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**AN UNIQUENESS PROOF FOR THE  
WULFF THEOREM**

by

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# AN UNIQUENESS PROOF FOR THE WULFF THEOREM

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## 1. INTRODUCTION.

Equilibrium problems for materials that may change phase usually lead to the minimization of functionals involving bulk and interfacial energies. For solid crystals with sufficiently small grains, HERRING [11] claims that the bulk contribution is negligible with respect to the surface tension. In this case, the energy reduces essentially to its surface energy component which, due to its anisotropy, plays a definite role in determining the shape of a crystal approaching an equilibrium configuration of minimum energy. Assuming that interfaces are sharp, the surface tension considered by HERRING [11] was of the type

$$\int_{\partial E} \Gamma(n_E(x)) dH_{N-1}(x) \quad (1.1)$$

where  $E$  is a smooth subset of  $\mathbb{R}^N$ ,  $n_E$  is the outward unit normal to its boundary and  $\Gamma$  denotes the anisotropic free energy density per unit area.

In this paper we obtain uniqueness (up to translations and sets of measure zero) of the solution for the geometric variational problem

(P) Minimize (1.1) subject to the volume constraint  $\text{meas}(E) = \text{constant}$ .

Clearly, when  $\Gamma$  is constant the problem (P) reduces to the classical isoperimetric inequality. For anisotropic  $\Gamma$ , one of the first attempts to solve this question is due to WULFF [16] in the early 1900's. His work was followed by that of DINGHAS [4], who proved formally that among convex polyhedra the *Wulff set* (or *crystal of  $\Gamma$* )

$$W_\Gamma := \{x \in \mathbb{R}^N \mid x \cdot n \leq \Gamma(n), \text{ for all } n \in S^{N-1}\}$$

is the shape having the least surface integral for the volume it contains. The key idea of this proof is the use of the Brunn-Minkowski inequality. Later, using the same argument and geometric measure theory tools, TAYLOR [13], [14] and [15] rendered DINGHAS's [4] proof precise, obtaining existence and uniqueness of a solution for (P) among measurable sets of finite perimeter. Recently, DACOROGNA & PFISTER [3] presented a completely different proof in  $\mathbb{R}^2$ , which does not involve the Brunn-Minkowski theorem and is purely analytical. This approach, however, cannot be extended to higher dimensions and the minimization is only carried out over a certain subclass of the class  $\mathcal{C}$  of all measurable sets with finite perimeter.<sup>1</sup> Finally, in FONSECA [8] existence of solution for (P) in  $\mathcal{C}$  is obtained using the theory of functions of bounded variation, hopefully rendering this problem more accessible to analysts. This proof relies on the Brunn-Minkowski Theorem and on the parametrized indicator measures (see FONSECA [7], RESHETNYAK [12]). These probability measures are very helpful to handle oscillating weakly converging sequences of surfaces and continuity and lower semicontinuity of functionals of the

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<sup>1</sup> TAYLOR [13], [14], [15] considers only bounded sets.

type (1.1). They are a refined version of the generalized surfaces of YOUNG [17] and they were studied by ALMGREN [2] (see also ALLARD [1]) under the name of varifolds.

In Section 2 we review some concepts of the theory of functions of bounded variation and we recall briefly some of the results obtained in FONSECA [8] which are relevant for this work. In Section 3 we obtain the proof of uniqueness within the class  $\bar{C}$ . As in TAYLOR [14], our proof is based on the Brunn-Minkowski Theorem and on the existence of an inverse for the Radon transform (see GELFAND, GRAEV & VILENKIN [9]). The main new idea is to use a sharpened version of the Brunn-Minkowski inequality, see Lemma 3.5.

## 2. PRELIMINARIES.

We recall briefly some results of the theory of functions of bounded variation (see EVANS & GARIEPY [5], FEDERER [6], GIUSTI [10], ZIEMER [18]). Let  $\Omega \subset \mathbb{R}^N$  be an open set and define  $S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ .

### Definition 2.1.

A function  $u \in L^1(\Omega)$  is said to be a *function of bounded variation* ( $u \in BV(\Omega)$ ) if

$$\int_{\Omega} |\nabla u(x)| \, dx := \sup \left\{ \int_{\Omega} u(x) \cdot \operatorname{div} \varphi(x) \, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \|\varphi\|_{\infty} \leq 1 \right\} < +\infty.$$

A particular case of a function of bounded variation is the characteristic function of a set of finite perimeter.

### Definition 2.2.

If  $A$  is a measurable subset of  $\mathbb{R}^N$  then the *perimeter of  $A$  in  $\Omega$*  is defined by

$$\operatorname{Per}_{\Omega}(A) := \int_{\Omega} |\nabla \chi_A(x)| \, dx = \sup \left\{ \int_A \operatorname{div} \varphi(x) \, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \|\varphi\|_{\infty} \leq 1 \right\},$$

where  $\chi_A$  denotes the characteristic function of  $A$ .

If  $A$  has finite perimeter in  $\mathbb{R}^N$  then for any borel set  $E$

$$\|\nabla \chi_A\|(E) = H_{N-1}(\partial^* A \cap E),$$

where  $H_{N-1}$  denotes the  $N-1$  dimensional Hausdorff measure,  $\partial^* A$  is the reduced boundary of  $A$  and  $\|\nabla \chi_A\|$  is the total variation measure of the vector-valued measure  $\nabla \chi_A$ . Also, there exists a  $\|\nabla \chi_A\|$ -measurable map  $n_A : \partial^* A \rightarrow S^{N-1}$  such that  $n_A(x)$  is the outward normal to  $\partial^* A$  at  $x$ ,

$$-n_A \|\nabla \chi_A\| = \nabla \chi_A \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

and the *generalized Green-Gauss theorem* holds, namely

$$\begin{aligned}\int_A \operatorname{div} \varphi(x) \, dx &= \int_{\mathbb{R}^N} \varphi(x) \cdot n_A(x) \, d\|\nabla \chi_A\| \\ &= \int_{\partial^* A} \varphi(x) \cdot n_A(x) \, dH_{N-1}(x)\end{aligned}$$

for all  $\varphi \in C_0^1(\mathbb{R}^N; \mathbb{R}^N)$ .

**Lemma 2.3.**

Let  $E$  be a set of finite perimeter in  $\mathbb{R}^N$ , let  $\xi \in S^{N-1}$  and set  $A(s) := \{x \in \mathbb{R}^N \mid x \cdot \xi < s\}$ ;  $E(s) = E \cap A(s)$ .

Then for almost all  $s \in \mathbb{R}$

$$\partial^* E(s) = ((\partial^* E) \cap A(s)) \cup (E(s) \cap \partial A(s))$$

up to a set of  $H_{N-1}$  measure zero.

**Proof.** The result is well known but as we are not aware of a precise reference we include a proof for the convenience of the reader. First,  $E(s)$  has finite perimeter for a.e.  $s$  (see e.g. [18, Lemma 5.5.3]). Secondly, one deduces as in [18, Lemma 5.5.2] that for  $f \in \mathcal{D}(\mathbb{R}^N)$  and for a.e.  $s$

$$\begin{aligned}\int_{E \cap A(s)} D_i f \, dx &= - \int_{A(s)} f \, d(D_i \chi_E) + \int_{E \cap \partial A(s)} f(y) \xi_i \, dH_{N-1}(y) \\ &= \int_{\partial^* E \cap A(s)} f(y) (n_E(y))_i \, dH_{N-1}(y) + \int_{E \cap \partial A(s)} f(y) \xi_i \, dH_{N-1}(y).\end{aligned}$$

Applying the Gauss - Green formula to the term on the left hand side one has

$$\begin{aligned}\int_{\partial^*(E \cap A(s))} f(y) (n_{E \cap A(s)}(y))_i \, dH_{N-1}(y) &= \\ &= \int_{\partial^* E \cap A(s)} f(y) (n_E(y))_i \, dH_{N-1}(y) + \int_{E \cap \partial A(s)} f(y) \xi_i \, dH_{N-1}(y).\end{aligned}$$

This identity holds for almost every  $s$ , simultaneously for a countably family of  $f$ 's and hence for all continuous  $f$  with compact support. The desired assertion follows and one finds that moreover

$$n_{E \cap A(s)} = n_E \text{ on } \partial^* E \cap A(s), \quad n_{E \cap A(s)} = \xi \text{ on } E \cap \partial A(s).$$

We will use the change of variables formula

$$\int_{\mathbb{R}^N} u(x) |\det \nabla f(x)| dx = \int_{\mathbb{R}^p} \left( \int_{f^{-1}(y)} u(z) dH_{N-p}(z) \right) dy \quad (2.4)$$

where  $N \geq p$ ,  $f: \mathbb{R}^N \rightarrow \mathbb{R}^p$  is a Lipschitz function and  $u: \mathbb{R}^N \rightarrow \mathbb{R}$  is measurable, as well as the *Fleming-Rishel co-area formula*

$$\int_{\Omega} |\nabla u(x)| dx = \int_{-\infty}^{\infty} \text{Per}_{\Omega} \{x \in \mathbb{R}^N \mid u(x) > t\} dt \quad (2.5)$$

for  $u \in \text{BV}(\Omega)$ . The next lemma shows that a bounded set of finite perimeter can be approached in BV by a sequence of  $C^\infty$  sets with the same volume. The proof can be found in FONSECA [8].

**Lemma 2.6.**

Let  $E \subset \mathbb{R}^N$  be a bounded set of finite perimeter. There exists a sequence of open, bounded sets  $E_n \subset \mathbb{R}^N$  such that

- (i)  $\partial E_n \in C^\infty$  and  $E_n, E \subset B(0, R)$  for some  $R > 0$ ;
- (ii)  $\chi_{E_n} \rightarrow \chi_E$  in  $L^1(\mathbb{R}^N)$ ;
- (iii)  $\text{Per}(E_n) \rightarrow \text{Per}(E)$ ;
- (iv)  $\text{meas}(E_n) = \text{meas}(E)$ .

Now we summarize some of the results obtained in FONSECA [8] concerning the Wulff set. In what follows  $\Gamma: S^{N-1} \rightarrow [0, +\infty)$  denotes the surface free energy of a solid. For crystalline materials, HERRING [11] proposes some constitutive hypotheses for  $\Gamma$  based on molecular considerations where surface energies arise from interatomic interactions of finite range. It turns out that for ordered materials (i. e. materials with a lattice structure)  $\Gamma$  is not differentiable with respect to certain crystallographically simple directions. In this case, if we plot  $\Gamma$  radially as a function of the direction  $n$ , this plot will present cusped minima in certain directions corresponding to surfaces of particular simple structure with respect to the lattice. At each point of this polar plot construct a plane perpendicular to the radius vector at that point. Then the volume  $W_\Gamma$  which can be reached from the origin without crossing any of the planes is the *Wulff set*. Precisely, assuming that  $\Gamma$  is continuous and bounded away from zero, i. e. there exists  $\alpha > 0$  such that

$$\Gamma(n) \geq \alpha \quad \text{for all } n, \|n\| = 1, \quad (2.7)$$

we have

**Definition 2.8.**

The *Wulff set* (or *crystal of  $\Gamma$* ) is the set  $W_\Gamma := \{x \in \mathbb{R}^N \mid x \cdot n \leq \Gamma(n) \text{ for all } n \in S^{N-1}\}$ .



Clearly, if  $\Gamma \equiv 1$  then  $W_\Gamma$  is the closed unit ball. Also, using HERRING's [11] idea it is easy to show that for solid crystals the lack of differentiability of  $\Gamma$  implies that its crystal is a polyhedron.

**Proposition 2.9.**

- (i)  $W_\Gamma$  is convex, closed and bounded;
- (ii)  $\Gamma^{**}(x) = \sup \{y \cdot x \mid y \in W_\Gamma\}$ , where  $\Gamma^{**}$  is the lower convex envelope of  $\Gamma$  ( $\Gamma$  being extended to  $\mathbb{R}^N$  as a homogenous function of degree 1);
- (iii) if  $x \in \partial W_\Gamma$  and if  $n$  is normal to  $W_\Gamma$  at  $x$  then  $x \cdot n = \Gamma(n) = \Gamma^{**}(n)$ ;
- (iv) the crystal of  $\Gamma^{**}$  is the equal to the crystal of  $\Gamma$ ;
- (v)  $0 \in \text{int}(W_\Gamma)$ .

It turns out that the Wulff set minimizes (1.1) among all sets that have the same volume.

**Theorem 2.10.**

Let  $E \subset \mathbb{R}^N$  be a set with finite perimeter and such that  $\text{meas}(E) = \text{meas}(W_\Gamma)$ . Then

$$\int_{\partial^* E} \Gamma(n_E(x)) \, dH_{N-1}(x) \geq \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x)) \, dH_{N-1}(x).$$

Changing variables, it follows immediately that

**Corollary 2.11.**

The dilation  $\lambda W_\Gamma$  minimizes the surface energy functional (1.1) among all sets of finite perimeter with volume equal to  $\lambda^N \text{meas}(W_\Gamma)$ .

The key idea of the proof of Theorem 2.10 is the use of the Brunn-Minkowski inequality<sup>2</sup>. This was exploited formally by DINGHAS [4] and later made precise in the context of geometric measure theory by TAYLOR<sup>3</sup> [13], [15]. For two sets  $A$  and  $B$  in  $\mathbb{R}^N$  we let

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

**Brunn-Minkowski Theorem 2.12.**

If  $A$  and  $B$  are nonempty sets of  $\mathbb{R}^N$  then

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<sup>2</sup> In DACOROGNA & PFISTER [3] existence is obtained for a certain class of sets in  $\mathbb{R}^2$  without using the Brunn-Minkowski Theorem.

<sup>3</sup> We generalize TAYLOR's [37] result to unbounded sets.

$$\text{meas}(A + B) \geq (\text{meas}(A)^{1/N} + \text{meas}(B)^{1/N})^N.$$

The other fundamental tool used in the proof of Theorem 2.10 is the notion of indicator measures (see FONSECA [7], RESHETNYAK [12]). They allow one to establish continuity and lower semicontinuity properties for energies of the type (1.1).

**Theorem 2.13.**

Let  $E_\epsilon \subset \mathbb{R}^N$  be a sequence of bounded sets with finite perimeter in  $\mathbb{R}^N$ . If  $\{\text{meas}(E_\epsilon) + \text{Per}(E_\epsilon)\}$  is bounded and if  $\chi_{E_\epsilon} \rightarrow \chi_E$  in  $L^1(\mathbb{R}^N)$  then

$$\int_{\partial^* E} F(x, n_E(x)) \, dH_{N-1}(x) \leq \liminf_{\epsilon \rightarrow 0} \int_{\partial^* E_\epsilon} F(x, n_{E_\epsilon}(x)) \, dH_{N-1}(x)$$

for all nonnegative, continuous functions  $F$  such that  $F(x, \cdot)$  is convex and homogeneous of degree one for all  $x \in \mathbb{R}^N$ . Moreover, equality holds for an arbitrary  $F \in C(\mathbb{R}^N \times \mathbb{R}^N)$  with compact support in the first variable if  $\text{Per}(E_\epsilon) \rightarrow \text{Per}(E)$ .

In order to prove Theorem 2.10, a lower bound for the relaxed energy was obtained in FONSECA [8].

**Lemma 2.14.**

Let  $E$  be a  $C^\infty$ , open, bounded domain. Then

$$\int_{\partial^* E} \Gamma^{**}(n_E(x)) \, dH_{N-1}(x) \geq \liminf_{\epsilon \rightarrow 0} \frac{\text{meas}(E + \epsilon W_\Gamma) - \text{meas}(E)}{\epsilon}.$$

**3. THE WULFF THEOREM: UNIQUENESS.**

We show that the Wulff set or translations of it are, up to sets of measure zero, the only solutions for the variational problem

(P) Minimize  $\int_{\partial^* E} \Gamma(n_E(x)) \, dH_{N-1}(x)$

among all measurable sets  $E \subset \mathbb{R}^N$  of finite perimeter with  $\text{meas}(E) = \text{meas}(W_\Gamma)$ . This result was first obtained by TAYLOR [14] using geometrical arguments, in particular a delicate approximation of sets of finite perimeter by polyhedra. DACOROGNA & PFISTER [3] provided a proof in  $\mathbb{R}^2$  which is entirely analytical but it concerns a more restrictive class of sets  $E$  and it cannot be extended to higher dimensions. As in TAYLOR [14], the proof presented is based on the existence of an inverse for the Radon transform (see GELFAND, GRAEV & VILENKIN [9]). Also, as the proof of Theorem 2.10, it relies on the Brunn-Minkowski Theorem 2.12, on the lower

semicontinuity results of Theorem 2.14 and on Lemma 2.15. The main new ingredient is Lemma 3.5.

Let  $\xi \in S^{N-1}$  and let  $E \subset \mathbb{R}^N$  be a measurable set. In what follows we use the notation :

$$E_\xi(s) := \{x \in E \mid x \cdot \xi < s\},$$

$$g_{E,\xi}(s) = \text{meas}(\{x \in E \mid x \cdot \xi < s\}) / \text{meas}(E)$$

and

$$h_{E,\xi}(s) := H_{N-1}(\{x \in E \mid x \cdot \xi = s\}) / \text{meas}(E).$$

We first show that solutions of (P) must be bounded (up to sets of measure zero).

**Theorem 3.1.**

Let  $E \subset \mathbb{R}^N$  be a measurable set of finite perimeter. If  $E$  is a solution of (P) then  $E = E_1 \cup E_2$  where  $E_1 \cap E_2 = \emptyset$ ,  $\text{meas}(E_2) = 0$ ,  $H_{N-1}(\partial^* E_2) = 0$  and  $E_1$  is bounded. In addition, for all  $\xi \in S^{N-1}$  the function  $g_{E_1,\xi}$  is strictly increasing on the set  $\{s \mid 0 < g_{E_1,\xi}(s) < 1\}$ .

**Proof of Theorem 3.1.** Assume that  $E$  is a solution of (P), fix  $\xi \in S^{N-1}$  and set

$$g := g_{E,\xi} \text{ and } h := h_{E,\xi}.$$

Let  $-\infty \leq s_0 = s_0(\xi) := \sup \{s \mid g(s) = 0\}$  and  $s_1 = s_1(\xi) := \inf \{s \mid g(s) = 1\} \leq +\infty$ . By Lemma 2.3 and Theorem 2.10, for almost all  $s_0 < s < s_1$  we have

$$\begin{aligned} \int_{\partial^* E(s)} \Gamma(n_{E(s)}) dH_{N-1} &= g(s)^{(N-1)/N} \int_{\partial^* \left( \left( \frac{1}{g(s)} \right)^{1/N} E(s) \right)} \Gamma(n) dH_{N-1} \\ &\geq g(s)^{(N-1)/N} \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}) dH_{N-1}. \end{aligned}$$

In a similar way, with  $E(s)' := \{x \in E \mid x \cdot \xi \geq s\} = E \setminus E(s)$ ,

$$\int_{\partial^* E(s)'} \Gamma(n_{E(s)'}) dH_{N-1} \geq (1 - g(s))^{(N-1)/N} \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}) dH_{N-1}.$$

Adding up these two inequalities and using Lemma 2.3 yields

$$\begin{aligned} \int_{\partial^* E} \Gamma(n_E) dH_{N-1} + 2 \int_{\{x \in E \mid x \cdot \xi = s\}} \Gamma(\xi) dH_{N-1} &\geq \\ &\geq \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}) dH_{N-1} [g(s)^{(N-1)/N} + (1 - g(s))^{(N-1)/N}], \end{aligned}$$

and so, by Theorem 2.10 and the fact that  $E$  is a solution of (P)

$$h(s) \geq C^* [g(s)^{(N-1)/N} + (1 - g(s))^{(N-1)/N} - 1] \tag{3.2}$$

where  $C^* := \frac{1}{2M \text{meas}(W_\Gamma)} \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}) dH_{N-1}$  and  $M := \max_{v \in S^{N-1}} \Gamma(v)$ . By the co-area formula

(2.5) and by Fubini's theorem,  $g$  is absolutely continuous and

$$g'(s) = h(s) \text{ for a. e. } s \in (s_0, s_1) \tag{3.3}$$

which implies by (3.2) that

$$g \text{ is strictly increasing in the interval } (s_0, s_1). \quad (3.4)$$

Let

$$F(s) := s^{(N-1)/N} + (1-s)^{(N-1)/N} - 1.$$

By (3.2) and (3.3) it follows that

$$\begin{aligned} C^*(s_1 - s_0) &\leq \int_{s_0}^{s_1} \frac{g'(s)}{F(g(s))} ds \\ &= 2 \int_0^{1/2} \frac{ds}{F(s)} =: C. \end{aligned}$$

Let  $\{e_1, e_2, \dots, e_N\}$  be the canonical orthonormal basis of  $\mathbb{R}^N$  and consider a  $a \in \mathbb{R}^N$  such that

$$B(a, C/C^*) \supset \{x \in \mathbb{R}^N \mid s_0(e_i) \leq x \cdot e_i \leq s_1(e_i), i = 1, \dots, N\}.$$

Hence

$$\begin{aligned} \text{meas}(E \setminus B(a, C/C^*)) &\leq \text{meas}(E \setminus \{x \in \mathbb{R}^N \mid s_0(e_i) \leq x \cdot e_i \leq s_1(e_i), i = 1, \dots, N\}) \\ &\leq \sum_{i=1}^N \text{meas}(E \setminus \{x \in \mathbb{R}^N \mid s_0(e_i) \leq x \cdot e_i \leq s_1(e_i)\}) \\ &= 0. \end{aligned}$$

Setting  $E_1 := E \cap B(a, C/C^*)$  and  $E_2 := E \setminus E_1$  one has  $\text{meas}(E_2) = 0$ , which by Definition 2.2 implies that  $H_{N-1}(\partial^* E_2) = 0$ .

The following sharpened version of the Brunn-Minkowski inequality will be useful.

**Lemma 3.5.**

Let  $\xi \in S^{N-1}$  and let  $A$  and  $B$  be bounded sets such that  $\text{meas}(A) = \text{meas}(B)$  and the functions  $g_{A,\xi}$  and  $g_{B,\xi}$  are strictly increasing on the sets, respectively,  $\{s \mid 0 < g_{A,\xi}(s) < 1\}$  and  $\{s \mid 0 < g_{B,\xi}(s) < 1\}$ . Then for all  $\varepsilon > 0$

$$\text{meas}(A + \varepsilon B) \geq \text{meas}(A) \int_0^1 \left( 1 + \varepsilon \left( \frac{\gamma_{B,\xi}(t)}{\gamma_{A,\xi}(t)} \right)^{1/(N-1)} \right)^{N-1} \left( 1 + \varepsilon \frac{\gamma_{A,\xi}(t)}{\gamma_{B,\xi}(t)} \right) dt$$

where  $\gamma_{A,\xi}(t) := h_{A,\xi}(g_{A,\xi}^{-1}(t))$ .

**Proof.** For simplicity of notation we set

$$g_A := g_{A,\xi}, \quad h_A := h_{A,\xi} \text{ and } \gamma_A := \gamma_{A,\xi}.$$

By the co-area formula (2.5) and by Fubini's theorem,  $g_A$  is absolutely continuous and

$$g'_A(s) = h_A(s) \text{ for a. e. } s \in (s_0, s_1)$$

where  $s_0 := \sup \{s \mid g_A(s) = 0\}$  and  $s_1 := \inf \{s \mid g_A(s) = 1\}$ . As  $A$  is bounded

$$-\infty < s_0 < s_1 < +\infty,$$

and by hypothesis  $g_A$  admits an inverse  $g_A^{-1} : (0, 1) \rightarrow (s_0, s_1)$ . Setting

$$\gamma_A(t) := h_A(g_A^{-1}(t))$$

we obtain

$$\frac{dg_A^{-1}}{dt}(t) = \frac{1}{\gamma_A(t)} \text{ for a. e. t.} \quad (3.6)$$

We can assume, without loss of generality, that  $\xi = e_1$  and write  $x = (x_1, x')$ . Let

$$A_t := \{x' \in \mathbb{R}^{N-1} \mid (x_1, x') \in A \text{ and } x_1 = g_A^{-1}(t)\}, \text{ for } t \in (0, 1).$$

As

$$\{g_A^{-1}(t) + g_B^{-1}(t)\} \times (A_t + B_t) \subset A + B,$$

setting  $z(t) := g_A^{-1}(t) + g_B^{-1}(t)$  by (3.6) we have

$$\begin{aligned} \text{meas}(A + B) &\geq \int_{z(0)}^{z(1)} H_{N-1}(A_{z^{-1}(s)} + B_{z^{-1}(s)}) ds \\ &= \int_0^1 H_{N-1}(A_t + B_t) z'(t) dt \\ &= \int_0^1 H_{N-1}(A_t + B_t) \left( \frac{1}{\gamma_A(t)} + \frac{1}{\gamma_B(t)} \right) dt. \end{aligned}$$

By the Brunn-Minkowski Theorem (see Theorem 2.12)

$$\begin{aligned} H_{N-1}(A_t + B_t)^{1/(N-1)} &\geq H_{N-1}(A_t)^{1/(N-1)} + H_{N-1}(B_t)^{1/(N-1)} \\ &= (\gamma_A(t) \text{meas}(A))^{1/(N-1)} + (\gamma_B(t) \text{meas}(B))^{1/(N-1)} \end{aligned}$$

and so

$$\text{meas}(A + B) \geq \int_0^1 [(\gamma_A(t) \text{meas}(A))^{1/(N-1)} + (\gamma_B(t) \text{meas}(B))^{1/(N-1)}]^{N-1} \left( \frac{1}{\gamma_A(t)} + \frac{1}{\gamma_B(t)} \right) dt. \quad (3.7)$$

It is easy to verify that

$$h_{\varepsilon B, \xi}(s) = h_{B, \xi}(s/\varepsilon)/\varepsilon, \quad g_{\varepsilon B, \xi}(s) = g_{B, \xi}(s/\varepsilon) \text{ and } \gamma_{\varepsilon B, \xi}(t) = \gamma_{B, \xi}(t)/\varepsilon$$

which, together with (3.7) and the assumption  $\text{meas } A = \text{meas } B$  imply

$$\begin{aligned} \text{meas}(A + \varepsilon B) &\geq \text{meas}(A) \int_0^1 [\gamma_A^{1/(N-1)}(t) + \varepsilon \gamma_B^{1/(N-1)}(t)]^{N-1} \left( \frac{1}{\gamma_A(t)} + \frac{\varepsilon}{\gamma_B(t)} \right) dt \\ &= \text{meas}(A) \int_0^1 \left( 1 + \varepsilon \left( \frac{\gamma_B(t)}{\gamma_A(t)} \right)^{\frac{1}{N-1}} \right)^{N-1} \left( 1 + \varepsilon \frac{\gamma_A(t)}{\gamma_B(t)} \right) dt. \end{aligned}$$

**Theorem 3.8.**

If  $E$  is a solution of (P) then  $\|\chi_{E+c} - \chi_{W_\Gamma}\|_{L^1} = 0$ , where

$$c := \frac{1}{\text{meas}(W_\Gamma)} \left( \int_{W_\Gamma} x \, dx - \int_E x \, dx \right).$$

**Proof.** Let  $E$  be a solution of (P) and consider the translated sets  $E' := E - a$  and  $W' := W - b$ , where

$$a := \frac{1}{\text{meas}(W_\Gamma)} \int_E x \, dx \quad \text{and} \quad b := \frac{1}{\text{meas}(W_\Gamma)} \int_{W_\Gamma} x \, dx. \quad (3.9)$$

By Theorem 3.1 we can suppose that  $E$  is bounded and that for all  $\xi \in S^{N-1}$  the function  $g_{E,\xi}$  is strictly increasing on the set  $\{s \mid 0 < g_{E,\xi} < 1\}$ . Hence, by Lemma 2.6 there exists a sequence of smooth, open, bounded sets  $E_n \subset \mathbb{R}^N$  such that  $E_n, E' \subset B(0, R)$  for some  $R > 0$ ,  $\text{meas}(E_n) = \text{meas}(E')$ ,  $\text{Per}(E_n) \rightarrow \text{Per}(E')$  and  $\text{meas}(E_n \setminus E') + \text{meas}(E' \setminus E_n) \rightarrow 0$ . In addition (see FONSECA [8])  $g_{E_n,\xi}$  are strictly increasing on the sets  $\{s \mid 0 < g_{E_n,\xi}(s) < 1\}$  for all  $\xi \in S^{N-1}$ ,  $n \in \mathbb{N}$ . Fix  $\xi \in S^{N-1}$ . As in the proof of Lemma 3.5 we set

$$g_E := g_{E,\xi}, \quad h_E := h_{E,\xi} \quad \text{and} \quad \gamma_E := \gamma_{E,\xi}.$$

By Lemma 2.14, by Lemma 3.5 and by Fatou's Lemma we have

$$\begin{aligned} \int_{\partial^+ E_n} \Gamma(n_{E_n}(x)) \, dH_{N-1}(x) &\geq \liminf_{\varepsilon \rightarrow 0} \frac{\text{meas}(E_n + \varepsilon W') - \text{meas}(W')}{\varepsilon} \\ &\geq \text{meas}(W_\Gamma) \liminf_{\varepsilon \rightarrow 0} \int_0^1 \frac{\left(1 + \varepsilon \left(\frac{\gamma_{W'}(t)}{\gamma_{E_n}(t)}\right)^{\frac{1}{N-1}}\right)^{N-1} \left(1 + \varepsilon \frac{\gamma_{E_n}(t)}{\gamma_{W'}(t)}\right) - 1}{\varepsilon} \, dt \\ &\geq \text{meas}(W_\Gamma) \int_0^1 \left[ (N-1) \left(\frac{\gamma_{W'}(t)}{\gamma_{E_n}(t)}\right)^{1/(N-1)} + \frac{\gamma_{E_n}(t)}{\gamma_{W'}(t)} \right] dt. \end{aligned}$$

As  $h_{E_n}(s) = 0$  if  $|s| > R$ , setting  $t = g_{E_n}(s)$  we obtain (recall  $\gamma = h \circ g^{-1}$ ,  $g' = h$ )

$$\begin{aligned} \int_{\partial^+ E_n} \Gamma(n_{E_n}(x)) \, dH_{N-1}(x) &\geq \\ &\geq \text{meas}(W_\Gamma) \int_{-R}^R \left[ (N-1) \left(\frac{\gamma_{W'}(g_{E_n}(s))}{h_{E_n}(s)}\right)^{\frac{1}{N-1}} + \frac{h_{E_n}(s)}{\gamma_{W'}(g_{E_n}(s))} \right] h_{E_n}(s) \, ds. \end{aligned} \quad (3.10)$$

On the other hand, as  $\text{meas}(E_n \setminus E') + \text{meas}(E' \setminus E_n) \rightarrow 0$  we have  $\|h_{E_n} - h_E\|_{L^1} \rightarrow 0$  and  $\|g_{E_n} - g_E\|_\infty \rightarrow 0$  and so, by Theorem 2.13, (3.10) and Fatou's Lemma we conclude that

$$\begin{aligned} & \int_{\partial^* E'} \Gamma(n_{E'}(x)) dH_{N-1}(x) \geq \\ & \geq \text{meas}(W_\Gamma) \int_{-R}^R \left[ (N-1) \left( \frac{\gamma_{W'}(g_{E'}(s))}{h_{E'}(s)} \right)^{\frac{1}{N-1}} + \frac{h_{E'}(s)}{\gamma_{W'}(g_{E'}(s))} \right] h_{E'}(s) ds. \end{aligned}$$

As  $g_{E'}$  is strictly increasing in  $(s_0(E', \xi), s_1(E', \xi)) \subset (-R, R)$ , by the change of variables formula (2.4), by Theorem 2.10, by Proposition 2.9 (iii) and by the generalized Gauss-Green theorem we have

$$\begin{aligned} N \text{meas}(W_\Gamma) &= \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x)) dH_{N-1}(x) = \int_{\partial^* E'} \Gamma(n_{E'}(x)) dH_{N-1}(x) \geq \\ &\geq \text{meas}(W_\Gamma) \int_0^1 \left[ (N-1) \left( \frac{\gamma_{W'}(t)}{\gamma_{E'}(t)} \right)^{1/(N-1)} + \frac{\gamma_{E'}(t)}{\gamma_{W'}(t)} \right] dt. \end{aligned} \quad (3.11)$$

However  $(N-1)a^{1/(N-1)} + 1/a \geq N$  and equality holds only if  $a = 1$ . Thus (3.11) implies that

$$\gamma_{E'}(t) = \gamma_{W'}(t) \text{ for almost all } t \in (0, 1)$$

which, by (3.6) and (3.9) yields

$$g_{E'}^{-1}(t) = g_{W'}^{-1}(t) + C \text{ for some constant } C \text{ and for all } t.$$

Hence

$$g_{E'}(s + C) = g_{W'}(s) \text{ for all } s$$

which, after differentiating, implies that

$$h_{E'}(s + C) = h_{W'}(s) \text{ for a. e. } s \in \mathbb{R}. \quad (3.12)$$

We claim that  $C = 0$ . Indeed, by (3.9) and (3.12)

$$\begin{aligned} 0 &= \int_{E'} x \cdot \xi dx = \int_{-\infty}^{+\infty} s h_{E'}(s) ds = \int_{-\infty}^{+\infty} (s + C) h_{E'}(s + C) ds \\ &= \int_{-\infty}^{+\infty} s h_{E'}(s + C) ds + C \int_{-\infty}^{+\infty} h_{E'}(s + C) ds \\ &= \int_{W'} x \cdot \xi dx + C \text{meas}(W_\Gamma) = C \text{meas}(W_\Gamma). \end{aligned}$$

Thus, and returning to the original notation,

$$h_{E', \xi}(s) = h_{W', \xi}(s) \text{ for a. e. } s \in \mathbb{R} \text{ and for all } \xi \in S^{N-1},$$

and so, due to the existence of the inverse of the Radon transform (see GELFAND, GRAEV & VILENKIN [9]) we conclude that  $\|\chi_{E'} - \chi_{W'}\|_1 = 0$ .

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