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by

Bernard Dacorogna
Department of Mathematics
Ecole Polytechnique Federale de Lausanne
1015 Lausanne, Switzerland

and

Irene Fonseca
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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BERNARD DACOROGNA
Department of Mathematics
Ecole Polytechnique Federate de Lausanne
1015 Lausanne, Switzerland

and

IRENE FONSECA
Department of Mathematics, Carnegie Mellon University
Pittsburgh, Pennsylvania 15213, U. S. A.

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1. INTRODUCTION.

DAVINI [4] and DAVINI & **PARRY** [5, 6] introduced a model for slightly defective crystals where non-elastic defect-preserving deformations are called *neutral* and generally they involve some kind of rearrangement representing the slip mechanisms of the classic phenomenological plasticity theories. Neutral deformations can be factorized into components which are exclusively elastic at the macroscopic level or exclusively slip at the microscopic level. Essentially, a neutral change of state of a perfect crystal corresponds to a lattice matrix

L(u(x)) = Vu(x){Vv(x)}-i,

where u is the elastic deformation of the reference configuration *Q Q* [R^3 into u(Q) and v represents the slip or plastic deformation with det Vv = 1 a. e. in *Cl.*

Taking the viewpoint that equilibria correspond to some kind of variational principle, in DAVINI & PARRY [4, 5] and in FONSECA & PARRY [10] the implications of including in the class of admissible variations the neutral changes of state were analyzed. Precisely, FONSECA & PARRY [10] considered the minimization of the total stored energy functional

E(u, v) := $\int_\Omega$ w(Vu(x){Vv(x)}-^1) dx (1.1)

where W represents the strain energy density in the class of admissible pairs

& := {(u, v) G W^^DR^W ^1 * - ^^ ^3) ! u = uo on 3Q, det Vu > 0 and det Vv = 1 a. e. in Q}.

Of course, *S&* includes the elastic deformations in the case where v is the identity map. Formally, minimizing E(.,.) in *s&* involves variations of the reference domain ; indeed, setting co := u (v^1) the integral (1.1) becomes

$\int_{v(\Omega)}$ W(Vco(y))dy.

As it is well known, the bulk energy W for ordered materials is not quasiconvex (see ERICKSEN [7], , FONSECA [8], KINDERLEHRER [11]) and so, the functional E(.,.) is not lower semicontinuous. Hence, we cannot use the direct methods of the calculus of variations in order to obtain existence of minimizers of the energy and in general, such minimizers exist only in the generalized sense. Using the parametrized probability measures of YOUNG and the theory of compensated compactness of MURAT & TARTAR (see TARTAR [12]), FONSECA & PARRY [10] examined the behavior of minimizing sequences for defective crystals and their state functions.

In this paper we study the existence and regularity properties for minimizers of (1.1) where W satisfies some convexity assumption. It should be pointed out immediatly that the direct methods of the calculus of variations fail to apply to this problem. Indeed, sequential weak lower semicontinuous of E(.,.) (see Propositions 3.8 and 3.10) is not sufficient to ensure existence of minimizers. Precisely, setting

$$W(X) = \|X\|^r$$

where $\|X\|^2 := \sum_{i,j=1}^{N} X_{ij}^2$, we shall establish that minimizers exist if and only if $r \geq N$ (See Theorem 2.2 and Proposition 4.1). This is in sharp contrast with the usual Dirichlet problem of minimizing

$$\inf\{ \int_\Omega \|\nabla u(x)\|^r \, dx \mid u = u_0 \text{ on } \partial\Omega, \, u \in W^{1,r}(\Omega)\}$$

which has solutions for every $r > 1$. Surprisingly the problem behaves in fact very similarly to

$$\text{(Q)} \qquad \inf\{ \int_\Omega |\det \nabla u(x)|^{r/N} \, dx \mid u = u_0 \text{ on } \partial\Omega\}$$

(cf. Corollary 2.5 and Proposition 4.3). This is in agreement with the continuum theory for elastic crystals where it can be shown that, due to the crystallographic material symmetries, the relaxation of the bulk energy depends only on the determinant of the deformation gradient (see ERICKSEN [7], FONSECA [8], KINDERLEHRER [11]).

Another interesting feature of this problem is that, under some convexity-type hypotheses on W satisfied by $W(X) = \|X\|^r$, $r \geq N$, there are solutions (u^*, v^*) verifying

$$\nabla u^*(x)\{\nabla v^*(x)\}^{-1} = X_0 \quad \text{for every } x \in \Omega,$$

where X_0 is a constant matrix. In the case where $W(X) = \|X\|^r$, $r \geq N$, it turns out that $X_0 = \lambda R$ where R is an orthogonal transformation and $\lambda^N = \text{meas } u_0(\Omega) / \text{meas } \Omega$.

2. THE CASE $W(X) = ||X||^r$.

Although the results obtained in this section are srtictly included on the next, we present them beforehand for the sake of clarity. We start by introducing some notations.

Notations : i) M^{NxN} denotes the set of NxN matrices and if X € M^{NxN} then adj X denotes the matrix of cofactors. In particular, if A is invertible then

$$A^{-1} = \frac{(\mathrm{adj}X)^T}{\det X}$$

and (2.1)

$$<X, \mathrm{adj}\ X> = N \det X$$

where

$$<X, Y> := \sum_{i,j=1}^{N} X_{ij} Y_{ij}.$$

ii) Let Q C [R^N be a bounded, open set with strongly Lipschitz boundary 3Q. If $k \geq 1$ is an integer and if $0 \leq a \leq 1$ then by $\mathrm{Diff}^{k,a}(\overline{i2})$ we mean the set of diffeomorphisms $u : \overline{\Omega} \to \overline{u(Q)}$ such that u, u^{-1} € $C^{k,a}(\overline{Q}, \overline{u(Q)})$, where $C^{k,a}$ stands for the usual set of Hölder continuous functions. In the case k = ~ we shall write $\mathrm{Diff}^{\infty}(\overline{Q})$.

iii) With the above notations, if uo e $\mathrm{Diff}^{k,a}(\overline{Q})$ with det Vuo > 0 in $\overline{Q}$ is given we let

£^k,a := {(u, v) G Diff^$\overline{Q}$) x Diffc^a($\overline{Q}$) I u = uo on dQ and det Vv(x) = 1 in *£1}*

and in the case k = «> we write simply £#oo. Finally, for $r \geq 1$ consider the problem

(P) inf {E(u, v) := JllVu(x)(Vv(x))-^{1}ll^ dx I (u, v) e #Z$_{k,a}$}-

Remark 2.1. We note that, formally, problem (P) reduces to the minimization of functional where both the domain and the deformation are varying. Indeed, if v was invertible then

H(u \f) *"~ I "V11'Y)'V'VI'Y) ' $^{-1}$H^r MY

•L$_{(\Omega)}$ IIVu*(y)H^r dx,

where $u^*(y) := u(v^{-1}(y))$.

Theorem 2.2

Let k ≥ 1 be an integer, 0 < a < 1, let *Cl Q* [R^N be a bounded, open set with $C^{k+3,a}$ boundary and let uo e Diff^^{a}($\overline{Q}$), det Vuo > 0 in $\overline{Q}$. Then (P) attains its minimum at every (u*, v*) € *S#±$_a$* such that

Vu*(x)(Vv*(x))-^{1} = *X* R inQ

where X^N = meas uo(n)/meas *Q* and R is an orthogonal transformation. Thus

inf {E(u, v) I (u, v) G *sa$_{Ka}$}* =E(u*, v*) = N^ meas (Q)i*/N (meas u_o(Q))$^{r/N}$.

The proof of Theorem 2.2 is based on the following lemma.

Lemma 2,3

If A G $M^{N\times N}$ then IIAII^N ≥ NW Idet Al. Furthermore, the equality holds if and only if A = A,R, for some *X e* IR and some orthogonal transformation R.

Proof. If det A = 0, then the inequality is trivially valid. Suppose that det A > 0. Using the polar decomposition, we can write

A = RU,

where U = U^7 > 0 and R is an orthogonal transformation, i. e. R^T R = R R^T = 11, det R = 1. Thus, U = Q^T diag(?ii,..., XN) Q, where Q is an orthogonal transformation and A,i,..., ^N > 0 and so

$$\|A\| = \|RU\| = \|U\| = \Big(\sum_{i=1}^{N} \lambda_i^2\Big)^{1/2}. \qquad (2.2)$$

As In is a concave function, we have

$$\ln(\det A) = \ln\Big(\prod_{i=1}^{N} \lambda_i\Big) = \frac{1}{2}\sum_{i=1}^{N} \ln(\lambda_i^2) \le \frac{N}{2} \ln\Big(\frac{1}{N}\sum_{i=1}^{N} \lambda_i^2\Big) \qquad (2.3)$$

hence, by (2.2) and (2.3)

$$\det A \le \Big(\sum_{i=1}^{N} \lambda_i^2/N\Big)^{N/2} = \frac{\|A\|^N}{N^{N/2}}.$$

Finally, if $\det A < 0$ choose R' to be an orthogonal transformation such that $\det R' = -1$. Then, as $\det (R'A) > 0$, by the first part of the proof we have

$$\|A\|^N = \|R'A\|^N \geq N^{N/2} |\det (R'A)| = N^{N/2} |\det A|.$$

Due to the strict concavity of the logarithmic function, it follows immediatly from (2.3) that equality holds if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_N$ in which case A is proportional to an orthogonal transformation.

Remark 2.4.

By abuse of language we shall call a matrix A such that $\|A\|^N = N^{N/2} |\det A|$ *harmonic.* In $\mathbb{R}^2$, a matrix A such that $\|A\|^2 = 2 |\det A|$ is of the form

either $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

Proof of Theorem 2.2. If $(u^*, v^*) \in \mathcal{A}_a$ then, as $\det \nabla v^*(x) = 1$ in Ω, as $r \geq N$ by Lemma 2.3 and by Hölder's inequality we have

$$E(u^*,v^*) := \int_\Omega \|\nabla u^*(x)(\nabla v^*(x))^{-1}\|^r dx$$

$$\geq \text{meas}(\Omega)^{1-r/N} \left(\int_\Omega \|\nabla u^*(x)(\nabla v^*(x))^{-1}\|^N dx\right)^{r/N}$$

$$\geq \text{meas}(\Omega)^{1-r/N} \left(N^{N/2} \int_\Omega |\det \nabla u^*(x)| \, dx\right)^{r/N}$$

$$\geq \text{meas}(\Omega)^{1-r/N} \left(N^{N/2} \int_\Omega \det \nabla u^*(x) \, dx\right)^{r/N}$$

$$= N^{r/2} \, \text{meas}(\Omega)^{1-r/N} \left(\int_\Omega \det \nabla u_0(x) dx\right)^{r/N}. \tag{2.4}$$

Let

$$\lambda := \frac{\text{meas } \Omega}{\text{meas } u_0(\Omega)}, \quad \tilde{v}_0 := \lambda^{1/N} u_0,$$

As $\tilde{v}_0 \in \text{Diff}^{k+3,\alpha}(\overline{\Omega})$, $\text{meas } \tilde{v}_0(\Omega) = \text{meas } \Omega$ and since $\partial\Omega \in C^{k+3,\alpha}$, by Theorem 1 in DACOROGNA & MOSER [3] there exists $v \in \text{Diff}^{k+1,\alpha}(\overline{\Omega}, \overline{\tilde{v}_0(\Omega)})$ such that

$$\begin{cases} \det \nabla v(x) = 1 & \text{in } \Omega, \\ v(x) = \tilde{v}_0(x) & \text{on } \partial\Omega \end{cases}$$

and define

$$u := \frac{v}{\lambda^{1/N}} \in C^{k,\alpha}(\overline{\Omega}).$$

Clearly

$$u = uo \text{ on } \partial\Omega$$

and we have

$$E(u,v) := \int_{\Omega} \|\nabla u(x)(\nabla v(x))^{-1}\|^{r}\, dx$$

$$= \text{J\_f,IIll|N dx}$$

$$= \frac{N^{r/2}\text{meas (fl)}}{\lambda^{r/N}}$$

$$= N^{r/2} \text{ meas } (Q)^{1}\text{-}\wedge \text{ (meas } u_o(Q))^{r/N}$$

which, together with (2.4), finishes the proof.

Corollary 2.5

Under the hypotheses of Theorem 2.2, and in particular if r ≥ N, then

inf {E(u, v) I (u, v) G £0_{tot}} = $N^{r/2}$ inf $\Big\{\int_{\Omega}$ Idet Vu(x)l$^{r/N}$ dx I u = ^ on *dQ* and u $\in \mathrm{Diff}^{k,\alpha}(\overline{\Omega})\Big\}$.

Moreover, if (u*, v*) is a solution then there exist a rotation R(.) and a scalar *X(.)* such that

Vu*(x) = *X(x)* R(x)Vv*(x) for every x e Q .

Proof. As det Vuo > 0, for all u eDiffc^{a}($\overline{Q}$) with u = uo on 3Q we have

$\int_{Ja}$ Idet Vu(x)l dx ≥ $\int_{JQ}$ 11 det Vu(x) dxl = $\int_{Jft}$ det $Vu_o(x)$ dx ≥ a

where

a := inf {Jldet Vu(x)l dx I u = uo on dQ and u € Diff^^{a}($\overline{Q}$)}.

Thus, by Theorem 2.2 we obtain

inf {E(u, v) I (u, v) G *S&^a)* = N'tf measCQ)^{1}-^ *aJt"*. (2.5)

On the other hand, as r ≥ N using Hölder's inequality we deduce that

$$\alpha^{r/N} = \inf\left\{\left(\int_\Omega |\det \nabla u(x)|\, dx\right)^{r/N} \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in \mathrm{Diff}^{k,\alpha}(\bar\Omega)\right\}$$

$$\leq \mathrm{meas}(\Omega)^{r/N-1} \inf\left\{\int_\Omega |\det \nabla u(x)|^{r/N}\, dx \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in \mathrm{Diff}^{k,\alpha}(\bar\Omega)\right\}$$

which, together with (2.5) implies that

$$\inf\{E(u, v) \mid (u, v) \in W_{k,\alpha}\} \leq N^{r/2} \inf\left\{\int_\Omega |\det \nabla u(x)|^{r/N}\, dx \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in \mathrm{Diff}^{k,\alpha}(\bar\Omega)\right\}$$

and the reverse inequality follows immediatly from Lemma 2.3.

Finally, by Lemma 2.3 if (u^*, v^*) is a solution then

$$\int_\Omega |\det \nabla u^*(x)|^{r/N}\, dx \geq \inf\left\{\int_\Omega |\det \nabla u(x)|^{r/N}\, dx \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in \mathrm{Diff}^{k,\alpha}(\bar\Omega)\right\}$$

$$= N^{-r/2} \inf\{E(u, v) \mid (u, v) \in W_{k,\alpha}\}$$

$$= N^{-r/2} \int_\Omega \|\nabla u^*(x)(\nabla v^*(x))^{-1}\|^r dx$$

$$\geq \int_\Omega |\det \nabla u^*(x)|^{r/N} dx$$

and so

$$\int_\Omega \{|\det \nabla u^*(x)|^{r/N} - N^{-r/2} \|\nabla u^*(x)(\nabla v^*(x))^{-1}\|^r\}\, dx = 0 \qquad (2.6)$$

which, together with Lemma 2.3, implies that

$$\|\nabla u^*(x)\, (\nabla v^*(x))^{-1}\|^N = N^{N/2}\, |\det (\nabla u^*(x)\, (\nabla v^*(x))^{-1})| \quad \text{a. e. in } \Omega.$$

Thus

$$\nabla u^*(x) = \lambda(x)\, R(x) \nabla v^*(x) \quad \text{a. e. in } \Omega$$

for some rotation $R(.)$ and some scalar $\lambda(.)$. From (2.6), Theorem 2.2 and using Hölder's inequality we deduce that

$$(\mathrm{meas}\,\Omega)^{1-N/r} \left(\int_\Omega |\det \nabla u^*(x)|^{r/N}\, dx\right)^{N/r} =$$

$$= N^{-r/2}\, [(\mathrm{meas}\,\Omega)^{-1+r/N} \inf\{E(u, v) \mid (u, v) \in W_{k,\alpha}\}]^{N/r}$$

$$= \int_\Omega \det \nabla u_0(x)\, dx$$

$$= \int_\Omega \det \nabla u^*(x)\, dx \leq \int_\Omega |\det \nabla u^*(x)|\, dx$$

$$\leq (\mathrm{meas}\,\Omega)^{1-N/r} \left[\int_\Omega |\det \nabla u^*(x)|^{r/N} dx\right]^{N/r}.$$

Hence

f [detVu*(x)dx-ldetVu*(x)l]dx = 0

which implies that det Vu* > 0 in *Q*.

If uo is affine then we can obtain existence of minimizers under less restrictive hypothesis on *BQ*, namely

Proposition 2.6.

Let *Q* be a bounded, open set with Lipschitz boundary. Let uo(x) = Ax + b where A € M^{NxN} with det A > 0 and b € $[R^N$. Then (P) admits a solution (u, v) *e S&**.

This result relies on the fact that any affine deformation is harmonic up to a volume preserving transformation. Precisely

Lemma 2.7.

If det A * 0 then there exists a matrix B such that det B = 1 and $IIABII^N = N^N$# Idet Al.

Proof. Suppose that det A > 0. As in the proof of Lemma 2.3, we can write

A = RQ^Tdiag(^i,..., ^N)Q> where R and Q are orthogonal transformations and *X*,..., *X^* > 0. Set

B:=QTdiag(pi,...,p_N)

where

ft fll - ^N$)^{1/N}$
Pi := Xi '

Then det B = 1 and

IIABIIN = H^ ... ^_N)i/N m|N = *fa... X_N)* $H11II^N$ = NN/2 det A.

If det A < 0, it suffices to multiply A by an orthogonal transformation R^1 with det R' = -1 and to apply the previous case to the matrix R^fA.

Proof of Proposition 2.6. Setting u* : = uo and **Vv*** := $\mathbf{B^{-1}}$, as in the proof of Theorem 2.2 it follows that (u*, v*) is a solution for (P).

3. EXISTENCE AND REGULARITY RESULTS.

Now we show that the results of Section 2 can be generalized in the following way.

Hypothesis (H): Let W : $M^{N\times N}$ -» [0, +<») be continuous and such that there exist *Xe* OR and Xo e $M^{N\times N}$ with

$$\det X_0 = \frac{\text{meas } u_0(Q)}{\text{meas} Q}$$

and

$$W(X) - X \det X \geq W(X_0) - X \det X_0 \quad \text{for every } X \in M^{N\times N}.$$

Remark 3.1. (i) In some sense the *X* appearing in (H) can be seen as a Lagrange multiplier.

(ii) If $W \in C^1(M^{N\times N})$ then

$$\begin{cases} \dfrac{\partial W}{\partial X}(X_0) = \lambda \operatorname{adj} X_0 \\ \\ \lambda := \dfrac{\langle \frac{\partial W}{\partial X}(X_0), X_0\rangle \text{ meas}(Q)}{N \text{meas } u_0(Q)}. \end{cases} \tag{3.1}$$

Indeed, as Xo is a minimum for W(X) - *X* det X we have

$$0 = \frac{\partial}{\partial X}(W(X) - X \det X)|_{X_0}$$

$$= \frac{\partial W}{\partial X}(X_0) - \lambda \operatorname{adj} X_0$$

and so, by (2.1) and (H)

$$\langle \frac{\partial W}{\partial X}(X_0), X_0\rangle = X N \det X_0$$

$$= \lambda N \frac{\text{meas } u_0(Q)}{\text{meas} Q}$$

which proves (3.1).

Consider the problem

(P) minimize in $\mathscr{A}_{k,\alpha}$ the functional

$$E(u, v) := \int_\Omega W(\nabla u(x)(\nabla v(x))^{-1})\, dx$$

where the class of admissible pairs is defined by

$$\mathscr{A}_{k,\alpha} := \{(u, v) \in \mathrm{Diff}^{k,\alpha}(\overline{\Omega}) \times \mathrm{Diff}^{k,\alpha}(\overline{\Omega}) |\ u = u_0 \text{ on } \partial\Omega \text{ and } \det \nabla v(x) = 1 \text{ a. e. in } \Omega\}$$

and, as in Section 2, $u_0 \in \mathrm{Diff}^{k,\alpha}(\overline{\Omega})$ is such that $\det \nabla u_0 > 0$ in $\overline{\Omega}$.

Before stating the main result of this section, we give examples of functions satisfying the condition (H).

Proposition 3.2

The following functions $W : M^{N\times N} \to \mathbb{R}$ verify (H).

i) Let $N \geq 2$, let $g : \mathbb{R}^2 \to \mathbb{R}$ be convex, C^1 and is increasing with respect to the first variable and set

$$W(X) = g(\|X\|^N, \det X).$$

In particular, (H) holds for

$$W(X) = \|X\|^r \text{ if and only if } r \geq N.$$

ii) Let $N = 2$ and let

$$W(X) = \sum_{i,j,k,l=1,2} a_{ijkl}\, X_{ij}\, X_{kl}$$

with $a_{ijkl} = a_{klij}$ and W a strictly rank one convex function, i. e.

$$W(\lambda\otimes\mu) \geq \alpha\, \|\lambda\otimes\mu\|^2$$

for some $\alpha > 0$ and for all $\lambda, \mu \in \mathbb{R}^2$, where $(\lambda\otimes\mu)_{ij} := \lambda_i\, \mu_j$ for $i, j = 1, 2$.

Remark 3.3.

There are other examples of functions satisfying (H), namely for $N = 2$
$W(X) = \|X\|^4 - 2(\det X)^2$ and $W(X) = \frac{1}{2}(X_{11}^2 + X_{12}^2 + X_{21}^2) + \frac{1}{4}X_{22}^4$.

Proof of Proposition 3.2. i) Set

XQ := £R with $\wedge^N$ = meas uo(£2)/ meas(Q) and R a rotation.

As g = g(t, s) is convex we have

$$W(X) - W(Xo) \geq |f(HXoll^N, \det Xo)(HXII^N - IIXoll^N) + \wedge IIXoll^N, \det Xo)(\det X - \det X_o),$$

and so, as $\frac{\partial}{\partial s} \geq 0$ and since by Lemma 2.3

$$HX\|N \geq {}_N N/2 \det x \quad \text{and} \quad IIX_0H^N = N^N\# \det Xo,$$

we conclude that

$$W(X) - W(X_0) \geq X(\det X - \det X_o)$$

where

$$X := NN/2 \wedge(IIX_OII^N, \det X_o) + \frac{\partial f}{\partial s}(IIXoll^N, \det X_o).$$

If $W(X) = IIXII^r$ then (H) is violated if r < N (see also Proposition 4.1). Indeed, in this case (3.1) reduces to

$$r\|X_0\|^{r-2} X_0 = \lambda \operatorname{adj} X_0$$

and, as Xo must be parallel to its adjugate matrix, Xo is a harmonic matrix and

Xo = % R for some % € [R, % > 0 and some rotation R.

If in (H) we set X = p R, with p € [R, p > £, then we obtain

$$p^r N^1/^2 - £^r N^1/^2 > X(\$F - £^N) \tag{3.2}$$

and so, either $X \leq 0$ and then (3.2) fails for p < £, or $X > 0$ and (3.2) is false for p large enough,

(ii) Since W is rank one convex and as N = 2, then W is polyconvex (see DACOROGNA [2]) and so

$$\sup\{\wedge\wedge l \det Y < O\} \leq \inf\{\wedge : l \det Y > 0\}. \tag{3.3}$$

Choose

$$\lambda = \inf\{\frac{W(Y)}{\det Y} \mid \det Y > 0\} \tag{3.4}$$

and observe that the infimum is attained. Indeed, since W is quadratic there is no loss of generality in assuming that a minimizing sequence has norm 1 and so, up to the extraction of a subsequence, we have

$$\frac{W(Y_n)}{\det Y_n} \to \lambda,\ Y_n \to X,\ \det Y_n > 0 \ \text{ and } \|Y_n\| = 1. \tag{3.5}$$

Note that $\det X > 0$ otherwise $X = a \otimes b$ for some unit vectors a, b, and using the strict rank one convexity and (3.5) we would have for every $\varepsilon > 0$

$$\frac{W(Y_n)}{\varepsilon + \det Y_n} \to \frac{W(X)}{\varepsilon + \det X} \geq \frac{\alpha}{\varepsilon}.$$

Letting $\varepsilon \to 0^+$ we would obtain

$$\frac{W(Y_n)}{\det Y_n} \to +\infty$$

which contradicts (3.3), (3.4), (3.5). Hence $\det X > 0$ and setting

$$X_0 := \xi X \text{ where } \xi^2 := \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega} \frac{1}{\det X} = \frac{\lambda}{W(X)} \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega}$$

it follows from (3.3) that

$$W(Y) - \lambda \det Y \geq 0 = W(X_0) - \lambda \det X_0.$$

and

$$\det X_0 = \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega}.$$

Theorem 3.4.

Let $k \geq 1$ be an integer, $0 < \alpha < 1$, Ω a bounded, open set with $C^{k+3,\alpha}$ boundary and let $u_0 \in \text{Diff}^{k,\alpha}(\overline{\Omega})$ with $\det \nabla u_0 > 0$ in $\overline{\Omega}$. If (H) holds then (P) admits a solution $(u^*, v^*) \in \mathcal{A}_{k,\alpha}$ such that

$$\nabla u^*(x)\, (\nabla v^*(x))^{-1} = X_0 \quad \text{for every } x \in \Omega$$

and

$$\inf \{ \int_\Omega W(\nabla u(x)(\nabla v(x))^{-1})\, dx \mid (u, v) \in \mathcal{A}_{k,\alpha} \} = W(X_0) \text{ meas } \Omega.$$

Remarks 3.5.

(i) As it will become clear in Section 4, in some sense the condition (H) is optimal to guarantee existence of solution.

(ii) The set $S\&^{\wedge}_a$ of admissible pairs of functions (u, v) was chosen so as to give immediatly a regularity result as well as existence of solution.

(iii)If

$$W(X) = \|X\|^r$$

for r ≥ N, by Proposition 3.2 i) we can take Xo = XR where R is a rotation and

$$y_N = \frac{\text{meas } up(Q)}{\text{meas } Cl}$$

Then, according to Theorem 3.4 we can find a minimizer (u*, v*) such that

$$Vu^*(x)(Vv^*(x))^{-1} = X R \quad \text{a. e. in } Q$$

and the minimum value of the energy functional is given by

$$W(X_0) \text{ meas } Q = \|XR\|^T \text{ meas } Q$$

$$= X^T N^r \# \text{ meas } Cl$$

$$= N^{\wedge} \text{ meas } Cl^{1\wedge} (\text{meas } uo(ft))^{r/N}$$

which is in agreement with Theorem 2.2 and Corollary 2.5.

Before giving the proof of Theorem 3.4 we state a theorem which is proved exactly as the preceding one but requires less regularity on dQ (see also Proposition 2.6).

Theorem 3.6.

Let Q. be a bounded, open, Lipschitz domain, let uo(x) = Ax + b where A e $M^{N\times N}$ and b € $\mathbb{R}^N$ and assume that W satisfies (H). If det A * 0 then (P) admits a solution (u*, v*) € Stf^* with u* = uo on dQ and det Vv*(x) = 1 in Q. Furthermore, if det A = 0 and if W(X) -> 0 when 11X11 -> 0 then (P) has no solution unless A = 0.

Proof of Theorem 3.4.

Let X_o e $M^{N\times N}$ be a matrix for which (H) holds. By Theorem 1 in DACOROGNA & MOSER [3], we find a mapping u* e Diffc^$\overline{Q}$) such that

$$\begin{cases} \det Vu^*(x) = \det X_o \text{ in } Q \\ u^*(x) = UQ(X) \quad \text{on } 3Q. \end{cases}$$

Setting

$$v^* := \wedge\ u^*.$$

we have

$$\det V v^*(x) = 1 \text{ in } Q,\ Vu^*(x)\,(Vv^*(x))\text{-!} = Xo \quad \text{in } Q,$$

and, by (H), if (u, v) € £0k,a we have

JwtVuCxXVvtx))-^{1}) dx ≥
d
fw(Vu*(x)(Vv*(x))-^{1}) dx + *X* J[det Vu(x) - det Vu*(x)] dx. **(3.6)**

As u = u* on *dQ*. we obtain

f (detVu(x)-detVu*(x))dx = O

which, together with (3.6) implies that

f W(Vu(x) (VvCx))"^{1}) dx ->f W(Vu*(x)(Vv*(x)r^{1})dx = meas(Q)W(X_0).

The proof of Theorem 3.6 requires the following lemma.

Lemma 3.7.

If det A = 0 then there exists a family of matrices $B_£$ such that det $B_£$= 1 and IIAB$_£$II -» 0 as

$\varepsilon \to 0^+$.

Proof. Using the polar decomposition for A we can assume that

A = R diag(^i, ..., *X-$_u$* ..., λ_N)

where R is a rotation and *X{* = 0. Set

$B_£$:= diag(bi(e),..., b_N(e))

where

$$b_j(\varepsilon) := \begin{cases} 1 & \text{if } \lambda_j = 0, \\ \varepsilon/\lambda_j & \text{if } \lambda_j \neq 0 \end{cases}$$

if $j \neq i$ and

$$b_i(\varepsilon) := \frac{1}{b_1(\varepsilon)\ldots b_{i-1}(\varepsilon) b_{i+1}(\varepsilon)\ldots b_N(\varepsilon)}.$$

Clearly det B_e = 1 and IIAB$_E$II ≤ $(N\text{-}l)^1/^2$e.

Proof of Theorem 3.6. Suppose that det A * 0. Setting u* := uo and v*(x) := sign (det A) XQ^1 U*, by (H) it foUows that

det Xo = Idet Al

and so

det Vv* = 1 in Q.

As in the proof of Theorem 3.4, we conclude that

inf {J\V(Vu(x)(Vv(x))-^{1}) dx I (u, v) e ^ $_{t\ a}$ } = W(X_0) meas Q = E(u*, v*).

Finally, if det A = 0 with A * 0, by Lemma 3.7 consider a sequence {B_e} such that det $B_£$= 1 and IIAB$_E$II -> 0 as e -» 0+. Setting $u_£$ = uo and Vv_E = B"$^1_£$ we obtain

E(u_e, $v_£$) = meas (Q) W($AB_£$) -^ 0.

It is clear that in this case (P) admits no solution since, if

E(u*, v*) = inf {E(u, v) I (u, v) e *S#$_{Ka}$*} = 0,

then Vu*(x) (Vv*(x))"^{1} = 0 in Q, i. e. Vu*(x) = 0 in £1 Hence u* must be constant and as A * 0, this is in contradiction with the condition u* = uo on $\partial\Omega$.

Finally, we conclude this section with a result on the weak lower semicontinuity of E(.,.). However, we insist that this property is not sufficient to ensure existence since, in general, no weak compactness can be obtained for the minimizing sequences regardless of the coercivity of W.

Proposition 3.8

Let *Q* ⊂ [RN b$_{e\ a}$ bounded, open set. Let p >_1, q ≥_N and 1/p + (N-l)/q ≤_1. If

(u$_£$, v$_£$) -* (u, v) weakly in W^1^ x W^1^

and if det Vv$_e$ = 1 a. e. in *Cl* then det Vv = 1 a. e. in *Q*, and

Vu$_e$(x) (Vve(x))-^{1} -> Vu(x) (Vv(x))-i weakly in IA

Consequently, if W : M$^{N\times N}$ -» [0, -H*>) is convex then

$$\int_\Omega W(\nabla u(x)(\nabla v(x))^{-1})\, dx \le \liminf \text{JwCVueCxXVv^x))}^{-1})\ dx.$$

Conjecture.

In Proposition 3.8 we used the convexity of W to obtain the weak lower semicontinuity of the energy functional E(.,.). As, formally, Vu (Vv)-^{1} is the gradient of uov^1, we conjecture that if W is quasiconvex and if (u$_e$, v$_e$) converges weakly to (u, v) then

$$\int_d \text{w}(\text{Vu}(x)(\text{Vv}(x))^{-1})\, dx \le \liminf \int_d \text{w}(\text{Vu}_e(x)(\text{Vv}_e(x))\text{-i})\, dx.$$

Proof of Proposition 3.8. As det Vv = 1 a. e. in **£2,** we have

(Vu(x) (Vv(x))-i)ij = (Vu(x) (adj Vv(x))T)$_{ij}$ = m(x). ^(x)

where r|i is the gradient of the i* component of u and £j is the j^A row of adj Vv. Hence

curl r|i = 0 and div £j = 0

and by the div-curl lemma (see TARTAR [12]) we conclude that if p ≥ 1, q ≥ N and if 1/p + (N-l)/q≤lthen

$$\text{Vu}_£(x)\ (\text{Vv^x}))^{-1} \to \text{Vu}(x)\ (\text{Vv}(x))^{-1} \text{ weakly in } L^1 \tag{3.7}$$

whenever

- (u$_£$, v$_£$) -» (u, v) weakly in W^1* x W^1^.

Finally, if W is a convex, nonegative function then by (3.7) the functional (u,v) -» E(u,v) is lower semicontinuous (see DACOROGNA [2]).

Remark 3.9.

Let $p \geq 1$, $q \geq N$ and $1/p + (N-1)/q \leq 1$ and consider the class of admissible pairs to be given by

$$\mathcal{A}_{p,q} := \{(u, v) \in W^{1,p}(\Omega; \mathbb{R}^N) \times W^{1,q}(\Omega; \mathbb{R}^N) \mid u = u_0 \text{ on } \partial\Omega,\ \det \nabla v(x) = 1 \text{ a. e. in } \Omega$$
$$\text{and } \int_\Omega v(x)\,dx = 0 = \int_\Omega x\,dx\}.$$

Suppose that W is convex, $W(X) \geq C_1\|X\|^r - C_0$ with $C_1 > 0$. By Proposition 3.8 (P) has a solution in $\mathcal{A}_{p,q}$ if there is a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ bounded in $W^{1,p} \times W^{1,q}$. Suppose that $r \geq p\,\frac{q}{q-1}$ and that $\{v_\varepsilon\}$ is bounded in $W^{1,q}$. Let s be such that

$$\frac{q}{q-1} \leq s \leq \frac{r}{p}.$$

By Hölder's inequality

$$\int_\Omega \|\nabla u_\varepsilon(x)\|^p dx = \int_\Omega \|\nabla u_\varepsilon(x)(\nabla v_\varepsilon(x))^{-1}\nabla v_\varepsilon(x)\|^p dx$$
$$\leq \left(\int_\Omega \|\nabla u_\varepsilon(x)\,(\nabla v_\varepsilon(x))^{-1}\|^{ps}\, dx\right)^{1/s} \left(\int_\Omega \|\nabla v_\varepsilon(x)\|^{ps'} dx\right)^{1/s'}$$
$$\leq \text{Const.} \left(\int_\Omega \|\nabla u_\varepsilon(x)(\nabla v_\varepsilon(x))^{-1}\|^r dx\right)^{1/r} \left(\int_\Omega \|\nabla v_\varepsilon(x)\|^q\, dx\right)^{1/q}$$

and so $\{u_\varepsilon\}$ is bounded in $W^{1,p}$. We conclude that if there exists a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ where $\{v_\varepsilon\}$ is bounded in $W^{1,q}$ then (P) admits a solution in $\mathcal{A}_{p,q}$.

We next show that the set of solutions is weakly closed.

Proposition 3.10.

Let Ω be an open, bounded, Lipschitz domain, let W be a convex function, let $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ and let $p \geq 1$, $q \geq N$, $1/p + (N-1)/q \leq 1$ and $r \geq 1$. If $\{(u_n, v_n)\}$ is a sequence of solutions of (P) in $\mathcal{A}_{p,q}$ and if (u_n, v_n) converges weakly to (u, v) in $W^{1,p}(\Omega; \mathbb{R}^N) \times W^{1,q}(\Omega; \mathbb{R}^N)$ then (u, v) is also a solution of (P).

Proof. As q $\geq$ N standard results imply that

adj Vv_n converges weakly in L ^ - ^{1}) to adj Vv.

Moreover, as

det Vv_n converges in the sense of distributions to det Vv,

we must have

det Vv = 1 a. e. in *Q*.

and so

(u, v) € $Stf_{p,q}$.

Finally, using the div-curl lemma we deduce that

$Vu_n(x)$ (Vvn(x))-^{1} ->Vu(x) (Vv(x))-i weakly in L^1

and as W is convex we conclude that

J\V(Vu(x)(Vv(x))-^{1}) dx $\leq$ lim inf $Jw(Vu_n(x)(Vv_n(x))$-^{1}) dx
d ***d***

= inf {E(u, v) I (u, v) € ^ $_{p, q}$} .

4. NON EXISTENCE RESULTS.

In this section we present two types of non existence results showing that, despite the resemblance of our problem to the classic Dirichlet problem of minimizing Jll VUIIP, problem (P) is in fact very different in nature. It turns out to be much closer to

(Q) inf **{ fldet** Vu(x)lP dx I u = uo on 3Q, u € **Diff$^{k,\alpha}(\bar{\Omega})$}**

as already seen in Corollary 2.5 and as it will be illustrated bellow. Indeed, restricting our attention to

W(X) = IIXII^r,r>l,

Theorem 3.6 provides a first type of non existence result. Namely, if uo(x) = Ax for some A e $M^{N\times N}$, A * 0 with det A = 0, then (P) does not admit a solution. This is in sharp contrast with the minimization of Jll VUIIP.

We have seen in Theorem 2.2 and Corollary 2.5 that if $r \geq N$ then (P), as well as (Q) (with $p = r/N \geq 1$), admit solutions. Now we show the second type of non existence result, proving that if $r < N$ then neither (P) (see Proposition 4.1) nor (Q) (see Proposition 4.3) have solutions.

Proposition 4.1.

Let ***Q.*** = {x € [R^{2}I llxll < 1}, let uo(x) = x and let 0 < r < 2 = N. Then

inf{ JllVu(x)(Vv(x))-^{1}ll^{r}dx I (u, v) ***e*** W^1--(Q)xW^1--(Q)$_f$ u = uo on 9Q, det Vv = 1 a. e. in *Q}* =

0 and hence the infimum is not attained.

Remarks 4.2.

i) In order to avoid some technicalities, in the previous proposition we considered u and v in **$W^{1,\infty}(\Omega)$**. However, the result remains valid if instead we assume that the admissible pairs (u, v) **e** **$\mathrm{Diff}^{k,\alpha}(\overline{\Omega})\mathrm{x}\mathrm{Diff}^{k,\alpha}(\overline{\Omega})$**.

ii) Similarly, we take the boundary condition uo(x) = x just for the sake of illustration, since it could be replaced by any UQ € Diffc^a(**$\overline{\Omega}$**).

Proof of Proposition 4.1. Using polar coordinates we define

$$u_n(x, y) := \begin{cases} \frac{1}{\varepsilon}(x,y) & \text{if re } (0, e) \\ \frac{1}{r}(x, y) & \text{if re } (e, 1), \end{cases}$$

where **e** := (2n)^k/*-^{2}) -> 0, and

^{v}n00 •=~$\frac{r}{\backslash 2n}$==-(cos 2n9, sin 2n9)

where r = $\sqrt{x^2 + y^2}$. A direct computation gives

detVv$_n$(x)^l,

$$\|\nabla u_n(x)(\nabla v_n(x))^{-1}\|^k = \begin{cases} \frac{1}{\varepsilon^k}\Big(1 \quad ,f.,\ n < r < e \\ \\ \frac{1}{r^k}\left(\frac{1}{2n}\right)^{k/2} \quad \text{if } \ e < r < 1, \end{cases}$$

and

$$E(u», v_n) = n(2n + \_\wedge)^{k/2}\,(2n)^{-r} + \pi\frac{2(2n)^{-k/2}}{2 - r}(1 - (2n)^{-k}) \to 0 \text{as } n \to +\infty.$$

Finally, we conclude this section with a similar result on problem (Q).

Proposition 4.3.

If an e $C^{3,a}$ for some $0 < a < 1$, if uo € DifiP^C$\overline{Q}$) with det Vuo > 0 in $\overline{Q}$ and if $0 < \beta <$ 1 then for all $p \geq 1$

$$\inf \Big\{ \int_\Omega |\det \nabla u(x)|^p\, dx \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in \mathrm{Diff}^{1,a}(\overline{\Omega}) \Big\} = 0$$

and thus the infimum is not attained[1].

Proof. Let xo e Q and let $\overline{B}$(xo, 2e) C *Q*. Let φ_n be a family of smooth functions such that $0 \leq \varphi_n \leq 1$ and

$$\varphi_n(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ \\ 0 & \text{if } t \geq e^{1/n} \end{cases}$$

and define

$$f_n(x) := \varphi_n\left(\left|\frac{x - x_0}{\varepsilon}\right|^2\right)\left(1 + \left|\frac{x - x_0}{\varepsilon}\right|^{2n}\right) + 1 - \varphi_n\left(\left|\frac{x - x_0}{\varepsilon}\right|^2\right).$$

Clearly, $f_n \geq 0$, f_n are smooth and $f_n \geq 1$. In addition,

$$\int_\Omega f_n(x)\,dx \geq \int_{|x - x_0| < \varepsilon} \left|\frac{x - x_0}{\varepsilon}\right|^{2n} dx \geq \mathrm{Const.}\left(\int_0^1 R^{N-1} R^{2n}\,dx\right)^{\beta}$$

---

[1]The same result holds for $u \in W^{1,*}$, with $p \geq PN$.

$$= \text{Const.} \frac{1}{(N + 2n)^{\beta}} \tag{4.1}$$

and

$$\int_{\Omega} f_n(x)^{\beta}\, dx \le \text{Const.} + \int_{|x-x_0|<\varepsilon} \left|\frac{x - x_0}{\varepsilon}\right|^{2n\beta} dx$$
$$= \text{Const.} + \text{Const.} \frac{1}{N + 2n\beta},$$

and so, from (4.1) we conclude that

$$\lim_{n \to \infty} \frac{\int_{\Omega} f_n(x)^{\beta}\, dx}{\left(\int_{\Omega} f_n(x)\, dx\right)^{\beta}} \le \lim_{n \to \infty} \text{Const.}\left(1 + \frac{1}{N + 2n\beta}\right)(N + 2n)^{\beta} = 0. \tag{4.2}$$

Using Theorem 1 in DACOROGNA & MOSER [3], we find a sequence $u_n \in \mathrm{Diff}^{1,\alpha}(\overline{\Omega})$ such that

$$\begin{cases} \det \nabla u_n(x) = \dfrac{\text{meas } u_0(\Omega)}{\int_{\Omega} f_n(x)\, dx} f_n(x) & \text{in } \Omega \\ \\ u_n(x) = u_0(x) & \text{if } x \in \partial\Omega. \end{cases}$$

From (4.2) it follows that

$$\lim_{n \to \infty} \int_{\Omega} |\det \nabla u_n(x)|^{\beta}\, dx = \text{meas } u_0(\Omega)^{\beta} \lim_{n \to \infty} \frac{\int_{\Omega} f_n(x)^{\beta}\, dx}{\left(\int_{\Omega} f_n(x)\, dx\right)^{\beta}} = 0.$$

5. QUALITATIVE PROPERTIES.

We remark that if (P) has one solution then, if $\partial\Omega$ is sufficiently smooth[2], there are uncountably many solutions. In fact, if

min $E(u, v) = E(u_1, v_1)$

---

[2]If the class of admissible functions is $\mathcal{A}_{k,\alpha}$ then $\partial\Omega$ must be $C^{k+3,\alpha}$. If we are considering the set $\mathcal{A}_{p,q}$ then we assume that $\partial\Omega$ is Lipschitz.

and if f is such that[3]

$$\begin{cases} \det \nabla f(x) = 1 & \text{in } \Omega \\ \\ f(x) = x & \text{on } \partial\Omega, \end{cases}$$

then $(u_1 \circ f, v_1 \circ f)$ is admissible and, as $f(\Omega) = \Omega$ we obtain

$$\begin{aligned} E(u_1 \circ f, v_1 \circ f) &= \int_\Omega W(\nabla u_1(f(x)) \nabla f(x)\, (\nabla f(x))^{-1} (\nabla v_1(f(x)))^{-1})\, dx \\ &= \int_\Omega W(\nabla u_1(f(x)) (\nabla v_1(f(x)))^{-1})\, dx \\ &= \int_{f(\Omega)} W(\nabla u_1(y) (\nabla v_1(y))^{-1})\, dx \\ &= E(u_1, v_1). \end{aligned}$$

In Remark 3.9 we noted that if $W(X) = \|X\|^r$, $r \geq N$, and if there exists a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ where $\{v_\varepsilon\}$ is bounded in $W^{1,q}$ then (P) admits a solution in $\mathcal{A}_{p,q}$. By the preceeding remark, it would suffice to show that given a sequence $\{v_\varepsilon\}$ in $W^{1,q}$ then there exists a sequence $f_\varepsilon \in W^{1,\infty}(\Omega, \Omega)$ such that

$$\begin{cases} \det \nabla f_\varepsilon(x) = 1 & \text{in } \Omega \\ \\ f_\varepsilon(x) = x & \text{on } \partial\Omega, \end{cases}$$

and $\{v_\varepsilon \circ f_\varepsilon\}$ is bounded in $W^{1,q}$. However, such sequence $\{f_\varepsilon\}$ may fail to exist since (P) has no solution if $u_0(x) = Ax + b$, $b \in \mathbb{R}^N$, $A \in M^{N \times N}$, $\det A = 0$ and $A \neq 0$ (see Theorem 3.6).

As we mentioned before, the minimization of (P) corresponds, formally, to a minimization of a functional where the domain is varying. Theorem 3.4 provides a sufficient condition under which there is existence of solution. Here, $v(\Omega)$ becomes the domain of the solution. It is natural to ask what type of domains may correspond to solutions of (P). The following proposition partially answers that question.

[3]Here $f \in C^{k,\alpha}$ in the case where the class of admissible functions is $\mathcal{A}_{k,\alpha}$ and f is Lipschitz if we are considering the class $\mathcal{A}_{p,q}$.

Proposition 5.1.

Let $k \geq 1$ be an integer, $0<cc < 1$, let Q ^ [R^2 be a bounded, open set with $C^{k+3,ct}$ boundary and let u_0 € Diff^C$\overline{Q}$) with det $Vu_0 > 0$ in $\overline{Q}$. Let W(X) = IIXII^2 and assume that 3uo(f2) is an analytic Jordan curve. If Y C IR^2 is such that meas Y = meas *Q* and if 9Y is an analytic Jordan curve then there exists a minimizer (u, v) of E(.,.) on £#k,a such that *v(Q)* is a translation of Y.

Proof. By the Riemann Mapping Theorem there exists a conformal equivalence f *e* Diff^~($\overline{Y}$, $\overline{u_o(£2)}$). Thus we have f = (fi, *fi)* where

$$
\begin{cases}
\dfrac{\partial f_1}{\partial y_1} = \dfrac{3f_2}{dy_2} \\[2ex]
\dfrac{\partial f_1}{\partial y_2} = -\dfrac{3f_2}{\partial y_1}.
\end{cases}
\tag{5.1}
$$

Set

vorsf^uoiQ^BCO, R).

As v_0 € Diff^{k,ot}($\overline{ii}$), we have

meas vo(Q) = meas Y = meas Q

and since dQ e C^{k+3}-^{a}, by Theorem 1 in DACOROGNA & MOSER [3] there exists vi € $\mathrm{Diff}^{k,\alpha}$(S; $\overline{Y}$) such that

$$
\begin{cases}
\det Vv!(x) = l & \text{in} Q \\[2ex]
v_2(x) = v_o(x) & \text{on } \partial\Omega.
\end{cases}
$$

Finally, set

v(x) := vi(x) + C, where the constant C is such that

$$\int v(x)dx = 0,$$

and define

u := fo(v - C) € $C^{k,\alpha}(\Omega)$.

Clearly

u = uo on $\partial\Omega$

and as v is invertible (see BALL [1], Theorems 1 **and** 2), we have

E(u, v) := I HVu(x)(Vv(x)) $^{-1}$II^2 dx

= f **IIVuGOOfyGOrVdx**

$$= \int \|\nabla u(v_1^{-1}(y))\, \nabla v_1^{-1}(y)\|^2\, dy$$

= f **HVftvjtv^ty))) Vv$_1$(v7^1(y))Vv7^1(y)ll^2 dy**

= f **IIVf(y)ll^2 dy.**

Jv,(Q)

Therefore, by (5.1) we deduce that

$$\mathbf{E(u, v) = 2} \quad \left[\left(\frac{\partial f_1}{\partial y_1}(y)\right)^2 + \left(\frac{\partial f_1}{\partial y_2}(y)\right)^2\right] dy$$

= 2 f det Vf(y) dy

= 2 f det Vu(v7*(y)) det Vv^ty) dy

= 2 f det Vu(x) dx

= 2 fdet Vu$_o$(x) dx. (5.2)

Ω

From (2.4) and (5.2) we deduce that (u, v) is a solution of (P) and

v(Q) = v$_o$(Q) + C = Y + C.

Next, and pursuing the discussing of the nature of the set of solutions of (P), we give some uniqueness results.

Proposition 5.2.

Let *Q* be an open, bounded, Lipschitz domain in [R^N, let r ≥ N and let p ≥ r ≥ N, q ≥ N. W(X) = IIXII^r and if (ui, v) and (U2, v) are solutions of (P) in Ptf$_{p,q}$ then ui = U2 a. e. in *Q*.

Proof. Clearly, if 6 e (0, 1) then (0ui + 8u_2, v) is admissible and

JllV(eui+eu2)(x)(Vv(x))-^{1}ll^{r}dx < 6 JllVui(x)(Vv(x))-^{1}ll^{r}dx

d *d*

+ (1 - 0) JllVu_2(x)(Vv(x))-i|l^{r}dx

d

unless Vui(x)= Vu_2(x) a. e. in Q, and so, as ui = u_2 on *dQ.* we conclude that ui = u_2 a. e. in Q.

Proposition 5.3.

Let Q be an open, bounded, Lipschitz domain in [R^N, let r ≥ N and let p ≥ r ≥ N, q > N. If W(X) = IIXII^r and if (u, vi) is a solution of (P) in £0pq such that vi is invertible and *vi(Q)* is a Lipschitz domain[4], then (u, v_2) is another solution of (P) if and only if there exist a constant rotation R and a constant C € [R^N such that v_2(x) = Rvi(x) + C a. e. in *Cl.*

Proof. Suppose that v_2(x) = Rvi(x) + C a. e. in Q. By Corollary 2.5

∇u(x) = λ(x) Q(x) Vvi(x) a.e. in Q

for some rotation Q(.) and some scalar *X(.).* Hence,

Vu(x) = *X(x)* Q(x)R^TVv_2(x) a. e. in *Q*

and so

JllVu(x)(Vv_2(x))-i|l^r dx = N^r/2 JlA.(x)l' dx

= JllVu(x)(Vvi(x))-i|l'dx

and so, (u, v_2) is also a minimizer. Conversely, if (u, vi) and (u, v_2) are solutions of (P) then by Corollary 2.5 we must have det Vu > 0, Vu(x) = Xi(x)Qi(x)Vvi(x) and Vu(x)=X_2(x)Q2(x)Vv_2(x) a. e. in £2, where *X*, X_2 € [R and Qi, Q2 are rotations. Thus *X*$_9$ *X2* > 0,

---

[4]Here we will use the fact that if v e W^1*, q > N, v is invertible, v(ft) is a strongly Lipschitz domain and if detVv = 1 a. e. then

(i) v^1 € W^1^^{1}), *Vv^l(y)* = (Vv(x))"^{1} a. e., where y = v(x);**

(ii) Wev € W^1-^{1} and V(w_ov)(x) = Vw(v(x)) Vv(x) a. e. in Q, whenever w € W^1*, p ≥ q/(q-l).

Vvi(x) = $\wedge^{1}$(x)X_2(x)Qy(x)Q2(x)Vv_2(x) **a.** e. in ***Q***

and as det Vvi(x) = 1 we have

X\(x) = ^200 a. e. in *Q*.

We conclude that

$$V v_2(x)(V_{V1}(x))\text{-}i = Ro(x) \tag{5.3}$$

for some rotation R(.). Setting

0)$_2$(y) :=v_2(v;^{1}(y)) and $\tilde{R}_0$(y) := RoCv^Cy)) ,

(5.3) reduces to

Vo>2(y) = $\tilde{R}$o(y) a. e. y € vi(Q)

and we conclude that (see FONSECA [9], Proposition A.I)

$\tilde{R}_o$(.), and therefore Ro, must be constantly equal to a fixed rotation R

which, together with (5.3) implies that

v_2(x) = Rvi(x) + C a. e. in *Q*.

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Existence, regularity properties and uniqueness of solutions for a minimization problem involving variation of the domain are obtained. This model arises in the study of defective solid crystals and it is shown that direct methods of the Calculus of Variations cannot be applied.

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