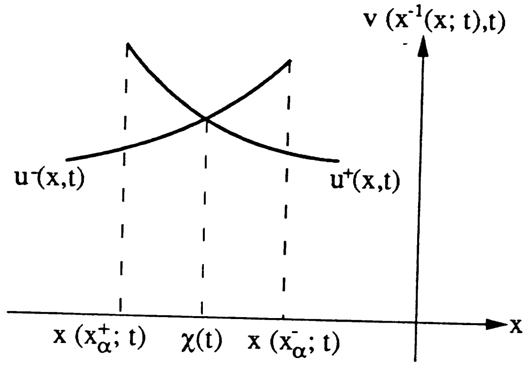

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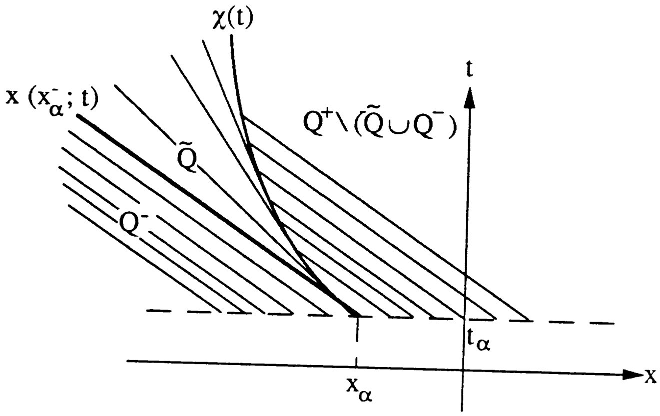

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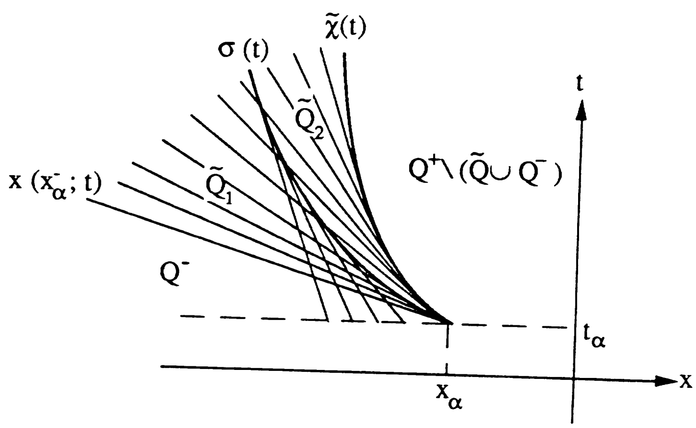

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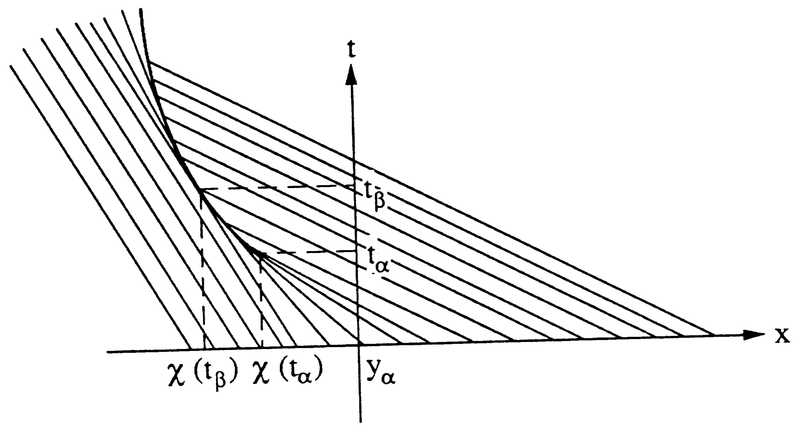

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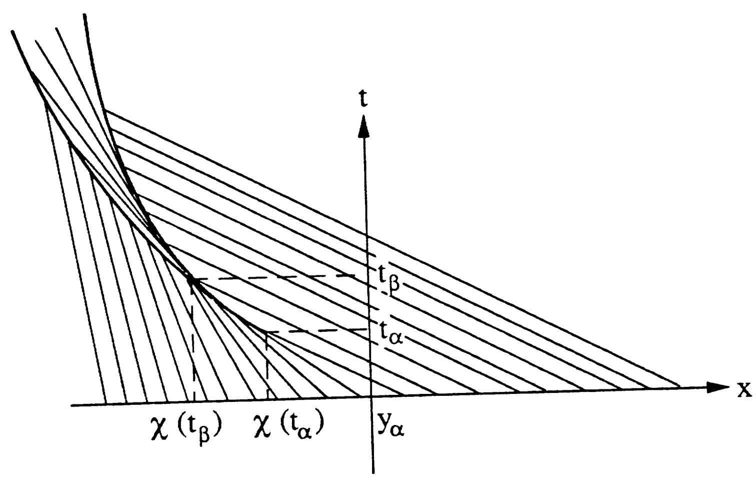

Figure 10

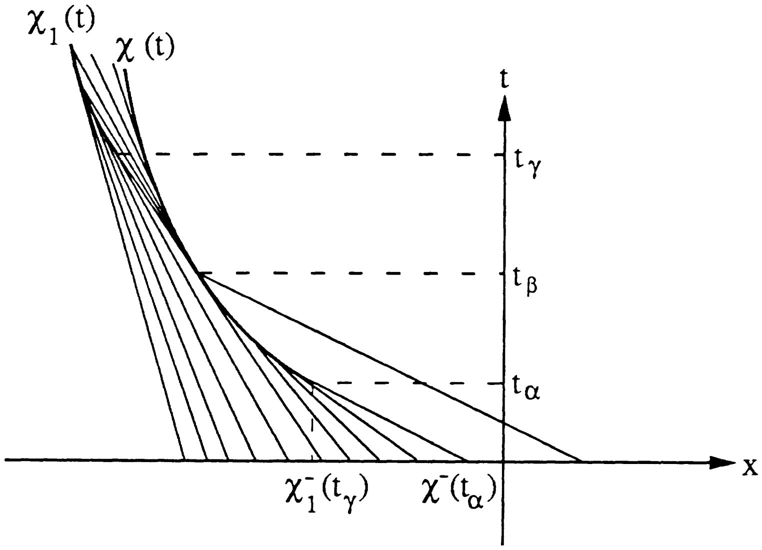

Figure 11

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13. ABSTRACT (*Maximum 200 words*)

In this work we study the generation and propagation of singularities (shock waves) of the solution of the Cauchy problem for Hamilton-Jacobi equations in one space variable, under no assumption on convexity or concavity of the hamiltonian. We study the problem in the class of viscosity solutions, which are the correct class of weak solutions. We obtain the exact global structure of the shock waves by studying the way the characteristics cross. We construct the viscosity solution by either selecting a single-valued branch of the multi-valued function given as a solution by the method of characteristics or constructing explicitly the proper rarefaction waves.

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