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THE WEDGE ENTRY PROBLEM

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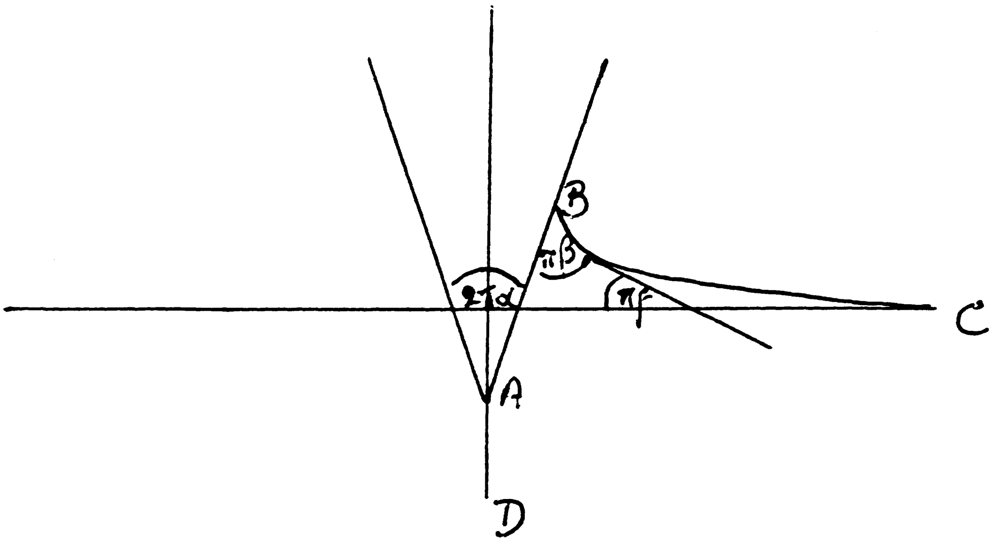
The problem is one of the classical problems in fluid mechanics. It is to study the entry of a wedge which is descending with uniform velocity $V$and point downwards into a bath of water whose surface is initially horizontal. The wedge is assumed to be symmetrical about a plane of symmetry, and this plane is perpendicular to the surface of the water. As the wedge descends into the water, the problem is to determine the free surface.



The problem first appeared in print in 1932 when Wagner [1] formulated the problem and realized that it could, by complex variable techniques, be transformed into one with fixed boundary. If we take the x-axis to be the original surface of the water and the y-axis the axis of symmetry of the wedge, and assume (in the simplest form of the problem) that the wedge is infinite and also the water in both width and depth, then there is no natural length scale in the problem and so we can look for similarity solutions of the form

$$z = z(\sigma, t) = VtZ(\sigma/Vt).$$

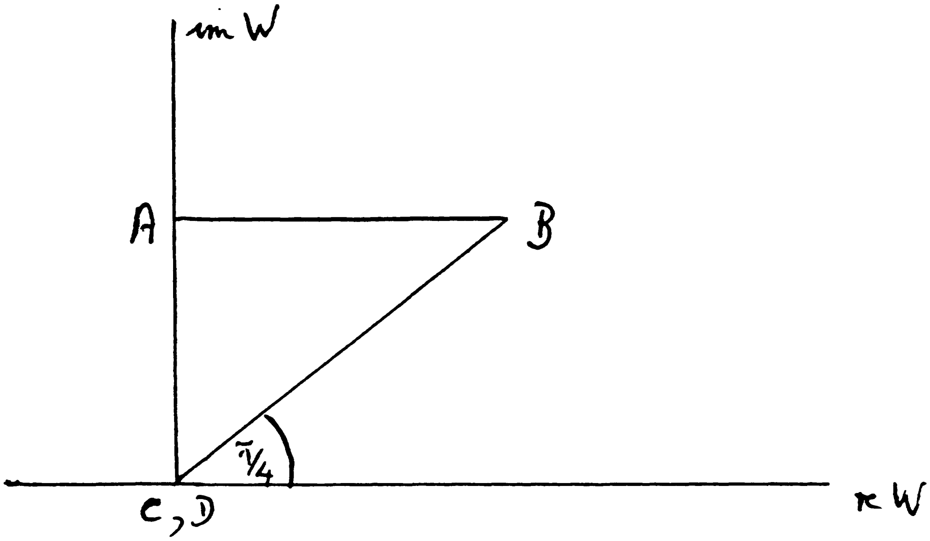
where $z = x + iy$ gives the position at time t of the particle on the free surface distant σ from the wedge at $t = 0$. Since the governing equation in the water is Laplace's equation, we can introduce the potential function ϕ and its harmonic conjugate the stream function ψ, and write

$$\chi = \phi + i\psi$$

as an analytic function of z. Clever involvement of the hydrodynamical boundary conditions led Wagner to observe that the function $W(z)$ defined by

$$W = \int_{\infty}^{z} (\chi''(\zeta))^{1/2}\, d\zeta$$

maps the region $\mathcal{D}$ in the water to the right of the y-axis and the wedge and below the free surface on to an isosceles right-angled triangle, with the points A,B, C, D corresponding as shown.



Coupling this with a standard mapping that maps the triangle on to the upper half-plane, Dobrovol'skaya [2] was able to map the original region $\mathcal{D}$ on to the upper half-plane and consequently write down an integral equation for the free surface. If πf denotes the angle of the free surface to the horizontal, then with a suitable choice of independent variable the integral equation takes the form

$$f'(t) = A(f')\, \frac{(1-t)^{-1/2-\beta}\, exp[-\int_0^1 f'(\tau)\, log|\tau - t| d\tau]}{\int_t^1 \sigma^{-3/2}(1-\sigma)^{\beta-1}\, exp[\int_0^1 f'(\tau)\, log|\tau-\sigma| d\tau] d\sigma}. \tag{1}$$

The variable t runs from 0 (at C) to 1 (at B). The equation is an equation for f', but f can then be obtained by integration and using the fact that $f(0) = 0$. The quantity $\pi\beta$ is the angle between the free surface and the wedge at the point of contact. A piece of elementary geometry shows that

$$\beta + f(1) = \alpha + \frac{1}{2},$$

where $2\pi\alpha$ is the wedge angle. The quantity $A(f')$ is a constant which depends in an explicit way on f'. The fact that it does depend on f' adds difficulty to the problem.

This problem has been studied by many authors, some references being [3] - [8], but two questions remain unanswered. The first is the purely analytical one of whether there exists a solution to the above problem, hopefully including some qualitative information on the nature of the solution; and the second is a rigorous discussion of the behaviour of the solution as $\alpha \longrightarrow \frac{1}{2}$. If one regards the wedge entry problem as a model for the situation that arises as the bow of a ship plunges up and down in the ocean, then since the bow of a ship is not a sharp V, the case of practical interest is $\alpha \longrightarrow \frac{1}{2}$, and there has been recent interest in the problem for that reason.

We can make two important observations with regard to the problem wich enable us to answer the questions above. The first is that if we consider the set of functions S, where

$$S = \{f : f'(t)(1-t)^{2\beta+1/2} \text{ is non-decreasing}\},$$

and if we insert such a function into the right-hand side of (1), denoted by $T(f')$, then $T(f') \in S$. Since (1) states that $f' = T(f')$, we can set the problem up so that we work always in the set S, and this has two consequences. The first is that the final solution will have this property, giving useful qualitative information. The second is that, if we organize an existence proof by a degree or continuation method, continuing the solution in the parameter β, then one property we shall need is an *a priori* bound on possible solutions, and this is much easier in the space S than in some more general space.

The second requirement for a continuation method is that there should be some value of the parameter β for which there is a unique solution (in S). (This requirement can be relaxed, but something of this sort is required.) The second observation is that, as $\beta \longrightarrow 0$, which one can associate with $\alpha \longrightarrow \frac{1}{2}$, the solution must tend with a suitable scaling to a solution of

$$X' = \frac{2}{\pi} X\hat{X}, \tag{2}$$

where $\hat{X}$ denotes the Hilbert transform of X. (It is not surprising that the Hilbert transform makes its appearance since the logarithmic potential is very evident in (1).) The equation (2) has the unique solution

$$X(s) = \frac{\pi^2}{\pi^2 + s^2},$$

and this gives both the uniqueness that is required for the continuation argument, and also the asymptotic behaviour of the solution as $\alpha \longrightarrow \frac{1}{2}$.

This work is being carried on in collaboration with Professor L.E. Fraenkel of the University of Bath, England.

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