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RELAXATION OF MULTIPLE INTEGRALS IN THE SPACE $BV(\Omega;\mathbb{R}^p)$

by

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Abstract. A characterization of the surface energy density for the relaxation in $BV(\Omega;\mathbf{R}^p)$ of the functional

$$u \to \int_\Omega f(x, u(x), \nabla u(x))dx$$

is obtained. A "slicing" technique is used allowing to patch together sequences without increasing their total energy.

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1. Introduction

Recently much attention has been devoted to the study of phase transitions problems involving singular perturbations. A typical example is the case where equilibria correspond to minima of a certain energy functional

$$E(u) = \int_\Omega W(u(x))dx$$

and the bulk energy density W supports two or more wells. Depending on the boundary conditions or constrains, in general there are more than one solution and to resolve this non-uniqueness one may consider a family of singular perturbations

$$E_\epsilon(u) = \int_\Omega W(u(x))dx + \epsilon^2 \int_\Omega g^2(\nabla u(x))dx,$$

expecting that, when $\epsilon \to 0$, minimizers of $E_\epsilon(\cdot)$ will select the physically reasonable solution for $E(\cdot)$. The isotropic version of this model, where $g = |\cdot|$, was first introduced by Cahn [C] and Carr, Gurtin & Slemrod [CGS] analyzed it in the case where $u: \Omega \subset \mathbf{R}^N \to \mathbf{R}^p$, $N = p = 1$. For $p = 1$ and $N \geq 1$ Modica [M] identified the limiting energy for the rescaled functionals $J_\epsilon := \frac{1}{\epsilon}E_\epsilon$, Kohn & Sternberg [KS], Owen [O1], [O2], Sternberg [S] studied the problem in the scalar-valued case and Baldo [Ba], Bouchitté [Bo] and Fonseca & Tartar [FT1] solved the isotropic vector-valued case.

In 1988 Fonseca and Tartar [FT2] initiated the analysis of the interesting case corresponding to a change of phase in three-dimensional nonlinear elasticity. Here u is a 3×3 matrix representing the deformation gradient of a body with reference configuration Ω, W has two potential wells of equal depth at a and b which in order to meet kinematic compatibility conditions, differ by a rank-one matrix, i.e.

$$a - b = c \otimes \nu$$

for some $c \in \mathbf{R}^3$, $\nu \in S^2$. Considering the isotropic penalization

$$E_\epsilon(u) = \int_\Omega W(u(x))dx + \epsilon^2 \int_\Omega \|\nabla u(x)\|^2 dx$$

together with the constraint

$$\text{curl } u = 0,$$

Fonseca and Tartar [FT2] conjectured that the Γ-limit of the rescaled functional

$$J_\epsilon(u) = \frac{1}{\epsilon} E_\epsilon(u)$$

is given by

$$J_0(u) = \begin{cases} C \ \mathrm{Per}_\Omega\{x \in \Omega : u(x) = a\} & \text{if curl} u = 0,\ u \in \{a, b\} \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

The constant C is defined as follows. Let $\{\nu_1, \nu_2, \nu_3 = \nu\}$ be an orthonormal basis of $\mathbf{R}^3$ and let

$$Q_\nu := \{x \in \mathbf{R}^3 : |x \cdot \nu_i| < \frac{1}{2},\ i = 1, 2, 3\}.$$

Then

$$C = \inf\{\int_{Q_\nu} W(\nabla\xi(y)) + \|D^2\xi(y)\|^2 : \xi \in \mathcal{A}\} \tag{1.1}$$

where $\mathcal{A}$ is the subset of $W^{2,2}(\Omega; \mathbf{R}^p)$ consisting of functions ξ such that

(a) the trace of $\nabla\xi$ on $y_3 = \frac{1}{2}$ is equal to b;
(b) the trace of $\nabla\xi$ on $y_3 = -\frac{1}{2}$ is equal to a;
(c) $\nabla\xi$ is periodic in the directions of e_1, e_2 with period 1.

We recall that if curl $u = 0$ and if $u \in \{a, b\}$ a.e. then the interface $\partial\{x \in \Omega : u(x) = a\} \cap \Omega$ must be planar with normal ν (see Ball and James [BJ]). Some results on the later constrained problem have been already obtained by Fonseca and Tartar [FT2]. However, for solids that may undergo a change of phase we expect the surface energy density to be anisotropic (see Fonseca [Fo], Taylor [T1], [T2], Wulff [W]) and so the ultimate goal is to understand the asymptotic behavior of $E_\epsilon(\cdot)$ when g is any convex function.

Owen and Sternberg [OS] showed that the Γ-limit in the anisotropic, unconstrained, scalar-valued case reduces to the Wulff shape (see Fonseca [Fo], Fonseca and Müller [FM], Taylor [T1], [T2]). In the vector-valued, unconstrained case the problem is still unsolved. In order to find a lower bound for the Γ-limit $J_0(\cdot)$ we apply the Cauchy-Schwarz inequality and obtain

$$J_\epsilon(u) \geq \int_\Omega f(x, u(x), \nabla u(x))dx$$

where

$$f(x, u, \nabla u) := 2\sqrt{W(u)}g(\nabla u).$$

Assuming, in a first instance, that f is nondegenerate, and that $f(x, u, \cdot)$ has linear growth at infinity we want to identify

$$\mathcal{F}[u] := \inf\{\liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx : u_n \in W^{1,1}(\Omega; \mathbf{R}^p),\quad u_n \to u\ L^1(\Omega; \mathbf{R}^p)\}$$

where

$$u \in \{a, b\} \quad \text{a.e. in}\Omega$$

and the interface, or jump set Σ, is planar. Aviles and Giga [AG] obtained a lower bound for $\mathcal{F}[u]$, but it is not clear that the resulting functional is indeed the greatest lower bound. Under the *isotropy condition* (see Definition 2.5) Ambrosio, Mortola and Tortorelli [AMT] and Aviles and Giga [AG] identified the integral representation for $\mathcal{F}[\cdot]$ which turns out to be a generalization of a result of Dal Maso [DM] for the scalar-valued case. Unfortunately, the isotropy condition is so strong that it implies, essentially, the isotropy of the surface energy density (see Proposition 5.1).

Motivated by (1.1) we identify $\mathcal{F}[u]$ for $u: \Omega \to \{a, b\} \subset \mathbf{R}^p$ such that

$$u(x) = \begin{cases} b & \text{if } x \cdot \nu > 0 \\ a & \text{if } x \cdot \nu > 0. \end{cases} \tag{1.2}$$

We show that

$$\mathcal{F}[u] = \int_\Omega f(x, u(x), 0)dx + \int_\Sigma K(x, a, b, \nu)dH_{N-1}(x)$$

where $\Sigma := \{x \in \Omega: x \cdot \nu = 0\}$,

$$K(x, a, b, \nu) := \inf\{\int_{Q_\nu} f^\infty(x, \xi(y), \nabla\xi(y))dy: \xi \in \mathcal{A}(a, b, \nu)\}$$

where

$$\begin{aligned} \mathcal{A}(a, b, \nu) :=& \{\xi \in W^{1,1}(Q_\nu; \mathbf{R}^p) : \xi(y) = a \text{ if } y \cdot \nu = -1/2, \quad \xi(y) = b \text{ if } y \cdot \nu = 1/2 \\ & \text{and } \xi \text{ is periodic, width period 1, in the } \nu_1, \ldots, \nu_{N-1} \text{ directions}\}, \end{aligned}$$

and $f^\infty(x, u, \cdot)$ is the recession function of $f(x, u, \cdot)$.

Recently, we became aware of the work by Ambrosio and Pallara [AP] in which they obtain an abstract integral representation for $\mathcal{F}[u]$ for every $u \in BV(\Omega; \mathbf{R}^p)$ (see Theorem 5.2). It turns out that our Propositions 3.1 and 4.1 together with Theorem 5.2 provide a full characterization of the integral representation of $\mathcal{F}[\cdot]$ in $BV(\Omega; \mathbf{R}^p)$ (see Theorem 5.3).

In Section 2 we state some results on functions of bounded variation. A general discussion on this subject can be found in Evans and Gariepy [EG], Giusti [G], Federer [F], Ziemer [Z]. Also, we analyze the implications of our hypotheses on f with respect to f^∞ and to K (see Lemmas 2.3 and 2.4). In Proposition 2.6 and 2.7 we study the isotropy condition in some detail and we conclude that it forces the resulting relaxed surface energy density to be isotropic.

Setting

$$I(u) := \int_\Omega f(x, u(x), 0)dx + \int_\Sigma K(x, a, b, \nu)dH_{N-1}(x),$$

where u is a function as in (1.2), in Section 3 we show that

$$\mathcal{F}[u] \geq I(u).$$

To this end, in Lemma 3.2 we introduce a "slicing" technique which allows us to modify a sequence $u_n \to u$ in $L^1(Q_\nu; \mathbf{R}^p)$ in such a way that the new sequence w_n agrees with u_n in most of the cube Q_ν and there are slices near the top and bottom of Q_ν where w_n is a convex combination of u_n and, respectively b and a,

$$w_n(x) = \begin{cases} b & \text{if } x \cdot \nu = 1/2 \\ a & \text{if } x \cdot \nu = -1/2 \end{cases}$$

and concentrations are avoided so that

$$\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx \geq \liminf_{n\to+\infty}\int_\Omega f(x,w_n(x),\nabla w_n(x))dx.$$

A similar slicing procedure is possible in order to render u_n periodic with respect to the remaining $N-1$ directions. We used the same idea again in Section 4, where we show that

$$\mathcal{F}[u] = I(u)$$

by constructing a sequence $u_n \to u$ in L^1 such that

$$\int_{Q_\nu} f(x,u_n(x),\nabla u_n(x))dx \to I(u).$$

In Section 5 we conclude that our results, namely our characterization of the surface density K, together with Ambrosio and Pallara's theorem yield the integral representation for $\mathcal{F}[u]$, for all $u \in BV(\Omega;\mathbf{R}^p)$.

2. Preliminaries. The isotropy condition.

In what follows $\Omega \subset \mathbf{R}^N$ is an open, bounded, strongly Lipschitz domain, $p, N \geq 1$, and $\{e_1,\ldots,e_N\}$ is the standard orthonormal basis of $\mathbf{R}^N$.

Definition 2.1 A function $u \in L^1(\Omega;\mathbf{R}^p)$ is said to be of *bounded variation* ($u \in BV(\Omega;\mathbf{R}^p)$) if for all $i \in \{1,\ldots,p\}$, $j \in \{1,\ldots,N\}$ there exists a Radon measure $\mu_{i,j}$ such that

$$\int_\Omega u_i(x)\frac{\partial\phi}{\partial x_j}(x)dx = -\int_\Omega \phi(x)d\mu_{i,j}$$

for all $\phi \in C_0^1(\Omega)$.

The distributional derivative Du is the matrix-valued measure with components $\mu_{i,j}$. If $u \in BV(\Omega;\mathbf{R}^p)$ then Du can be represented as

$$Du = \nabla u dx + (u^+ - u^-)\otimes \nu dH_{N-1}\lfloor\Sigma + C(u) \tag{2.1}$$

where ∇u is the density of the absolutely continuous part of Du and H_{N-1} is the $N-1$ dimensional Hausdorff measure. Here u^+ and u^- denote, respectively, the *approximate upper* and *lower limits* of u, i.e. for all $i \in \{1,\ldots,p\}$

$$u_i^+(x) := \inf\{t \in \mathbf{R} : \lim_{\epsilon\to 0+}\frac{1}{\epsilon^N}\mathcal{L}_N[\{u_i > t\}\cap B_\epsilon(x)] = 0\}$$

and

$$u_i^-(x) := \sup\{t \in \mathbf{R} : \lim_{\epsilon\to 0+}\frac{1}{\epsilon^N}\mathcal{L}_N[\{u_i < t\}\cap B_\epsilon(x)] = 0\}$$

where $B_\epsilon(x)$ is the open ball centered at x and with radius ϵ. The three measures in (2.1) are mutually singular. In fact, if $H_{N-1}(B) < +\infty$ then $\|C(u)\|(B) = 0$ and there exists a Borel set E such that

$$\mathcal{L}_N(E) = 0 \quad \text{and} \quad \|C(u)\|(B) = \|C(u)\|(B \cap E)$$

for any Borel set B, where $\mathcal{L}_N$ is the $N-$dimensional Lebesgue's measure. $\Sigma(u)$ is called the *singular set of* u *or jump set* and is defined by

$$\Sigma(u) = \bigcup_{i=1}^{p} \{x \in \Omega : u_i^-(x) < u_i^+(x)\}.$$

$\Sigma(u)$ is the complement of the Lebesgue set, i.e.

$$\Sigma(u) = \{x \in \Omega : \forall z \in \mathbf{R} \lim_{\epsilon \to 0+} \frac{1}{\mathcal{L}_N(B_\epsilon(x))} \int_{B_\epsilon(x)} |u(y) - z| dy \neq 0\}.$$

It is well known that $\Sigma(u)$ is countably $N-1$ rectifiable, i.e.

$$\Sigma(u) = \bigcup_{n \in \mathbf{N}} K_n \cup N$$

where $H_{N-1}(N) = 0$ and K_n is a compact subset of a C^1 hypersurface. Also, for H_{N-1} a.e. $x \in \Sigma(u)$ there exists a unit vector $\nu(x) \in S^{N-1}$, *normal to* $\Sigma(u)$ *at* x, such that

$$u^+(x) = \lim_{\epsilon \to 0+} u(x + \epsilon\nu(x)),$$

$$u^-(x) = \lim_{\epsilon \to 0+} u(x - \epsilon\nu(x))$$

and

$$\begin{aligned} &\lim_{\epsilon \to 0+} \frac{1}{\epsilon^N} \int_{\{y \in B_\epsilon(x) : (y-x)\cdot\nu(x) > 0\}} |u(y) - u^+(x)|^{N/N-1} dy \\ &+ \lim_{\epsilon \to 0+} \frac{1}{\epsilon^N} \int_{\{y \in B_\epsilon(x) : (y-x)\cdot\nu(x) < 0\}} |u(y) - u^-(x)|^{N/N-1} dy = 0. \end{aligned} \tag{2.2}$$

For a detailed study of the spaces $BV(\Omega; \mathbf{R}^p)$ we refer the reader to Evans and Gariepy [EG], Federer [Fe], Giusti [G], Ziemer [Z].

Let $M^{p\times N}$ denote the space of $p \times N$ real matrices and if $A \in M^{p\times N}$ let

$$\|A\| := (tr(A^T A))^{1/2}.$$

Let $f: \Omega \times \mathbf{R}^p \times M^{p\times N} \to [0, +\infty)$ be a continuous function such that

(H1) $f(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbf{R}^p$;

(H2) There exist $c'_1 \in \mathbf{R}$ and there exist $c_1, c_2 > 0$ such that

$$c_1\|A\| - c'_1 \leq f(x, u, A) \leq c_2(1 + \|A\|)$$

for all $(x, u, A) \in \Omega \times \mathbf{R}^p \times M^{p\times N}$;

(H3) For all $x_0 \in \Omega$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x, u, A) - f(x_0, u, A)| \leq \epsilon C(1 + |u| + \|A\|)$ for every $(u, A) \in \mathbf{R}^p \times M^{p\times N}$ where $C > 0$ is a constant independent of x_0;

(H4) There exist $0 < m \leq 1$, $t_0 > 0$, $c_4 > 0$ such that if $(x, u, A) \in \Omega \times \mathbf{R}^p \times M^{p\times N}$, $\|A\| = 1$, $t > t_0$ then

$$\left|\frac{f(x, u, tA)}{t} - f^\infty(x, u, A)\right| \leq \frac{c_4}{t^m}.$$

Remark 2.2 If f satisfies (H1) and (H2) then f is globally Lipschitz, i.e.

$$|f(x, u, A) - f(x, u, B)| \leq c_3\|A - B\|.$$

Let $f^\infty : \Omega \times \mathbf{R}^p \times M^{p\times N} \to [0, +\infty)$ be the *recession function*, i.e.

$$f^\infty(x, u, A) := \lim_{t\to+\infty} \frac{f(x, u, tA)}{t}.$$

Note that, fixing $(x, u, A) \in \Omega \times \mathbf{R}^p \times M^{p\times N}$ and setting $g(t) := f(x, u, tA) - f(x, u, 0)$, then g is a convex function, $g(0) = 0$ and so

$$t \longrightarrow g(t)/t$$

is increasing. Thus

$$\begin{aligned} f^\infty &= \sup_{t\to+\infty} g(t)/t \\ &= \lim_{t\to+\infty} \frac{f(x, u, tA) - f(x, u, 0)}{t} \qquad (2.3). \\ &= \lim_{t\to+\infty} f(x, u, tA)/t \end{aligned}$$

It is well-known that $f^\infty(x, u, \cdot)$ is a convex function, homogeneous of degree one. In addition we have

Lemma 2.3 Under the hypotheses (H2) and (H3) the following hold:

(i) $c_1\|A\| \leq f^\infty(x,u,A) \leq c_2\|A\|$, for every $(x,u,A) \in \Omega \times \mathbf{R}^p \times M^{p\times N}$;

(ii) For all $x_0 \in \Omega$, $\epsilon > 0$ there exists $\delta > 0$ such that for every $(u,A) \in \mathbf{R}^p \times M^{p\times N}$, $|x - x_0| < \delta$ implies

$$|f^\infty(x,u,A) - f^\infty(x_0,u,A)| \leq \epsilon C\|A\|.$$

Proof.

(i) By (2.3) and (H2) we have

$$f^\infty(x,u,A) \geq f(x,u,A) - f(x,u,0) \geq c_1\|A\| - c'_1 - c_2$$

and so, for all $t > 0$

$$tf^\infty(x,u,A) = f^\infty(x,u,tA) \geq c_1\|tA\| - c'_1 - c_2.$$

Dividing by t and letting t go to infinity, we conclude that

$$f^\infty(x,u,A) \geq c_1\|A\|.$$

Also,

$$f^\infty(x,u,A) = \lim_{t\to+\infty} \frac{f(x,u,tA)}{t} \leq \liminf_{t\to+\infty} \frac{c_2(1+t\|A\|)}{t} = c_2\|A\|.$$

(ii) Fix $x_0 \in \Omega$, $\epsilon > 0$ and let $\delta > 0$ be such that

$$|x - x_0| < \delta \Rightarrow |f(x,u,A) - f(x_0,u,A)| \leq \epsilon C(1 + |u| + \|A\|).$$

If $0 < t < t'$, then by (H3)

$$\begin{aligned}
&\frac{f(x,u,tA) - f(x,u,0)}{t} - \frac{f(x_0,u,t'A) - f(x_0,u,0)}{t'} \\
\leq &\frac{f(x,u,tA) - f(x,u,0)}{t} - \frac{f(x_0,u,tA) - f(x_0,u,0)}{t} \\
= &\frac{f(x,u,tA) - f(x_0,u,tA)}{t} + \frac{f(x_0,u,0) - f(x,u,0)}{t} \\
\leq &\epsilon C\frac{1 + |u| + \|tA\|}{t} + \frac{f(x_0,u,0) - f(x,u,0)}{t}
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{f(x,u,tA) - f(x,u,0)}{t} \\
\leq &\frac{f(x_0,u,t'A) - f(x_0,u,0)}{t'} + \epsilon C\frac{1 + |u| + \|tA\|}{t} + \frac{f(x_0,u,0) - f(x,u,0)}{t}.
\end{aligned}$$

Letting $t' \to +\infty$ we obtain

$$\begin{aligned}&\frac{f(x,u,tA)-f(x,u,0)}{t}\\ \le& f^\infty(x_0,u,A)+\epsilon C\frac{1+|u|+\|tA\|}{t}+\frac{f(x_0,u,0)-f(x,u,0)}{t}\end{aligned}$$

and if $t \to +\infty$ then

$$f^\infty(x,u,A) \le f^\infty(x_0,u,A)+\epsilon C\|A\|.$$

In a similar way we obtain

$$f^\infty(x_0,u,A) \le f^\infty(x,u,A)+\epsilon C\|A\|$$

and we conclude that

$$|f^\infty(x,u,A) - f^\infty(x_0,u,A)| \le \epsilon C\|A\|.$$

■

We want to find an integral representation for

$$\mathcal{F}[u] := \inf\left\{\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx : u_n \in W^{1,1},\ u \in BV,\ u_n \to u \text{ in } L^1\right\},$$

when u takes only two values across a planar interface.

Given $\nu \in S^{N-1}$, Q_ν is the open unit cube centered at the origin with respect to an orthonormal basis $\{\nu_1,\dots,\nu_{N-1},\nu\}$ of $\mathbf{R}^N$, i.e.

$$Q_\nu := \{x \in \mathbf{R}^N : |x\cdot\nu_i| < 1/2,\ |x\cdot\nu| < 1/2,\ i=1,\dots,N-1\}.$$

For $(a,b,\nu) \in \mathbf{R}^p \times \mathbf{R}^p \times S^{N-1}$ we define the class of admissible functions

$$\begin{aligned}\mathcal{A}(a,b,\nu) :=&\{\xi \in W^{1,1}(Q_\nu;\mathbf{R}^p) : \xi(y)=a \text{ if } y\cdot\nu=-1/2, \quad \xi(y)=b \text{ if } y\cdot\nu=1/2\\ &\text{and } \xi \text{ is periodic, width period 1, in the } \nu_1,\dots,\nu_{N-1} \text{ directions}\},\end{aligned}$$

where the boundary value of ξ is understood in the sense of trace. A function ξ is said to be *periodic with period 1 in the direction of* ν_i if

$$\xi(y) = \xi(y+k\nu_i)$$

for all $k \in \mathbf{Z}$, $y \in Q_\nu$. Our surface energy density $K : \Omega \times \mathbf{R}^p \times \mathbf{R}^p \times S^{N-1} \longrightarrow [0,+\infty)$, is defined by

$$K(x,a,b,\nu) := \inf_{\xi\in\mathcal{A}(a,b,\nu)}\int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))dy \tag{2.4}$$

and our candidate for the relaxation $\mathcal{F}[\cdot]$ is given by

$$I(u) := \int_\Omega f(x, u(x), 0)dx + \int_{\Sigma(u)} K(x, a, b, \nu(x))dH_{N-1}(x). \tag{2.5}$$

We examine some continuity properties of the surface energy density K.

Proposition 2.4 Under the hypotheses (H1) and (H2) the following hold:

(i) $0 \leq K(x, a, b, \nu) \leq C|b-a|$ for all $(x, a, b, \nu) \in \Omega \times \mathbf{R}^p \times \mathbf{R}^p \times S^{N-1}$;

(ii) For all $x_0 \in \Omega$, $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |K(x, a, b, \nu) - K(x_0, a, b, \nu)| \leq \epsilon C(1 + |b-a|);$$

(iii) For some constant $c > 0$ and for all (x, a, b, ν), $(x, a, b, \nu') \in \Omega \times \mathbf{R}^p \times \mathbf{R}^p \times M^{p\times N}$

$$|K(x, a, b, \nu) - K(x, a, b, \nu')| \leq c\,|\nu - \nu'|.$$

Proof.
(i) Fix $(a, b, \nu) \in \mathbf{R}^p \times \mathbf{R}^p \times S^{N-1}$ and let

$$\xi(y) := (b-a)(y\cdot \nu) + \frac{a+b}{2}.$$

Clearly $\xi \in \mathcal{A}(a, b, \nu)$ and so, by Lemma 2.3 (i)

$$0 \leq K(x, a, b, \nu) \leq \int_{-1/2}^{1/2} f^\infty(x, (b-a)t + (a+b)/2, (b-a)\otimes\nu)dt \leq c_2|b-a|.$$

(ii) Fix $x_0 \in \Omega$, $\epsilon > 0$ and by Lemma 2.3 (ii) choose $\delta > 0$ such that

$$|x - x_0| \Rightarrow |f^\infty(x, u, A) - f^\infty(x_0, u, A)| \leq \epsilon C\|A\|.$$

For all $n \in \mathbf{N}$ we choose $\xi_n \in \mathcal{A}(a, b, \nu)$ such that

$$\int_{Q_\nu} f^\infty(x_0, \xi_n(y), \nabla\xi_n(y))dy \leq K(x_0, a, b, \nu) + \frac{1}{n}.$$

By Lemma 2.3 (i) and by part (i) above we have

$$\int_{Q_\nu} \|\nabla\xi_n(y)\|dy \leq \frac{K(x_0, a, b, \nu) + 1}{c_1} \leq C(|b-a| + 1)$$

hence, if $|x - x_0| < \delta$ by Lemma 2.3 (ii)

$$\begin{aligned}
&K(x,a,b,\nu) - K(x_0,a,b,\nu) \leq \\
&\leq \int_{Q_\nu} [f^\infty(x,\xi_n(y),\nabla\xi_n(y)) - f^\infty(x_0,\xi_n(y),\nabla\xi_n(y))]dy + \frac{1}{n} \\
&\leq \int_{Q_\nu} \epsilon C \|\nabla\xi_n(y)\| dy + \frac{1}{n} \\
&\leq \epsilon C(|b-a|+1) + \frac{1}{n}.
\end{aligned}$$

Letting $n \to +\infty$, we obtain

$$K(x,a,b,\nu) - K(x_0,a,b,\nu) \leq \epsilon C(|b-a|+1).$$

On the other hand, if $|x - x_0| < \delta$ and if $g_n \in \mathcal{A}(a,b,\nu)$ is such that

$$\int_{Q_\nu} f^\infty(x,g_n(y),\nabla g_n(y))dy \leq K(x,a,b,\nu) + \frac{1}{n},$$

then due to part (i) and Lemma 2.3 (i) we have

$$\int_{Q_\nu} \|\nabla\xi_n(y)\| dy \leq C(|b-a|+1)$$

and similarly

$$\begin{aligned}
&K(x_0,a,b,\nu) - K(x,a,b,\nu) \leq \\
&\leq \int_{Q_\nu} [f^\infty(x_0,g_n(y),\nabla g_n(y)) - f^\infty(x,g_n(y),\nabla g_n(y))]dy + \frac{1}{n} \\
&\leq \epsilon C(|b-a|+1) + \frac{1}{n}
\end{aligned}$$

which implies that

$$K(x_0,a,b,\nu) - K(x,a,b,\nu) \leq \epsilon C(|b-a|+1).$$

(iii) We take $\xi_n \in \mathcal{A}(a,b,\nu)$ such that

$$\int_{Q_\nu} f^\infty(x,\xi_n(y),\nabla\xi_n(y))dy \leq K(x,a,b,\nu) + \frac{1}{n}.$$

By part (i) and by Lemma 2.3 (i) we have

$$\int_{Q_\nu} \|\nabla\xi_n(y)\| dy \leq C(1+|b-a|). \tag{2.6}$$

Let $\nu' \in S^{N-1}$ and choose R a rotation of $\mathbf{R}^N$ such that $R\nu = \nu'$ and $Q_{\nu'} = RQ_\nu$. Setting $\xi'_n(y) := \xi_n(R^T y)$, it is clear that

$$\xi' \in \mathcal{A}(a, b, \nu')$$

and so, by Remark 2.2 and by (2.6)

$$\begin{aligned}
&K(x,a,b,\nu') - K(x,a,b,\nu) \\
\leq &\int_{Q_{\nu'}} f^\infty(x, \xi_n(R^T y), \nabla \xi_n(R^T y) R^T) dy - \int_{Q_\nu} f^\infty(x, \xi_n(y), \nabla \xi_n(y)) dy + \frac{1}{n} \\
= &\int_{Q_\nu} f^\infty(x, \xi_n(y), \nabla \xi_n(y) R^T) dy - \int_{Q_\nu} f^\infty(x, \xi_n(y), \nabla \xi_n(y)) dy + \frac{1}{n} \\
\leq &C(|b-a|+1)|\nu - \nu'| + \frac{1}{n}.
\end{aligned}$$

If $n \to +\infty$ then the previous inequality yields (iii). ∎

Next, we study some implications of the isotropy condition.

Definition 2.5 A function $f : \Omega \times \mathbf{R}^p \times M^{p\times N} \to [0, +\infty)$ is said to satisfy the *isotropy condition* if

$$f(x, u, A) \geq f(x, u, An \otimes n)$$

for all $A \in M^{p\times N}$, $n \in S^{N-1}$.

Ambrosio, Mortola and Tortorelli [AMT] and Aviles and Giga [AG] showed that under the isotropy condition

$$\begin{aligned}
\mathcal{F}[u] = &\int_\Omega f(x, u(x), \nabla u(x)) dx + \int_{\Sigma(u)} D(x, u^-(x), u^+(x), \nu(x)) dH_{N-1} \\
&+ \int_\Omega f^\infty(x, u(x), \frac{dC(u)}{d\|C(u)\|}(x)) d\|C(u)\|(x)
\end{aligned}$$

for all $u \in BV(\Omega; \mathbf{R}^p)$; here we used the decomposition (2.1) and (2.2), $\|C(u)\|$ denotes the total variation of $C(u)$, i.e.

$$\|C(u)\|(B) = \sup\{\int_\Omega \phi \; dC(u) : \; \phi \in C_0(B; M^{p\times N}), \; \|\phi\|_\infty \leq 1\}$$

for all Borel sets $B \subset \Omega$, and

$$D(x, a, b, \nu) = \inf\{\int_0^1 f^\infty(x, \gamma(t), \gamma'(t) \otimes \nu(x)) dt : \gamma \in W^{1,\infty}, \; \gamma(0) = a, \; \gamma(1) = b\}.$$

D is the obvious extension of the energy density obtained by Dal Maso [DM] to the vector-valued case. Note that (see Clarke and Vinter [CV])

$$D(x,a,b,\nu) = \inf\{\int_0^1 f^\infty(x,\gamma(t),\gamma'(t)\otimes\nu(x))dt : \gamma \in W^{1,1},\ \gamma(0)=a,\ \ \gamma(1)=b\}.$$

We extend $I(\cdot)$, introduced in (2.5), to $BV(\Omega;\mathbf{R}^p)$ as

$$I(u) = \int_\Omega f(x,u(x),\nabla u(x))dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))dH_{N-1}$$
$$+ \int_\Omega f^\infty(x,u(x),\frac{dC(u)}{d\|C(u)\|}(x))d\|C(u)\|(x).$$

In the next proposition we compare $D(x,a,b,\nu)$ and $K(x,a,b,\nu)$.

Proposition 2.6 If (H1) holds then
(i) $D(x,a,b,\nu) \geq K(x,a,b,\nu)$;
(ii) If $f = f(x,A)$ then $D(x,a,b,\nu) = K(x,a,b,\nu) = f^\infty(x,(a-b)\otimes\nu)$ and $\mathcal{F}[u] = I(u)$.

(iii) If f satisfies the isotropy condition then

$$D(x,a,b,\nu) = K(x,a,b,\nu)$$

and $\mathcal{F}[u] = I(u)$.

Proof.
(i) Choose $\gamma \in W^{1,1}([0,1];\mathbf{R}^p)$, $\gamma(0)=a$, $\gamma(1)=b$ and set $\xi : Q_\nu \to \mathbf{R}^N$,

$$\xi(y) := \gamma(y\cdot\nu + 1/2).$$

Then

$$\xi(y) = \begin{cases} \gamma(0) = a & \text{if } y\cdot\nu = -1/2 \\ \gamma(1) = b & \text{if } y\cdot\nu = 1/2 \end{cases}$$

and

$$\xi(y + k\nu_i) = \gamma(y\cdot\nu + k\nu_i\cdot\nu + 1/2) = \gamma(y\cdot\nu + 1/2) = \xi(y)$$

for $i = 1,\ldots,N-1$. Thus

$$\begin{aligned} K(x,a,b,\nu) &\leq \int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))dy \\ &= \int_{Q_\nu} f^\infty(x,\gamma(y\cdot\nu+1/2),\gamma'(y\cdot\nu+1/2)\otimes\nu)dy \\ &= \int_0^1 f^\infty(x,\gamma(t),\gamma'(t)\otimes\nu)dt. \end{aligned}$$

Taking the infimum in γ, we conclude that

$$K(x,a,b,\nu) \leq D(x,a,b,\nu).$$

(ii) Suppose that $f = f(x,A)$ and let $\xi \in \mathcal{A}(a,b,\nu)$. By Jensen's inequality

$$\int_{Q_\nu} f^\infty(x,\nabla\xi(y))dy \geq f^\infty(x,\int_{Q_\nu} \nabla\xi(y))dy = f^\infty(x,(b-a)\otimes\nu) \tag{2.7}$$

since ξ is periodic in the $\nu_1,\ldots,\nu_{N-1}$ directions. On the other hand, if

$$\gamma(t) := t(b-a)+a \qquad 0 \leq t \leq 1,$$

then

$$D(x,a,b,\nu) \leq \int_0^1 f^\infty(x,\gamma'(t)\otimes\nu)dt = f^\infty(x,(b-a)\otimes\nu)$$

which, together with (i) yields

$$D(x,a,b,\nu) = K(x,a,b,\nu) = f^\infty(x,(b-a)\otimes\nu).$$

In this case, it is well-known that (see Goffman & Serrin [GS], Giaquinta, Modica, Souček [GMS], Reshetnyak [R])

$$\begin{aligned}\mathcal{F}[u] &= \int_\Omega f(x,u(x),\nabla(x))dx + \int_{\Sigma(u)} f^\infty(x,(u^+(x)-u^-(x))\otimes\nu(x))dH_{N-1}\\ &+ \int_\Omega f^\infty(x,u(x),\frac{dC(u)}{d\|C(u)\|})\,d\|C(u)\|(x)\\ &= I(u).\end{aligned}$$

(iii) If f satisfies the isotropy condition and if $\xi \in W^{1,1}(Q_\nu;\mathbf{R}^p)$ is such that $\xi(y) = a$ if $y\cdot\nu = -1/2$, $\xi(y) = b$ if $y\cdot\nu = 1/2$ and $\xi(y+k\nu_i) = \xi(y)$, $k \in \mathbf{Z}$, $i = 1,\ldots,N-1$, then writing $y = (y',y_{N-1}) \in Q' \times (-1/2,1/2)$ with respect to the orthonormal basis $\{\nu_1,\ldots,\nu_{N-1},\nu\}$,

$$\begin{aligned}\int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))dy &= \int_Q \left\{\int_{-1/2}^{1/2} f^\infty(x,\xi(y),\nabla\xi(y))dy_N\right\}dy'\\ &\geq \int_Q \left\{\int_{-1/2}^{1/2} f^\infty(x,\xi(y),\nabla\xi(y)\nu\otimes\nu)dy_N\right\}dy'.\end{aligned}$$

If $y' \in Q'$ is fixed and if we set

$$\gamma(t) := \xi(y',t-1/2),$$

then $\gamma \in W^{1,1}([0,1];\mathbf{R}^p)$, $\gamma(0) = \xi(y',-1/2) = a$, $\gamma(1) = \xi(y',1/2) = b$ and $\gamma'(t) = \nabla\xi\nu$. Hence

$$\int_{-1/2}^{1/2} f^\infty(x,\xi(y),\nabla\xi(y)\nu\otimes\nu)dy_N = \int_0^1 f^\infty(x,\gamma(t),\gamma'(t)\otimes\nu)dt \geq D(x,a,b,\nu)$$

and we conclude that

$$K(x,a,b,\nu) = \inf_{\xi\in\mathcal{A}(a,b,\nu)} \int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y)\nu\otimes\nu)dy \geq$$
$$\geq \int_{Q'} D(x,a,b,\nu)dy' = D(x,a,b,\nu)$$

which, together with part (i), implies that

$$D(x,a,b,\nu) = K(x,a,b,\nu).$$

■

We are particularly interested in the characterization of the surface energy density $K(x,a,b,\nu)$ (or $D(x,a,b,\nu)$), since its anisotropic nature may give some insight in the geometrical structure of interfaces for phase transition variational problems. Indeed, considering a family of singular perturbations for a nonconvex bulk energy functional and using the Cauchy-Schwartz inequality, we obtain a lower bound for the perturbed energies exhibiting the same structure as $\mathcal{F}[\cdot]$ (see Modica [M], Owen [O1], [O2], Owen and Sternberg [OS], Sternberg [S], Baldo [Ba], Bouchitté [Bo], Fonseca and Tartar [FT1]). For solid materials, anisotropy plays an important role in the selection of equilibrium states and so it becomes crucial to analyze functionals $\mathcal{F}[\cdot]$ for which the surface energy density is genuinely anisotropic. As it turns out, for homogeneous materials the isotropy condition renders the surface tension isotropic, namely if

$$f = f(u,A)$$

then

$$\int_{\Sigma(u)} D(x,u^-(x),u^+(x),\nu(x))dH_{N-1}(x) = C\ \mathrm{Per}_\Omega(\Sigma(u)). \tag{2.8}$$

Indeed,

Proposition 2.7 Let $g : M^{p\times N} \to \mathbf{R}_+$ be differentiable in $M^{p\times N}\setminus\{0\}$, $g(A) = 0$ only if $A = 0$, g is convex and homogeneous of degree one and $g(A) \geq g(An\otimes n)$ for all $A \in M^{p\times N}$, $n \in S^{N-1}$. Then there exists $\phi : \mathbf{R}^p \to \mathbf{R}_+$ convex, homogeneous of degree one such that $g(a\otimes b) = \phi(a)|b|$, for all $a \in \mathbf{R}^p$, $b \in \mathbf{R}^N$.

Hence, if $f = f(u,A)$ satisfies the isotropy condition then by the previous proposition we have

$$f^\infty(u,a\otimes b) = \phi(u,a)|b|$$

and so

$$D(x,a,b,\nu)=\inf\{\int_0^1 \phi(\gamma(t),\gamma'(t))dt:\ \gamma\in W^{1,1},\ \gamma(0)=a,\ \gamma(1)=b\}=:C$$

asserting (2.8). The proof of Proposition 2.7 is based on the following result

Lemma 2.8 Let $\xi:\mathbf{R}^N\to\mathbf{R}_+$ be differentiable in $\mathbf{R}^N\setminus\{0\}$, ξ is convex and homogeneous of degree one, $\xi(u)=0$ only if $u=0$ and

$$\xi(u)\geq\xi((u\cdot n)n)\qquad\forall n\in S^{N-1},\ u\in\mathbf{R}^N.$$

Then there exists $\phi>0$ such that $\xi(u)=\phi|u|$, for all $u\in\mathbf{R}^N$.

Proof. Let $u\neq 0$. As ξ is convex, due to the isotropy hypothesis we have for all $n\in S^{N-1}$

$$\xi(u)\geq\xi((u\cdot n)n)\geq\xi(u)+\nabla\xi(u)[(u\cdot n)n-u],$$

hence

$$\nabla\xi(u)\cdot u\geq(\nabla\xi(u)\cdot n)(u\cdot n).\tag{2.9}$$

As ξ is homogeneous of degree one,

$$\xi(u)=\nabla\xi(u)\cdot u$$

which, together with (2.9), implies that

$$\xi(u)\geq(\nabla\xi(u)\cdot n)(u\cdot n).\tag{2.10}$$

Set $G(x):=\xi(u)-[\nabla\xi\cdot\frac{x}{|x|}](u\cdot\frac{x}{|x|})$ for $x\neq 0$. By (2.10) we have

$$G(x)\geq 0$$

and

$$G(u)=\xi(u)-\nabla\xi(u)\cdot u\frac{(u\cdot u)}{|u|^2}=0$$

hence

$$\nabla G(u)=0.$$

Since $\nabla G(u)=-\nabla\xi(u)+\frac{\xi(u)u}{|u|^2}$ we conclude that

$$\frac{\nabla\xi(u)}{\xi(u)}=\frac{u}{|u|^2}$$

that is

$$\nabla[\ln\xi(u)]=\nabla(\ln|u|)$$

and so

$$\xi(u) = \phi|u|.$$

■

Proof of Proposition 2.7 Fix $a \in \mathbf{R}^p \setminus \{0\}$ and set $\xi(u) := g(a \otimes u)$. Then $\xi(u) > 0$ if $u \neq 0$, ξ is convex, homogeneous of degree one and

$$\begin{aligned}\xi(u) = g(a \otimes u) &\geq g((a \otimes u)n \otimes n) \\ &= g(a(u \cdot n) \otimes n) \\ &= g(a \otimes (u \cdot n)n) \\ &= \xi((u \cdot n)n).\end{aligned}$$

By Lemma 2.8 there exists $\phi = \phi(a)$ such that

$$g(a \otimes u) = \xi(u) = \phi(a)|u|.$$

It is clear that ϕ is convex and homogeneous on degree one, and $\phi(0) = 0$ as $\phi(a) = g(a \otimes \frac{u}{|u|})$.

■

3. A lower semicontinuity result

In this section we prove lower semicontinuity for the functional $I(\cdot)$ when the limiting functions take only two values across a planar interface.

Proposition 3.1 If (H1)-(H4) hold and if $u_n \in W^{1,1}(\Omega; \mathbf{R}^p)$ converges in $L^1(\Omega; \mathbf{R}^p)$ to a function u such that

$$u(x) = \begin{cases} b & \text{if } x \cdot \nu > \alpha \\ a & \text{if } x \cdot \nu < \alpha \end{cases}$$

for some $\alpha \in \mathbf{R}$, a, $b \in \mathbf{R}^p$, $\nu \in S^{N-1}$, then

$$\int_{\{x \in \Omega : x \cdot \nu = \alpha\}} K(x, a, b, \nu) dH_{N-1}(x) \leq \liminf_{n \to +\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x)) dx.$$

If in addition $f(x, u, A) \geq f(x, u, 0)$ for all $(x, u, A) \in \Omega \times \mathbf{R}^p \times M^{p \times N}$ then

$$I(u) \leq \liminf_{n \to +\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x)) dx.$$

To prove this result we start by showing that we can modify slightly a sequence u_n converging to u strongly in L^1 on a cube so that $u_n = b$ on the top, $u_n = a$ on the bottom and u_n becomes periodic with respect to the remaining directions. This is achieved by selecting thin slices were concentrations of $\|\nabla u_n\|$ and of the average of u_n are avoided.

Lemma 3.2 Assume that f satisfies (H2). Let $Q = \{x \in \mathbf{R}^N: |x \cdot e_i| \leq 1/2, \quad i = 1, \ldots, N\}$ and let

$$u(x) = \begin{cases} b & \text{if } x_N > 0 \\ a & \text{if } x_N < 0 \end{cases}.$$

If $u_n \in W^{1,1}(Q, \mathbf{R}^p)$ converges to u in $L^1(Q; \mathbf{R}^p)$ then there exists a sequence $w_n \in \mathcal{A}(a, b, e_N)$ such that w_n tends to u in $L^1(Q; \mathbf{R}^p)$ and

$$\liminf_{n \to +\infty} \int_Q f(x, u_n(x), \nabla u_n(x))dx \geq \liminf_{n \to +\infty} \int_Q f(x, w_n(x), \nabla w_n(x)))dx.$$

In particular, if f does not depend on x and if $f(u, \cdot)$ is homogeneous of degree one then

$$\liminf_{n \to +\infty} \int_Q f(u_n(x), \nabla u_n(x))dx \geq K(a, b, e_N).$$

Proof. First step. We modify the sequence u_n in order to meet the boundary conditions on $\{x \in Q: x_N = \pm 1/2\}$. We can assume, without loss of generality, that

$$\liminf_{n \to +\infty} \int_Q f(x, u_n, \nabla u_n(x))dx = \lim_{n \to +\infty} \int_Q f(x, u_n, \nabla u_n(x))dx < +\infty$$

and so by (H2) there exists $0 < C < +\infty$ such that

$$\int_Q \|\nabla u_n(x)\| dx \leq C$$

for all n. Choose an integer $m > 2C$ and a partition

$$\{x \in Q: \frac{1}{4} \leq x_N < \frac{1}{2}\} = \bigcup_{i=1}^{m} S_i$$

where $S_i = \{x \in Q: \alpha_i \leq x_N < \alpha_{i+1}\}$, $i = 1, \ldots, m$

$$\frac{1}{4} = \alpha_1 < \alpha_2 < \ldots < \alpha_{m+1} = \frac{1}{2}$$

and $\mathcal{L}_N(S_i) = \alpha_{i+1} - \alpha_i = \frac{1}{4m}$ (see Figure 1). Then

$$\sum_{i=1}^{m} \int_{S_i} \|\nabla u_n(x)\| dx \leq C$$

and so, for all $n \in \mathbf{N}$ there exists a slice S_i such that

$$\int_{S_i} \|\nabla u_n\| \leq \frac{1}{2}.$$

Since there are only m such slices, there must be a slice $S(2)$ such that

$$\int_{S(2)} \|\nabla u_n(x)\| dx \leq \frac{1}{2}$$

for infinitely many indices n. On the other hand

$$\lim_{n \to +\infty} \frac{1}{\mathcal{L}_N(S(2))} \int_{S(2)} |u_n(x) - u(x)| dx = 0$$

thus, there exists $n(2)$ such that if $n \geq n(2)$ then

$$\frac{1}{\mathcal{L}_N(S(2))} \int_{S(2)} |u_n(x) - u(x)| dx \leq 1/2.$$

Let n_2 be the smallest integer such that $n_2 \geq n(2)$ and

$$\int_{S(2)} \|\nabla u_{n_2}(x)\| dx \leq \frac{1}{2}, \qquad \frac{1}{\mathcal{L}_N(S(2))} \int_{S(2)} |u_n(x) - u(x)| dx \leq \frac{1}{2}.$$

By induction, if $k \geq 5$ let

$$\{x \in Q \colon \frac{1}{2} - \frac{1}{k} < x_N < \frac{1}{2}\} = \bigcup_{i=1}^{P_k} S_i$$

where P_k is an integer, $P_k > kC$ and S_i are mutually disjoint slices of the type

$$\{x \in Q \colon \alpha_i \leq x_N < \alpha_{i+1}\}$$

with measure

$$\mathcal{L}_N(S_i) = \frac{1}{kP_k} < \frac{1}{k^2 C}.$$

There exists a slice $S(k)$ such that for a subsequence

$$\int_{S(k)} \|\nabla u_n(x)\| dx \leq \frac{1}{k}, \quad \frac{1}{\mathcal{L}_N(S(k))} \int_{S(k)} |u_n(x) - u(x)| dx \leq \frac{1}{k}.$$

Choose $n_k > n_{k-1}$ such that

$$\int_{S(k)} \|\nabla u_{n_k}(x)\| dx \leq \frac{1}{k}, \qquad \frac{1}{\mathcal{L}_N(S(k))} \int_{S(k)} |u_{n_k}(x) - u(x)| dx \leq \frac{1}{k}. \tag{3.1}$$

Suppose that

$$S(k) = \{x \in Q \colon \gamma_k < x_N < \beta_k\},$$

where

$$\frac{1}{2}-\frac{1}{k}\leq\gamma_k<\beta_k\leq\frac{1}{2}.$$

Let $\theta_k\in C^\infty(\mathbf{R};[0;1])$ be a smooth cut-off function such that

$$\theta_k(t)=\begin{cases}1, & \text{if } t<\gamma_k\\ 0, & \text{if } t\geq\beta_k\end{cases}$$

and

$$\|\theta_n'\|_{L^\infty}\leq\frac{C}{\mathcal{L}_N(S(k))}\leq Ck^2.$$

Define

$$v_k(x):=\begin{cases}b & \text{if } x_N\geq\beta_k\\ \theta_k(x_N)u_{n_k}(x)+(1-\theta_k(x_N))b & \text{if } \gamma_k<x_N<\beta_k\\ u_{n_k} & \text{if } x_N\leq\gamma_k.\end{cases}$$

Clearly $v\in W^{1,1}$, $v_k(x)=b$ if $x_N=\frac{1}{2}$ and

$$\begin{aligned}&\|v_k-u\|_{L^1(Q)}=\\ =&\int_{Q\cap\{x_N\leq\gamma_k\}}|u_{n_k}(x)-u(x)|dx+\int_{S(k)}|\theta_k(x_N)(u_{n_k}(x)-u(x))|dx\\ \leq&2\|u_{n_k}-u\|_{L^1(Q)}\to 0\quad\text{as } n\to+\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\lim_{n\to+\infty}\int_Q f(x,u_n(x),\nabla u_n(x))dx\geq\\ &\liminf_{k\to+\infty}\int_{Q\cap\{x_N\leq\gamma_k\}}f(x,u_{n_k}(x),\nabla u_{n_k}(x))dx\geq\\ &\liminf_{k\to+\infty}[\int_Q f(x,v_k(x),\nabla v_k(x))dx-\int_{S(k)}f(x,v_k(x),\nabla v_k(x))dx+\\ -&\int_{Q\cap\{x_N\geq\beta_k\}}f(x,b,0)dx]\\ =&\liminf_{k\to+\infty}\int_Q f(x,v_k(x),\nabla v_k(x))dx\end{aligned}$$

because by (H2)

$$\int_{Q\cap\{x_N\geq\beta_k\}}f(x,b,0)dx\leq const.(\frac{1}{2}-\beta_k)\to 0\quad\text{as } k\to+\infty$$

and due to (H1) and (3.1)

$$\begin{aligned}
&\int_{S(k)} f(x, v_k(x), \nabla v_k(x))dx \leq C \int_{S(k)} (\|\nabla v_k(x)\| + 1)dx \\
\leq &\frac{C}{k^2} + \int_{S(k)} \{\theta_k(x_N)\|\nabla u_{n_k}(x)\| + |\theta_k'(x_N)||u_{n_k}(x) - u(x)|\}dx \\
\leq &\frac{C}{k^2} + \{\int_{S(k)} \|\nabla u_{n_k}(x))\| + \frac{C}{\mathcal{L}_N(S(k))} \int_{S(k)} |u_{n_k}(x) - u(x)|dx\} \\
\leq &C(\frac{1}{k^2} + \frac{1}{k}).
\end{aligned}$$

In order to meet the boundary conditions at $x_N = -\frac{1}{2}$, we modify the sequence $\{v_k\}$ using the same slicing procedure on $\{x \in Q: -\frac{1}{2} < x_N \leq -\frac{1}{2} + \frac{1}{k}\}$, $k \in \mathbf{N}$.

Second step. We transform the sequence $\{u_n\}$ into a periodic sequence with respect to the directions $\{e_1, e_2, \ldots, e_{N-1}\}$.

By the first step, we can assume that

$$\liminf_{n \to +\infty} \int_Q f(x, u_n, \nabla u_n(x))dx = \lim_{n \to +\infty} \int_Q f(x, u_n, \nabla u_n(x))dx < +\infty$$

where

$$u_n(x) = \begin{cases} b & \text{if } x_N = \frac{1}{2} \\ a & \text{if } x_N = -\frac{1}{2}. \end{cases}$$

Let $g \in BV((-1;1);\mathbf{R}^p)$ be given by

$$g(t) = \begin{cases} b & \text{if } t > 0 \\ a & \text{if } t < 0. \end{cases}$$

Let $h_n \in BV((-1;1);\mathbf{R}^p) \cap C^\infty((-1;1);\mathbf{R}^p)$ be such that

$$h_n \to g \quad \text{in } L^1((-1;1);\mathbf{R}^p) \qquad \text{and} \qquad \int_{-1}^{1} |h_n'(t)|dt \leq const,$$

and let θ, $\hat{\theta} \in C^\infty(\mathbf{R};[0;1])$ be smooth functions such that

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 1/4 \\ 0 & \text{if } t \leq 1/8 \end{cases} \qquad \text{and} \qquad \hat{\theta}(t) = \begin{cases} 1 & \text{if } t \leq -1/4 \\ 0 & \text{if } t \geq -1/8. \end{cases}$$

We define

$$g_n(t) := \begin{cases} b & \text{if } t > \frac{1}{4} \\ \theta(t)b + (1-\theta(t))h_n(t) & \text{if } \frac{1}{8} < t < \frac{1}{4} \\ h_n(t) & \text{if } |t| \leq \frac{1}{8} \\ \hat{\theta}(t)a + (1-\hat{\theta}(t))h_n(t) & \text{if } -\frac{1}{4} < t < -\frac{1}{8} \\ a & \text{if } t < -\frac{1}{4}. \end{cases}$$

Clearly, the g_n's are in $W^{1,1}((-1,1);\mathbf{R}^p)$,

$$g_n(t) = \begin{cases} a & \text{if } t < -\frac{1}{4} \\ b & \text{if } t > \frac{1}{4}, \end{cases}$$

g_n converge to g in $L^1((-1,1);\mathbf{R}^p)$ and

$$\int_{-1}^{1} |g_n'(t)|dt \leq const.$$

We set

$$G_n(x) := g_n(x_N).$$

It follows immediately that G_n is periodic in the directions e_i, $i = 1, \ldots, N-1$,

$$\int_Q |G_n(x) - u(x)|dx = \int_{Q'} \{\int_{-1/2}^{1/2} |g_n(t) - g(t)|dt\}dx' \to 0 \quad \text{as } n \to +\infty$$

and

$$\int_Q \|\nabla G_n(x)\|dx = \int_Q \|g_n'(x_N) \otimes e_N\|dx = \int_{-1/2}^{1/2} |g_n'(t)|dt \leq const,$$

where

$$Q' = \{x \in Q : x_N = 0\}.$$

As a result of (H2) we have

$$\int_Q \{\|\nabla u_n(x)\| + \|\nabla G_n(x)\|\}dx \leq const = C \tag{3.2}$$

for all n and so, choosing $P_1 \in \mathbf{N}$ such that$P_1 > 2C$ we decompose (see Figure 2)

$$\{x \in Q : -\frac{1}{2} < x_1 \leq 0\} = \bigcup_{i=1}^{P_1} S_i^-$$

where

$$S_i^- = \{x \in Q : \alpha_i^- < x_1 \leq \alpha_{i+1}^-\} \qquad -\frac{1}{2} = \alpha_1^- < \ldots < \alpha_{P_1+1}^- = 0,$$

and

$$\mathcal{L}_N(S_i^-) = \alpha_{i+1}^- - \alpha_i^- = \frac{1}{2P_1}.$$

It follows from (3.2) that there exists a slice $S^-(2)$ and a subsequence $\{(u_{n'}, G_{n'})\}$ such that

$$\int_{S^-(2)} \{\|\nabla u_{n'}(x)\| + \|\nabla G_{n'}(x)\|\}dx \leq \frac{1}{2} \tag{3.3}$$

for all n'. Similarly, writing

$$\{x \in Q: 0 \leq x_1 < \frac{1}{2}\} = \bigcup_{i=1}^{P_1} S_i^+$$

$$S_i^+ = \{x \in Q: \alpha_i^+ < x_1 \leq \alpha_{i+1}^+\}, \qquad \mathcal{L}_N(S_i^+) = \alpha_{i+1}^+ - \alpha_i^+ = \frac{1}{2P_1},$$

there exists a slice $S^+(2)$ such that

$$\int_{S^+(2)} \{\|\nabla u_{n'}(x)\| + \|\nabla G_{n'}(x)\|\} dx \leq \frac{1}{2} \qquad \text{for infinitely many indices } n'. \tag{3.4}$$

On the other hand, if n is large enough then

$$\frac{1}{\mathcal{L}_N(S^\pm(2))} \int_{S^\pm(2)} |u_n(x) - u(x)| dx \leq \frac{1}{2}, \quad \frac{1}{\mathcal{L}_N(S^\pm(2))} \int_{S^\pm(2)} |G_n(x) - u(x)| dx \leq \frac{1}{2}. \tag{3.5}$$

By (3.3), (3.4) and (3.5) we can find n_2 such that

$$\int_{S^\pm(2)} \{\|\nabla u_{n_2}(x)\| + \|\nabla G_{n_2}(x)\|\} dx \leq \frac{1}{2},$$

$$\frac{1}{\mathcal{L}_N(S^\pm(2))} \int_{S^\pm(2)} |u_{n_2}(x) - u(x)| dx \leq \frac{1}{2}, \quad \frac{1}{\mathcal{L}_N(S^\pm(2))} \int_{S^-(2)} |G_{n_2}(x) - u(x)| dx \leq \frac{1}{2}.$$

By induction, if $k \geq 3$ then

$$\{x \in Q: -\frac{1}{2} < x_1 \leq -\frac{1}{2} + \frac{1}{k}\} = \bigcup_{i=1}^{P_k} S_i^-$$

with

$$\mathcal{L}_N(S_i^-) = \frac{1}{kP_k} < \frac{1}{k^2 C},$$

$P_k > kC$, $P_k \in \mathbf{N}$,

$$\{x \in Q: \frac{1}{2} - \frac{1}{k} \leq x_1 < \frac{1}{2}\} = \bigcup_{i=1}^{P_k} S_i^+, \qquad \mathcal{L}_N(S_i^+) = \frac{1}{kP_k}$$

and we choose slices $S^\pm(k)$ and $n_k > n_{k-1}$ such that

$$\begin{gathered} \int_{S^\pm(k)} \{\|\nabla u_{n_k}(x)\| + \|\nabla G_{n_k}(x)\|\} dx \leq \frac{1}{k}, \\ \frac{1}{\mathcal{L}_N(S^\pm(k))} \int_{S^\pm(k)} |u_{n_k}(x) - u(x)| dx \leq \frac{1}{k}, \\ \frac{1}{\mathcal{L}_N(S^\pm(k)} \int_{S^-(k))} |G_{n_k}(x) - u(x)| dx \leq \frac{1}{k}. \end{gathered} \tag{3.6}$$

Let

$$S^{\pm}(k) = \{x \in Q \colon \gamma_k^{\pm} < x_1 < \beta_k^{\pm}\},$$

where

$$-\frac{1}{2} \le \gamma_k^- < \beta_k^- \le -\frac{1}{2} + \frac{1}{k}, \qquad \frac{1}{2} - \frac{1}{k} \le \gamma_k^+ < \beta_k^+ \le \frac{1}{2}, \qquad |\beta_k^{\pm} - \gamma_k^{\pm}| < \frac{1}{k^2 C}.$$

Let us consider θ_k^-, θ_k^+ smooth cut-off functions such that

$$\theta_k^-(t) = \begin{cases} 1, & \text{if } t \ge \beta_k^- \\ 0, & \text{if } t \le \gamma_k^- \end{cases} \qquad \theta_k^+(t) = \begin{cases} 1, & \text{if } t \le \gamma_k^+ \\ 0, & \text{if } t \ge \beta_k^+ \end{cases}$$

and define

$$v_k(x) = \begin{cases} G_{n_k}(x) & \text{if } x_1 < \gamma_k^- \\ \theta_k^-(x_1) u_{n_k}(x) + (1 - \theta_k^-(x_1)) G_{n_k}(x) & \text{if } x \in S^-(k) \\ u_{n_k}(x) & \text{if } \beta_k^- < x_1 < \gamma_k^+ \\ \theta_k^+(x_1) u_{n_k}(x) + (1 - \theta_k^+(x_1)) G_{n_k}(x) & \text{if } x \in S^+(k) \\ G_{n_k}(x) & \text{if } x_1 > \beta_k^+ . \end{cases}$$

Because

$$G_n(x) = \begin{cases} b & \text{if } x_N > \frac{1}{4} \\ a & \text{if } x_N < -\frac{1}{4}, \end{cases}$$

it follows that

$$v_k \in W^{1,1}, \quad v_k(x) = \begin{cases} b & \text{if } x_N = \frac{1}{2} \\ a & \text{if } x_N = -\frac{1}{2} \end{cases} .$$

In addition, if $x_1 = \pm\frac{1}{2}$ then

$$v_k(x_1, \ldots, x_N) = G_{n_k}(x_1, \ldots, x_N) = G_{n_k}(-x_1, x_2, \ldots, x_N) = v_k(-x_1, \ldots, x_N)$$

i.e. v_k is periodic in the e_1 direction. Also,

$$\begin{aligned} &\|v_k - u\|_{L^1(Q)} \le \\ &\le \int_{Q \cap \{\beta_k^- < x_1 < \gamma_k^+\}} |u_{n_k}(x) - u(x)| dx + \int_{S^{\pm}(k)} (|u_{n_k}(x) - u(x)| + |G_{n_k}(x) - u(x)|) dx \\ &\quad + \int_{Q \cap \{x_1 < \gamma_k^-\}} |G_{n_k}(x) - u(x)| dx + \int_{Q \cap \{x_1 > \beta_k^+\}} |G_{n_k}(x) - u(x)| dx \\ &\le \|u_{n_k} - u\|_{L^1(Q)} + \|G_{n_k} - u\|_{L^1(Q)} \to 0 \quad \text{as } k \to +\infty. \end{aligned}$$

We show that

$$\lim_{n \to +\infty} \int_Q f(x, u_n(x), \nabla u_n(x)) dx \ge \liminf_{k \to +\infty} \int_Q f(x, v_k(x), \nabla v_k(x)) dx.$$

Indeed, as f is nonnegative

$$\begin{aligned}
&\lim_{n\to+\infty}\int_Q f(x,u_n(x),\nabla u_n(x))dx \geq \\
\geq &\liminf_{k\to+\infty}\int_{Q\cap\{b_k^-<x_1<\gamma_k^+\}} f(x,u_{n_k}(x),\nabla u_{n_k}(x))dx \\
= &\liminf_{k\to+\infty}\{\int_Q f(x,v_k(x),\nabla v_k(x))dx - \int_{S^\pm(k)} f(x,v_k(x),\nabla v_k(x))dx \\
&-\int_{Q\cap\{x_1<\gamma_k^-\}} f(x,G_{n_k}(x),\nabla G_{n_k}(x)dx - \int_{Q\cap\{\beta_k^+<x_1\}} f(x,G_{n_k}(x),\nabla G_{n_k}(x))dx\} \\
= &\liminf_{k\to+\infty}\int_Q f(x,v_k(x),\nabla v_k(x)dx
\end{aligned}$$

because by (H2)

$$\begin{aligned}
&\int_{Q\cap\{x_1<\gamma_k^-\}} f(x,G_{n_k}(x),\nabla G_{n_k}(x))dx \leq C(\gamma_k^- + \frac{1}{2})\{1+\int_{-\frac{1}{2}}^{\frac{1}{2}}|g'_{n_k}(t)|dt\} \\
\leq &C(\gamma_k^- + \frac{1}{2}) \to 0 \quad \text{as } k\to+\infty,
\end{aligned}$$

and by (H2) and (3.6)

$$\begin{aligned}
&\int_{S^\pm(k)} f(x,v_k(x),\nabla v_k(x))dx \leq C\int_{S^\pm(k)}(1+\|\nabla v_k(x)\|)dx \\
\leq &C\left\{\int_{S^\pm(k)}\theta_k^\pm(x_1)(\|\nabla u_{n_k}\|+\|\nabla G_{n_k}(x)\|)dx + \int_{S^\pm(k)}|\theta_k^\pm(x_1)'||u_{n_k}(x)-G_{n_k}|dx\right\} \\
&+C\frac{1}{k^2} \\
\leq &C\{\frac{1}{k^2}+\int_{S^\pm(k)}(\|\nabla u_{n_k}\|+\|\nabla G_{n_k}(x)\|)dx \\
&+\frac{1}{\mathcal{L}_N(S^\pm(k))}\int_{S^\pm(k)}(|u_{n_k}(x)-u(x)|+|u_(x)-G_{n_k}(x)|)dx\} \\
\leq &O(1/k).
\end{aligned}$$

Denoting the sequence $\{v_k\}$ thus constructed by $\{u_n^{(1)}\}$ and using a similar slicing procedure in the x_2-direction, we construct a sequence $\{u_k^{(1,2)}\}$ as

$$u_k^{(1,2)} = \begin{cases} G_{n_k}(x) & \text{if } x_2<\gamma_k^- \\ \theta_k^-(x_2)u_{n_k}^{(1)}(x)+(1-\theta_k^-(x_2))G_{n_k}(x) & \text{if } x\in S^-(k) \\ u_{n_k}^{(1)}(x) & \text{if } \beta_k^-<x_2<\gamma_k^+ \\ \theta_k^+(x_2)u_{n_k}^{(1)}(x)+(1-\theta_k^+(x_2))G_{n_k}(x) & \text{if } x\in S^+(k) \\ G_{n_k}(x) & \text{if } x_2>\beta_k^+. \end{cases}$$

Because $u^{(1)}_{n_k}$ and G_{n_k} are periodic in the x_1-direction, then $u^{(1,2)}_k$ is periodic in the x_1 and x_2-directions and

$$u^{(1,2)}_k(x) = \begin{cases} b & \text{if } x_N = \frac{1}{2} \\ a & \text{if } x_N = -\frac{1}{2} \end{cases}.$$

By induction we repeat the process to construct a sequence $w'_n := u^{(1,2,\dots,N-1)}_n$ periodic in the directions $e_1, \dots, e_{N-1}$,

$$w_n(x) = \begin{cases} b & \text{if } x_N = \frac{1}{2} \\ a & \text{if } x_N = -\frac{1}{2} \end{cases},$$

$$w_n \to u \quad \text{in } L^1(Q)$$

and

$$\liminf_{n\to+\infty} \int_Q f(x, u_n(x), \nabla u_n(x))dx \geq \liminf_{n\to+\infty} \int_Q f(x, w_n(x), \nabla w_n(x))dx.$$

Proof of Proposition 3.1 Without loss of generality we assume that $\nu = e_N$, $\alpha = 0$. Suppose that we show that

$$\liminf_{n\to+\infty} \int_\Omega f(x, w_n(x), \nabla w_n(x))dx \geq \int_{\Omega\cap\{x_N=0\}} K(x, a, b, e_N)dH_{N-1}. \tag{3.7}$$

If we set

$$g(x, u, A) := f(x, u, A) - f(x, u, 0)$$

then g satisfies (H1)-(H3). If in addition g is nonnegative, obviously we have

$$g^\infty(x, u, A) = f^\infty(x, u, A)$$

and the assumptions (H2) and (H4) on f yield

$$\begin{aligned} &|\frac{g(x,u,tA)}{t} - g^\infty(x,u,tA)| \\ &\leq |\frac{f(x,u,tA)}{t} - f^\infty(x,u,tA)| + \frac{f(x,u,0)}{t} \\ &\leq C\{\frac{1}{t^m} + \frac{1}{t}\} \end{aligned}$$

for $t \geq t_0$; note that as $m \leq 1$

$$\frac{1}{t^m} + \frac{1}{t} \leq \frac{2}{t^m}$$

and so g verifies (H4). Hence, by (3.7)

$$\liminf_{n\to+\infty} \int_\Omega g(x, u_n(x), \nabla u_n(x))dx \geq \int_{\Omega\cap\{x_N=0\}} K(x, a, b, e_N)dH_{N-1} \tag{3.8}$$

and, as $u_n \to u$ in L^1 strong and by (H2)

$$0 \le f(x,u,0) \le c_2;$$

by Lebesgue's Dominated Convergence Theorem we obtain

$$\int_\Omega f(x,u_n(x),0)dx \to \int_\Omega f(x,u(x),0)dx,$$

which, together with (3.8), yields

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \ge \int_\Omega f(x,u(x),0)dx + \int_{\Omega\cap\{x_N=0\}} K(x,a,b,e_N)dH_{N-1}.$$

Next, we prove (3.7). We assume that

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx = \lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx < +\infty$$

which, by (H2) implies that

$$\int_\Omega \|\nabla u_n(x)\|dx \le const. \tag{3.9}$$

Fix $\epsilon > 0$ and consider the open subset of $\mathbf{R}^{N-1}$

$$\Omega' := \{x \in \Omega : x_N = 0\}.$$

By (H3) if $x \in \Omega'$ then there exists $\delta(x) > 0$ such that

$$x' \in x + \delta(x)Q \Rightarrow |f(x,u,A) - f(x',u,A)| \le \epsilon C(1 + |u| + \|A\|) \tag{3.10}$$

for every (x,u,A), $(x',u,A) \in \Omega \times \mathbf{R}^p \times M^{p\times N}$, where $Q = (-\frac{1}{2},\frac{1}{2})^N$. Since for all $k \in \mathbf{N}$

$$\Omega' = \bigcup_{x\in\Omega'} \bigcup_{\substack{0<r<\min\{\delta(x),\frac{1}{k}\} \\ x+rQ\subset\Omega}} (x + rQ'),$$

where $Q' := \{y \in Q : y_N = 0\}$, by Vitali's Covering Theorem there exists a countable disjoint subcollection such that

$$H_{N-1}(\Omega' \setminus \bigcup_{q=1}^{\infty}(x_q^k + \delta_q^k Q')) = 0.$$

As f is nonnegative we have

$$\begin{gathered}\lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \ge \\ \liminf_{n\to+\infty} \sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} f(x_q^k,u_n(x),\nabla u_n(x))dx + O(\epsilon)\end{gathered} \tag{3.11}$$

because, by (3.9), (3.10) and as $\|u_n\|_{L^1} \leq C$,

$$\sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} |f(x, u_n(x), \nabla u_n(x)) - f(x_q^k, u_n(x), \nabla u_n(x))| dx$$
$$\leq \epsilon C \sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} (1 + |u_n(x)| + \|\nabla u_n(x)\|) dx \leq$$
$$\leq \epsilon C \int_{\Omega} (1 + |u_n(x)| + \|\nabla u_n(x)\|) dx \leq \epsilon C.$$

By (3.11) we deduce that

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx \geq \\ &\liminf_{n\to+\infty} \sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} f^{\infty}(x_q^k, u_n(x), \nabla u_n(x)) dx \\ +&\liminf_{n\to+\infty} \sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} |f(x_q^k, u_n(x), \nabla u_n(x)) - f^{\infty}(x_q^k, u_n(x), \nabla u_n(x))| dx \\ &+ O(\epsilon). \end{aligned} \tag{3.12}$$

Defining

$$F_k := \bigcup_{q=1}^{\infty} (x_q^k + \delta_q^k Q),$$

clearly

$$F_k \subset \{x \in \Omega : dist(x, \Omega') \leq \frac{1}{k}\}$$

and so

$$\mathcal{L}_N(F_k) = \sum_{q=1}^{\infty} (\delta_q^k)^N \to 0.$$

Indeed,

$$\mathcal{L}_N(F_k) \leq \frac{1}{k} \sum_{q=1}^{\infty} (\delta_q^k)^{N-1} = \frac{1}{k} H_{N-1}(\Omega'). \tag{3.13}$$

By (H2), (H4), Lemma 2.3 (i), (3.9), (3.13) and Hölder's inequality we have

$$\sum_{q=1}^{\infty} \int_{x_q^k+\delta_q^k Q} |f(x_q^k, u_n(x), \nabla u_n(x)) - f^{\infty}(x_q^k, u_n(x), \nabla u_n(x))| dx \leq$$
$$c_2(1+2t_0)\mathcal{L}_N(\{x \in F_k : \|\nabla u_n(x)\| \leq t_0\} +$$
$$\sum_{q=1}^{\infty} \int_{V_q^k} \|\nabla u_n(x)\| | \frac{1}{\|\nabla u_n(x)\|} f(x_q^k, u_n(x), \|\nabla u_n(x)\| \frac{\nabla u_n(x)}{\|\nabla u_n(x)\|})$$
$$- f^{\infty}(x_q^k, u_n(x), \|\nabla u_n(x)\| \frac{\nabla u_n(x)}{\|\nabla u_n(x)\|}) | dx$$
$$\leq O(\frac{1}{k}) + \int_{F_k} C_4 \|\nabla u_n(x)\| \frac{1}{\|\nabla u_n(x)\|^m} dx$$
$$\leq O(\frac{1}{k}) + C_4 (\int_{F_k} \|\nabla u_n(x)\| dx)^{1-m} (\mathcal{L}_N(F_k))^m$$
$$\leq O(\frac{1}{k}) + C(\mathcal{L}_N(F_k))^m \leq O(\frac{1}{k^m})$$

where

$$V_q^k = (x_q^k + \delta_q^k Q) \cap \{x \in F_k : \|\nabla u_n(x)\| \geq t_0\}.$$

Thus, (3.12) reduces to

$$\lim_{n\to+\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx \geq$$
$$\liminf_{n\to+\infty} \sum_{q=1}^{\infty} (\delta_q^k)^{N-1} \int_Q f^{\infty}(x_q^k, v_{n,q}^k(y), \nabla v_{n,q}^k(y)) dy + O(\frac{1}{k^m}) + O(\epsilon),$$

where

$$v_{n,q}^k(y) := u_n(x_q^k + \delta_q^k y).$$

Since, for fixed $x_q^k \in \Omega'$, it is clear that

$$v_{n,q}^k(y) \to \begin{cases} b & \text{if } y_N > 0 \\ a & \text{if } y_N < 0 \end{cases} \qquad \text{as } n \to +\infty$$

in $L^1(Q, \mathbf{R}^p)$, by lemma 3.2 we have

$$\lim_{n\to+\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx \geq$$
$$\sum_{q=1}^{\infty} (\delta_q^k)^{N-1} \liminf_{n\to+\infty} \int_Q f^{\infty}(x_q^k, v_{n,q}^k(y), \nabla v_{n,q}^k(y)) dy + O(\frac{1}{k^m}) + O(\epsilon) \geq$$
$$\sum_{q=1}^{\infty} (\delta_q^k)^{N-1} K(x_q^k, a, b, e_N) + O(\frac{1}{k^m}) + O(\epsilon).$$

By Proposition 2.4 (ii), the function $x \mapsto K(x, a, b, \nu)$ is continuous and so, letting $k \to +\infty$ we conclude that

$$\lim_{n \to +\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx \geq \int_{\Omega \cap \{x_N = 0\}} K(x, a, b, e_N) dH_{N-1}(x) + O(\epsilon).$$

Now, it suffices to let $\epsilon \to 0^+$. ■

4. A continuity result.

In section 3 we showed that if u takes only two values across a planar surface $\Sigma(u)$ then

$$I(u) \leq \mathcal{F}[u].$$

Next we prove that the equality holds.

Proposition 4.1 Let $f: (a_0 + \lambda Q_\nu) \times \mathbf{R}^p \times M^{p \times N} \to [0, +\infty)$ satisfy (H1)-(H4), where $a_0 \in \mathbf{R}^N$, $\lambda > 0$, $\nu \in S^{N-1}$, and let

$$u(x) = \begin{cases} b & \text{if } (x - a_0) \cdot \nu > 0 \\ a & \text{if } (x - a_0) \cdot \nu < 0. \end{cases}$$

Then there exists a sequence $u_n \in W^{1,1}(a_0 + \lambda Q_\nu; \mathbf{R}^p)$ such that $u_n \to u$ in $L^1(a_0 + \lambda Q_\nu; \mathbf{R}^p)$ and

$$\lim_{n \to +\infty} \int_{a_0 + \lambda Q_\nu} f(x, u_n(x), \nabla u_n(x)) dx = I(u).$$

We start by considering the case where f does not depend on x.

Lemma 4.2 Assume that $f: \mathbf{R}^p \times M^{p \times N} \to [0, +\infty)$ satisfies (H1) and (H2). Let $\nu \in S^{N-1}$, $\lambda \in \mathbf{R}$, $a_0 \in \mathbf{R}^N$,

$$u(x) = \begin{cases} b & \text{if } (x - a_0) \cdot \nu > \alpha \\ a & \text{if } (x - a_0) \cdot \nu < \alpha\ . \end{cases}$$

There exists a sequence $u_n \in W^{1,1}(a_0 + \lambda Q_\nu; \mathbf{R}^p)$ such that

$$u_n(x) = \begin{cases} a & \text{if } x \cdot \nu = -\alpha/2 \\ b & \text{if } x \cdot \nu = \alpha/2 \end{cases}$$

$$u_n(x) = u_n(x + k\alpha\nu_i), \quad i = 1, \ldots, N-1, \ k \in \mathbf{Z}$$

where $\{\nu_1, \ldots, \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbf{R}^N$,

$$u_n \to u \quad \text{in} \quad L^1(a_0 + \lambda Q_\nu; \mathbf{R}^p)$$

and

$$\int_{a_0+\lambda Q_\nu} f(u_n(x), \nabla u_n(x))dx \to I(u) = \int_{a_0+\lambda Q_\nu} f(u,0)dx + \lambda^{N-1}K(a,b,\nu)$$

Proof of Lemma 4.2. Step 1. We assume that $a_0 = 0$, $\lambda = 1$ and, without loss of generality, we set $\nu = e_N$. We claim that for all $\xi \in \mathcal{A}(a,b,e_N)$ there exists a sequence $\xi_n \in \mathcal{A}(a,b,e_N)$ such that

$$\|\xi_n - u\|_{L^1(Q)} \to 0 \quad \text{as } n \to +\infty$$

and

$$\int_Q f(\xi_n(x), \nabla\xi_n(x))dx \to \int_Q f(u,0)dx + \int_Q f^\infty(\xi(x), \nabla\xi(x))dx. \tag{4.1}$$

We denote by Σ the set $\{x \in Q : x_N = 0\}$. For $k \in \mathbf{N}$ we label the elements of $(\mathbf{Z} \cap [-k,k])^{N-1} \times \{0\}$ by $\{a_i\}_{i=1}^{(2k+1)^{N-1}}$ and we observe

$$(2k+1)\overline{\Sigma} = \bigcup_{i=1}^{(2k+1)^{N-1}} (a_i + \overline{\Sigma})$$

with

$$(a_i + \Sigma) \cap (a_j + \Sigma) = \emptyset \quad \text{if } i \neq j.$$

(See Figure 3). Extending $\xi(\cdot, x_N)$ to $\mathbf{R}^{N-1}$ by periodicity we define

$$\xi_{2k+1}(x) := \begin{cases} b & \text{if } x_N > 1/(2(2k+1)) \\ \xi((2k+1)x) & \text{if } |x_N| < 1/(2(2k+1)) \\ a & \text{if } x_N < -1/(2(2k+1)) \ . \end{cases}$$

Clearly $\xi_{2k+1} \in \mathcal{A}(a,b,e_N)$ and

$$\|\xi_{2k+1} - u\|_{L^1(Q)} = \int_{-\frac{1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_\Sigma |\xi((2k+1)x) - u(x)|dx'dx_N =$$

$$\frac{1}{2k+1}\int_{\frac{-1}{2}}^{0}\int_\Sigma |\xi((2k+1)x', x_N) - a|dx'dx_N$$

$$+\frac{1}{2k+1}\int_0^{\frac{1}{2}}\int_\Sigma |\xi((2k+1)x', x_N) - b|dx'dx_N.$$

Due to the periodicity,

$$\int_{\frac{-1}{2}}^{0}\int_\Sigma |\xi((2k+1)x', x_N) - a|dx'dx_N \to \int_{\frac{-1}{2}}^{0}\int_\Sigma |\xi(x) - a|dx'dx_N$$

and so we conclude that

$$\xi_{2k+1} \to u \quad \text{in } L^1(Q;\mathbf{R}^p).$$

Also,

$$\begin{aligned}
&\int_Q f(\xi_{2k+1}(x), \nabla \xi_{2k+1}(x))dx \\
&= \int_{-\frac{1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_\Sigma f(\xi((2k+1)x),(2k+1)\nabla\xi((2k+1)x)dx \\
&+ \int_{\frac{1}{2(2k+1)}}^{\frac{1}{2}} \int_\Sigma f(b,0)dx'dx_N + \int_{-\frac{1}{2}}^{\frac{-1}{2(2k+1)}} \int_\Sigma f(a,0)dx'dx_N \qquad (4.2)\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{(2k+1)\Sigma} \frac{1}{(2k+1)^N} f(\xi(y),(2k+1)\nabla\xi(y))dy'dy_N \\
&+ \int_{\frac{1}{2(2k+1)}}^{\frac{1}{2}} \int_\Sigma f(b,0)dx'dx_N + \int_{-\frac{1}{2}}^{\frac{-1}{2(2k+1)}} \int_\Sigma f(a,0)dx'dx_N.
\end{aligned}$$

On the other hand, due to periodicity of ξ, by (H2) and by Lebesgue's Dominated Convergence Theorem we have that

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(2k+1)^N} \int_{(2k+1)\Sigma} f(\xi(y),(2k+1)\nabla\xi(y))dy'dy_N \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(2k+1)^N} \sum_{i=1}^{(2k+1)^{N-1}} \int_{a_i+\Sigma} f(\xi(y),(2k+1)\nabla\xi(y))dy'dy_N \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(2k+1)} \int_\Sigma f(\xi(y),(2k+1)\nabla\xi(y))dy'dy_N \\
&\to \int_Q f^\infty(\xi(y),\nabla\xi(y))dy \quad \text{as } k \to +\infty
\end{aligned}$$

which, together with (4.2) yields

$$\int_Q f(\xi_{2k+1}(x),\nabla\xi_{2k+1}(x))dx \to \int_Q f(u,0)dx + \int_Q f^\infty(\xi(y),\nabla\xi(y))dy$$

thus proving (4.1). Next, let $\{\eta_n\} \subset \mathcal{A}(a,b,e_N)$ be a minimizing sequence for $K(a,b,e_N)$, i.e.

$$K(a,b,e_N) = \lim_{n\to+\infty} \int_Q f^\infty(\eta_n(y),\nabla\eta_n(y))dy.$$

By (4.1), for all n we can choose $u_n \in \mathcal{A}(a,b,e_N)$ such that

$$\|u_n - u\|_{L^1(Q)} < \frac{1}{n}$$

and

$$\left|\int_Q f(u_n(x),\nabla u_n(x))dx - \int_Q f(u(x),0)dx - \int_Q f^\infty(\eta_n(x),\nabla\eta_n(x))dx\right| < \frac{1}{n}.$$

By Theorem 3.1 we conclude that

$$\begin{aligned}
I(u) &\le \liminf_{n\to+\infty}\int_Q f(u_n(x),\nabla u_n(x))dx \le \limsup_{n\to+\infty}\int_Q f(u_n(x),\nabla u_n(x))dx\\
&\le \lim_{n\to+\infty}\left\{\int_Q f(u(x),0)dx + \int_Q f^\infty(\eta_n(x),\nabla\eta_n(x))dx + \frac{1}{n}\right\}\\
&= \int_Q f(u(x),0)dx + K(a,b,e_N) = I(u)
\end{aligned}$$

and so

$$I(u) = \lim_{n\to+\infty}\int_Q f(u_n(x),\nabla u_n(x))dx.$$

Step 2. Let $\lambda > 0$, and define

$$f_\lambda(u,A) := f(u,\frac{A}{\lambda}).$$

Setting

$$u_0(x) := \begin{cases} b & \text{if } x\cdot e_N > 0\\ a & \text{if } x\cdot e_N < 0\end{cases},$$

by Step 1 there exists $v_n \in \mathcal{A}(a,b,e_N)$ such that

$$v_n \to u_0 \quad \text{in} \quad L^1(Q;\mathbf{R}^p)$$

and

$$\int_{Q_\nu} f_\lambda(v_n(x),\nabla v_n(x))dx \to \int_Q f_\lambda(u_0(x),0)dx + K_\lambda(a,b,\nu).$$

Let $a_0 \in \mathbf{R}^N$ and set

$$u_n(x) := v_n(\frac{x-a_0}{\lambda}), \quad x \in a_0 + \lambda Q.$$

It is clear that u_n meets the boundary conditions, u_n is periodic in the $e_1,\ldots,e_{N-1}$ directions with period λ,

$$\|u_n - u\|_{L^1(a_0+\lambda Q;\mathbf{R}^p)} \to 0 \quad \text{as} \quad n\to+\infty$$

and

$$\begin{aligned}
&\int_{a_0+\lambda Q} f(u_n(x),\nabla u_n(x))dx\\
=&\int_{a_0+\lambda Q} f(v_n(\frac{x-a_0}{\lambda}),\frac{1}{\lambda}\nabla v_n(\frac{x-a_0}{\lambda}))dx\\
=&\lambda^N\int_{Q_\nu} f_\lambda(v_n(y),\nabla v_n(y))dy \to \lambda^N\int_{Q_\nu} f_\lambda(u_0(y),0)dy + \lambda^N K_\lambda(a,b,\nu).
\end{aligned}$$

However,

$$\lambda^N \int_{Q_\nu} f_\lambda(u_0(y), 0) dy = \int_{a_0+\lambda Q} f(u(x), 0) dx$$

and since

$$K_\lambda(a, b, \nu) = \frac{1}{\lambda} K(a, b, \nu),$$

we deduce that

$$\int_{a_0+\lambda Q} f(u_n(x), \nabla u_n(x)) dx \to I(u) = \int_{a_0+\lambda Q} f(u(x), 0) dx + \lambda^{N-1} K(a, b, \nu).$$

■

Proof of Proposition 4.1. Without loss of generality, we may assume that $a_0 = 0$, $\lambda = 1$, $\nu = e_N$. In the subsequent constructions we will use (H3). In order to make sure that this property is satisfied uniformly we will work on compact subsets of Q. Since as the proof will show it suffices to construct the desired sequence only on compact sets in a thin neighborhood of the set

$$\Sigma = \{x \in Q : x_N = 0\}.$$

Fix $\epsilon > 0$ and let

$$\Sigma_\epsilon = \{x \in \Sigma : |x_i| \leq \frac{1-\epsilon}{2}, \quad i = 1, \ldots, N-1\}.$$

Since Σ_ϵ is compact, by a standard argument we can find $\delta > 0$ such that (H3) is satisfied uniformly in $\Sigma_\epsilon \times [-\delta/2, \delta/2]$ i.e.

$$x', x \in \Sigma_\epsilon \times [-\delta/2, \delta/2], |x - x'| < \delta \Rightarrow |f(x, u, A) - f(x', u, A)| \leq \epsilon C(1 + |u| + \|A\|) \quad (4.3)$$

for every $(u, A) \in \mathbf{R}^p \times M^{p \times N}$. Let $k \in \mathbf{N}$ be such that

$$\frac{1-\epsilon}{k} < \delta \tag{4.4}$$

and partition Σ_ϵ into k^{N-1} $(N-1)$-dimensional cubes, aligned according to the coordinate axis and with mutually disjoint interiors

$$\Sigma_\epsilon = \bigcup_{i=1}^{k^{N-1}} (a_i + \frac{1-\epsilon}{k} \overline{\Sigma}).$$

Note that $\Sigma \setminus \Sigma_\epsilon$ can be covered by at most

$$\sigma(N) \epsilon k^{N-1} \tag{4.5}$$

non-overlapping $(N-1)$-dimensional cubes of size $\frac{1-\epsilon}{k}$. After we set

$$Q_i := a_i + \frac{1-\epsilon}{k} Q, \qquad \eta := \frac{1-\epsilon}{k},$$

Lemma 4.2 guarantees the existence of a sequence $\{\xi_n^{(i)}\} \subset W^{1,1}(Q_i;\mathbf{R}^p)$ such that

$$\begin{aligned}
&\xi_n^{(i)}(x) = \begin{cases} b & \text{if } (x-a_i)\cdot e_N = -\frac{\eta}{2} \\ a & \text{if } (x-a_i)\cdot e_N = \frac{\eta}{2} \end{cases} \\
&\xi_n^{(i)}(x + k\eta e_j) = \xi_n^{(i)}(x) \; k \in \mathbf{Z}, j = 1, \ldots, N-1, \\
&\xi_n^{(i)} \to u \text{ in } L^1(Q_i;\mathbf{R}^p)
\end{aligned} \tag{4.6}$$

and

$$\int_{Q_i} f(a_i, \xi_n^{(i)}(x), \nabla \xi_n^{(i)}(x))dx \to \int_{Q_i} f(a_i, u(x), 0)dx + \eta^{N-1} K(a_i, a, b, e_N).$$

We can assume that for all n and for all $i \in \{1, \ldots, k^{N-1}\}$

$$\|\xi_n^{(i)} - u\|_{L^1(Q_i;\mathbf{R}^p)} < \frac{\epsilon}{k^{N-1}} \tag{4.7}$$

and

$$\left| \int_{Q_i} f(a_i, \xi_n^{(i)}(x), \nabla \xi_n^{(i)}(x))dx - \int_{Q_i} f(a_i, u(x), 0)dx - \eta^{N-1} K(a_i, a, b, e_N) \right| < \frac{\epsilon}{k^{N-1}}. \tag{4.8}$$

By (4.8), (H2) and Proposition 2.4 (i) there exists a constant $c^\star$ such that

$$\int_{Q_i} \|\nabla \xi_n^{(i)}(x)\| dx \leq c^\star \eta^{N-1}. \tag{4.9}$$

We want to piece together the functions $\xi_n^{(i)}$ in order to obtain a sequence on the cube converging to $I(u)$. Firstly, we show that due to the periodicity of $\xi_n^{(i)}$ we can avoid concentrations near the boundaries of the $(N-1)$-dimensional cubes $a_i + \eta\Sigma$.

Step 1. Fix $i \in \{1, \ldots, k^{N-1}\}$. We claim that it is always possible to assume the existence of slices $S_{n,\pm}^{(i)}$ such that (see Figure 4)

$$\begin{aligned}
&S_{n,+}^{(i)} = \{x \in Q_i : \frac{\eta}{2} - \alpha_n < x_1 - a_{i_1} < \frac{\eta}{2}\}, \\
&S_{n,-}^{(i)} = \{x \in Q_i : -\frac{\eta}{2} < x_1 - a_{i_1} < -\frac{\eta}{2} + \alpha_n\}, \\
&\mathcal{L}_N(S_{n,\pm}^{(i)}) \leq \frac{c}{kn} \\
&\int_{S_{n,\pm}^{(i)}} \|\nabla \xi_n^{(i)}(x)\| dx \leq \frac{1}{k^{N-1} n}, \\
&\frac{1}{\mathcal{L}_N(S_{n,\pm}^{(i)})} \int_{S_{n,\pm}^{(i)}} |\xi_n^{(i)}(x) - u(x)| dx \leq \frac{1}{k^{N-1} n}.
\end{aligned} \tag{4.10}$$

Moreover, the width of the slices is the same for all $i = 1, \dots, k^{N-1}$. Indeed, let $Q^\star := Q_i \cup Q'_i$ where Q'_i is a translation of Q_i in the x_1-direction, (see Figure 5),

$$Q'_i = (a_i + \eta e_1 + Q_i),$$

and by (4.6) extend $\xi_n^{(i)}$ periodically to Q'_i. Choose $M > 2c^\star(1-\epsilon)^{N-1}$, $M \in \mathbf{N}$ and slice $Q^\star$ orthogonally to the x_1 direction into M slices of width $\frac{2\eta}{M}$. By (4.9) there exists a slice S_1 such that

$$\int_{S_1} \|\nabla \xi_n^{(i)}(x)\| dx < \frac{1}{k^{N-1}}, \quad \mathcal{L}_N(S_1) = \frac{2\eta}{M} < \frac{1}{c^\star k(1-\epsilon)^{N-2}} \leq \frac{c}{k},$$

for infinitely many indices n. Assume that $\{\xi_n^{(i)}\}$ is the subsequence thus extracted and choose n_1 large enough such that

$$\frac{1}{\mathcal{L}_N(S_1)} \int_{S_1} |\xi_{n_1}^{(i)}(x) - u(x)| dx \leq \frac{1}{2k^{N-1}}.$$

By induction, let $M \in \mathbf{N}$ be such that $M > 2mc^\star(1-\epsilon)^{N-1}$ and partition $Q^\star$ orthogonally to the x_1-direction, into M slices of width $\frac{2\eta}{M}$. There exists a slice S_m such that

$$\int_{S_m} \|\nabla \xi_j^{(i)}(x)\| dx < \frac{1}{mk^{N-1}}$$

for infinitely many indices j and so, we can find

$$n_m > n_{m-1} > \dots > n_2 > n_1$$

such that

$$\int_{S_m} \|\nabla \xi_{n_m}^{(i)}(x)\| dx < \frac{1}{mk^{N-1}}, \quad \frac{1}{\mathcal{L}_N(S_m)} \int_{S_m} |\xi_{n_m}^{(i)}(x) - u(x)| dx < \frac{1}{2mk^{N-1}}, \tag{4.11}$$

with

$$\mathcal{L}_N(S_m) = \frac{2\eta}{M} < \frac{1}{mc^\star k(1-\epsilon)^{N-2}} \leq \frac{c}{mk}, \quad S_m = \{x \in Q^\star : \alpha_m < x_1 < \beta_m\}.$$

Set

$$\gamma_m := \frac{\alpha_m + \beta_m}{2}$$

and define

$$\begin{aligned}
w_m^{(i)}(x) &:= \xi_{n_m}^{(i)}(x + \lambda_m e_1), \quad x \in Q_i, \\
\lambda_m &:= \gamma_m - (a_{i_1} + \frac{\eta}{2}), \\
S_{m,-}^{(i)} &:= \{x \in Q_i : a_{i_1} - \frac{\eta}{2} < x_1 < a_{i_1} - \frac{\eta}{2} + \frac{\beta_m - \alpha_m}{2}\}, \\
S_{m,+}^{(i)} &:= \{x \in Q_i : a_{i_1} + \frac{\eta}{2} - \frac{\beta_m - \alpha_m}{2} < x_1 < a_{i_1} + \frac{\eta}{2}\}.
\end{aligned}$$

It is clear that $w_m^{(i)}$ satisfies $(4.6)_{1,2,3}$. Also, since u does not depend on x_1 and $\xi_n^{(i)}(\cdot, x_N)$ is periodic in the directions of $e_1, \ldots, e_{N-1}$ with period η,

$$\int_{Q_i} |w_m^{(i)}(x) - u(x)|dx = \int_{Q_i} |\xi_{n_m}^{(i)}(x + \lambda e_1) - u(x + \lambda e_1)|dx$$
$$= \int_{Q_i} |\xi_{n_m}^{(i)}(x) - u(x)|dx \to 0 \quad \text{as } m \to +\infty.$$

Similarly,

$$\int_{Q_i} f(a_i, w_m^{(i)}(x), \nabla w_m^{(i)}(x))dx = \int_{Q_i} f(a_i, \xi_{n_m}^{(i)}(x), \nabla \xi_{n_m}^{(i)}(x))dx.$$

Finally, if $x \in S_{m,-}^{(i)} - \lambda e_1$ then $\alpha_m < \gamma_m < x_1 < \beta_m$, and if $x \in S_{m,+}^{(i)} - \lambda e_1$ then $\alpha_m < x_1 < \gamma_m < \beta_m$ and so, by (4.11)

$$\int_{S_{m,\pm}^{(i)}} \|\nabla w_m^{(i)}(x)\|dx \leq \int_{S_m} \|\nabla \xi_{n_m}^{(i)}(x)\|dx < \frac{1}{mk^{N-1}}$$

and

$$\frac{1}{\mathcal{L}_N(S_{m,\pm}^{(i)})} \int_{S_{m,\pm}^{(i)}} |w_m^{(i)}(x) - u(x)| = \frac{2}{\mathcal{L}_N(S_m)} \int_{S_{m,\pm}^{(i)}} |\xi_{n_m}^{(i)}(x + \lambda e_1) - u(x)|dx$$
$$\leq \frac{2}{\mathcal{L}_N(S_m)} \int_{S_m} |\xi_{n_m}^{(i)}(x) - u(x)|dx \leq \frac{1}{mk^{N-1}}.$$

Step 2. We consider $\{\xi_n^{(i)}\}$ as in (4.10) and we are going to piece them together row by row, in the x_1-direction. Suppose that the first row in the x_1-direction is

$$R_1 = \bigcup_{i=1}^{k} (a_i + \eta Q).$$

We define (see Figure 6)

$$v_n^{(1)}(x) := \begin{cases} \xi_n^{(1)}(x) & \text{if } -\frac{1}{2} < x_1 < a_1 - \frac{\eta}{2} \\ \theta_{1,n}(x_1)\xi_n^{(1)}(x) + (1 - \theta_{1,n}(x_1))\xi_n^{(2)}(x) & \text{if } x \in S_{n,+}^{(1)} \cup S_{n,-}^{(2)} \\ \xi_n^{(2)}(x) & \text{if } x \in Q_2 \setminus (S_{n,-}^{(2)} \cup S_{n,+}^{(2)}) \\ \vdots & \vdots \\ \xi_n^{(k)}(x) & \text{if } a_k + \frac{\eta}{2} < x_1 < \frac{1}{2}, \end{cases}$$

where $0 \leq \theta_{i,n} \leq 1$, $\|\theta'_{i,n}\|_{L^\infty} \leq Ckn$. Clearly, the $v_n^{(1)}(x)$'s are periodic in $x_2, \ldots, x_{N-1}$ and

$$\sum_{i=1}^{k} \int_{Q_i} |v_n^{(1)}(x) - u(x)|dx \leq \sum_{i=1}^{k} \int_{Q_i} |\xi_n^{(i)}(x) - u(x)|dx + \sum_{i=1}^{k-1} \int_{S_{n,+}^{(i)}} |\xi_n^{(i)}(x) - u(x)|dx$$

and so by (4.7) and (4.10) we obtain

$$\sum_{i=1}^{k}\int_{Q_i}|v_n^{(1)}(x)-u(x)|dx \le \frac{\epsilon}{k^{N-2}}+\frac{C}{k^{N-1}n^2}\le\frac{2\epsilon}{k^{N-2}} \tag{4.13}$$

for n large. Also, by (4.8), (4.3), (4.4), (H2) and Proposition 2.4 (i), (ii)

$$\begin{aligned}
&|\sum_{i=1}^{k}\int_{Q_i} f(x,v_n^{(1)}(x),\nabla v_n^{(1)}(x))dx-\sum_{i=1}^{k}\int_{Q_i} f(x,u(x),0)dx\\
&-\sum_{i=1}^{k}\int_{a_i+\eta\Sigma} K(x,a,b,e_N)dH_{N-1}|\\
\le&\left|\sum_{i=1}^{k}\int_{Q_i} f(a_i,\xi_n^{(i)}(x),\nabla\xi_n^{(i)}(x))dx-\sum_{i=1}^{k}\int_{Q_i} f(a_i,u(x),0)dx-\sum_{i=1}^{k}\eta^{N-1}K(a_i,a,b,e_N)\right|\\
&+\sum_{i=1}^{k}\int_{Q_i}|f(x,\xi_n^{(i)}(x),\nabla\xi_n^{(i)}(x))-f(a_i,\xi_n^{(i)}(x),\nabla\xi_n^{(i)}(x)|dx\\
&+\sum_{i=1}^{k}\int_{S^{(i)}_{n,\pm}}|f(x,v_n^{(1)}(x),\nabla v_n^{(1)}(x))-f(x,\xi_n^{(i)}(x),\nabla\xi_n^{(i)}(x))|dx\\
&+\sum_{i=1}^{k}\int_{Q_i}|f(a_i,u(x),0)-f(x,u(x),0)|dx\\
&+\sum_{i=1}^{k}\int_{a_i+\eta\Sigma}|K(x,a,b,e_N)-K(a_i,a,b,e_N)|dH_{N-1}\\
\le&\frac{\epsilon}{k^{N-2}}+\sum_{i=1}^{k}C\epsilon\int_{Q_i}(1+|\xi_n^{(i)}(x)|+\|\nabla\xi_n^{(i)}(x)\|)dx\\
&+\sum_{i=1}^{k}\int_{S^{(i)}_{n,\pm}}C(1+\|\nabla v_n^{(1)}(x)\|+\|\nabla\xi_n^{(i)}(x)\|)dx+\sum_{i=1}^{k}C\epsilon\int_{Q_i}(1+|u(x)|)dx\\
&+C\epsilon H_{N-1}(\Sigma)(1+|b-a|).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{S^{(i)}_{n,\pm}}(\|\nabla v_n^{(1)}(x)\|+\|\nabla\xi_n^{(i)}(x)\|)dx\\
\le&2\int_{S^{(i)}_{n,\pm}}\|\nabla\xi_n^{(i)}(x)\|dx+\frac{2}{\mathcal{L}_N(S^{(i)}_{n,\pm})}\int_{S^{(i)}_{n,\pm}}|\xi_n^{(i)}(x)-u(x)|dx
\end{aligned}$$

by (4.7), (4.9), (4.10) we conclude that

$$
\begin{aligned}
&|\sum_{i=1}^{k}\int_{Q_i} f(x,v_n^{(1)}(x),\nabla v_n^{(1)}(x))dx-\sum_{i=1}^{k}\int_{Q_i} f(x,u(x),0)dx\\
&-\sum_{i=1}^{k}\int_{a_i+\eta\Sigma} K(x,a,b,e_N)dH_{N-1}|\\
&\leq\frac{\epsilon}{k^{N-2}}+C\epsilon\left(2k(\frac{1-\epsilon}{k})^{N-1}+\frac{\epsilon}{k^{N-2}}+C\frac{(1-\epsilon)^{N-1}}{k^{N-2}}\right)+\\
&+C\left(\frac{k}{kn}+\frac{k}{k^{N-1}n}+\frac{k}{k^{N-1}n}\right)+\frac{C\epsilon}{k^{N-1}}+\frac{C\epsilon}{k^{N-2}}\\
&\leq\frac{2\epsilon}{k^{N-2}}
\end{aligned}
\tag{4.14}
$$

for n sufficiently large.

Step 3. Having pieced together the functions $\{\xi_n^{(i)}\}$ in every row R_i, $i=1,\dots,k$ corresponding to $x_1 = constant$, we obtain as in step 4 the sequences

$$\{v_n^{(1)}\},\ \{v_n^{(2)}\},\dots,\{v_n^{(k)}\}.$$

Now, we connect them in the x_2-direction (see Figure 7). Since $\{v_n^{(i)}\}$ is periodic in the x_2-direction, using a similar argument as in (4.7), (4.8), (4.9) and (4.10) which involves extending $v_n^{(i)}$ to R_{i+1} by periodicity, we can assume that there are slices $S_{n,\pm}^{(i)}$ orthogonal to the x_2-direction, where concentrations are avoided,

$$\mathcal{L}_N(S_{n,\pm}^{(i)})\leq\frac{C}{kn},$$

$$\int_{S_{n,\pm}^{(i)}}\|\nabla v_n^{(i)}(x)\|dx\leq\frac{1}{k^{N-2}n},\quad \frac{1}{\mathcal{L}_N(S_{n,\pm}^{(i)})}\int_{S_{n,\pm}^{(i)}}|v_n^{(i)}(x)-u(x)|dx\leq\frac{1}{k^{N-2}n}.$$

Using convex combinations of $v_n^{(i)}(x)$ and $v_n^{(i+1)}$ across $S_{n,\pm}^{(i)}$ in the x_2-direction, we construct w_n such that

$$\sum_{i=1}^{k^2}\int_{Q_i}|w_n(x)-u(x)|dx\leq\frac{3\epsilon}{k^{N-3}}$$

and

$$
\begin{aligned}
&|\sum_{i=1}^{k^2}\int_{Q_i} f(x,w_n(x),\nabla w_n(x))dx-\sum_{i=1}^{k^2}\int_{Q_i} f(x,u(x),0)dx\\
&-\sum_{i=1}^{k^2}\int_{a_i+\eta\Sigma} K(x,a,b,e_N)dH_{N-1}|\leq\frac{3\epsilon}{k^{N-3}}.
\end{aligned}
$$

By induction, we obtain finally a sequence

$$v_n\colon \bigcup_{i=1}^{k^{N-1}} Q_i \to \mathbf{R}^p$$

such that $v_n(x) = b$ if $x_N = \eta$, $v_n(x) = a$ if $x_N = -\eta$,

$$\sum_{i=1}^{k^{N-1}} \int_{Q_i} |v_n(x) - u(x)|dx \leq \frac{N\epsilon}{k^{N-N}} = N\epsilon, \tag{4.15}$$

and

$$\begin{aligned} &|\sum_{i=1}^{k^{N-1}} \int_{Q_i} f(x, v_n(x), \nabla v_n(x))dx - \sum_{i=1}^{k^{N-1}} \int_{Q_i} f(x, u(x), 0)dx \\ &- \sum_{i=1}^{k^{N-1}} \int_{a_i+\eta\Sigma} K(x, a, b, e_N)dH_{N-1}(x)| < N\epsilon. \end{aligned} \tag{4.16}$$

We now extend v_n to the whole cube Q. Using the periodicity assumption, we consider $\xi_n^{(i)}(\cdot, x_N)$ defined in $\mathbf{R}^{N-1} \times (-\frac{\eta}{2}, \frac{\eta}{2})$ and we set

$$u_n(x) = \begin{cases} b & \text{if } x_N \geq \frac{\eta}{2} \\ v_n(x) & \text{if } x \in \Sigma_\epsilon \times (-\frac{1}{2}\eta, \frac{1}{2}\eta) \\ \hat{v}_n(x) & \text{if } \Sigma \setminus \Sigma_\epsilon \times (-\frac{1}{2}\eta, \frac{1}{2}\eta) \\ a & \text{if } x_N \leq \frac{\eta}{2}, \end{cases}$$

where $\hat{v}_n$ is the natural extension of v_n as a $W^{1,1}$ function on the layer $\Sigma \setminus \Sigma_\epsilon \times (-\frac{1}{2}\eta, \frac{1}{2}\eta)$. Using Figure 8 as a reference, by (4.5) and (4.10) we can see that $\Sigma \setminus \Sigma_\epsilon \times (-\frac{1}{2}\eta, \frac{1}{2}\eta)$ is formed by at most $\sigma(N)\epsilon k^{N-1}$ cubes on which $\hat{v}_n$ is equal to some $\xi_n^{(i)}$ and by slices of the total measure of order $\frac{1}{n}$ where $\hat{v}_n$ is a convex combination of some $\xi_n^{(i)}$'s. Thus, by (4.7), (4.10), (4.15) we have

$$\begin{aligned} &\int_Q |u_n(x) - u(x)|dx \\ &\leq \sum_{i=1}^{k^{N-1}} \int_{Q_i} |v_n(x) - u(x)|dx + \sigma(N)\epsilon k^{N-1} \frac{\epsilon}{k^{N-1}} + O(\frac{1}{n}) = O(\epsilon) + O(\frac{1}{n}) \end{aligned} \tag{4.17}$$

and

$$\left|\int_Q f(x,u_n(x),\nabla u_n(x))dx - \int_Q f(x,u(x),0)dx - \int_\Sigma K(x,a,b,e_N)dH_{N-1}(x)\right|$$
$$\leq \Big| \sum_{i=1}^{k^{N-1}} \int_{Q_i} f(x,v_n(x),\nabla v_n(x))dx - \sum_{i=1}^{k^{N-1}} \int_{Q_i} f(x,u(x),0)dx$$
$$- \sum_{i=1}^{k^{N-1}} \int_{a_i+\eta\Sigma} K(x,a,b,e_N)dH_{N-1}(x)\Big|$$
$$+ \sum_{l=1}^{N-1} \int_{(\Sigma\setminus\Sigma_\epsilon)\times(-\eta/2,\eta/2)} f(x,\hat{v}_n(x),\nabla\hat{v}_n(x))dx + \int_{(\Sigma\setminus\Sigma_\epsilon)\times(-\eta/2,\eta/2)} f(x,u(x),0)dx$$
$$+ \int_{(\Sigma\setminus\Sigma_\epsilon)} K(x,a,b,e_N)dH_{N-1}(x). \tag{4.18}$$

From (H2) and Proposition 2.4 (i) it follows that

$$\int_{(\Sigma\setminus\Sigma_\epsilon)\times(-\eta/2,\eta/2)} f(x,u(x),0)dx + \int_{\Sigma\setminus\Sigma_\epsilon} K(x,a,b,e_N)dH_{N-1}(x) = O(\epsilon), \tag{4.19}$$

and by (4.9) and (4.10) we obtain

$$\int_{(\Sigma\setminus\Sigma_\epsilon)\times(-\eta/2,\eta/2)} f(x,\hat{v}_n(x),\nabla\hat{v}_n(x))dx \leq \sigma(N)\epsilon k^{N-1}\frac{(1-\epsilon)^{N-1}}{k^{N-1}} + O(\frac{1}{n})$$
$$= O(\epsilon) + O(\frac{1}{n}). \tag{4.20}$$

By (4.16)-(4.20) it suffices to choose $n = n(\epsilon)$ so large that

$$\int_Q |u_{n(\epsilon)}(x) - u(x)|dx = O(\epsilon)$$

and

$$\Big|\int_Q f(x,u_{n(\epsilon)}(x),\nabla u_{n(\epsilon)}(x))dx - I(u)\Big| \leq O(\epsilon).$$

■

5. Relaxation on $BV(\Omega;\mathbf{R}^p)$

As mentioned in Section 1, our initial goal was to find the greatest lower bound for

$$\int_\Omega f(x,u_n(x),\nabla u_n(x))dx$$

when $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$ and u takes only two values across a planar interface Σ. We accomplished this in Propositions 3.1 and 4.1 where we showed that

Proposition 5.1 Let f satisfy (H1)-(H4) and let

$$u(x)=\begin{cases} b & \text{if } (x-a_0)\cdot\nu>0 \\ a & \text{if } (x-a_0)\cdot\nu<0 \end{cases} \tag{5.1}$$

$a_0\in\mathbf{R}^N$, $a,\ b\in\mathbf{R}^p$, $\nu\in S^{N-1}$, $\lambda>0$.

(i) If $u_n\to u$ in $L^1(\Omega;\mathbf{R}^p)$ then

$$\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx \geq \int_\Sigma K(x,a,b,\nu)dH_{N-1}(x)$$

where $\Sigma=\{x\in\Omega\colon (x-a_0)\cdot\nu=0\}$;

(ii) There exists $u_n\in W^{1,1}(a_0+\lambda Q_\nu;\mathbf{R}^p)$ such that $u_n\to u$ in $L^1(a_0+\lambda Q_\nu;\mathbf{R}^p)$ and

$$\int_{a_0+\lambda Q_\nu} f(x,u_n(x),\nabla u_n(x))dx \to \int_{a_0+\lambda Q_\nu} f(x,u(x),0)dx$$
$$+\int_{(a_0+\lambda Q_\nu)\cap\Sigma} K(x,a,b,\nu)dH_{N-1}(x).$$

We recall that (see Section 2)

$$K(x,a,b,\nu)=\inf\{\int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))dx\colon \xi\in\mathcal{A}(a,b,\nu)\}.$$

In order to find the specific form of the surface energy density K, we drew our inspiration from the conjecture of I. Fonseca and L. Tartar [FT2] concerning the Γ-limit for a phase transition problem in nonlinear elasticity (see Section 1). Later, we became aware of the work by Ambrosio and Pallara [AP] where they proved

Theorem 5.2 Under the hypotheses (H1) - (H4), the relaxation $\mathcal{F}[\cdot]$ on $BV(\Omega;\mathbf{R}^p)$ admits the integral representation

$$\mathcal{F}[u]=\int_\Omega f(x,u_n(x),\nabla u_n(x))dx+\int_{\Sigma(u)}\gamma_f(x,u^+(x),u^-(x),\nu(x))dH_{N-1}(x)$$
$$\int_\Omega f^\infty(x,u(x),\frac{dC(u)}{d\|C(u)\|}(x))d\|C(u)\|(x),$$

where

$$\gamma_f(x,a,b,\nu) := \inf\{\int_{Q_\nu} f(x,u_n(y),\nabla u_n(y)dy : u_n \to u \quad \text{in } L^1(Q_\nu;\mathbf{R}^p)\}$$
$$- \int_{Q_\nu} f(x,u(y),0)dy,$$

where u is given by (5.1) for $a_0 = 0$. It is now clear that Proposition 5.1 and Theorem 5.2 provide the final integral representation for $\mathcal{F}[u]$.

Theorem 5.3 If f satisfies (H1) - (H4) then

$$\mathcal{F}[u] = \int_\Omega f(x,u(x),\nabla u(x))dx + \int_{\Sigma(u)} K(x,u^+(x),u^-(x),\nu(x))dH_{N-1}(x)$$
$$\int_\Omega f^\infty(x,u(x),\frac{dC(u)}{d\|C(u)\|}(x))d\|C(u)\|(x)$$

Proof. By Theorem 5.2, it suffices to show that

$$\gamma_f(x,a,b,\nu) = K(x,a,b,\nu).$$

Fix $x_0 \in \Omega$, $\epsilon > 0$ and by Theorem 5.2 let $u_n \in W^{1,1}(\epsilon Q_\nu;\mathbf{R}^p)$ be such that

$$\lim_{n\to+\infty} \int_{\epsilon Q_\nu} f(x_0,u_n(y),\nabla u_n(y))dy = \int_{\epsilon Q_\nu} f(x_0,u(y),0)dy + \gamma_f(x_0,a,b,\nu)\epsilon^{N-1}.$$

By Proposition 5.1 (i) we have

$$\epsilon^{N-1}K(x_0,a,b,\nu) \le \lim_{n\to+\infty} \int_{\epsilon Q_\nu} f(x_0,u_n(y),\nabla u_n(y))dy$$
$$= \frac{\epsilon^N}{2} f(x_0,b,0) + \frac{\epsilon^N}{2} f(x_0,a,0) + \epsilon^{N-1}\gamma_f(x_0,a,b,\nu).$$

Dividing the inequality by ϵ^{N-1} and letting $\epsilon \to 0^+$ yields

$$K(x_0,a,b,\nu) \le \gamma_f(x_0,a,b,\nu).$$

On the other hand, by Proposition 5.1 (ii) let $u_n \in W^{1,1}(Q_\nu;\mathbf{R}^p)$ be such that $u_n \to u$ in $L^1(Q_\nu;\mathbf{R}^p)$ and

$$\int_{Q_\nu} f(x_0,u_n(y),\nabla u_n(y))dy \to \int_{Q_\nu} f(x_0,u(y),0)dy + K(x_0,a,b,\nu).$$

By Theorem 5.2

$$\lim_{n\to+\infty} \int_{Q_\nu} f(x_0,u_n(x),\nabla u_n(x))dx \ge \int_{Q_\nu} f(x_0,u(y),0)dy + \gamma_f(x_0,a,b,\nu)$$

and so we conclude that

$$K(x_0, a, b, \nu) \geq \gamma_f(x_0, a, b, \nu).$$

■

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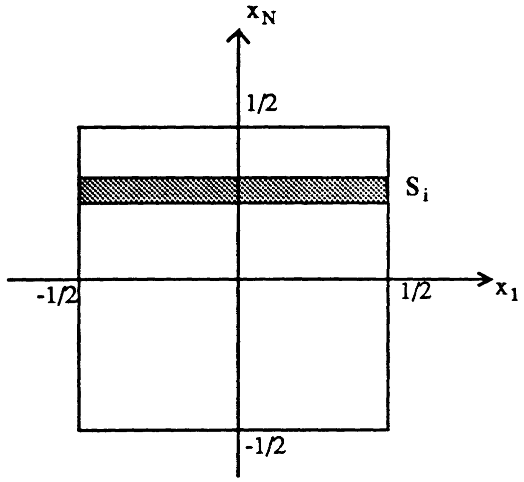


Fig. 1

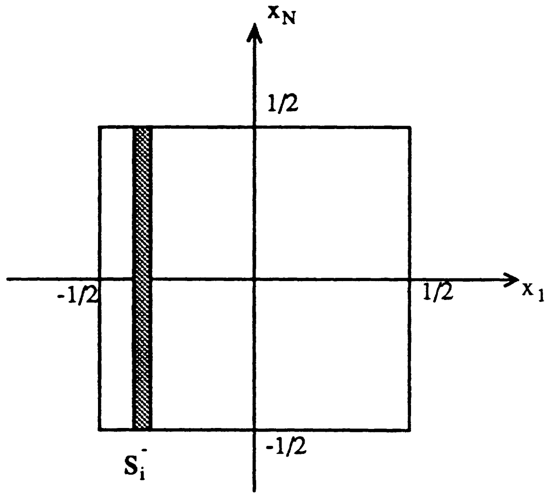
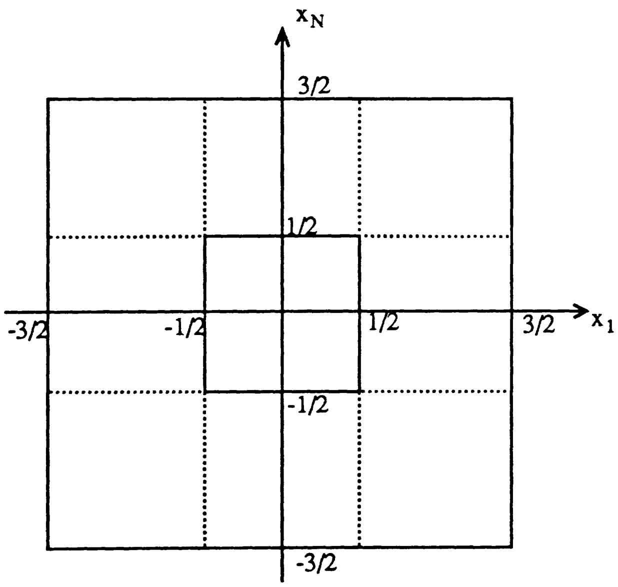


Fig. 2



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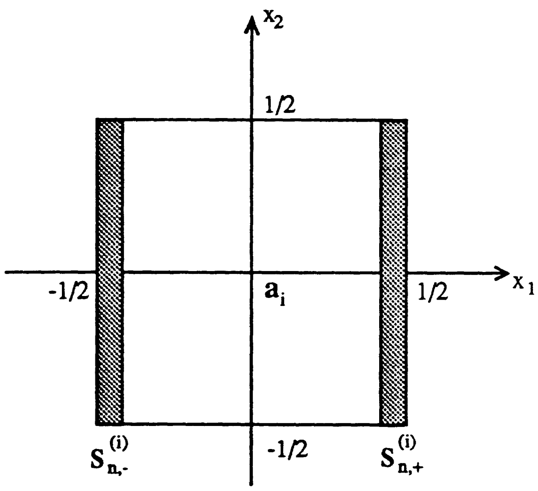

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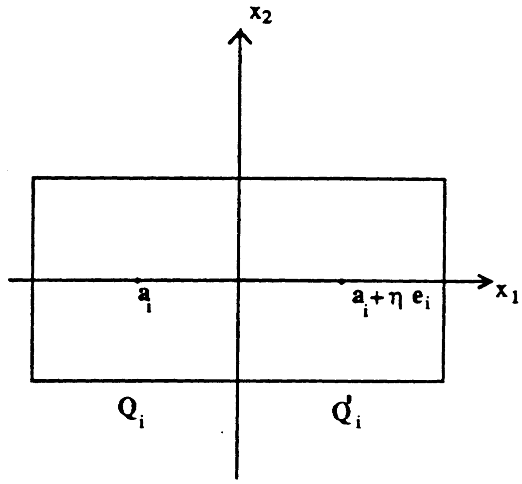

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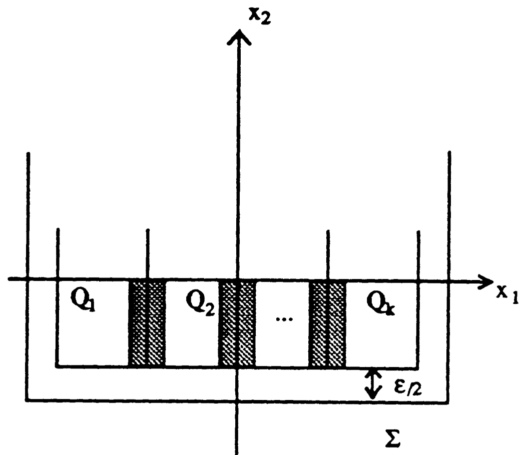

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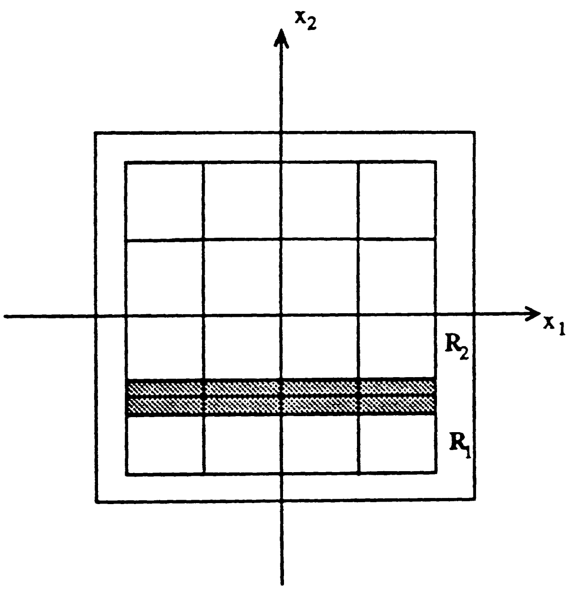

Fig. 7

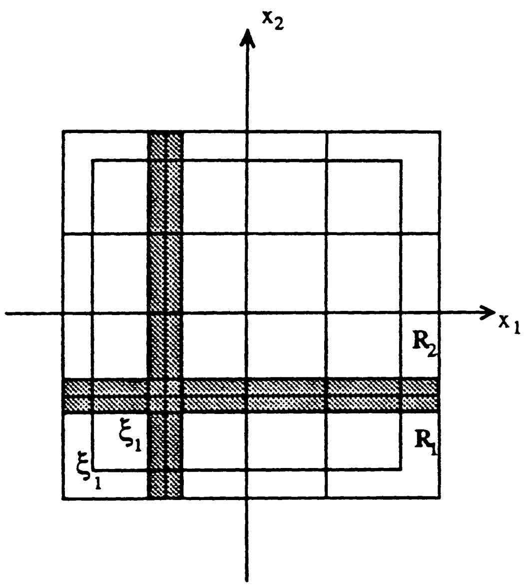

Fig. 8

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A characterization of the surface energy density for the relaxation in BV(Ω;R^p) of the functional

$$u \to \int_\Omega f(x,u(x), \nabla u(x))dx$$

is obtained. A "slicing" technique is used allowing to patch together sequences without increasing their total energy.

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