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# CHARACTERIZATIONS OF YOUNG MEASURES

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## Contents



## 1 *Introduction Variational Principles which lead to Parametrized Measure Solutions*

The use of variational methods to study equilibrium configurations of crystalline solids has led us to consider energy functionals which lack the property of lower semicontinuity.[1] In these circumstances the infimum of energy is achieved only in some generalized sense while a minimizing sequence may develop finer and finer oscillations, reminiscent of a finely twinned microstructure. The weak limit of a minimizing sequence for such a functional need not by itself characterize sufficiently many properties of the configuration, at least not in an obvious way. We trace the origins of this theory in thermoelasticity theory to Ericksen [24 - 35]. Our approach has

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[1] *Transitions and Defects in Ordered Materials.* Supported by the NSF and the ASOFR through grant DMS 871-8881 and the ARO through grant DAAL03 88 K 0110 at the University of Minnesota.

been to study the parametrized measures, or Young measures, generated by minimizing sequences whose function is to serve as an accounting device to summarize their oscillatory properties. The primary objective in this note is to characterize these measures.

The oscillatory properties of a weak* convergent sequence of gradients may be decoupled from its deformation properties, a localization property easily shown, cf. §2. Of greater interest is that oscillations may be coupled to a sequence and limit deformation assuming only a kinematic condition and technical condition. The possibility of assembling or coupling oscillations to a deformation in this generality was asked us by Richard James. This suggests the question of what measures, that is to say, ordinary measures not parametrized measures, may occur as limits of sequences of gradients. They are necessarily probability measures. It turns out that they may be characterized by a form of Jensen's inequality for a special class of quasiconvex functions, cf. §5. This is not quite the characterization we set out to prove, which would be in terms of continuous quasiconvex functions. The consequences of this disparity are of some interest in understanding the sort of approximations, or processes, which lead to complicated microstructures and are relevant to the nature of approximation by Lipschitz functions in general. We give a complete discussion of this at the end of §6.

The point of view we adopt here is similar to Ball and James [5,6] and [16,42]. Additional material about this and relaxation of functionals is given in Ball [3,4], Ball and Murat [7], Fonseca [37,38], and [45]. Ball and Zhang [9] have recently studied the relationship between lower semicontinuity and Young measures based on Chacon's Biting Lemma, cf. also Ball and Murat [8]. A complimentary point of view is adopted in [47], where the relationship between functional convergence and the Young measure representation is examined. Our investigation here was stimulated by the examination of Young measures which are supported on energy wells, which we discuss separately in [46]. In [55] parametrized measures are studied in connection to rank one convex and polyconvex functions. They are used to study questions about ferromagnetism in [43,44]. The relationship of Young measures to other types of compensation operators is discussed in [56]. Recent developments also include the role of Young measures in the numerical analysis of nonconvex functionals, especially by Luskin, Collins, and Chipot [15,18-21] and in the $\alpha$-$\beta$ transition in quartz by Matos [49]. Fonseca provides and interesting view of surface phenomena in [39,40]. Availaable microstructures and self accomodation in martensite are studied by Battacharya [11,12], who employs among his methods the averaging device we intorduce here in §2.

Young discovered that parametrized measures could serve as solutions to control problems which lacked classical solutions. There is an extensive literature about them, cf. Berliocchi and

Lasry [13], Warga [60], and Balder [2]. A recent application in control theory is given by Capuzzo Dolcetta and Ishii [14]. The use of Young measures in differential equations was first introduced by Tartar [57], especially to study scalar conservation laws, cf. also [58,59]. This subject has an extensive literature. Ball [4] gives a version of the existence theorem which is tailored for use in the calculus of variations. His paper also contains an historical introduction and references to some recent applications. A different version of the existence theorem appears in Evans [36].

Finally, we wish to remark that the methods of this investigation are completely elementary, relying on little more than Vitali's Covering Lemma and the Hahn Banach Theorem.

To introduce the Young measure in our context, we begin with the relationship between the minimization question for a functional $\mathcal{E}$ and its relaxation $\mathcal{E}^{\#}$. Given a bounded domain $\Omega \subset \mathbb{R}^n$, consider the functional

$$\mathcal{E}(v) = \int_\Omega W(\nabla v)\, dx, \quad v \in H^{1,\infty}(\Omega;\mathbb{R}^m). \tag{1.1}$$

Here we assume that $W$ is smooth and nonnegative.

The relaxation of $\mathcal{E}$ is given by the integrand

$$W^{\#}(A) = \inf_V \frac{1}{|\Omega|}\int_\Omega W(A + \nabla\zeta)\, dx, \quad A \in \mathbb{M}, \tag{1.2}$$
$$V = H^{1,\infty}_o(\Omega;\mathbb{R}^m),$$

where $\mathbb{M}$ denotes $m \times n$ matrices. It is known that $W^{\#}$ is continuous, quasiconvex, and independent of the choice of the domain $\Omega$, as long as we insist that $|\partial\Omega| = 0$, Dacorogna [22,23], Ball and Murat [7].

The notion of quasiconvexity to characterize integrands of lower semicontinuous functionals was introduced by Morrey [51]. A function $\varphi: \mathbb{M} \to \mathbb{R}$ is quasiconvex provided

$$\varphi(A) \le \frac{1}{|\Omega|}\int_\Omega \varphi(A + \nabla\zeta)\, dx \quad \text{for} \quad A \in \mathbb{M} \quad \text{and} \quad \zeta \in H^{1,\infty}_o(\Omega;\mathbb{R}^m). \tag{1.3}$$

Morrey proved that the functional

$$\Phi(v) = \int_\Omega \varphi(\nabla v)\, dx$$

with $\varphi$ continuous is sequentially lower semicontinuous in $H^{1,\infty}(\Omega;\mathbb{R}^m)$ in the weak* topology if and only if $\varphi$ is quasiconvex. In particular, the functional

$$\mathcal{E}^{\#}(v) = \int_\Omega W^{\#}(\nabla v)\, dx\,, \quad v \in H^{1,\infty}(\Omega;\mathbb{R}^m)\,, \tag{1.4}$$

is lower semicontinuous.

Under these hypotheses about W, one may deduce the relaxation property

$$\begin{aligned} &\inf_{\mathcal{A}} \int_\Omega W(\nabla v)\, dx = \inf_{\mathcal{A}} \int_\Omega W^{\#}(\nabla v)\, dx\,, \\ &\mathcal{A} = \mathcal{A}_\Omega(y_o) = \{\, v \in H^{1,\infty}(\Omega;\mathbb{R}^m) \colon v = y_o \text{ on } \partial\Omega \,\}\,, \end{aligned} \tag{1.5}$$

where $y_o \in H^{1,\infty}(\Omega;\mathbb{R}^m)$ is given. For the proof of this we refer to Dacorogna [23]. Extensions of Morrey's theorem under differing hypotheses about the smoothness of W or its dependence on other variables are discussed by Acerbi and Fusco [1], Ball and Murat [7], and also [48].

If W is not quasiconvex, some information is lost by seeking minima of $\mathcal{E}^{\#}$. It seems sensible to regard the Young measure as a means of summarizing the oscillatory properties of a minimizing sequence of (1.1), thus conserving at least some of that information.

For any sequence $(z^k) \subset L^\infty(\Omega;\mathbb{R}^N)$ with $\sup |z^k| \le C$ and $z^k \to z$ in $L^\infty(\Omega;K)$ weak*, where $K = \{\, \xi \in \mathbb{R}^N\colon |\xi| \le C \,\}$, we may find a family $\nu = (\nu_x)_{x \in \Omega}$ of probability measures such that whenever $\psi(\xi,x)$ is continuous in $\xi$ and bounded in x, and a subsequence of the $(z^k)$ which we do not relabel, such that

$$\psi(z^k(x),x) \to \bar{\psi}(x) \qquad \text{in } L^\infty(\Omega) \text{ weak*},$$

where

$$\bar{\psi}(x) = \int_K \psi(\xi,x)\, d\nu_x(\xi) \quad \text{a.e. in } \Omega. \tag{1.6}$$

The converse also holds. Given a family $(\nu_x)_{x\in\Omega}$ of probability measures in $M(K)$, there exists a sequence $(z^k) \subset L^\infty(\Omega;\mathbb{R}^N)$ with the property (1.12). Since

$$L^1(\Omega;C(K))' = L^\infty(\Omega;M(K)),$$

these remarks amount to characterizing the weak* closure of the measures

$$(\mu) = ((\mu_x)_{x\in\Omega}) \subset L^\infty(\Omega;M(K)) \quad \text{for which}$$
$$\mu_x = \delta_{f(x)} \quad \text{for some } f \in L^\infty(\Omega;K).$$

This is discussed in Dacorogna [22], Tartar [57], and Young [61], for example. One form of Jensen's inequality is that whenever $\varphi$ is convex

$$\int_\Omega \varphi(z)\,dx \le \int_\Omega \int_K \varphi(\xi)\,d\nu_x(\xi)\,dx$$

where $$z(x) = \int_K \xi\,d\nu_x(\xi), \quad x \in \Omega. \tag{1.7}$$

Jensen's Inequality characterizes proability measures: If (1.7) holds for all convex $\varphi$, then $\nu = (\nu_x)_{x\in\Omega}$ is a family of probability measures.

The measures we intend to consider here are distinguished by the constraint that they are limits of *gradients*. This places restrictions on their structure. Implicit in what we have written is a second constraint, which is that the sequence determining the measure is bounded in $H^{1,\infty}(\Omega;\mathbb{R}^m)$. Let us formalize this by agreeing that

$\nu = (\nu_x)_{x\in\Omega}$ is a *parametrized measure* or *Young measure* provided there is a sequence $(y^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$y^k \to y \quad \text{in } H^{1,\infty}(\Omega;\mathbb{R}^m) \text{ weak*},$$
$$F^k = \nabla y^k, \; F = \nabla y, \quad \text{and} \tag{1.8}$$
$(F^k)$ generates the parametrized measure $(\nu_x)_{x\in\Omega}$ in the sense that

$$\psi(F^k(x)) \to \bar{\psi}(x) = \int_M \psi(A)\,d\nu_x(A) \quad \text{in } L^\infty(\Omega) \text{ weak*}$$

whenever $\psi \in C(\mathbb{M})$.

If $\nu = (\nu_x)_{x \in \Omega}$ does not depend on $x \in \Omega$, we shall say the measure is *homogeneous* and refer to it simpy by $\nu$. We shall refer to $y(x)$ or $F(x) = \nabla y(x)$ as the *underlying deformation* for $\nu = (\nu_x)_{x \in \Omega}$. Since $F^k \rightarrow F$ in $L^\infty(\Omega)$ weak*, $F$ is the first moment of $\nu_x$, namely

$$F(x) = \langle \nu_x, A \rangle = \int_{\mathbf{M}} A \, d\nu_x(A) \quad \text{a.e. in } \Omega,$$

or $\psi(A) = A$ is a weak* continuous function on $H^{1,\infty}(\Omega;\mathbb{R}^m)$. We remind the reader that other known weak* continuous functions are the minors of the matrix of $A$.

An immediate consequence of (1.8) is a version of Jensen's Inequality for quasiconvex functions. If $\varphi$ is quasiconvex in Morrey's sense, then, for any subset $D \subset \Omega$,

$$\int_D \varphi(F) \, dx \le \int_D \int_{\mathbf{M}} \varphi(A) \, d\nu_x(A) dx \,,$$

where (1.9)

$$F(x) = \int_{\mathbf{M}} A \, d\nu_x(A) \,, \quad x \in \Omega.$$

Our major objective here is to understand the manner in which (1.9) characterizes parametrized measures generated by sequences of gradients. The principal results are stated in THEOREM 5.1 and THEOREM 6.1.

Another consequence of (1.8) is that

$$\operatorname{supp} \nu_x \subset K \,, \qquad x \in \Omega,$$

for any compact $K$ with $F^k(x) \in K$ for all k. It will be useful for us to keep in mind the converse of this statement.

PROPOSITION 1.2 *Let* $(u^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ *satisfy*

$$\limsup_{k \to \infty} \int_\Omega \psi(\nabla u^k) \, dx < \infty \qquad \textit{whenever} \qquad \psi \in C(\mathbf{M}) \,. \tag{1.10}$$

*Then* $(u^k)$ *are bounded in* $H^{1,\infty}(\Omega;\mathbb{R}^m)$.

Hence there is a subsequence $(u^k)$ of the $(u^k)$ which generates a parametrized measure $(\nu_x)_{x\in\Omega}$ and a compact $K \subset M$ with $\operatorname{supp}\nu_x \subset K, x \in \Omega$. Thinking slightly differently, we may know for some reason that

$$\lim_{k\to\infty} \int_\Omega \psi(\nabla u^k)\, dx = \int_K \psi(A)\, d\mu(A), \text{ whenever } \psi \in C(M),$$

for some measure $\mu$ with compact support K. The proposition then informs us that the sequence $(u^k)$ is bounded in $H^{1,\infty}(\Omega;\mathbb{R}^m)$.

PROOF Assume that

$$\limsup_{k\to\infty} \|\nabla u^k\|_{L^\infty(\Omega)} = \infty.$$

By choosing a subsequence and relabeling, we may assume that $\|\nabla u^k\|_{L^\infty(\Omega)} > k$. Let $E^k = \{ |\nabla u^k| > k \}$ and $\alpha_k = |E^k|$. Select $\varphi \in C(M)$, any function satisfying

$$\lim_{k\to\infty} \alpha_k\, \varphi(k) = \infty.$$

Then by hypothesis, there is a constant $C = C(\varphi)$ such that

$$\alpha_k\, \varphi(k) \le \int_\Omega \varphi(\nabla u^k)\, dx \le C \quad \text{for all } k,$$

which is a contradiction. QED

We raise this issue to distinguish between what we have called Young measures or parametrized measures in (1.8) and families of probability measures which arise in the same way but may satisfy (1.8) only for $\psi \in C_0(M)$, that is $\psi$ such that $\lim_{|A|\to\infty} \psi(A) = 0$. These latter are also called Young measures in some of the literature.

For any $u \in H^{1,\infty}(\Omega;\mathbb{R}^m)$ we may define the the measure $\delta_{\nabla u(x)}$ and regard it as an element of the dual space $L^1(\Omega;C_0(M))'$. Given a sequence $(u^k)$, the set of measures $(\delta_{\nabla u^k})$

has a weak* convergent subsequence with limit, say, $(\nu_x)_{x\in\Omega}$. Under the very mild condition that

$$\lim_{r\to\infty} \sup_k \left| \{ |\nabla u^k| > r \} \right| = 0,$$

$(\nu_x)_{x\in\Omega}$ is even a family of probability measures, cf. Ball [4], and may also all have the same compact support. Unfortunately, it may be difficult to recover much information about the nature of oscillations of $\nabla u^k$ if we cannot verify the formula in (1.8) for a sufficiently wide class of functions $\psi$. Necessary for this is that the sequence $(\psi(\nabla u^k))$ converges weakly in $L^1(\Omega)$.

One situation of interest here is simply the case of a sequence $(u^k)$ bounded in $H^{1,p}(\Omega)$. Such a sequence defines some sort of Young measure, the function $\psi(A) = |A|^p$ is integrable with respect to this measure, but $|\nabla u^k|^p$ need not converge weakly and the representation formula (1.8) may fail. However, if $(u^k)$ is a *minimizing sequence* weakly convergent to $u \in H^{1,p}(\Omega)$ for a functional of the form (1.1) with the property that

$$0 \le W(A)$$
$$c|A|^p - 1 \le W(A) \le C|A|^p + 1,$$

one may indeed show that (for a subsequence)

$$W(\nabla u^k) \to W^\#(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly} \quad \text{and}$$
$$\overline{W}^\#(x) = \overline{W}(x) = W^\#(F(x)),$$

cf. [47] and Matos [50].

There are aspects of our work which may be applied to other compensation conditions as well. By this we mean parametrized measures which may arise as the weak or weak* limits of vector valued functions $u(x)$ satisfying

$$Eu = 0 \qquad \text{in } \Omega,$$

$$E^i u = \sum_{j,k} a^{ijk} \frac{\partial u^j}{\partial x_k}, \qquad i = 1, \dots, N.$$

Murat and Tartar have written extensively about this [52 - 54,58 - 60].

## 2 *Averaging and localization of parametrized measures*

Among the elementary devices for analyzing parametrized measures are averaging and localization. Localization is the decoupling mentioned in the introduction. We discuss them in turn. Recall that our measures are constrained in the sense that they arise from gradients.

THEOREM 2.1 *Suppose that* $\Omega$ *and* D *are domains in* $\mathbb{R}^m$ *with* $|\partial\Omega| = |\partial D| = 0$. *Let* $\nu = (\nu_x)_{x\in\Omega}$ *be a parametrized measure with underlying deformation* $y(x), x \in \Omega$, *which has the properties*

$$\operatorname{supp}\nu_x \subset K, \qquad a.e.\ in\ \Omega\ for\ a\ fixed\ compact\ K \subset \mathbb{M},\ and$$
$$y(x) = y_o(x) = F_o x \quad on\ \partial\Omega,$$

*where* $F_o$ *is a fixed* $m \times n$ *matrix. Then the family of measures* $(\bar{\nu}_x)_{x\in D}$ *given by* $\bar{\nu}_x = \bar{\nu}$, *where*

$$\langle \bar{\nu}, \psi \rangle = \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\nu_x(A)\, dx \quad , \tag{2.1}$$

*is a parametrized measure. Moreover,*

$$\operatorname{supp}\bar{\nu} \subset K \quad and \quad F_o = \langle \bar{\nu}, A \rangle . \tag{2.2}$$

PROOF Suppose that

$$y^k \to y \quad \text{in } H^{1,\infty}(\Omega;\mathbb{R}^m) \text{ weak*}, \qquad y^k = y_o \quad \text{on } \partial\Omega, \text{ and}$$
$$(y^k) \quad \text{generates the parametrized measure} \quad \nu = (\nu_x)_{x\in\Omega}.$$

We may suppose that $0 \in \Omega$. Given an integer $k$, the collection of sets $\{ a + \varepsilon\bar{\Omega}:\ a \in D,\ \varepsilon < k^{-1} \}$ is a covering of D from which, by the Vitali covering theorem, we may select a countable or finite subset $\{ a_i + \varepsilon_i\bar{\Omega}: i = 1,2,3 \ldots\}$ of pairwise disjoint sets such that

$$\bar{D} = \bigcup (a_i + \varepsilon_i \bar{\Omega}) \cup N, \quad |N| = 0.$$

Note that $\Sigma (\varepsilon_i)^n |\Omega| = |D|$. Define

$$u^k(x) = \begin{cases} \varepsilon_i y^k(\frac{x - a_i}{\varepsilon_i}) + F_o a_i & x \in a_i + \varepsilon_i \Omega \\ y_o(x) & \text{otherwise} \end{cases}, \quad x \in D,$$

and let $\psi \in C(K)$ and $\zeta \in C(D)$. We compute the integral

$$\int_D \psi(\nabla u^k)\, \zeta\, dx = \Sigma_i \int_{a_i + \varepsilon_i \Omega} \psi(F^k(\frac{x - a_i}{\varepsilon_i}))\, \zeta(x)\, dx$$

$$= \Sigma_i\, \varepsilon_i^{-n} \int_\Omega \psi(F^k(\xi))\, \zeta(a_i + \varepsilon_i \xi)\, d\xi\,,$$

for $x = a_i + \varepsilon_i \xi$. By the mean value theorem, there are points $\xi_i \in \Omega$ such that

$$\int_D \psi(\nabla u^k)\, \zeta\, dx = \frac{1}{|\Omega|} \int_\Omega \psi(F^k(\xi))\, d\xi\, \Sigma_i\, \zeta(a_i + \varepsilon_i \xi_i)\, \varepsilon_i^n\, |\Omega|\,.$$

Observing that the sum in the last term is a Riemann sum for the integral of $\zeta$, we see easily that for a subsequence of $k$, which we do not distinguish from the original sequence,

$$\lim_{k \to \infty} \int_D \psi(\nabla u^k)\, \zeta\, dx = \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\nu_x(A) dx \int_D \zeta\, d\xi\,.$$

Thus the sequence $u^k$ determines the Young measure $(\bar{\nu}_x)_{x \in D}$ with the property

$$\int_\Omega \int_K \psi(A) \zeta(x)\, d\bar{\nu}_x(A) dx = \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\nu_x(A) dx \int_D \zeta\, d\xi\,.$$

To show that this Young measure is homogeneous, given $a \in \Omega$, let $\zeta_\rho = |B_\rho|^{-1} \chi_{B_\rho}$ and compute that

$$\int_K \psi(A)\, d\bar{\nu}_a(A) = \lim_{\rho\to 0} \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\nu_x(A)dx \int_D \zeta_\rho \, d\xi$$

$$= \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\nu_x(A)dx \;,$$

for almost every a, which is independent of a .

Finally note that for any $y \in H^{1,\infty}(D;\mathbb{R}^m)$ with $y = y_o$ on $\partial D$,

$$\int_D \nabla y \, dx = \int_{\partial D} y\cdot n \, dS = \int_{\partial D} y_o\cdot n \, dS = F_o \, |D| .$$

In particular,

$$\langle \bar{\nu}, A \rangle |D| = \lim_{k\to\infty} \int_D \nabla u^k \, dx = F_o \, |D| .$$

QED

We remark that $\bar{\nu}$ is generated by a sequence $(u^k)$ with

$$\text{range } \nabla u^k \subset G \quad \text{whenever} \quad \bigcup \text{range } \nabla y^j \subset G.$$

In (1.8) we wrote

$$\bar{\psi}(x) = \int_K \psi(A)\, d\nu_x(A)$$

as the weak* limit of $\psi(F^k)$. In this way we may rewrite formula (2.1) as

$$\langle \bar{\nu}, \psi \rangle = \frac{1}{|\Omega|} \int_\Omega \bar{\psi}(x)\, dx \;.$$

We caution the reader again that the average Young measure $\bar{\nu}$ is different from the original Young measure $\nu$. In particular for a product $\psi(A)\zeta(x)$,

$$\int_\Omega \langle \nu_x, \psi\zeta \rangle\, dx \;=\; \int_\Omega \int_K \psi(A)\zeta(x)\, d\nu_x(A)dx$$

whereas

$$\int_D \langle \bar{\nu}_\xi, \psi\zeta \rangle\, d\xi \;=\; \frac{1}{|\Omega|}\int_\Omega \int_K \psi(A)\, d\nu_x(A)dx \int_D \zeta\, d\xi\ .$$

Averaging a family of measures defines a mapping between spaces of measures. If $\mu \in L^\infty(\Omega;\mathcal{M}(K))$, then its average

$$\mathrm{Av}\,\mu \;=\; \bar{\mu} \;\in\; \mathcal{M}(K)$$

where

$$\langle \bar{\mu}, \psi \rangle \;=\; \frac{1}{|\Omega|}\int_\Omega \int_K \psi(A)\, d\mu_x(A)\, dx \quad , \ \psi \in C(K). \tag{2.3}$$

Let $d^*$ denote the distance in the unit ball of $\mathcal{M}(K)$ and $\delta^*$ denote the distance in the unit ball of $L^\infty(\Omega;\mathcal{M}(K))$. For example, let $\{\eta_j\}$ be a dense sequence in $L^1(\Omega)$ and $\{\psi_j\}$ be a dense sequence in $C(K)$ satisfying

$$\int_\Omega |\eta_j|\, dx \;=\; 1, \quad \eta_1(x) \equiv 1, \quad \text{and} \quad \sup|\psi_j| \;=\; 1.$$

Then we may write

$$d^*(\mu,\mu') \;=\; \sum 2^{-j}\, |\langle \mu - \mu', \psi_j \rangle|, \quad \mu, \mu' \in \mathcal{M}(K)$$

and

$$\delta^*(\mu,\mu') \;=\; \sum 2^{-(j+k)}\, |\langle \mu - \mu', \eta_j \otimes \psi_k \rangle|, \quad \mu, \mu' \in L^\infty(\Omega;\mathcal{M}(K)).$$

Since $\eta_1(x) \equiv 1$,

$$d^*(\mu,\mu') \;\le\; \delta^*(\bar{\mu}, \bar{\mu}')\ .$$

Hence averaging is continuous. We summarize this using THEOREM 2.1. Introduce the notations

$$Y(\Omega;\mathcal{M}(K)) \;=\; \text{the parametrized measures } \nu \text{ with } \operatorname{supp}\nu_x \subset K, \text{ and}$$

$$\bar{Y}(K) = \text{the homogeneous parametrized measures } \bar{\nu} \text{ with supp } \bar{\nu}_x \subset K,$$

where $K \subset \mathbb{M}$ is compact. Recall that for homogeneous Young measures, we do not have to specify the domain $\Omega$ since any (reasonable) $\Omega$ may be chosen as the domain of a sequence of functions which generates $\bar{\nu}$.

PROPOSITION 2.2 *The mapping*

$$\begin{aligned} Av:\ Y(\Omega;\mathcal{M}(K)) &\to \bar{Y}(K) \\ \nu &\to \bar{\nu} \end{aligned}$$

*defined by* (2.3) *is continuous.*

Note that we do not claim that these sets of parametrized measures are closed.

Localization will enable us to interpret the family of measures $(\mu_x)_{x \in Q}$ given by $\mu_x = \nu_a$, for a fixed $a \in \Omega$, as a parametrized measure for almost every $a$. In other words, $\nu_a$ is a homogeneous parametrized measure for almost every $a$. Here $Q$ is the unit cube in $\mathbb{R}^n$ with center at $x = 0$. This decouples the oscillatory properties of the sequence which determines the parametrized measure from the underlying limit deformation. This and other localization properties are based on two elementary facts: translation is continuous in $L^1(\Omega)$ and the spaces $C(K)$ and $L^1(\Omega)$ are separable. We refer also to [16].

THEOREM 2.3 *Let* $\nu$ *be a parametrized measure. Then* $\nu_a$ *is a homogeneous parametrized measure for almost every* $a \in \Omega$. *If* $(y^k) \subset H^{1,\infty}(\Omega)$ *is a sequence which generates* $\nu$ *and* $\|\nabla y^k\|_{L^\infty(\Omega)} \le M$, *for all* k, *then there is a sequence* $(u_a^k) \subset H^{1,\infty}(\Omega)$ *which generates* $\nu_a$ *with the property that* $\|\nabla u_a^k\|_{L^\infty(\Omega)} \le M$, *for all* $k$.

PROOF Note that given any $f \in L^\infty(\Omega)$,

$$f(a + \varepsilon x) \to f(a) \quad \text{in } L^1(Q), \text{ a.e. in } \Omega. \tag{2.4}$$

This is a restatement of the translation property, namely, given $\zeta \in L^\infty(Q)$,

$$\left| \int_\Omega \int_Q (f(a + \varepsilon x) - f(a))\, \zeta(x)\, dx da \right| = \left| \int_Q \int_\Omega (f(a + \varepsilon x) - f(a))\, \zeta(x)\, da dx \right|$$

$$\leq \ \sup_x \| f(\cdot + \varepsilon x) - f(\cdot) \|_{L^1(\Omega)} \| \zeta \|_{L^\infty(Q)} \ \to \ 0 \,,$$

as $\varepsilon \to 0$. Hence by Lebesgue's Theorem,

$$\int_Q f(a + \varepsilon x)\, \zeta(x)\, dx \quad \to \quad f(a) \int_Q \zeta(x)\, dx \ \text{ as } \ \varepsilon \to 0, \ a \in \Omega \text{ a.e.}$$

Suppose now that $( \nu_x )_{x \in \Omega}$ is a parametrized measure. Introduce the local spatial average

$$\langle \nu^k_{a,\rho}, \psi \rangle \ = \ \rho^{-n} \int_{a + \rho Q} \psi(F^k)\, dx$$

$$= \int_Q \psi(F^k(a + \rho x))\, dx \ ,$$

for $a \in \Omega$ and $\rho > 0$. By the weak* convergence,

$$\lim_{k \to \infty} \langle \nu^k_{a,\rho}, \psi \rangle \ = \ \int_Q \int_K \psi(A)\, d\nu_{a + \rho x}(A) dx$$

$$= \int_Q \bar{\psi}(a + \rho x)\, dx \ .$$

By (2.3), or simply because almost every $a \in \Omega$ is a Lebesgue point of $\bar{\psi}$,

$$\lim_{\rho \to 0} \lim_{k \to \infty} \langle \nu^k_{a,\rho}, \psi \rangle \ = \ \bar{\psi}(a) \ = \ \langle \nu_a, \psi \rangle, \quad \text{a.e. in } \Omega. \tag{2.5}$$

What we wish to point out is that a subsequence of $(k,\rho)$ may be chosen so that $( \mu_x )_{x \in Q}$ with $\mu_x = \nu_a$ for each $x \in Q$ is a parametrized measure. It suffices to choose a subsequence of the functions

$$\eta^{k,\rho}(x) \ = \ \frac{1}{\rho}\, (y^k(a + \rho x) - y^k(a))\,,$$
$$\nabla \eta^{k,\rho}(x) \ = \ F^k(a + \rho x). \tag{2.6}$$

One merely observes that for $a \in \Omega$ satisfying (2.5),

$$\lim_{\rho \to 0} \lim_{k \to \infty} \int_Q \psi(\nabla \eta^{k,\rho}) \, \zeta \, dx$$

$$= \lim_{\rho \to 0} \lim_{k \to \infty} \int_Q \psi(F^k(a + \rho x)) \, \zeta(x) \, dx$$

$$= \lim_{\rho \to 0} \int_Q \bar{\psi}(a + \rho x) \, \zeta(x) \, dx$$

$$= \bar{\psi}(a) \int_Q \zeta(x) \, dx \; .$$

Hence for a subsequence $( u_a^k )$ of the $( \eta^{k,\rho} )$,

$$\lim_{j \to \infty} \int_Q \psi(\nabla u_a^k) \, \zeta \, dx = \int_K \psi(A) \, d\nu_a(A) \int_Q \zeta(x) \, dx \; .$$

QED

## 3 *Coupling of measures to parametrized measures*

In this section, our objective is to show how a family of measures may be assembled or coupled to become a Young measure. This will be the converse of THEOREM 2.3.

THEOREM 3.1 *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with* $|\partial\Omega| = 0$. *Let* $(\nu_a)_{a \in \Omega}$ *be a family of measures on* $\mathbb{M}$ *with the properties*

(i) *there is a* $y \in H^{1,\infty}(\Omega;\mathbb{R}^m)$ *such that*

$$\nabla y(x) = \int_{\mathbf{M}} A \, d\nu_x(A) , \quad x \in \Omega \; a.e., \tag{3.1}$$

(ii) $\nu_a$ *is a homogeneous parametrized measure for* $a \in \Omega$, *a.e., and*

(iii) *there are sequences* $(u_a^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ *which generate* $\nu_a$ *such that*

$$\| u_a^k \|_{H^{1,\infty}(\Omega)} \leq M \quad \text{for all } k,\ a \in \Omega \text{ a.e.} \tag{3.2}$$

*Then* $\nu = (\nu_x)_{x \in \Omega}$ *is a parametrized measure with underlying deformation* $y(x)$.

Knowing the sequences $(u_a^k)$ permits us to construct a sequence which generates $\nu$, but this sequence is not unique. Two sequences may generate the same Young measure but their difference need not converge in measure.

LEMMA 3.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with* $|\partial\Omega| = 0$ *and let* $N \subset \Omega$ *be a null set. Given a countable family* $\{f_j\} \subset L^1(\Omega)$ *and functions* $r_k: \Omega - N \to \mathbb{R}^+$, *there is a set of points* $\{a_{ki}\} \subset \Omega - N$ *and positive numbers* $\varepsilon_{ki}$, $\varepsilon_{ki} < r_k(a_{ki})$, *such that*

$\{a_{ki} + \varepsilon_{ki}\bar{\Omega}\}$ *are pairwise disjoint for each* $k$,

$\bar{\Omega} = \bigcup\{a_{ki} + \varepsilon_{ki}\bar{\Omega}\} \cup N_k$, *where* $|N_k| = 0$, *and*

$$\int_\Omega f_j\,dx = \lim_{k\to\infty} \Sigma_i f_j(a_{ki})\,|\varepsilon_{ki}\bar{\Omega}|, \quad \textit{for each } j.$$

PROOF Let $D \subset \Omega$ be the intersection of the sets of Lebesgue points of the $f_j$'s and set $E = D \setminus N$. For each $k$ the family

$$\mathcal{F}_k = \Big\{ a + \varepsilon\bar{\Omega}:\ a \in D\setminus N,\ \varepsilon \leq r_k(a),\ \frac{1}{|\varepsilon\Omega|}\int_{a+\varepsilon\Omega} |f_j(x) - f_j(a)|\,dx < \frac{1}{k},$$
$$1 \leq j \leq k \text{ and } a + \varepsilon\bar{\Omega} \subset \Omega \Big\}$$

covers $E$ in the Vitali sense by the Lebesgue Differentiation Theorem. Hence we may write

$$D\setminus N = \bigcup\{a_{ki} + \varepsilon_{ki}\bar{\Omega}\} \cup N_k', \quad |N_k'| = 0,$$

or

$$\Omega = = \bigcup\{a_{ki} + \varepsilon_{ki}\bar{\Omega}\} \cup N_k, \text{ where } |N_k| = 0.$$

Now for fixed j and $k \geq j$,

$$\left| \int_\Omega f_j \, dx - \Sigma_i f_j(a_{ki}) \, | \varepsilon_{ki} \bar{\Omega} | \right| = \left| \Sigma_i \int_{a_{ki} + \varepsilon_{ki}\bar{\Omega}} ( f_j(x) - f_j(a_{ki}) ) \, dx \right|$$

$$\leq \Sigma_i \int_{a_{ki} + \varepsilon_{ki}\bar{\Omega}} | f_j(x) - f_j(a_{ki}) | \, dx$$

$$\leq \frac{1}{k} \Sigma_i \, | \varepsilon_{ki} \bar{\Omega} |$$

$$= \frac{1}{k} \, | \Omega | .$$

This proves the LEMMA. QED

PROOF of THEOREM 3.1 Step 1 To prove that a sequence $( y^k ) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ generates $\nu$, it suffices to verify the formula

$$\int_\Omega \int_M \zeta(x) \, \psi(A) \, d\nu_x(A) \, dx = \lim_{k \to \infty} \int_\Omega \zeta \psi(\nabla y^k) \, dx \tag{3.3}$$

for a countable set of products whose linear combinations are dense in $L^1(\Omega;C(\bar{B}_M))$, $\bar{B}_M = \{ |A| \leq M \} \subset \mathbb{M}$. We may suppose that $\zeta \in C(\bar{\Omega})$ and $\psi \in C(\bar{B}_M)$ and that $\sup |\zeta| \leq 1$ and $\sup |\psi| \leq 1$. We denote by $C$ the set of all such products $\zeta(x)\psi(A)$. Let

$$\bar{\psi}(x) = \int_M \psi(A) \, d\nu_x(A) \in L^1(\Omega) . \tag{3.4}$$

Step 2 Vitali Theorem and application of the LEMMA. Set $F(a) = \nabla y(a)$. By Rademacher's Theorem, y is differentiable in $\Omega$ a.e. So let N be the null set where y fails to be differentiable and where (3.1) fails. For $a \in \Omega - N$ and k a natural number, there is an $r_k(a) > 0$ such that

$$| y(a + \varepsilon z) - y(a) - \varepsilon F(a) z | \leq \frac{\varepsilon}{k} \quad \text{for } z \in \Omega \text{ and } \varepsilon < r_k(a). \tag{3.5}$$

We apply the Lemma to the set of $f = \zeta \, \bar{\psi}$, $\zeta\psi \in C$, and $r_k(a)$ as above. Here, $\bar{\psi}$ is defined by (3.4). Thus there are $\{ a_{ki} \} \subset \Omega - N$ and $\varepsilon_{ki} \leq r_k(a_{ki})$ such that

$$\int_\Omega \zeta\ \bar{\psi}\, dx \;=\; \lim_{k\to\infty} \Sigma_i\, \zeta(a_{ki})\ \bar{\psi}(a_{ki})\,|\,\varepsilon_{ki}\Omega\,|\,,\quad \zeta\otimes\psi\in \mathcal{C}. \tag{3.6}$$

Step 3 Construction of $y^k$. Choose a sequence $\eta^k$ of smooth cut-off functions such that

$$\eta^k = 0 \quad \text{in } \Omega_k = \{\,\mathrm{dist}(x,\partial\Omega) \ge 1/k\,\},$$
$$\eta^k = 1 \quad \text{on } \partial\Omega, \text{ and}$$
$$|\,\nabla\eta^k\,| \le 2k\,.$$

Now $\nu_{a_{ki}}$ is a homogeneous Young measure since $a_{ki} \notin N$. Let $(\,u^h_{ki}\,)$ denote a sequence which generates $\nu_{a_{ki}}$ and satisfies

$$|\,\nabla u^h_{ki}\,| \le M \quad \text{in } \Omega \quad \text{and}$$
$$u^h_{ki} \to F(a_{ki})z \quad \text{in } H^{1,\infty}(\Omega) \text{ weak*}.$$

Define

$$y^k(x) = \Big(y(a_{ki}) + \varepsilon_{ki}\, u^h_{ki}\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big)\Big(1-\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big) + \eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\, y(x) \qquad x\in a_{ki}+\varepsilon_{ki}\Omega \tag{3.7}$$
$$y^k(x) = y(x) \quad \text{otherwise,}$$

where $h = h(k,i)$ will be chosen later. Thus

$$\begin{aligned}
\nabla y^k(x) &= \nabla u^h_{ki}\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big(1-\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big) + \eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\, F(x) \\
&\quad + \frac{1}{\varepsilon_{ki}}\Big[y(x) - y(a_{ki}) - \varepsilon_{ki} u^h_{ki}\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big] \otimes \nabla\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big) \\
&= \nabla u^h_{ki}\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big(1-\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big) + \eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\, F(x) \\
&\quad + \frac{1}{\varepsilon_{ki}}\Big[y(x) - y(a_{ki}) - \varepsilon_{ki} F(a_{ki})\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big] \otimes \nabla\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big) \\
&\quad + \Big[F(a_{ki})\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big) - u^h_{ki}\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big)\Big] \otimes \nabla\eta^k\big(\tfrac{x-a_{ki}}{\varepsilon_{ki}}\big) \\
&= I_k + II_k + III_k + IV_k\,.
\end{aligned} \tag{3.8}$$

We first show that the $|\nabla y^k|$ are bounded for suitable choice of the sequence $h = h(k,i)$. From the choice of $(u^h_{ki})$,

$$|I_k| \le M \quad \text{and} \quad |II_k| \le M.$$

Since $a_{ki} \in \Omega - N$, by (3.5)

$$|III_k| \le \frac{1}{\varepsilon_{ki}} \frac{\varepsilon_{ki}}{k} 2k = 2.$$

Finally, $u^h_{ki}(z) \to F(a_{ki})z$ uniformly in $\overline{\Omega}$, so for $h = h(k,i)$ sufficiently large,

$$|u^h_{ki}(z) - F(a_{ki})z| \le \frac{1}{k}. \tag{3.9}$$

Hence

$$|IV_k| \le \frac{1}{k} 2k = 2. \tag{3.10}$$

Step 4 Generation of $\nu_x$. Since $(y^k)$ is bounded in $H^{1,\infty}(\Omega)$, it generates a parametrized measure, which we must show is $\nu$. For each $\zeta \otimes \psi \in C$ and $\varepsilon > 0$, for $k$ sufficiently large,

$$\begin{aligned}
\int_\Omega \zeta\psi(\nabla y^k)\,dx &= \sum_i \int_{a_{ki}+\varepsilon_{ki}\Omega} \zeta\psi(\nabla y^k)\,dx \\
&= \sum_i \varepsilon^n_{ki} \int_\Omega \zeta(a_{ki}+\varepsilon_{ki}z)\psi(\nabla y^k(a_{ki}+\varepsilon_{ki}z))\,dz \\
&= \sum_i \varepsilon^n_{ki}\,\zeta(a_{ki}+\varepsilon_{ki}z_{ki}) \int_\Omega \psi(\nabla y^k(a_{ki}+\varepsilon_{ki}z))\,dz \\
&= \sum_i \varepsilon^n_{ki}\,\zeta(a_{ki}+\varepsilon_{ki}z_{ki}) \int_{\Omega_k} \psi(\nabla u^h_{ki}(z))\,dz + \frac{C}{k}
\end{aligned}$$

$$= \sum_i \varepsilon^n_{ki}\, \zeta(a_{ki} + \varepsilon_{ki} z_{ki}) \int_\Omega \psi(\nabla u^h_{ki}(z))\, dz + \frac{C}{k}$$

$$= \sum_i \varepsilon^n_{ki}\, \zeta(a_{ki}) \int_\Omega \psi(\nabla u^h_{ki}(z))\, dz + \frac{C}{k} + \varepsilon . \tag{3.11}$$

Here we have used the continuity of $\zeta$, the boundedness of $\zeta$ and $\psi$, and the smallness of $|\Omega - \Omega_k|$. Choose an index $\iota = \iota(k)$ so that

$$\sum_{i \geq \iota} \varepsilon^n_{ki} \leq \frac{1}{k} . \tag{3.12}$$

For $i \leq \iota$, choose $h = h(k,i)$ so large that the weak star distance in $C(B_M)'$ satisfies

$$d^*(\mu^h_{ki}, \nu_{a_{ki}}) \leq \frac{1}{k}, \text{ where } \mu^h_{ki} \text{ is the average of } \delta_{\nabla u^h_{ki}} .$$

Thus if $\psi$ is the Nth function in the list of the $\psi$'s, then

$$|\langle \mu^h_{ki}, \psi \rangle - \langle \nu_{a_{ki}}, \psi \rangle| \leq \frac{1}{k} 2^N . \tag{3.13}$$

Using (3.12) and (3.13) in (3.11) yields that

$$\int_\Omega \zeta \psi(\nabla y^k)\, dx = \sum_i \varepsilon^n_{ki}\, \zeta(a_{ki})\ \bar{\psi}(a_{ik}) + \frac{C}{k} + \varepsilon + \frac{1}{k} 2^N .$$

Invoking Step 2, we conclude that

$$\lim_{k \to \infty} \int_\Omega \zeta \psi(\nabla y^k)\, dx = \int_\Omega \zeta\ \bar{\psi}\, dx .$$

QED

## 4 *Convex combinations of parametrized measures*

Our characterization of measures which are generated by sequences of gradients relies on the Hahn Banach Theorem. For this reason it is useful to understand convex combinations of parametrized measures. Throughout we let $\Omega$ be a domain with $|\partial\Omega| = 0$.

THEOREM 4.1 *Let* $\nu$ *and* $\nu^*$ *be homogeneous parametrized measures with the same underlying affine deformation* $y(x)$ *and with*

$$\operatorname{supp}\nu \subset K \quad \textit{and} \quad \operatorname{supp}\nu^* \subset K,$$

*for some compact* $K \subset \mathbb{M}$. *Then for each* $\lambda \in [0,1]$, *the measure* $(1-\lambda)\nu + \lambda\nu^*$ *is a parametrized measure with underlying deformation* $y(x)$ *and*

$$\operatorname{supp}[(1-\lambda)\nu + \lambda\nu^*] \subset K.$$

*If* $(y^k)$ *and* $(y^{*k})$ *are sequences which generate* $\nu$ *and* $\nu^*$ *respectively and satisfy*

$$\|\nabla y^k\|_{L^\infty(\Omega)} \leq M \quad \textit{and} \quad \|\nabla y^{*k}\|_{L^\infty(\Omega)} \leq M,$$

*then there is a sequence* $(u^k)$ *which generates* $(1-\lambda)\nu + \lambda\nu^*$ *which satisfies*

$$\|\nabla u^k\|_{L^\infty(\Omega)} \leq M.$$

We first prove a simple lemma.

LEMMA 4.2 *Let* $D \subset\subset \Omega$ *have smooth boundary and let* $\nu, \nu^*$ *be homogeneous parametrized measures with the same underlying deformation* $y \in H^{1,\infty}(\Omega;\mathbb{R}^m)$. *Then the measure* $\mu = (\mu_x)_{x\in\Omega}$ *defined by*

$$\mu_x = \begin{cases} \nu_x & x \in D \\ \nu^*_x & x \in \Omega - D \end{cases} \tag{4.1}$$

*is a parametrized measure and has underlying deformation* $y$.

PROOF This follows from THEOREM 3.1 since there are only two sequences to consider in (iii). If $(y^k)$ and $(y^{*k})$ be sequences generating $\nu$ and $\nu^*$ respectively, then $u^k_a = y^k$ if

$a \in D$ and $u_a^k = y^{*k}$ if $a \in \Omega - D$. Obviously (i) and (ii) are satisfied. QED

PROOF of THEOREM 4.1. Given $\nu, \nu^*$ and $0 < \lambda < 1$, choose $D \subset \Omega$ with smooth boundary and $|D| = \lambda |\Omega|$. Define $(\nu_x)_{x \in \Omega}$ by

$$\mu_x = \begin{cases} \nu & x \in D \\ \nu^* & x \in \Omega - D \end{cases},$$

which is a parametrized measure according to the lemma. Since the underlying deformation of $\mu$ is $y(x) = Fx$, affine, we may apply THEOREM 2.1. Here $\bar{\nu}$ is given by

$$\langle \bar{\nu}, \psi \rangle = \frac{1}{|\Omega|} \int_\Omega \int_K \psi(A)\, d\mu_x(A)\, dx$$

$$= \frac{1}{|\Omega|} \int_D \int_K \psi(A)\, d\nu(A)\, dx + \frac{1}{|\Omega|} \int_{\Omega - D} \int_K \psi(A)\, d\nu^*(A)\, dx$$

$$= \lambda \int_K \psi(A)\, d\nu(A) + (1 - \lambda) \int_K \psi(A)\, d\nu^*(A)$$

QED

Let us briefly return to some implications of LEMMA 4.2. Given parametrized measures $\nu$ and $\nu^*$ with underlying deformations $y(x)$ and $y^*(x)$ satisfying

$$y = y^* = y_o \quad \text{on } \partial\Omega \text{ where } \quad y_o(x) = F_o x,$$

then, for $0 < \lambda < 1$,

$$\mu = (1 - \lambda)\, \bar{\nu} + \lambda\, \bar{\nu}^* \tag{4.2}$$

is a homogeneous parametrized measure with underlying deformation $y_o$. Specifically, it is given by the formula

$$\langle \mu, \psi \rangle = (1 - \lambda) \int_\Omega \int_K \psi(A)\, d\nu_x(A)\, dx + \lambda \int_\Omega \int_K \psi(A)\, d\nu_x^*(A)\, dx . \tag{4.3}$$

Now consider the situation where $\nu$ is the delta function at $F = \nabla y$, that is

$$\nu_x = \delta_{F(x)} \quad \text{or} \quad \langle \nu_x, \psi \rangle = \psi(F(x)) .$$

Here

$$\langle \bar{\nu}, \psi \rangle = \frac{1}{|\Omega|} \int_\Omega \psi(F(x))\, dx . \tag{4.4}$$

If also $\nu^*$ is the delta function at $F^* = \nabla y^*$ , then

$$\mu = (1-\lambda)\, \bar{\nu} + \lambda\, \bar{\nu}^*$$

is a parametrized measure with

$$\langle \mu, \psi \rangle = \frac{1-\lambda}{|\Omega|} \int_\Omega \psi(F(x))\, dx + \frac{\lambda}{|\Omega|} \int_\Omega \psi(F^*(x))\, dx . \tag{4.5}$$

## 5 *Characterization in terms of special quasiconvex functions*

In this section we shall characterize parametrized measures by a form of Jensen's inequality. For $\varphi \in C(\mathbb{M})$ the functional

$$\int_\Omega \varphi(\nabla u)\, dx$$

is (sequentially) lower semicontinuous with respect to weak* convergence in $H^{1,\infty}(\Omega;\mathbb{R}^m)$ if and only if $\varphi$ is quasiconvex, which means that

$$\varphi(A) \le \frac{1}{|\Omega|} \int_\Omega \varphi(A + \nabla\zeta)\, dx , \quad \text{for } \zeta \in H^{1,\infty}_o(\Omega;\mathbb{R}^m) \text{ and } A \in \mathbb{M} . \tag{5.1}$$

A version of Jensen's inequality follows from this, as mentioned in the introduction. Given a parametrized measure $\nu = (\nu_x)_{x \in \Omega}$ , using the notations of (2.1), we have that for quasiconvex $\varphi$

$$\int_E \varphi(F)\, dx \;\le\; \lim_{k\to\infty} \int_E \varphi(F^k)\, dx \;=\; \int_E \bar{\varphi}\, dx \;=\; \int_E \int_K \varphi(A)\, d\nu_x(A) dx$$

for every measureable $E \subset \Omega$. Hence

$$\varphi(F(x)) \;\le\; \int_K \varphi(A)\, d\nu_x(A) \quad \text{where} \quad F(x) \;=\; \int_K A\, d\nu_x(A) \ \text{ a.e. in } \Omega. \qquad (5.2)$$

The analogous fact for unconstrained parametrized measures, those of (1.12), holds for $\varphi$ convex and has a well known converse, Tartar [57], Young [61], Dacorogna [22]: A family of probability measures $(\nu_x)_{x\in\Omega}$ is a parametrized measure associated to some sequence $(z^k)$ which converges in $L^\infty(\Omega)$ weak* provided Jensen's inequality holds,

$$\varphi\Big( \int_K \xi\, d\nu_x(\xi) \Big) \;\le\; \int_K \varphi(\xi)\, d\nu_x(\xi) \quad \text{for } \varphi \text{ convex.}$$

We shall give an analogous characterization in terms of a special class of quasiconvex functions

$$\mathcal{Q}_M \;=\; \{\, \varphi\colon \mathbb{M} \to \mathbb{R} \cup \{\infty\}\colon \varphi \text{ is quasiconvex and } \varphi \in C(\bar{B}_M) \text{ and } \varphi \;=\; +\infty \text{ in } \mathbb{M} - \bar{B}_M \}, \qquad (5.3)$$

where $B_M = \{ A \in \mathbb{M}\colon |A| < M \}$. Our characterization may also be applied to the unconstrained case mentioned above. We do this in COROLLARY 5.3.

Two remarks about parametrized measures and the class $\mathcal{Q}_M$ are in order. First, if $\nu = (\nu_x)_{x\in\Omega}$ is a Young measure, then there is an $M$ such that (5.2) holds for $\varphi \in \mathcal{Q}_M$. This is because of the local nature of the proof of Morrey's Theorem, which requires information about the function $\varphi$ only in a convex neighborhood of the ranges of the sequence $(\nabla u^k)$.

In §7, we point out that the relaxation $\hat{\psi}^\#$ of a function $\psi \in C(\bar{B}_M)$ extended to $+\infty$ outside $\bar{B}_M$ is in $\mathcal{Q}_M$. This is shown in PROPOSITION 7.2. Other detailed properties of the class $\mathcal{Q}_M$ are also given in §7. Obviously, it will suffice to set $M = 1$. We set $B = B_1$ and $\mathcal{Q} = \mathcal{Q}_M$.

THEOREM 5.1 *Let* $K \subset B \subset \mathbb{M}$ *be compact and* $F_o \in \mathbb{M}$. *Then* $\mu \in M(K)$ *satisfies*

$$\varphi(F_o) \leq \int_K \varphi(A)\, d\mu(A) \quad \textit{whenever} \quad \varphi \in Q, \textit{ where}$$

$$F_o = \int_K A\, d\mu(A)\,, \tag{5.4}$$

*if and only if* $\mu$ *is a homogeneous parametrized measure and is generated by a sequence* $(u^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ *such that*

$$\| \nabla u^k \|_{L^\infty(\Omega)} \leq 1.$$

Let us introduce the set $\mathcal{M}$ of homogeneous parametrized measures $\nu$ with

$$\begin{aligned} &\operatorname{supp} \nu \subset \bar{B}, \\ &F_o = \int_{\bar{B}} A\, d\nu(A)\,, \text{ and} \\ &\nu \text{ is generated by a sequence } (u^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m) \text{ such that} \\ &\qquad \| \nabla u^k \|_{L^\infty(\Omega)} \leq 1. \end{aligned} \tag{5.5}$$

Note that $\mathcal{M} \neq \emptyset$ since $\delta_{F_o} \in \mathcal{M}$. For this subclass we may state

LEMMA 5.2 (i) *The set* $\mathcal{M}$ *is weak* closed and convex in* $M(\bar{B})$.

(ii) *Let* $y_o(x) = F_o x$ *and* $u \in H^{1,\infty}(\Omega;\mathbb{R}^m)$ *satisfy* $u = y_o$ *on* $\partial\Omega$ *and* $\| \nabla u \|_{L^\infty(\Omega)} \leq 1$. *Then* $\bar{\delta}_{\nabla u} \in \mathcal{M}$ *and the set of all such* $\bar{\delta}_{\nabla u} \in \mathcal{M}$ *is dense in* $\mathcal{M}$.

Recall that the action of the measure $\bar{\delta}_{\nabla u}$ is given by

$$\langle \bar{\delta}_{\nabla u}, \psi \rangle = \int_\Omega \psi(\nabla u)\, dx\ .$$

PROOF (i) $\mathcal{M}$ is convex by THEOREM 4.1. To check that it is closed, it suffices to show that $\nu \in M(\bar{B})$ with $\nu = \lim \nu^j$ is a parametrized measure. Thus we must produce a generating sequence satisfying (5.5). An appropriate subsequence of the $(u^{j,k})$, where for fixed j the sequence $(u^{j,k})$ generates $\nu^j$, will suffice.

(ii) Given u satisfying the hypotheses of (ii), then $\bar{\delta}_{\nabla u} \in \mathcal{M}$ by the averaging theorem, THEOREM 2.1. To show the density, choose a sequence $(u^k)$ which generates a given $\nu$ and apply PROPOSITION 2.2. QED

PROOF of THEOREM 5.1 Observe that (5.4) implies automatically that $\mu$ is a probability measure. To verify this, first take $\varphi(A) = \pm 1$ on B, so $\mu(K) = 1$. Now let $\psi \in C(B)$ be nonnegative. Then the convexification $\hat{\psi}^{**}(A)$ is nonnegative and quasiconvex. We apply (5.5) to it, so

$$0 \le \hat{\psi}^{**}(F_o) \le \int_K \hat{\psi}^{**}(A)\, d\mu(A) \le \int_K \psi(A)\, d\mu(A) .$$

We use the Hahn-Banach Theorem in the space $M(\bar{B})$ in the weak* topology. Let T be any weak* continuous linear functional with

$$\langle T, \nu \rangle \ge 0 \quad \text{for } \nu \in \mathcal{M} \tag{5.6}$$

Thus there is a $\psi \in C(\bar{B})$ such that

$$\langle T, \nu \rangle = \langle \nu, \psi \rangle = \int_{\bar{B}} \psi(A)\, d\nu(A) .$$

In particular, (5.6) implies that

$$\int_\Omega \psi(\nabla u)\, dx \ge 0$$

whenever u satisfies the conditions of hypothesis (ii) of LEMMA 5.2. Recalling the definition of $\hat{\psi}^{\#}(A)$, we have by (7.5) that

$$\psi(F_o) \ge \hat{\psi}^{\#}(F_o) = \inf_{u = y_o \text{ on } \partial\Omega,\ |\nabla u| \le 1} \int_\Omega \psi(\nabla u)\, dx \ge 0 .$$

In other words, (5.6) means that $\hat{\psi}^{\#}(F_o) \geq 0$. Now $\hat{\psi}^{\#} \in C(\bar{B})$, so we may compute, using the hypothesis on $\mu$,

$$\langle T, \mu \rangle = \int_{\bar{B}} \psi(A)\, d\mu(A)$$

$$\geq \int_{\bar{B}} \hat{\psi}^{\#}(A)\, d\mu(A)$$

$$\geq \hat{\psi}^{\#}(F_o) \geq 0.$$

Hence $\mu$ cannot be separated from the closed convex $\mathcal{M}$, whence $\mu \in \mathcal{M}$.

We have already noted in the remarks preceding the Theorem that the local nature of Morrey's Theorem shows that any parametrized measure satisfies (5.4). QED

One should take note that the set $\mathcal{M}$ does not consist exactly of averaged Dirac masses, that is, parametrized measures given by (4.4). Were this so, our theorem would be trivial. Disparate scaling of a sequence easily gives rise to Young measures which are not averaged Dirac masses, as Ball and Zhang [9] illustrate. Also, many of the Young measures used to describe complicated microstructures consist of sequences which have disparate scaling [42],[55].

Any probability measure, that is, any unconstrained homogeneous parametrized measure in the sense of (1.6), may be realized as the parametrized measure generated by a sequence of gradients. Some reflection shows that this holds for the Dirac masses. Since they are the extreme points of the probability measures, the conclusion follows. We provide a more entertaining proof based on our THEOREM 5.1.

COROLLARY 5.3 *Let* $K \subset B \subset \mathbb{R}^n$ *be compact and* $\mu \in M(K)$ *be a probability measure. Then* $\mu$ *is a homogeneous parametrized measure in the sense of* (1.8).

PROOF In this case quasiconvex becomes convex and the class $\mathcal{Q}$ reduces to

$$\mathcal{Q} = \{ \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} : \varphi \text{ is } \mathit{convex} \text{ and } \varphi \in C(\bar{B}) \text{ and } \varphi = +\infty \text{ in } \mathbb{R}^n - \bar{B} \},$$

For any convex $\varphi \in C(\mathbb{R}^n)$,

$$\hat{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \xi \in \bar{B} \\ \infty & \text{otherwise} \end{cases}$$

has the property $\hat{\varphi} \in Q$. Now if $\mu \in M(K)$ is a probability measure, then

$$\xi_o = \int_K \xi \, d\mu(\xi) \in \bar{B}$$

so

$$\hat{\varphi}(\xi_o) = \varphi(\xi_o) \leq \int_K \varphi(\xi) \, d\mu(\xi) = \int_K \hat{\varphi}(\xi) \, d\mu(\xi) .$$

The conclusion follows from THEOREM 5.1. QED

## 6 *Characterization in terms of continuous quasiconvex functions*

A natural class of functions to investigate is the set of continuous quasiconvex functions on $\mathbb{M}$,

$$Q_\infty = \{ \varphi \in C(\mathbb{M}) : \varphi \text{ is quasiconvex} \}. \tag{6.1}$$

If $\nu$ is a homogeneous parametrized measure and $\varphi \in Q_\infty$, by lower semicontinuity of the functional

$$\Phi(v) = \int_\Omega \varphi(\nabla u) \, dx, \quad v \in H^{1,\infty}(\Omega;\mathbb{R}^m),$$

we have that

$$\varphi(F_o) \leq \int_M \varphi(A) \, d\nu(A) \quad \text{when} \quad F_o = \int_M A \, d\nu(A) . \tag{6.2}$$

What is the relationship between parametrized measures and $Q_\infty$? We shall infer a result slightly weaker that THEOREM 5.1. The difference in the conclusion owes to the fact that $C(\mathbb{M})$ is not a Banach space and its dual

$$M(\mathbb{M}) = C(\mathbb{M})',$$

the Radon measures with compact support, is not a space where sequences suffice, even though closed bounded sets in the weak* topology are compact.

Given $K \subset \mathbb{M}$ compact and $F_o \in \mathbb{M}$, let $\mathcal{M}_\infty = \mathcal{M}_\infty(F_o,K)$ be the homogeneous parametrized measures $\nu$ satisfying

$$\operatorname{supp} \nu \subset K \quad \text{and} \quad F_o = \int_K A \, d\nu(A) . \tag{6.3}$$

According to THEOREM 4.1, $\mathcal{M}_\infty$ is convex but, unable to control the ranges of generating sequences, it is not clear that it is closed. It is easy to see that the special parametrized measures

$$\nu = \bar{\delta}_{\nabla u}, \; u \in H^{1,\infty}(\Omega;\mathbb{R}^m), \; u\big|_{\partial\Omega} = y_o,$$

where $y_o(x) = F_o x$, are dense in $\mathcal{M}_\infty$ in $M(\mathbb{M})$ weak*.

THEOREM 6.1 *Let* $K \subset \mathbb{M}$ *be compact and* $F_o \in \mathbb{M}$. *Then* $\mu \in M(K)$ *satisfies*

$$\varphi(F_o) \le \int_{\mathbb{M}} \varphi(A) \, d\mu(A) \quad \textit{where} \quad F_o = \int_{\mathbb{M}} A \, d\mu(A) , \tag{6.4}$$

*whenever* $\varphi \in Q_\infty$, *if and only if* $\mu \in \bar{\mathcal{M}}_\infty$, *the weak* closure of* $\mathcal{M}_\infty$.

PROOF As in the proof of THEOREM 5.1, (6.4) implies that $\mu$ is a probability measure. Define the weak* continuous linear functional $T$ by

$$\langle T, \nu \rangle = \langle \nu, \psi \rangle = \int_{\mathbb{M}} \psi(A) \, d\nu(A) , \quad \nu \in C(\mathbb{M})',$$

where $\psi \in C(\mathbb{M})$. Suppose that

$$\langle T, \bar{\nu} \rangle \geq 0 \quad \text{whenever} \quad \bar{\nu} \in \mathcal{M}_\infty , \tag{6.5}$$

which, by (3.6), means that

$$\int_\Omega \psi(\nabla u)\, dx \geq 0 \quad \text{whenever} \quad u \in H^{1,\infty}(\Omega;\mathbb{R}^m), \ u = y_o \ \text{on} \ \partial\Omega.$$

We shall show that

$$\langle T, \mu \rangle \geq 0 . \tag{6.6}$$

According to (6.5) and the definition of $\psi^\#$,

$$\psi^\#(F_o) = \inf_{H_o^{1,\infty}(\Omega)} \int_\Omega \psi(F_o + \nabla\zeta)\, dx \geq 0 , \tag{6.7}$$

thus $\psi^\#(F_o) \geq 0 > -\infty$. Hence by PROPOSITION 8.1, $\psi^\# \in C(\mathbb{M})$ and $\psi^\#$ is $\mu$ - integrable.

Using (6.5) and (6.4), we compute that

$$\begin{aligned} \langle T, \mu \rangle &= \int_M \psi(A)\, d\mu(A) \\ &= \int_K \psi(A)\, d\mu(A) \\ &\geq \int_K \psi^\#(A)\, d\mu(A) \\ &\geq \psi^\#(F_o) \geq 0 . \end{aligned}$$

This proves the claim. QED

We do not know if $\mathcal{M}_\infty$ is closed. If it were, THEOREM 6.1 would provide another characterization of parametrized measures generated by gradients. One approach to attacking this question would be to use the hypothesis (6.4) to prove the hypothesis (5.4). A means to accomplish this would be to prove that for any $\varphi \in \mathcal{Q}$,

$$\varphi(A) = \sup \left\{ f(A) : f \in \mathcal{Q}_\infty \text{ and } f \le \varphi \right\} . \tag{6.8}$$

It would be interesting to determine if $\mathcal{M}_\infty$ is closed. From the functional analytic point of view it would mean that all parametrized measures supported in a given compact set form a closed convex set of measures.

With regard to microstructure, we might be confronted with a sequence of complicated microstructures, not obviously finite rank laminates, which tend to an equilibrium microstructure whose parametrized measure has compact support. We might even know that this sequence is generated by a sequence of functions or a net of functions. Indeed, if it is generated by a single sequence, then PROPOSITION 1.1 informs us that the sequence is bounded. But in general we do not expect to know about existence of this sequence. If $\mathcal{M}_\infty$ is closed, we would know that this complicated process is generated by a sequence and that the sequence is bounded in $H^{1,\infty}$, i.e., has uniformly bounded gradients. Now, roughly speaking, when we have sequences of parametrized measures which tend to minimizing configurations, they spend most of their time near minimum enegy wells with their supports growing in an uncontrollable manner only when kinematic compatibility requires oscillations to connect these wells together. This behavior is associated with the formation of interfaces. If the limit configuration may be generated by a sequence with bounded gradients, grounds for limitations on the size and energy of these interfaces result.

Measures $\mu \in \bar{\mathcal{M}}_\infty$ do enjoy many properties. For example, given $f \in C(\mathbb{M})$, we may extract a sequence $(u^k) \subset H^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\int_\Omega \zeta f(\nabla u^k)\, dx \quad \to \quad \int_K f(A)\, d\mu(A) \int_\Omega \zeta\, dx \ , \quad \zeta \in L^\infty(\Omega) .$$

By the de la Valleé Poussin criterion, it follows that whenever

$$\frac{|\psi(A)|}{|f(A)|} \to 0 \quad \text{as} \quad |A| \to \infty ,$$

the sequence $(\psi(\nabla u^k))$ converges weakly in $L^1(\Omega)$, namely,

$$\int_\Omega \zeta \psi(\nabla u^k)\, dx \quad \to \quad \int_K \psi(A)\, d\mu(A) \int_\Omega \zeta\, dx \ , \quad \zeta \in L^\infty(\Omega) . \tag{6.9}$$

Choosing, for example, $f(A) = e^{|A|}$ implies in particular the existence of a sequence $(u^k)$ bounded in $H^{1,p}(\Omega;\mathbb{R}^m)$ for every $p$, $1 \le p < \infty$, with the property that (6.9) holds whenever $\psi$ has polynomial growth. Thus such measures $\mu$ have better properties than the $H^{1,p}$ Young measures we introduce in [55] and to which we alluded briefly in §1.

## 7 *The special class $\mathcal{Q}$ of quasiconvex functions*

In this section we explain some properties of the class $\mathcal{Q}_M$ of quasiconvex functions which was defined to be

$$\mathcal{Q}_M = \{ \varphi: \mathbb{M} \to \mathbb{R} \cup \{\infty\}: \varphi \text{ is quasiconvex and } \varphi \in C(\bar{B}_M) \text{ and } \varphi = +\infty \text{ in } \mathbb{M} - \bar{B}_M \}, \tag{7.1}$$

where $B_M = \{ A \in \mathbb{M}: |A| < M \}$. An interesting feature of $\varphi \in \mathcal{Q}_M$ is that the functional

$$\int_\Omega \varphi(\nabla v)\, dx \,, \quad v \in H^{1,\infty}(\Omega),$$

is lower semicontinuous. We thank John Ball for insisting on this point.

PROPOSITION 7.1 *Let* $\varphi \in \mathcal{Q}_M$ *and* $\Omega \subset \mathbb{R}^n$ *be a domain with* $|\partial\Omega| = 0$. *If* $u^k \to u$ *in* $H^{1,\infty}(\Omega)$ *weak*, then*

$$\int_\Omega \varphi(\nabla u)\, dx \le \liminf_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\, dx \,. \tag{7.2}$$

PROOF If the right hand side of (7.2) is infinite, then the conclusion holds. Suppose, then, without loss in generality, that

$$\lim_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\, dx < \infty \,.$$

Thus $\| \nabla u^k \|_{L^\infty(\Omega)} \le M$ and it follows by lower semicontinuity of the norm that $\| \nabla u \|_{L^\infty(\Omega)} \le M$.

The idea now is to use the proof of Morrey's Theorem, cf.[51], which requires, however, that $\varphi$ be continuous in a convex neighborhood of the ranges of the $(\nabla u^k)$. We overcome this by using the uniform continuity of $\varphi$ in $\bar{B}_M$. Let $\omega$ denote the modulus of continuity of $\varphi$. For each $t$, $0 < t < 1$,

$$tu^k \to tu \text{ in } H^{1,\infty}(\Omega) \text{ weak*}$$

and by [51],

$$\int_\Omega \varphi(t\nabla u)\,dx \le \liminf_{k\to\infty} \int_\Omega \varphi(t\nabla u^k)\,dx$$

$$= \liminf_{k\to\infty} \left\{ \int_\Omega \varphi(\nabla u^k)\,dx + \int_\Omega (\varphi(t\nabla u^k) - \varphi(\nabla u^k))\,dx \right\}$$

$$\le \liminf_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\,dx + \omega((1-t)M)\,.$$

Hence,

$$\int_\Omega \varphi(\nabla u)\,dx \le \liminf_{k\to\infty} \int_\Omega \varphi(\nabla u^k)\,dx + 2\omega((1-t)M)\,.$$

QED

PROPOSITION 7.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with* $|\Omega| = 1$ *and* $|\partial\Omega| = 0$. *Let* $\psi \in C(\bar{B}_M)$ *and set*

$$\hat{\psi}(A) = \begin{cases} \psi(A) & A \in \bar{B}_M \\ \infty & \text{otherwise} \end{cases}\,. \tag{7.3}$$

*Let*

$$\hat{\psi}^\#(A) = \inf_{H_0^{1,\infty}(\Omega)} \int_\Omega \hat{\psi}(A+\nabla\zeta)\,dx\,, \tag{7.4}$$

*the relaxation of* $\hat{\psi}$. *Then* $\hat{\psi}^{\#} \in Q_M$ *and* $\hat{\psi}^{\#} = \psi$ *on* $\partial B_M$.

Note in particular that $\hat{\psi}^{\#}(A) < \infty$ for a given $A$ implies that

$$\hat{\psi}^{\#}(A) = \inf_{|A+\nabla\zeta| \leq M} \int_\Omega \psi(A+\nabla\zeta)\,dx\ , \tag{7.5}$$

so that $\hat{\psi}^{\#}(A)$ is given either by (7.5) or is $+\infty$. This observation also makes it easy to check that $\hat{\psi}^{\#}$ is well defined, cf. the argument in [7] or [16], Lemma 3.2. The dependence of $\hat{\psi}^{\#}(A)$ on the radius $M$ will be useful.

LEMMA 7.3 *Given* $\psi \in C(\bar{B}_M)$, *continue to denote by* $\psi$ *any of its extensions to* $C_0(\mathbb{M})$. *Define*

$$\hat{\psi}^{\#}_\rho(A) = \inf_{|A+\nabla\zeta| \leq \rho} \int_\Omega \psi(A+\nabla\zeta)\,dx\ . \tag{7.6}$$

*Then*

$$\hat{\psi}^{\#}_R(A) \uparrow \hat{\psi}^{\#}_M(A)\ ,\ |A| \leq M,\ \text{as } R \downarrow M \ \textit{and} \tag{7.7}$$

$$\hat{\psi}^{\#}_r(A) \downarrow \hat{\psi}^{\#}_M(A),\ |A| < M, \text{as } r \uparrow M\ . \tag{7.8}$$

PROOF We give the proof of (7.7); that of (7.8) is analogous. First of all, it is obvious that

$$\hat{\psi}^{\#}_R(A) \leq \hat{\psi}^{\#}_M(A) \leq \hat{\psi}^{\#}_r(A) \quad \text{for } R \geq M \geq r.$$

Let $R_j \to M$ and let $\omega(s)$ be a modulus of continuity for $\psi$ in some ball of radius $R_o \geq \sup R_j$. For each $j$, choose $\zeta_j$ and $t_j$ such that

$$\int_\Omega \psi(A+t_j\nabla\zeta_j)\,dx \leq \hat{\psi}^{\#}_{R_j}(A) + \frac{1}{j} \quad \text{and}$$

$$|\nabla\zeta_j| \le M, \; t_j = \frac{R_j}{M} \to 1 .$$

Then

$$\int_\Omega |\psi(A+t_j\nabla\zeta_j) - \psi(A+\nabla\zeta_j)|\, dx \le \omega((t_j-1)M) = \omega_j \to 0 \text{ as } j \to \infty.$$

Thus

$$\hat{\psi}^{\#}_M(A) \le \int_\Omega \psi(A+\nabla\zeta_j)\, dx$$

$$\le \int_\Omega \psi(A+t_j\nabla\zeta_j)\, dx + \omega_j$$

$$\le \hat{\psi}^{\#}_{R_j}(A) + \omega_j + \frac{1}{j} .$$

QED

Proof of PROPOSITION 7.2. We proceed in steps. We may assume that $M = 1$ and denote by $B = B_1$.

Step 1. $\hat{\psi}^{\#}(A) = +\infty$ for $|A| > 1$.

This is obvious. Let $g(A)$ be any convex function satisfying

$$g(A) = \begin{cases} 0 & \bar{B} \\ > 0 & \text{in } \mathbb{M} - \bar{B} \end{cases} \qquad (7.9)$$

Thus

$$\hat{\psi}(A) \ge \psi(A) + \frac{1}{\varepsilon} g(A), \quad \varepsilon > 0.$$

By Jensen's inequality, for every $\varepsilon > 0$,

$$\hat{\psi}^{\#}(A) \ge \inf \psi + \frac{1}{\varepsilon}\int_\Omega g(A+\nabla\zeta)\, dx$$

$$\ge \inf \psi + \frac{1}{\varepsilon} g(A) .$$

Hence

$$\int_\Omega \hat{\psi}(A+\nabla\zeta)\,dx \;=\; \infty \quad \text{for } |A| > 1 \;\text{ and}$$

$$\hat{\psi}^{\#}(A) \;=\; \infty \quad \text{for } |A| > 1.$$

Step 2. $\hat{\psi}^{\#} \in C(B)$

Given $A$ and $A' \in B$, let $\varepsilon > 0$ and choose $\zeta$ such that

$$\varepsilon + \hat{\psi}^{\#}(A') \;\geq\; \int_\Omega \psi(A'+\nabla\zeta)\,dx\,.$$

Hence

$$\varepsilon + \hat{\psi}^{\#}(A') \;\geq\; \int_\Omega \psi(A+\nabla\zeta)\,dx \;+\; \int_\Omega (\psi(A'+\nabla\zeta)-\psi(A+\nabla\zeta))\,dx$$

$$\geq\; \hat{\psi}^{\#}_R(A) \;-\; \omega(|A-A'|)\,,\;\; R \;=\; |A-A'|+1.$$

According to the LEMMA, we may choose $|A - A'|$ so small that

$$\hat{\psi}^{\#}_R(A) \geq\; \hat{\psi}^{\#}(A) \;-\; \varepsilon,$$

whence

$$\hat{\psi}^{\#}(A) \;\leq\; \liminf_{A' \to A}\, \hat{\psi}^{\#}(A')\,.$$

Analogously,

$$\hat{\psi}^{\#}(A) \geq\; \limsup_{A' \to A}\, \hat{\psi}^{\#}(A')\,.$$

Step 3.

$$\hat{\psi}^{\#}(A) = \inf_{Aff} \int_\Omega \psi(A+\nabla\zeta)\, dx , \tag{7.10}$$

where Aff denotes piecewise affine functions $\zeta$ in $H_o^{1,\infty}(\Omega)$. The proof of this is standard.

Step 4. $\hat{\psi}^{\#}(A) = \psi(A)$ for $|A| = 1$.

Given $\zeta \in$ Aff, $\zeta \neq 0$, we may write

$$\nabla\zeta = \Sigma Z_i \chi_{D_i} , \quad \text{with} \quad \Sigma Z_i |D_i| = 0.$$

So the convex combination of the matrices $A + Z_i$,

$$\Sigma (A + Z_i)|D_i| = A ,$$

a point on the boundary of the ball B, hence at least one $Z_k$ has the property that $A + Z_k \notin B$. Consequently, since $\hat{\psi}(A)$ is bounded below,

$$\int_\Omega \hat{\psi}(A+\nabla\zeta)\, dx = +\infty .$$

Hence there is a unique $\zeta$ for which the integral is finite, namely $\zeta = 0$. This concludes Step 4.

Step 5. $\lim_{A_n \to A_o} \hat{\psi}^{\#}(A_n) = \psi(A_o)$ for $A_o \in \partial B$.
Let $A_n \to A_o \in \partial B$ with $r_n = |A_o - A_n| \to 0$. Given any $\zeta \in H_o^{1,\infty}(\Omega)$, suppose that

$$|A_n + \nabla\zeta| \leq 1 \tag{7.11}$$

and write

$$|\hat{\psi}^{\#}(A_n) - \psi(A_o)| \leq |\hat{\psi}^{\#}(A_n) - \int_\Omega \psi(A_n+\nabla\zeta)\, dx| + |\psi(A_o) - \psi(A_n)|$$
$$+ |\psi(A_n) - \int_\Omega \psi(A_n+\nabla\zeta)\, dx| . \tag{7.12}$$

The essence of the demonstration is to show that

$$\lim_{A_n \to A_o} \left| \psi(A_n) - \int_\Omega \psi(A_n+\nabla\zeta)\, dx \right| = 0 ,$$

uniformly in $\zeta$ subject to the constraint (7.11). Introduce the sets

$$H_\rho = \{ A \in \bar{B}\colon A \cdot A_o \le \rho^2 \}, \quad 0 < \rho < 1, \text{ and}$$
$$K_\rho = K_\rho(n) = \{ x \in \Omega\colon A_n+\nabla\zeta \in H_\rho \}.$$

Both $H_\rho$ and $\bar{B} - H_\rho$ are convex and

$$\int_{\Omega - K_\rho} |\psi(A_n+\nabla\zeta) - \psi(A_n)|\, dx \le \omega(2\sqrt{1-\rho^2}) . \tag{7.13}$$

In addition, the averages

$$A_n' = \frac{1}{|K_\rho|} \int_{K_\rho} (A_n+\nabla\zeta)\, dx \in H_\rho \quad \text{and}$$

$$A_n'' = \frac{1}{|\Omega - K_\rho|} \int_{\Omega - K_\rho} (A_n+\nabla\zeta)\, dx \in B - H_\rho .$$

Meanwhile, we may write $A_n$ as the convex combination

$$A_n = |K_\rho|\, A_n' + (1 - |K_\rho|)\, A_n'' ,$$

so that

$$A_n - A_o = |K_\rho|\,(A_n' - A_o) + (1 - |K_\rho|)\,(A_n'' - A_o) .$$

Now by elementary geometry, as $n \to \infty$,

$$|A_n'' - A_o| \to 0 \quad \text{and} \quad |A_n' - A_o| \ge 1 - \rho .$$

Thus $|K_\rho| \to 0$ as $n \to \infty$. Consequently, in view of (7.13), given $\varepsilon > 0$, we may choose $\rho$ and then $n_\rho$ so that

$$\left| \psi(A_n) - \int_\Omega \psi(A_n+\nabla\zeta)\, dx \right| \le \varepsilon \quad \text{for } n \ge n_\rho .$$

By the continuity of $\psi$ and the definition of $\hat{\psi}^{\#}(A)$, we may choose n so large and then $\zeta$ so that each of the first two terms in (7.12) is dominated by $\varepsilon$.

Step 6. To complete the proof of the PROPOSITION, we show that $\hat{\psi}^{\#}(A)$ is quasiconvex. Suppose that $A \in B$ and $\zeta \in H_o^{1,\infty}(\Omega)$. If

$$\| A + \nabla\zeta \|_{L^\infty(\Omega)} > 1 ,$$

then $|\{x \in \Omega : | A + \nabla\zeta | > 1 \}| > 0$, whence

$$\hat{\psi}^{\#}(A) < +\infty = \int_\Omega \hat{\psi}^{\#}(A+\nabla\zeta)\, dx .$$

So suppose that

$$\| A + \nabla\zeta \|_{L^\infty(\Omega)} \leq 1 .$$

We proceed in a standard manner. Given $\varepsilon > 0$, we may find a $\zeta_\varepsilon \in H_o^{1,\infty}(\Omega)$, which may be taken piecewise affine, such that

$$\| A + \nabla\zeta_\varepsilon \|_{L^\infty(\Omega)} \leq t < 1 \quad \text{and}$$

$$\int_\Omega | \hat{\psi}^{\#}(A+\nabla\zeta) - \hat{\psi}^{\#}(A+\nabla\zeta_\varepsilon) | \, dx < \varepsilon. \tag{7.14}$$

Writing

$$\nabla\zeta_\varepsilon = \Sigma Z_i \chi_{D_i} , \quad \text{with} \quad | A + Z_i | \leq t < 1,$$

and each $D_i$ a finite union of simplices, we may find $\eta_i \in H_o^{1,\infty}(D_i)$ such that

$$\hat{\psi}^{\#}(A + Z_i) \geq \frac{1}{| D_i |} \int_{D_i} \psi(A+Z_i+\nabla\eta_i)\, dx - \varepsilon. \tag{7.15}$$

Let $\eta \in H_o^{1,\infty}(\Omega)$ be defined by $\eta = \eta_i$ in $D_i$. From (7.14) and (7.15),

$$\int_\Omega \hat{\psi}^\#(A+\nabla\zeta)\,dx \;\geq\; \int_\Omega \hat{\psi}^\#(A+\nabla\zeta_\varepsilon)\,dx - \varepsilon$$

$$= \sum \hat{\psi}^\#(A + Z_i)\,|\,D_i\,| - \varepsilon$$

$$\geq \sum \int_{D_i} \psi(A+Z_i+\nabla\eta_i)\,dx - 2\varepsilon$$

$$= \int_\Omega \psi(A+\nabla(\zeta_\varepsilon+\eta))\,dx - 2\varepsilon$$

$$\geq \hat{\psi}^\#(A) - 2\varepsilon\,.$$

For $|\,A\,| = 1$, the proof of Step 4 shows that $\hat{\psi}^\#(A)$ is quasiconvex at A. Similarly, if $|\,A\,| > 1$, one easily verifies that

$$\int_\Omega \hat{\psi}^\#(A+\nabla\zeta)\,dx = +\infty \quad \text{for any } \zeta \in H^{1,\infty}_o(\Omega).$$

QED

## 8 *The class $\mathcal{Q}_\infty$ of continuous quasiconvex functions*

For $\psi \in C(\mathbb{M})$, we set

$$\psi^\#(A) = \inf_V \frac{1}{|\,\Omega\,|}\int_\Omega \psi(A + \nabla\zeta)\,dx\,, \qquad A \in \mathbb{M} \tag{8.1}$$

$$V = H^{1,\infty}_o(\Omega;\mathbb{R}^m)\,.$$

The facts of the proposition below are well known but appear scattered in the literature and under varying hypotheses. It is important for us not to assume that $\psi$ is bounded below. Obviously,

$$-\infty \;\le\; \inf \psi \;\le\; \psi^{\#}(A) \;\le\; \psi(A) \qquad \text{for } A \in \mathbb{M}.$$

PROPOSITION 8.1 *Let* $\psi \in C(\mathbb{M})$ *and define* $\psi^{\#}$ *by* (8.1). *Then*

(i) $\psi^{\#}$ *is independent of* $\Omega$, *i.e., is well defined,*

(ii) *and it is rank-one convex.*

(iii) *if* $\psi^{\#}(A_o) > -\infty$ *for some* $A_o \in \mathbb{M}$, *then* $\psi^{\#} \in C(\mathbb{M})$, *and*

(iv) *if* $\psi^{\#} \in C(\mathbb{M})$, *then it is quasiconvex.*

Proof (i) Following Ball and Murat [7], let us provisionally set $\psi^{\#}_{\Omega}$ and $\psi^{\#}_{D}$ the infima taken over $H^{1,\infty}_{o}(\Omega;\mathbb{R}^m)$ and $H^{1,\infty}_{o}(D;\mathbb{R}^m)$, respectively. As in the proof of THEOREM 2.1, given $\varepsilon_o > 0$, the collection of sets $\{ a + \varepsilon\bar{D}:\ a \in \Omega,\ \varepsilon < \varepsilon_o \}$ is a covering of $D$ from which, by the Vitali covering theorem, we may select a countable or finite subset $\{ a_i + \varepsilon_i\bar{D}: i = 1,2,3 \ldots\}$ of pairwise disjoint sets such that

$$\bar{\Omega} = \bigcup (a_i + \varepsilon_i\bar{D}) \cup N, \quad |N| = 0.$$

Note that $\Sigma (\varepsilon_i)^n |D| = |\Omega|$. Let $\zeta \in H^{1,\infty}_{o}(D;\mathbb{R}^m)$ and define

$$v(x) = \begin{cases} \varepsilon_i\, \zeta(\frac{x - a_i}{\varepsilon_i}) & x \in a_i + \varepsilon_i D \\ 0 & \text{otherwise} \end{cases} .$$

Then

$$\psi^{\#}_{\Omega}(A) \le \frac{1}{|\Omega|}\int_{\Omega} \psi(A + \nabla v)\, dx$$

$$= \frac{1}{|\Omega|}\sum \int_{a_i + \varepsilon_i D} \psi(A + \nabla\zeta(\frac{x - a_i}{\varepsilon_i}))\, dx$$

$$= \frac{1}{|\Omega|} \sum (\varepsilon_i)^n \int_D \psi(A + \nabla\zeta)\, dx$$

$$= \frac{1}{|D|} \int_D \psi(A + \nabla\zeta)\, dx \ .$$

Thus $\psi^{\#}_{\Omega}(A) \leq \psi^{\#}_{D}(A)$. Interchanging the roles of $\Omega$ and D gives that $\psi^{\#}$ is well defined.

(ii) To show that $\psi^{\#}$ is rank-one convex, we use the method of [16], Lemma 3.3. Choose $F \in \mathbb{M}$, $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, $|b| = 1$. Let $D \subset \mathbb{R}^n$ be a unit cube with one face perpendicular to b. For $\lambda \in (0,1)$, let $\chi(t)$ denote the characteristic function of $(0,\lambda) \subset (0,1)$ extended periodically so that $\chi(t + k) = \chi(t)$ for $k \in \mathbf{Z}$. Set $\chi^k(x) = \chi(kx \cdot b)$. Determine $u^k \in H^{1,\infty}(D;\mathbb{R}^m)$ such that

$$\nabla u^k = F + \chi^k\, a\otimes b \quad \text{and} \quad u^k \rightarrow y \quad \text{in } H^{1,\infty}(D;\mathbb{R}^m) \text{ weak }*,$$
$$\text{where } \ y(x) = (F + \lambda\, a\otimes b)x \ .$$

Since $u^k \rightarrow y$ uniformly, we may choose a subsequence of k and cut-off functions $\eta^k$ such that

$$\eta^k = 1 \quad \text{on a subcube } D^k \subset D \text{ with } |D - D^k| \rightarrow 0 \text{ as } k \rightarrow \infty,$$
$$\eta^k = 0 \quad \text{on } \partial D, \text{ and}$$
$$v^k = y + \eta^k(u^k - y) \rightarrow y \quad \text{in } H^{1,\infty}(D;\mathbb{R}^m) \text{ weak}*.$$

In particular, $|\nabla v^k|$ are bounded and with

$$D^k_o = \{ x \in D^k : \chi^k = 0 \} \quad \text{and} \quad D^k_1 = \{ x \in D^k : \chi^k = 1 \},$$

$$|D^k_o| \rightarrow 1 - \lambda \quad \text{and} \quad |D^k_1| \rightarrow \lambda \quad \text{as } k \rightarrow \infty .$$

Note that

$$\left| \int_{D - D_k} \psi(\nabla v^k)\, dx \right| \leq \varepsilon_k \rightarrow 0,$$

Choose $\zeta \in H^{1,\infty}(D;\mathbb{R}^m)$ with $\operatorname{supp}\zeta \subset D^k$ and note that

$$\nabla(v^k + \zeta) = F + \lambda\, a \otimes b + \nabla w \qquad \text{for some } w \in H_o^{1,\infty}(D;\mathbb{R}^m)\,.$$

Thus

$$\psi^{\#}(F + \lambda\, a \otimes b) \le \int_D \psi(\nabla v^k + \nabla\zeta)\, dx$$

$$= \int_{D_k} \psi(\nabla u^k + \nabla\zeta)\, dx + \int_{D - D_k} \psi(\nabla v^k)\, dx$$

$$\le \int_{D_k} \psi(\nabla u^k + \nabla\zeta)\, dx + \varepsilon_k$$

$$= \int_{D_o^k} \psi(F + \nabla\zeta) dx + \int_{D_1^k} \psi(F + a \otimes b + \nabla\zeta)\, dx + \varepsilon_k\,.$$

Now choose $\zeta$ so that $\zeta = 0$ on $\partial D_o^k \cup \partial D_1^k$. We may then vary $\zeta$ independently in the two sets $D_o^k$ and $D_1^k$, so that taking the infimum of the right hand side above gives that

$$\psi^{\#}(F + \lambda\, a \otimes b) \le |D_o^k|\, \psi^{\#}(F) + |D_1^k|\, \psi^{\#}(F + a \otimes b) + \varepsilon_k\,.$$

Passing to the limit as $k \to \infty$ gives the result.

(iii) Suppose that $\psi^{\#}(A_o) > -\infty$ for some $A_o \in \mathbb{M}$. Then for any a,b, $f(t) = \psi^{\#}(A_o + t\, a \otimes b)$ is convex by (ii) and continuous for $-\infty < t < \infty$ since it is finite at some point. Hence, of course, for any a',b', the function $g(s) = \psi^{\#}(A_o + t\, a \otimes b + s\, a' \otimes b')$ is convex and continuous in s, $-\infty < s < \infty$. Since any F may be connected to $A_o$ by a sequence of such paths it is easy to check that $\psi^{\#}$ is finite everywhere and continuous.

(iv) The proof of this is standard, cf. [23].

QED

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The oscillatory properties of a weak* convergent sequence of gradients may be decoupled from its deformation properties, a localization property easily shown. Of greater interest is that oscillations may be coupled to a sequence and limit deformation assuming only a kinematic condition and technical condition. This suggests the question of what measures, that is to say, ordinary measures not parametrized measures, may occur as limits of sequences of gradients. We show that they may be characterized by a form of Jensen's inequality for a special class of quasiconvex functions. The consequences of this disparity are of some interest in understanding the sort of approximations, or processes, which lead to complicated microstructures and are relevant to the nature of approximation by Lipschitz functions in general.

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