NAMS
91-11

# PHASE TRANSITIONS AND GENERALIZED MOTION BY MEAN CURVATURE

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Research Report No. 91-103-NAMS-11

February 1991

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**Abstract:** We study the limiting behavior of solutions to appropriately rescaled versions of the Allen–Cahn equation, a model for phase transition in polycrystalline material. We rigourously establish the existence in the limit of a phase–antiphase interface evolving according to mean curvature motion. This assertion is valid for all positive time, the motion interpreted in the generalized sense of Evans–Spruck and Chen–Giga–Goto after the onset of geometric singularities.

* Supported in part by NSF grant DMS–8903328.

** Supported in part by NSF grant DMS–9002249 and NSF–ARO Grant for Nonlinear Analysis and Continuum Mechanics in the Science of Materials.

*** Supported in part by NSF Grants DMS–8801208 and DMS–8657464 (PYI), ARO contract DAAL03–90–G–0012 and the Sloan Foundation.

## 1. Introduction

Allen and Cahn proposed in [AC 1979] the following semilinear parabolic PDE to describe the time evolution of an "order parameter" v determining the phase of a polycrystalline material:

$$v_t - 2\alpha\kappa\,\Delta v + \alpha\, f(v) = 0 \text{ in } R^3 \times (0,\infty). \tag{1.1}$$

Here $\alpha$ is a positive kinetic constant and $\kappa$ is a gradient energy coefficient. The nonlinearity is

$$f \equiv F', \tag{1.2}$$

F denoting the free energy per unit volume. We assume F to be a W–shaped potential, whose two wells, of equal depth, correspond to different stable material phases. The Ginzburg–Landau excess free energy is then

$$\int_{R^3} \kappa|Dv|^2 + F(v)dx,$$

the term $\kappa|Dv|^2$ corresponding to interfaces between stable regions. See Allen–Cahn [AC 1979], Cahn–Hilliard [CH 1958], and Caginalp [C 1988] for more explanation. (The PDE (1.1) is related also to the stochastic Ginzburg–Landau model, an equation for first–order phase transitions: see Gunton, San Miguel, Sahri [GSS 1983, p. 290].)

We are interested in the asymptotics of the Allen–Cahn equation in the limit $\varepsilon \to 0^+$ for

$$\alpha = \frac{1}{\varepsilon^2}, \quad \kappa = \frac{\varepsilon^2}{2}, \quad 2\alpha\kappa = 1. \tag{1.3}$$

This represents a rapid rescaling in time and a simultaneous diminution of the gradient energy term. We consequently expect the solution to converge at each point of $R^3 \times (0,\infty)$ to one of the two minima of F, creating thereby a sharp interface, the "antiphase boundary", between regions of different phases.

An interesting physical and mathematical problem is determining the motion of this antiphase boundary. Allen and Cahn [AC 1979] propose for the general problem (1.1) the motion by mean curvature rule

$$V = 2\alpha\kappa(k_1 + k_2), \tag{1.4}$$

V denoting the velocity of the interface and $k_1,k_2$ its principle curvatures. In his study of two phase continua [Gu 1988a,b], Gurtin has also derived the mean curvature type flow as a model for the motion of the interface, and later Angenent and Gurtin further developed this theory for perfect conductors [AG 1989]. The asymptotic limit (1.4) is also consistent with the stationary results of Modica [M 1987], Fonseca and Tartar [FT 1989], etc.: these authors have shown the Γ–limit of the problem of minimizing the excess free energy is the surface area minimization problem.

Our goal in this paper is a mathematically rigorous verification of the law of motion (1.4) in the asymptotic limit (1.3), for all times $t \geq 0$. This undertaking turns out to be rather subtle mathematically. The big problem is that a surface evolving according to the mean curvature evolution (1.4) can start out smooth and yet later develop singularities. For instance the boundary of a dumbell shaped region will after a time "pinch off": see for instance Grayson [Gr 1989], Sethian [S 1990], etc. From the viewpoint of classical differential geometry it is not so clear if, and how, it may be possible even to define the subsequent evolution of the surface after the onset of singularities.

There have been, to our knowledge, two general proposals for interpreting the mean curvature evolution of surfaces past singularities. K. Brakke [Br 1978] has exploited techniques of geometric measure theory to construct a (generally nonunique) varifold solution.

An alternate approach, initially suggested for numerical calculations by Sethian [S 1985], Osher & Sethian [OS 1988], and, for a first order model of flame propagation, by Barles [B 1985], represents the evolving surface as the level set of an auxiliary function solving an appropriate nonlinear PDE. This latter suggestion has been extensively developed by Evans & Spruck [ES 1989a,b] [ES 1990] and, independently, Chen, Giga and Goto [CGG 1989]. Their analysis made use of the theory of so–called viscosity solutions to fully nonlinear second order elliptic equation, as developed by Crandall & Lions [CL 1983], Crandall, Evans & Lions [CEL 1984], Lions [L 1983a,b], Ishii [I 1989], Jensen [J 1988], Jensen, Lions & Souganidis [JLS 1988], etc. etc. For a general review of the theory as well as an extensive list of references we refer to the *user's guide* by Crandall, Ishii and Lions [CIL 1990]. Recently, Soner [So 1990] has recast the definitions, constructions and uniqueness criterion of [ES 1989a], [CGG 1989] into a different and more intrinsic form using the distance function to the surface: this reformation is an important tool in our analysis below. A general theory for moving fronts using the distance function to the surface is developed in Barles, Soner and Souganidis [BSS 1991].

The level set approach uniquely defines a generalized mean curvature evolution $\{\Gamma_t\}_{t\geq 0}$, starting with a given compact surface $\Gamma_0 \subset R^n$. This flow exists for all time and agrees with the classical smooth differential geometric flow so long as the later exists. The geometric motion may, on the other hand, develop singularities, changing topological type and exhibit various other geometric pathologies.

In spite of these peculiarities the generalized motion $\{\Gamma_t\}_{t\geq 0}$ seems in many ways a strong candidate for being the "right" way to extend the classical motion past singularities. We are consequently led to conjecture that this generalized mean curvature motion governs asymptotic behavior for solutions of the Allen–Cahn equation (1.1) in the limit (1.3). Formal asymptotic expansions suggesting this have been carried out by Fife [F 1989 ], Rubinstein, Sternberg & Keller [RSK 1989], Pego [P 1989], and others. The radial case has been studied by Bronsard & Kohn [BK 1989], and De Mottoni & Schatzman [DS 1989] have given a complete proof for the case of a classical geometric motion. Chen [Ch 1990] has very recently generalized much of this work and given simpler proofs, as has N. Korevaar in unpublished work.

All these arguments require knowledge that the mean curvature flow be smooth, and consequently fail once geometric irregularities appear. The main accomplishment of this work is consequently our verification that the generalized motion $\{\Gamma_t\}_{t\geq 0}$ does indeed determine the antiphase boundary for all positive time, with the one proviso (discussed in §5) that the sets $\{\Gamma_t\}_{t\geq 0}$ do not develop interiors.

This assertion, by the way, provides an independent check on the reasonableness of the level set model of Evans–Spruck and Chen–Giga–Goto. The generalized motion $\{\Gamma_t\}_{t\geq 0}$ can behave in all sorts of odd ways (cf. Evans & Spruck [ES 1989a, §8]) and so it is reassuring to learn $\{\Gamma_t\}_{t\geq 0}$ nevertheless controls asymptotics for the scaled Allen–Cahn equation. Finally, we note that a general weak theory of moving fronts (by using the signed distance function) has been recently formulated by Barles, Soner & Souganidis [BBS 1991]. This theory provides a general framework to the study of front propogation.

We have organized this paper by first providing in §2 a quick review of the level set approach to mean curvature flow, followed by a detailed analysis of the distance function $d$ to the motion. The key assertion is that $d$ is a supersolution solution of the heat equation in the region $\{d > 0\}$, in the weak, that is, viscosity sense. This observation is at the heart of Soner's work [So 1990]. In §3 we build supersolutions of the scaled Allen–Cahn equations out of $d$ and the standing wave solution $q$ of the one–dimensional Allen–Cahn equation. Such change of variable have already been employed by Fife & McLeod [FM 1977], Barles, Bronsard & Souganidis [BBS 1990], Rubinstein, Sternberg & Keller [RSK 1989], etc. Our construction is thus deeply motivated by previous work, the new contribution being various adjustments such as cutting off $d$ near $\Gamma_t$ and adding a small positive term. Finally in §4 we extend the maximum principle to our general setting and prove solutions of the scaled Allen–Cahn equation lie everywhere beneath our supersolutions. An analogous assertion for subsolutions completes the proof.

In §5 we discuss the possibility the sets $\{\Gamma_t\}_{t \geq 0}$ may develop an interior. We do not know whether our assumptions in fact exclude this possibility.

## 2. The Distance function to a generalized motion by mean curvature

In this section we recall the level set construction in Evans–Spruck [ES 1989a,b] and Chen–Giga–Goto [CGG 1989] of a generalized evolution by mean curvature, and then study properties of the distance function to the motion.

Given a compact subset $\Gamma_0 \subset \mathbb{R}^n$, $n \geq 2$, choose a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\Gamma_0 = \{x \in \mathbb{R}^n \mid g(x) = 0\} \tag{2.1}$$

and

$$g \text{ is constant outside some ball.} \tag{2.2}$$

We consider then the *mean curvature evolution PDE*

$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j} & \text{in } \mathbb{R}^n \times (0,\infty) , \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2.3}$$

As explained in [ES 1989a] this PDE asserts on each level set of $u$ evolves according to mean curvature flow, at least in regions where $u$ is smooth and $Du \neq 0$. In addition there exists a unique, continuous weak solution of (2.3). See [ES 1989a], [CGG 1989] for the relevant definitions, proofs, etc. We accordingly *define* the compact sets

$$\Gamma_t \equiv \{x \in \mathbb{R}^n \mid u(x,t) = 0\} \quad (t \geq 0) \tag{2.4}$$

and call $\{\Gamma_t\}_{t \geq 0}$ the *generalized motion by mean curvature* starting from $\Gamma_0$. Consult [ES 1989a, §5], and [CGG 1989, Theorem 7.1] for a proof that the definition (2.4) does not depend on the choice of the particular function g verifying (2.1), (2.2).

Let $t^* = \inf\{t > 0 \mid \Gamma_t = \emptyset\}$ denote the *extinction time*. For each finite time $0 \le t \le t^*$, let us set

$$(2.5) \qquad d(x,t) \equiv \operatorname{dist}(x,\Gamma_t) \qquad (x \in R^n),$$

the distance of x to $\Gamma_t$ in $\mathbb{R}^n$. (Warning: We will later modify this definition, in (2.30)). Notice that the continuity of u implies $\Gamma_{t^*}$ is nonempty, and consequently the distance function is defined at $t^*$. Also, the function d is Lipschitz continuous in the spatial variable, but may be discontinuous in the time t.

First we verify that d is lower semicontinuous and continuous from below. (cf. Lemma 7.3 in Soner [So 1990].)

**Proposition 2.1** (i) *For each* $x \in R^n$ *and* $0 \le t \le t^*$,

$$(2.6) \qquad d(x,t) \le \liminf_{\substack{y \to x \\ s \to t}} d(y,s).$$

(ii) *For each* $x \in R^n$ *and* $0 < t \le t^*$,

$$d(x,t) = \lim_{\substack{y \to x \\ s \uparrow t}} d(y,s).$$

**Proof** 1. Choose $\{y_k\}_{k=1}^{\infty} \subset R^n$, $\{s_k\}_{k=1}^{\infty} \subset [0,t^*]$ so that $y_k \to x$, $s_k \to t$ and

$$d(y_k,s_k) \to \liminf_{\substack{y \to x \\ s \to t}} d(y,s).$$

As $\Gamma_{s_k}$ is compact and nonempty, there exists a point $z_k \in \Gamma_{s_k}$ for which

$$d(y_k,s_k) = \text{dist}(y_k,\Gamma_{s_k}) = |y_k - z_k| \quad (k = 1,2,\ldots)$$

We extract a subsequence $\{z_{k_j}\}_{j=1}^{\infty} \subset \{z_k\}_{k=1}^{\infty}$ and a point $z \in R^n$ so that $z_{k_j} \to z$. As $z_k \in \Gamma_{s_k}$, we have $u(z_k,s_k) = 0$ $(k = 1,\ldots)$; and consequently $u(z,t) = 0$. Thus $z \in \Gamma_t$. Hence

$$\begin{aligned} d(x,t) = \text{dist}(x,\Gamma_t) \leq |x-z| &= \lim_{j\to\infty} |y_{k_j} - z_{k_j}| \\ &= \lim_{j\to\infty} d(y_{k_j},s_{k_j}) \\ &= \liminf_{\substack{y\to x \\ s\to t}} d(y,s). \end{aligned}$$

This proves assertion (i).

2. To verify property (ii) suppose instead $d(x,t) < \limsup_{\substack{y\to x \\ s\uparrow t}} d(y,s)$ and choose $\{y_k\}_{k=1}^{\infty} \subset R^n$, $\{s_k\}_{k=1}^{\infty} \subset [0,t]$ satisfying $y_k \to x$, $s_k \uparrow t$ and $d(y_k,s_k) \to \limsup_{\substack{y\to x \\ s\uparrow t}} d(y,s)$. There exists a number $r \in R$ satisfying

$$(2.7) \qquad d(x,t) < r < d(y_k,s_k)$$

for all sufficiently large $k$, say $k \geq k_0$. In particular

$$(2.8) \qquad B(y_k,r) \subset [R^n \setminus \Gamma_{s_k}] \qquad (k \geq k_0).$$

Now set $B(y_k,r) = \Delta^k_{s_k}$ and let $\{\Delta^k_s\}_{s\geq s_k}$ denote the subsequent evolution of the ball $\Delta^k_{s_k}$ by the mean curvature flow. According to Evans–Spruck [ES 1989a] (2.8) implies $\Delta^k_s \cap \Gamma_s = \emptyset$

for all times $s \geq s_k$. But a direct computation [ES 1989a, §7.1] reveals $\Delta_s^k = B(y_k, r_k(s))$ $(s_k \leq s \leq t)$ for $r_k(s) \equiv (r^2 - 2(n-1)(s-s_k))^{1/2}$. As $\Delta_t^k \cap \Gamma_t = \emptyset$, we deduced $d(y_k,t) \geq r_k(t)$ $(k \geq k_0)$. Now send $k$ to infinity to discover $d(x,t) \geq r$, a contradiction to (2.7).

□

Next we verify that $d$ is a supersolution of the heat equation off the set $\Gamma = \{d = 0\}$. In whatever follows, the sub and supersolutions are interpreted in the "viscosity" sense of Crandall–Lions [CL 1983], Crandall–Evans–Lions [CEL 1984] and Lions [1983].

**Theorem 2.2** *Let $d$ be the distance function, as above. Then*

$$d_t - \Delta d \geq 0 \quad in \quad \{d > 0\} \subset R^n \times (0,t^*). \tag{2.9}$$

**Proof** 1. Fix a test function $\phi \in C^\infty (R^n \times (0,\infty))$ and suppose

$$d - \phi \text{ has a minimum at a point } (x_0,t_0) \in R^n \times (0,t^*) , \tag{2.10}$$

where

$$d(x_0,t_0) > 0. \tag{2.11}$$

We must demonstrate

$$\phi_t - \Delta\phi \geq 0 \text{ at } (x_0,t_0). \tag{2.12}$$

2. Adding if necessary a constant to $\phi$ we may assume

$$\phi(x_0,t_0) = d(x_0,t_0) \equiv \delta > 0. \tag{2.13}$$

Owing to (2.10) and (2.12) we have

$$d(x,t) \geq \phi(x,t) \quad (x \in R^n,\ 0 < t < t^*). \tag{2.14}$$

Choose $z_0 \in \Gamma_{t_0}$ so that

$$d(x_0,t_0) = |x_0 - z_0| = \delta. \tag{2.15}$$

Upon rotating coordinates we may assume

$$x_0 = z_0 + \delta e_n\ , \tag{2.16}$$

where $e_n = (0,\dots,0,1)$. Set

$$\psi(x,t) \equiv \phi(x + x_0 - z_0,t) - \delta \qquad (x \in \mathbb{R}^n,\ t > 0). \tag{2.17}$$

Then

$$\psi(z_0,t_0) = 0. \tag{2.18}$$

3. We now *claim*

$$\{\psi > 0\} \subseteq \{d > 0\}. \tag{2.19}$$

To verify this inclusion select any point $(x,t) \in R^n \times (0,t^*)$ where $\psi(x,t) > 0$. Then (2.14), (2.17) force $d(x + x_0 - z_0,t) \geq \phi(x + x_0 - z_0,t) > \delta$. Now if $d(x,t) = 0$ , then $\delta < d(x + x_0 - z_0,t) - d(x,t) \leq |x_0 - z_0| = \delta$ , a contradiction. Assertion (2.19) is proved.

4. For use later, let us pause to verify

$$D\phi(x_0,t_0) = e_n\ , \tag{2.20}$$

and

(2.21) $$\phi_{x_n x_n}(x_0,t_0) \le 0.$$

Indeed (2.13), (2.14) imply

$$\phi(x,t_0) - \phi(x_0,t_0) \le d(x,t_0) - d(x_0,t_0) \le |x - x_0| \qquad (x \in R^n).$$

Consequently $|D\phi(x_0,t_0)| \le 1$. On the other hand, let us consider next the scalar function $\Phi(s) = \phi(z_0 + se_n,t_0)$ $(s > 0)$. By (2.14) we have

$$\Phi(s) \le d(z_0 + se_n, t_0) \le s\ ,$$

since $z_0 \in \Gamma_{t_0}$. In addition $\Phi(\delta) = d(z_0 + \delta\, e_n,t_0) = d(x_0,t_0) = \delta$. Thus

$$\Phi'(\delta) = 1,\ \ \Phi''(\delta) \le 0\ ;$$

that is, $\phi_{x_n}(x_0,t_0) = 1$, $\phi_{x_n x_n}(x_0,t_0) \le 0$.

5. We return now to the main task at hand, verifying the inequality (2.12). Replacing $u$ by $|u|$ if necessary, we may assume

$$u \ge 0 \ \text{ in } \ R^n \times [0,\infty).$$

(Recall from Evans–Spruck [ES 1989a, §2.4] that $|u|$ is also a solution of the mean curvature evolution PDE.) Thus $\{d > 0\} = \{u > 0\}$ ; whence (2.19) implies

(2.22) $$\{\psi > 0\} \subseteq \{u > 0\}.$$

We next build a continuous function $\Psi : [0,\infty) \to [0,\infty)$ such that

$$\Psi(0) = 0,\ \Psi(z) > 0 \text{ if } z > 0 \tag{2.23}$$

and

$$\psi(z,t) \le \Psi(u(z,t)) \text{ for all } (z,t) \text{ near } (z_0,t_0). \tag{2.24}$$

To carry out this construction, define the compact sets

$$E_k \equiv \{x \in R^n, 0 < t < t^* \mid \psi(x,t) \ge \tfrac{1}{k},\quad |x - x_0| \le 1,\ |t - t_0| \le 1\}$$

for $k = 1, \ldots$. Write $\beta_k = \inf_{E_k} u$. Owing to (2.22) $\beta_1 \ge \ldots \ge \beta_k \ge \beta_{k+1} \ldots > 0$. Furthermore $\lim_{k\to\infty} \beta_k = 0$, since $u(z_0,t_0) = \psi(z_0,t_0) = 0$. Pass to a subsequence $\{\beta_{k_j}\}_{j=1}^{\infty} \subset \{\beta_k\}_{k=1}^{\infty}$ satisfying $\beta_{k_j} > \beta_{k_{j+1}}$ $(j = 1, \ldots)$ and define $\psi : [0,\infty) \to \mathbb{R}$ by

$$\begin{cases} \Psi(\beta_{k_{j+1}}) = \frac{1}{k_j} & (j = 1, \ldots,) \\ \Psi \text{ linear on } [\beta_{k_{j+1}}, \beta_{k_j}]. \end{cases}$$

Then if $(x,t) \in E_{k_{j+1}} \backslash E_{k_j}$,

$$\Psi(u(x,t)) \ge \Psi(\beta_{k_{j+1}}) = \frac{1}{k_j} > \psi(x,t).$$

Thus (2.24) is valid on the set

$$\bigcup_{j=1}^{\infty} E_{k_{j+1}} \backslash E_{k_j} = \{0 < \psi < \tfrac{1}{k_1}\}.$$

Since (2.24) is trivial on $\{\psi \le 0\}$, we deduce (2.24) is valid for all points near $(z_0,t_0)$.

6. Now $\Psi(u)$ is a solution of the mean curvature PDE according to Evans–Spruck [ES 1989a, §2.4]. As (2.18), (2.24) imply

$$\Psi(u) - \psi \text{ has a local minimum at } (z_0,t_0),$$

we have

$$\psi_t - (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2})\, \psi_{x_ix_j} \geq 0 \text{ at } (z_0,t_0).$$

Now

$$\psi_t(z_0,t_0) = \phi_t(x_0,t_0)\ ,\ D\psi(z_0,t_0) = D\phi(x_0,t_0)\ ,\ D^2\psi(z_0,t_0) = D^2\psi(x_0,t_0)\ ,$$

according to (2.17). Thus (2.20), (2.21) force

$$\phi_t - \Delta\phi = \phi_t - (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_ix_j} - \phi_{x_nx_n} \geq 0 \text{ at } (x_0,t_0).$$

This is inequality (2.12). □

Our proof has a geometric interpretation. In view of (2.17), (2.20) the set $\{\psi = 0\}$ is a smooth hypersurface S near $(z_0,t_0)$, and owing to (2.18), (2.19) this (smooth) surface is tangent to the (possibly nonsmooth) set $\Gamma$ at $(z_0,t_0)$.

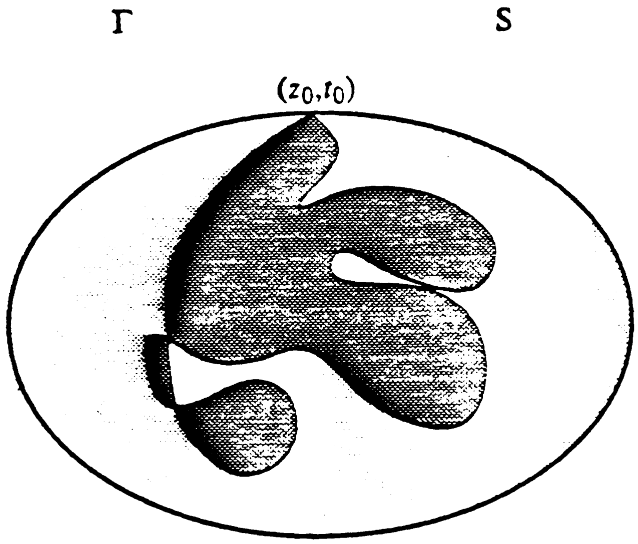


**Figure 1.** Cross-sections of S and Γ at time $t_0$.

It then follows from the definition of a solution for the mean curvature evolution PDE that

$$\psi_t - \left(\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2}\right)\psi_{x_ix_j} \geq 0 \quad \text{at } (z_0,t_0).$$

This means that the velocity of S at $(z_0,t_0)$ is greater than or equal to $(n-1)$ times the mean curvature of S at $(z_0,t_0)$. This interpretation is related to observations in Soner [So 1990, § 14A].

**Remark** In fact d is a supersolution of the heat equation all the way up to time $t^*$. In other words,

$$(2.25) \qquad d_t - \Delta d \geq 0 \ \text{ in } \{d > 0\} \subset R^n \times (0,t^*]$$

To verify this, we assume that for a $\phi$ as above

$$d - \phi \text{ has a minimum at a point } (x_0,t_0)$$

with $t_0 = t^*$ and $d(x_0,t_0) > 0$.

Upon modifying $\phi$ if necessary, we may assume that $d - \phi$ has a strict minimum at $(x_0,t_0)$. Finally, given $\varepsilon > 0$ we write

$$\phi^{\varepsilon}(x,t) \equiv \phi(x,t) + \frac{\varepsilon}{t-t^*} \quad (x \in R^n, 0 < t < t^*).$$

Since $d$ is lower semicontinuous and $\phi^{\varepsilon} = -\infty$ on $\{t = t^*\}$, $d - \phi^{\varepsilon}$ has a minimum at a point $(x_{\varepsilon},t_{\varepsilon}) \in R^n \times (0,t^*)$, with

(2.26) $$x_{\varepsilon} \to x_0 \text{ and } t_{\varepsilon} \to t_0 = t^* \text{ as } \varepsilon \to 0.$$

Since $d(x_0,t_0) > 0$ and $d$ is lower semicontinuous, we have $d(x_{\varepsilon},t_{\varepsilon}) > 0$ for sufficiently small $\varepsilon$. Consequently Theorem 2.2 implies

$$\phi^{\varepsilon}_t - \Delta\phi^{\varepsilon} \geq 0 \text{ at } (x_{\varepsilon},t_{\varepsilon}).$$

Now

$$\phi^{\varepsilon}_t(x,t) = \phi_t(x,t) - \frac{\varepsilon}{(t-t^*)^2} \leq \phi_t(x,t).$$

Thus

$$\phi_t - \Delta\phi \geq 0 \text{ at } (x_{\varepsilon},t_{\varepsilon}).$$

Now let $\varepsilon \to 0$. □

We conclude this section by modifying our notation, as follows. We will henceforth assume $\Gamma_0$ is the boundary of a bounded, open set $U \subset R$, and choose a continuous function $g$

so that

$$(2.27) \qquad g(x)\begin{cases} >0 & \text{if } x \in U \\ =0 & \text{if } x \in \Gamma_0 \\ <0 & \text{if } x \in R^n - \bar{U} \end{cases} .$$

We solve the mean curvature PDE (2.3), and then define

$$(2.28) \qquad I_t \equiv \{x \in R^n \mid u(x,t) > 0\}$$

and

$$(2.29) \qquad O_t \equiv \{x \in R^n \mid u(x,t) < 0\}. \qquad (t \geq 0)$$

In view of (2.4) and (2.27) we may informally regard $I_t$ as the "inside" and $O_t$ as the "outside" of the evolution at time $t$. We also write

$$(2.30) \qquad I \equiv \{(x,t) \in R^n \times (0,\infty) \mid u(x,t) > 0\}$$

and

$$(2.31) \qquad O \equiv \{(x,t) \in R^n \times (0,\infty) \mid u(x,t) < 0\}.$$

Let us now change notation, hereafter writing

$$(2.30) \qquad d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{if } x \in I_t \\ 0 & \text{if } x \in \Gamma_t \\ -\operatorname{dist}(x,\Gamma_t) & \text{if } x \in O_t. \end{cases}$$

for $x \in R^n$, $0 \leq t \leq t^*$. We henceforth call $d$ the *signed distance function.*

We recast Theorem 2.2 into the new notation.

**Theorem 2.3** *Let $d$ be the signed distance function, as above. Then*

$$(2.31) \qquad d_t - \Delta d \geq 0 \quad \textit{in } I \cap \{0 < t \leq t^*\},$$

*and*

(2.32) $$d_t - \Delta d \le 0 \quad \textit{in } \mathrm{O} \cap \{0 < t \le t^*\}\,.$$

In Soner [So 1990] a set valued map $\{C_t\}_{t\geq 0}$ is called a viscosity solution of the mean curvature flow problem if both (2.31) and (2.32) hold. Hence the above theorem establishes a connection between the level set solutions of Evans–Spruck and Chen–Giga–Goto, and that constructed in [So 1990]. In particular, these two definitions coincide if $\partial I_t = \partial O_t$ for all $t \neq t^*$. A more detailed discussion of this point is given in [So 1990, §11].

## 3. Supersolutions

We intend next to utilize the signed distance function $d$ to build sub– and supersolutions of the Allen–Cahn PDE.

For definiteness let us take the free energy per unit volume $F$ to be the quartic

$$F(z) = \frac{1}{2}(z^2 - 1)^2 \quad (z \in R), \tag{3.1}$$

so that

$$f(z) = F'(z) = 2(z^3 - z) \quad (z \in R). \tag{3.2}$$

(Our arguments however are still valid without significant change, if $F$ is any W–shaped potential, whose two wells are of equal depth.) For this free energy the ODE

$$\begin{cases} q''(s) = f(q(s)) \quad (s \in \mathbb{R}) \\ \lim_{s \to \pm\infty} q(s) = \pm 1 \end{cases} \tag{3.3}$$

has an explicit standing wave solution

$$q(s) = \tanh(s) = \frac{e^{2s} - 1}{e^{2s} + 1} \quad (s \in R).$$

We record for later use the equalities

$$\begin{cases} q'(s) = \operatorname{sech}^2(s) = \dfrac{4}{(e^s + e^{-s})^2} \\ q''(s) = -2 \operatorname{sech}^2(s) \tanh(s) \end{cases} \quad (s \in R). \tag{3.4}$$

Next fix $0 < \delta << 1$ and consider a smooth auxiliary function $\eta : R \to R$ satisfying

$$\text{(3.5)} \qquad \begin{cases} \eta(z) = -\delta & (-\infty < z \leq \delta/4) \\ \eta(z) = z - \delta & (z \geq \delta/2) \\ 0 \leq \eta' \leq C, \ |\eta''| \leq \frac{C}{\delta}, \end{cases}$$

where C is a constant, independent of $\delta$.

Suppose in addition $\{\Gamma_t\}_{t\geq 0}$ is a generalized motion by mean curvature, and d is the corresponding signed distance function.

**Lemma 3.1.** *There exists a constant* C, *independent of* $\delta$, *such that*

$$\text{(3.6)} \qquad \eta(d)_t - \Delta\eta(d) \geq -\frac{C}{\delta} \quad \textit{in} \ R^n \times (0,t^*]$$

*and*

$$\text{(3.7)} \qquad \eta(d)_t - \Delta\eta(d) \geq 0 \quad \textit{in} \ \{d > \frac{\delta}{2}\} \subseteq R^n \times (0,t^*]$$

**Proof** 1. Take $\phi \in C^\infty(R^n \times (0,\infty))$ and assume $\eta(d) - \phi$ has a strict minimum at point $(x_0,t_0) \in R^n \times (0,t^*)$.

2. Assume first $d(x_0,t_0) > 0$. Fix $\varepsilon > 0$, write

$$\eta_\varepsilon(z) = \eta(z) + \varepsilon z \qquad (z \in R),$$

and set

$$\rho_\varepsilon \equiv (\eta_\varepsilon)^{-1}.$$

Then $\eta_\varepsilon(d)$ is lower semicontinuous near $(x_0,t_0)$ and thus $\eta_\varepsilon(d) - \phi$ has a minimum at a point $(x_\varepsilon,t_\varepsilon) \in R^n \times (0,t^*)$, with

(3.8) $$x_\varepsilon \to x_0,\ t_\varepsilon \to t_0 \quad \text{as } \varepsilon \to 0.$$

Adding a constant to $\phi$ if necessary we may assume $\eta_\varepsilon(d) - \phi = 0$ at $(x_\varepsilon, t_\varepsilon)$. Thus $\eta_\varepsilon(d) \geq \phi$ and so

(3.9) $$d \geq \rho_\varepsilon(\phi) \equiv \psi^\varepsilon$$

for all $(x,t)$ near $(x_\varepsilon, t_\varepsilon)$, with equality at $(x_\varepsilon, t_\varepsilon)$. Since $d(x_0,t_0) > 0$ and $d$ is lower semicontinuous near $(x_0,t_0)$

$$d(x_\varepsilon, t_\varepsilon) > 0$$

for all small $\varepsilon > 0$. According to (3.9) and Theorem 2.2

$$\psi^\varepsilon_t - \Delta\psi^\varepsilon \geq 0 \ \text{ at } (x_\varepsilon, t_\varepsilon)\ ;$$

that is,

(3.10) $$\rho'_\varepsilon(\phi)(\phi_t - \Delta\phi) - \rho''_\varepsilon(\phi)|D\phi|^2 \geq 0 \ \text{ at } (x_\varepsilon, t_\varepsilon).$$

Now

$$\frac{\rho''_\varepsilon(\phi)}{\rho'_\varepsilon(\phi)} = -\eta''_\varepsilon(\psi^\varepsilon)\rho'_\varepsilon(\phi)^2$$

and so (3.10) yields

(3.11) $$\phi_t - \Delta\phi \geq -\eta''_\varepsilon(\psi^\varepsilon)\,\rho'_\varepsilon(\phi)^2|D\phi|^2 = -\eta''(\psi^\varepsilon)\,|D\psi^\varepsilon|^2 \geq -\frac{C}{\delta}$$

by (3.5) at $(x_\varepsilon,t_\varepsilon)$. We employed in this calculation the bound $|D\psi^\varepsilon| \leq 1$, which follows from (3.9). Sending $\varepsilon \to 0$ we deduce

$$\phi_t - \Delta\phi \geq -\frac{C}{\delta} \text{ at } (x_0,t_0). \tag{3.12}$$

3. Assume next $d(x_0,t_0) \leq 0$. Since d is continuous from below, we have $\eta(d) \equiv -\delta$ on the set $\{|x-x_0| \leq \sigma, t_0 - \sigma \leq t \leq t_0\}$ for some $\sigma > 0$. Thus

$$\phi_t(x_0,t_0) \geq 0, \ D^2\phi(x_0,t_0) \leq 0$$

and so

$$\phi_t - \Delta\phi \geq 0 \quad \text{at } (x_0,t_0). \tag{3.13}$$

4. If $\eta(d) - \phi$ has a minimum at a point $(x_0,t^*)$, we argue using the Remark after Theorem 2.2. Assertion (3.6) is proved.

5. To prove (3.7), suppose $d(x_0,t_0) > \delta/2$. Then for small $\varepsilon > 0$, $d(x_\varepsilon,t_\varepsilon) > \delta/2$. By (3.5) we conclude that $-\eta''(\psi^\varepsilon) = 0$ at $(x_\varepsilon,t_\varepsilon)$. Using this in (3.11), we arrive at (3.7). □

Our intention next is to build using q and d a supersolution of the scaled Allen–Cahn equation. For this let us take constants $\alpha,\beta > 0$ (to be selected later) and write

$$w^\varepsilon(x,t) \equiv q\left(\frac{\eta(d(x,t)) + \alpha t}{\varepsilon}\right) + \varepsilon\beta \quad (x \in R^n, 0 \leq t \leq t^*). \tag{3.14}$$

Since the cut–off function $\eta$ depends on the parameter $\delta$, so does the above function $w^\varepsilon$. However for notational simplicity we suppress this dependence in the notation.

**Theorem 3.2** *There exist* $\alpha = \alpha(\delta) > 0$, $\beta = \beta(\delta) > 0$ *and* $\varepsilon_0 = \varepsilon_0(\delta) > 0$ *such that*

(3.15)
$$w_t^\varepsilon - \Delta w^\varepsilon + \frac{1}{\varepsilon^2} f(w^\varepsilon) \geq 0 \quad \text{in } R^n \times (0,t^*]$$

*for all* $0 < \varepsilon \leq \varepsilon_0$. *In addition* $\alpha,\beta = 0(\delta)$ *as* $\delta \to 0$.

**Proof** 1. As usual choose $\phi \in C^\infty(R^n \times (0,\infty))$ and suppose

(3.16)
$$w^\varepsilon - \phi \text{ has a minimum at } (x_0,t_0) \in R^n \times (0,t^*]$$

with

(3.17)
$$w^\varepsilon - \phi = 0 \text{ at } (x_0,t_0).$$

We must demonstrate

(3.18)
$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq 0 \quad \text{at } (x_0,t_0),$$

provided $\varepsilon$ is sufficiently small, depending only on $\delta$ and not on $\phi$.

2. Write

$$q^{-1}(z) \equiv \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (-1 < z < 1)$$

and set $\psi(x,t) \equiv \varepsilon q^{-1}(\phi(x,t) - \varepsilon\beta)$. This function is defined near $(x_0,t_0)$ since $-1 < \phi(x_0,t_0) - \varepsilon\beta = q(\frac{\eta(d)+\alpha t}{\varepsilon}) < 1$. Owing to (3.14), (3.16), (3.17)

(3.19)
$$\eta(d) - (\psi - \alpha t) \text{ has a minimum at } (x_0,t_0),$$

with $\eta(d) - (\psi - \alpha t) = 0$ at $(x_0,t_0)$.

According to Lemma 3.1 we have

$$\psi_t - \Delta\psi \geq \alpha - \frac{C}{\delta} \quad \text{at } (x_0,t_0)\ , \tag{3.20}$$

and

$$\psi_t - \Delta\psi \geq \alpha \quad \text{at } (x_0,t_0) \ \text{ if } \ d(x_0,t_0) > \frac{\delta}{2}\,. \tag{3.21}$$

3. Since

$$\phi \equiv q(\tfrac{\psi}{\varepsilon}) + \varepsilon\beta\ ,$$

we can compute

$$\begin{aligned}\phi_t - \Delta\phi + \frac{1}{\varepsilon^2}f(\phi) &= \frac{1}{\varepsilon}q'(\tfrac{\psi}{\varepsilon})(\psi_t - \Delta\psi) - \frac{1}{\varepsilon^2}q''(\tfrac{\psi}{\varepsilon})|D\psi|^2 + \frac{1}{\varepsilon^2}f(q(\tfrac{\psi}{\varepsilon}) + \varepsilon\beta) \\ &= \frac{1}{\varepsilon}q'(\tfrac{\psi}{\varepsilon})(\psi_t - \Delta\psi) + \frac{1}{\varepsilon^2}q''(\tfrac{\psi}{\varepsilon})(1 - |D\psi|^2) + \frac{1}{\varepsilon^2}[f(q(\tfrac{\psi}{\varepsilon}) + \varepsilon\beta) - f(q(\tfrac{\psi}{\varepsilon}))]\end{aligned} \tag{3.22}$$

at the point $(x_0,t_0)$, where we utilized the ODE (3.3) to derive the last equality.

We now must estimate the various terms in (3.22).

**Case 1** $d(x_0,t_0) > \frac{\delta}{2}$.

In this situation $d > \frac{\delta}{2}$ near $(x_0,t_0)$ and so $\eta(d) = d - \delta$ near $(x_0,t_0)$. Then (3.19) implies

$$|D\psi(x_0,t_0)| = 1.$$

Thus (3.21) and (3.22) yield

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq q'(\tfrac{\psi}{\varepsilon}) \frac{\alpha}{\varepsilon} + \frac{f'(q(\frac{\psi}{\varepsilon}))\varepsilon\beta + O(\varepsilon^2)}{\varepsilon^2}$$

(3.23)

$$= \frac{1}{\varepsilon} [q'(\tfrac{\psi}{\varepsilon})\, \alpha + f'(q(\tfrac{\psi}{\varepsilon}))\beta] + O(1).$$

Fix $0 < \gamma < 1$ so that

$$\inf_{\gamma \leq |z| \leq 1} f'(z) \equiv a_1 > 0.$$

Then set

$$\inf_{|q(s)| \leq \gamma} q'(s) \equiv a_2 > 0,$$

define

(3.24) $$\alpha = \frac{\delta}{4t^*}, \quad \beta = a_2\alpha[2\|f'\|_{L^\infty((-1,1))}]^{-1}.$$

We consider two further possibilities:

**Subcase 1** $|q(\frac{\psi}{\varepsilon})| \geq \gamma$.

Then (3.23) implies

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq \frac{a_1\beta}{\varepsilon} + O(1) \geq 0 \text{ at } (x_0,t_0)$$

if $\varepsilon$ is small enough, depending only on $\delta$.

**Subcase 2** $|q(\frac{\psi}{\varepsilon})| \le \gamma$.

Then (3.23) implies

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq \frac{1}{\varepsilon}[a_2\alpha - \|f'\|_{L^\infty}\beta] + O(1)$$

$$= \frac{a_2\alpha}{2\varepsilon} + O(1) \geq 0 \text{ at } (x_0,t_0)$$

for small $\varepsilon$, depending on $\delta$.

Both subcases therefore yield (3.18).

**Case 2** $d(x_0,t_0) \le \frac{\delta}{2}$.

We use the same choices of $\alpha$ and $\beta$ as in the previous case. In this situation $\eta(d) \le -\frac{\delta}{2}$ and so

$$\eta(d) + \alpha t_0 \leq -\frac{\delta}{2} + \alpha t^* \leq -\frac{\delta}{4},$$

according to (3.24). Hence (3.19) yields the inequality

$$\psi \le -\frac{\delta}{4} \quad \text{at} \quad (x_0,t_0). \tag{3.25}$$

Statement (3.19) and the definition (3.5) of $\eta$ imply also $|D\psi| \le C$ at $(x_0,t_0)$.

We then compute utilizing (3.20), (3.22)

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \ge \frac{1}{\varepsilon}[q'(\tfrac{\psi}{\varepsilon})\alpha + f'(q(\tfrac{\psi}{\varepsilon}))\beta] + O(1) - \frac{C}{\varepsilon\delta} q'(\tfrac{\psi}{\varepsilon}) - \frac{C}{\varepsilon^2}|q''(\tfrac{\psi}{\varepsilon})|. \tag{3.26}$$

But since $q'' \geq 0$ on $(-\infty,0]$, (3.25) and (3.4) force

$$\frac{C}{\varepsilon\delta} q'(\frac{\psi}{\varepsilon}) \leq \frac{C}{\varepsilon\delta} q'(-\frac{\delta}{4\varepsilon}) \leq \frac{C}{\varepsilon\delta} e^{-\frac{\delta}{2\varepsilon}} = o(1) \text{ as } \varepsilon \to 0.$$

Similarly

$$\frac{C}{\varepsilon^2} |q''(\frac{\psi}{\varepsilon})| \leq \frac{C}{\varepsilon^2} e^{-\delta/2\varepsilon} = o(1) \text{ as } \varepsilon \to 0.$$

We analyze the remaining terms on the right hand side of (3.26) as in the two subcases of Case 1.

The conclusion is

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq 0 \quad \text{at } (x_0,t_0)$$

for all $0 < \varepsilon \leq \varepsilon_0(\delta)$, $\varepsilon_0(\delta)$ sufficiently small. As the constant appearing in the above argument is independent of $\phi$, the choice of $\varepsilon_0(\delta)$ does not depend on $\phi$.

□

## 4. Asymptotics for the Allen–Cahn equation

We at last turn to the scaled Allen–Cahn equation

$$(4.1)_\varepsilon \qquad \begin{cases} v^\varepsilon_t - \Delta v^\varepsilon + \dfrac{1}{\varepsilon^2} f(v^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v^\varepsilon = h^\varepsilon & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

the cubic $f$ given by (3.2) and the initial function $h^\varepsilon$ described below.

We intend to prove $v^\varepsilon \to 1$ in a region $I \subset R^n \times [0,\infty)$, $v^\varepsilon \to -1$ in another region $O \subset R^n \times [0,\infty)$, the "interface" $\Gamma$ between I (the "inside") and O (the "outside") being a generalized motion governed by mean curvature.

To induce this behavior, however, we must choose appropriate initial functions. More specifically, let $\Gamma_0$ henceforth denote the smooth boundary of a bounded, connected open set $U \subset R^n$. Let $d$ be the signed distance function to $\Gamma_0$, and set

$$(4.2) \qquad h^\varepsilon(x) \equiv q\left(\frac{d(x)}{\varepsilon}\right) \qquad (x \in R^n).$$

Thus $h^\varepsilon$ is approximately equal to 1 within U, is approximately equal to $-1$ within $R^n\backslash\bar{U}$, and has a transition layer of width approximately $\varepsilon$ across the surface $\Gamma_0$. Moreover, by the maximum principle, $-1 < v_\varepsilon < 1$ in $R^n \times [0,\infty)$.

We will show that $v^\varepsilon$ roughly maintains this form at later times, the transition layer forming across the generalized motion by mean curvature starting with $\Gamma_0$. To this end, we choose a continuous function $g : R^n \to R$ satisfying (2.27), solve the mean curvature evolution PDE (2.3), and define $\Gamma_t, I_t, O_t, I, O$ by (2.4), (2.28)–(2.31).

**Theorem 4.1** *We have*

$$(4.3)' \qquad v^{\varepsilon} \to 1 \text{ uniformly on compact subsets of } I$$

*and*

$$(4.4) \qquad v^{\varepsilon} \to -1 \text{ uniformly on compact subsets of } O.$$

**Remark** Assertions (4.3), (4.4) provide a great deal, but by no means all, of the desired information about the limiting behavior of the $\{v^{\varepsilon}\}_{\varepsilon>0}$. We note in particular it *is not known whether the "interface" Γ can develop an interior*: see the discussion following in §5.

**Proof** 1. As $\Gamma_0$ is smooth we may choose $g$ to be smooth, with $|Dg| = 1$ near $\Gamma_0$. Thus if $\delta > 0$ is small enough the set

$$(4.5) \qquad \Gamma_0^{\delta} \equiv \{x \in R^n \mid g(x) = d_0(x) = -2\delta\}$$

is smooth. We let

$$(4.6) \qquad \Gamma_t^{\delta} = \{x \in R^n \mid u(x,t) = -2\delta\} \quad (t \geq 0)$$

be the generalized evolution starting with $\Gamma_0^{\delta}$, and take $d^{\delta}$ to denote the signed distance function to $\Gamma_t^{\delta}$, $d_0^{\delta}$ being the signed distance function to $\Gamma_0^{\delta}$. Let $t_{\delta}^{*}$ be the extinction time for $\{\Gamma_t^{\delta}\}_{t\geq 0}$.

Choose $\eta(\cdot)$ as in §3 and set

$$(4.7) \qquad w^{\varepsilon,\delta}(x,t) \equiv q\left(\frac{\eta(d^{\delta}(x,t)) + \alpha t}{\varepsilon}\right) + \varepsilon\beta\,,$$

$\alpha$ and $\beta$ are given by (3.24), with $t^*_\delta$ replacing $t^*$. Then for $0 < \varepsilon < \varepsilon_0(\delta)$ we have

$$(4.8) \qquad w_t^{\varepsilon,\delta} - \Delta w^{\varepsilon,\delta} + \frac{1}{\varepsilon^2} f(w^{\varepsilon,\delta}) \geq 0 \text{ in } R^n \times (0,t^*_\delta) .$$

2. We first claim

$$(4.9) \qquad w^{\varepsilon,\delta}(x,0) \geq h^\varepsilon(x) \qquad (x \in R^n).$$

To verify this inequality it suffices in view of (4.2) to prove

$$\eta(d_0^\delta(x)) \geq d(x) \qquad (x \in R^n).$$

Now owing to (4.5) $d_0^\delta(x) \geq d(x) + 2\delta$ ; and so $\eta(d_0^\delta(x)) \geq \eta(d(x) + 2\delta)$ $(x \in R^n)$. It is therefore enough to show

$$(4.10) \qquad d(x) \leq \eta(d(x) + 2\delta) \qquad (x \in R^n).$$

But if $d(x) \geq -\frac{3\delta}{2}$, then $d(x) + 2\delta \geq \frac{\delta}{2}$; whence

$$\eta(d(x) + 2\delta) = d(x) + \delta \geq d(x).$$

On the other hand, if $d(x) \leq -\frac{3}{2}\delta$, (4.10) is obvious as $\eta \geq -\delta$.

3. Now write

$$(4.11) \qquad w \equiv e^{-\lambda t} w^{\varepsilon,\delta} \qquad (\lambda > 0).$$

We next claim

$$(4.12) \qquad w_t - \Delta w + \lambda w + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t} w) \geq 0 \text{ in } R^n \times (0,t_\delta^*] .$$

To check this, select as always $\phi \in C^\infty(R^n \times (0,\infty))$ and assume

$$w - \phi \text{ has a minimum at a point } (x_0,t_0) \in \mathbb{R}^n \times (0,t_\delta^*]$$

with $w - \phi = 0$ at $(x_0,t_0)$. Then

$$e^{-\lambda t} w^{\varepsilon,\delta} = w \geq \phi \text{ in } R^n \times (0,t_\delta^*],$$

with equality at $(x_0,t_0)$. Hence

$$w^{\varepsilon,\delta} \geq \psi \text{ in } R^n \times (0,t_\delta^*],$$

with equality at $(x_0,t_0)$, for $\psi \equiv e^{\lambda t}\phi$. Assertion (4.8) then implies

$$\psi_t - \Delta \psi + \frac{1}{\varepsilon^2} f(\psi) \geq 0 \text{ at } (x_0,t_0).$$

We rewrite the last inequality to read

$$\phi_t - \Delta \phi + \lambda \phi + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t}\phi) \geq 0 \text{ at } (x_0,t_0).$$

This establishes (4.12).

4. We hereafter set

$$\lambda = \lambda_\varepsilon \equiv \frac{2\|f'\|_{L^\infty((-1,1))}}{\varepsilon^2} .$$

Then for each $t$ the mapping

$$z \mapsto \lambda z + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t} z) \text{ is strictly increasing.} \tag{4.13}$$

5. We now assert

$$w^{\varepsilon,\delta} \geq v^\varepsilon \quad \text{in } R^n \times [0,t^*_\delta]. \tag{4.14}$$

Indeed if not, then

$$w^{\varepsilon,\delta} < v^\varepsilon \quad \text{somewhere in } R^n \times [0,t^*_\delta]$$

and consequently

$$w < v \text{ somewhere in } R^n \times [0,t^*_\delta],$$

for $w = e^{-\lambda t} w^{\varepsilon,\delta}$, $v = e^{-\lambda t} v^\varepsilon$. The function $w$ is lower semicontinuous. In addition

$$w \geq v \text{ on } R^n \times [t = 0],$$

and

$$\lim_{|x|\to\infty} w \geq e^{-\lambda t}(-1 + \varepsilon\beta) > \lim_{|x|\to\infty} v = -e^{-\lambda t}.$$

Hence there exists a point $(x_0,t_0) \in R^n \times (0,t^*_\delta]$ such that

$$(4.15)\qquad (w-v)(x_0,t_0) = \min_{R^n\times[0,t^*_\delta]} (w-v) \equiv b < 0.$$

Now (4.14) and $(4.1)_\varepsilon$ yield

$$(4.16)\qquad v_t - \Delta v + \lambda v + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t}v) = 0 \text{ in } R^n \times (0,t^*_\delta].$$

If

$$(4.17)\qquad \phi \equiv v + b,$$

then $\phi \in C^\infty(R^n \times [0,\infty))$ and (4.15) says

$$w - \phi \text{ has a minimum at } (x_0,t_0)$$

with $w - \phi = 0$ at $(x_0,t_0)$. According to step 3 above, we conclude

$$(4.18)\qquad \phi_t - \Delta\phi + \lambda\phi + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t}\phi) \geq 0$$

at $(x_0,t_0)$. However since $b < 0$, $\phi < v$. Consequently (4.13), (4.17), (4.18) imply

$$v_t - \Delta v + \lambda v + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t}v) > 0$$

at $(x_0,t_0)$. This contradicts (4.16) and thereby proves (4.14).

6. Utilizing (4.14) and the definition (4.7) of the auxiliary function $w^{\varepsilon,\delta}$, we discover

$$(4.19)\qquad q\left(\frac{\eta(d^\delta(x,t)) + \alpha t}{\varepsilon}\right) + \varepsilon\beta \geq v^\varepsilon(x,t)$$

for $x \in R^n$, $0 \leq t \leq t^*_\delta$. Now if

$$x \in O^\delta_t \equiv \{x \in R^n \mid u(x,t) < -2\delta\}, \;\; 0 \leq t \leq t^*_\delta ,$$

we have $d^\delta(x,t) < 0$ and so

$$\begin{aligned} \eta(d^\delta(x,t)) + \alpha t &\leq -\delta + \alpha t^*_\delta \\ &\leq -\tfrac{3}{4}\delta \qquad \text{by (3.24) (with } t^*_\delta \text{ replacing } t^*). \end{aligned}$$

Thus

$$\lim_{\varepsilon \to 0} q\left(\frac{\eta(d^\delta(x,t)) + \alpha t}{\varepsilon}\right) + \varepsilon\beta = -1.$$

In view of (4.19) we have

$$\lim_{\varepsilon \to 0} v^\varepsilon(x,t) = -1 ,$$

uniformly on $O^\delta \equiv \{(x,t) \in R^n \times [0,t^*_\delta] \mid u(x,t) < -2\delta\}$. In particular,

(4.20) $$\lim_{\varepsilon \to 0} v^\varepsilon(x,t) = -1$$

uniformly on compact subsets of $O^\delta$ for sufficiently small $\delta$. For large $\delta > 0$, a minor modification of the above proof yields (4.20). Since

$$O = \bigcup_{\delta > 0} O^\delta,$$

the proof of (4.4) is now complete. A similar argument proves (4.3).

□

**5. Uniqueness?**

In this concluding section we elaborate upon the remark following Theorem 4.1. Let us return to the scaled Allen–Cahn PDE and calculate the time derivative of the scaled excess free energy:

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{\varepsilon}{2}|Dv^\varepsilon|^2 + \frac{1}{\varepsilon}F(v^\varepsilon)dx = \int_{\mathbb{R}^n} \varepsilon Dv^\varepsilon\cdot Dv_t^\varepsilon + \frac{1}{\varepsilon}f(v^\varepsilon)v_t^\varepsilon\, dx$$

$$= \int_{\mathbb{R}^n} v_t^\varepsilon(-\varepsilon\Delta v^\varepsilon + \frac{1}{\varepsilon}f(v^\varepsilon))dx$$

$$= -\varepsilon\int_{\mathbb{R}^n} (v_t^\varepsilon)^2\, dx \leq 0.$$

Thus

$$\text{(5.1)} \qquad \sup_{0\leq t\leq T}\int_{\mathbb{R}^n} \frac{\varepsilon}{2}|Dv^\varepsilon|^2 + \frac{1}{\varepsilon}F(v^\varepsilon)dx + \varepsilon\int_0^T\int_{\mathbb{R}^n}(v_t^\varepsilon)^2 dxdt \leq \int_{\mathbb{R}^n}\frac{\varepsilon}{2}|Dh^\varepsilon| + \frac{1}{\varepsilon}F(h^\varepsilon)dx$$

$$\leq C < \infty,$$

in view of the special form (4.2) for the initial function $h^\varepsilon$. Since this inequality implies

$$\int_0^T\int_{\mathbb{R}^n} F(v^\varepsilon)dx \leq O(\varepsilon)$$

as $\varepsilon \to 0$ for each $T > 0$, we deduce

$$\text{(5.2)} \qquad (v^\varepsilon)^2 \to 1 \text{ a.e. in } \mathbb{R}^n \times [0,\infty).$$

In addition if we set $G(z) \equiv \frac{z^3}{3} - z$ and write

$$\tilde{v}^{\varepsilon} = G(v^{\varepsilon}) ,$$

we have (cf. Bronsard–Kohn [BK 1989])

$$\int_{R^n} |D\tilde{v}^{\varepsilon}| \, dx = \int_{R^n} |(v^{\varepsilon})^2 - 1| \, |Dv^{\varepsilon}| \, dx$$

$$\leq \int_{R^n} \frac{\varepsilon}{2} |Dv^{\varepsilon}|^2 + \frac{F(v^{\varepsilon})}{\varepsilon} dx \leq C < \infty$$

and

$$\int_0^T \int_{R^n} |\tilde{v}_t^{\varepsilon}| \, dx = \int_0^T \int_{R^n} |(v^{\varepsilon})^2 - 1| \, |v_t^{\varepsilon}| \, dxdt$$

$$\leq \int_0^T \int_{R^n} \varepsilon(v_t^{\varepsilon})^2 + \frac{1}{2\varepsilon} F(v^{\varepsilon}) dxdt \leq C < \infty.$$

Thus $\{\tilde{v}^{\varepsilon}\}_{\varepsilon>0}$ is bounded in $BV(R^n \times (0,T))$ for each $T > 0$, and so is precompact in $L^1_{loc}(R^n \times (0,T))$. It follows that $\{v^{\varepsilon}\}_{\varepsilon>0}$ is precompact in $L^1_{loc}(R^n \times (0,T))$. Consequently, passing if necessary to a subsequence we have

(5.3) $$v^{\varepsilon_j} \to \pm 1 \text{ in } R^n \times [0,\infty).$$

Our Theorem 4.1 augments this simple fact with the assertion

(5.4) $$v^{\varepsilon} \to 1 \text{ in } I, \; v^{\varepsilon} \to -1 \text{ in } O.$$

However *we do not know*

$$\Gamma \equiv R^n \times [0,\infty)\backslash[I \cup 0]$$

has (n+1)–dimensional Lebesgue measure zero, and consequently (5.4) does *not* imply (5.2), (5.3). The problem is that the sets $\{\Gamma_t\}_{t\geq 0}$ could conceivably develop an interior for times $t^* \geq t \geq t_*$, $t_*$ denoting the first time the classical evolution by mean curvature has a singularity. See [ES 1989a, §8] for an example of a nonsmooth 1–dimensional compact set $\Gamma_0 \subset R^2$ for which $\Gamma_t$ has an interior for times $t > 0$.

On the other hand Evans–Spruck [ES 1990 ] have recently proved for smooth $\Gamma_0$ that

$$H^{n-1}(\Gamma_t^*) < \infty \quad (t \geq 0) ,$$

where $H^{n-1}$ is (n–1)–dimensional Hausdorff measure and $\Gamma_t^* = \partial\Gamma_t$. Thus $\Gamma_t$ has positive n–dimensional Lebesgue measure if and only if $\Gamma_t$ has an interior. Finally, [BSS 1991] gives a general but by no means sharp geometric condition which guarantees no interior. This condition is used by Soner–Souganidis [So Sou 1991] to show that rotationally symmetric surfaces which look like the torus, do not develop interior.

Now if in fact $\text{int}(\Gamma_t) \neq \emptyset$ in $R^n$ for some time $t_* \leq t \leq t^*$, then $\text{int}(\Gamma) \neq \emptyset$ in $R^n \times [0,\infty)$. In this case assertion (5.3) tells us that for some subsequence $v^{\varepsilon_j} \to \pm 1$ a.e. within $\Gamma$, whereas (5.4) provides no information at all regarding $v^\varepsilon$ inside $\Gamma$.

Should this be possible, it seems most likely that the regions when $v^{\varepsilon_j} \to 1$ and $v^{\varepsilon_j} \to -1$ would be separated by an "interface" evolving by mean curvature in the sense of Soner [So 1990]. Such a motion is generally nonunique. And perhaps different subsequences correspond to different interfaces, or the initial profile picks the particular interface to which the solutions convergence. At present it is unclear whether these circumstances can arise and, if so, how the solutions $v^\varepsilon$ of the scaled Allen–Cahn equation would behave within the interior of $\Gamma$.

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# REPORT DOCUMENTATION PAGE

*Form Approved*
*OMB No. 0704-0188*



**1. AGENCY USE ONLY** *(Leave blank)*

**2. REPORT DATE**
February 1991

**3. REPORT TYPE AND DATES COVERED**

**4. TITLE AND SUBTITLE**
Phase transitions and generalized motion by mean curvature

**5. FUNDING NUMBERS**

**6. AUTHOR(S)**
L.C. Evans, H.M. Soner and P.E. Souganidis

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA 15213

**8. PERFORMING ORGANIZATION REPORT NUMBER**
NAMS-11

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

**10. SPONSORING/MONITORING AGENCY REPORT NUMBER**

**11. SUPPLEMENTARY NOTES**
The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

**12a. DISTRIBUTION/AVAILABILITY STATEMENT**
Approved for public release; distribution unlimited.

**12b. DISTRIBUTION CODE**

**13. ABSTRACT** *(Maximum 200 words)*

We study the limiting behavior of solutions to appropriately rescaled versions of the Allen-Cahn equation, a model for phase transition in polycrystalline material. We rigourously establish the existence in the limit of a phase-antiphase interface evolving according to mean curvature motion. This assertion is valid for all positive time, the motion interpreted in the generalized sense of Evans-Spruck and Chen-Giga-Goto after the onset of geometric singularities.

**14. SUBJECT TERMS**
phase transitions, mean curvature flow, viscosity solutions

**15. NUMBER OF PAGES**
39

**16. PRICE CODE**

| 17. SECURITY CLASSIFICATION OF REPORT | 18. SECURITY CLASSIFICATION OF THIS PAGE | 19. SECURITY CLASSIFICATION OF ABSTRACT | 20. LIMITATION OF ABSTRACT |
|---|---|---|---|
| UNCLASSIFIED | UNCLASSIFIED | UNCLASSIFIED | UL |