WEAK CONVERGENCE OF INTEGRANDS AND THE YOUNG MEASURE REPRESENTATION

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Abstract   Validity of the Young measure representation is useful in the study of microstructure of ordered solids. Such a Young measure, generated by a minimizing sequence of gradients converging weakly in L^p, often needs to be evaluated on functions of pth power polynomial growth. We give a sufficient condition for this in terms of the variational principle. The principal result concerns lower semicontinuity of functionals integrated over arbitrary sets, THEOREM 1.2. The question arose in the numerical analysis of configurations. Several applications are given. Of particular note, Young measure solutions of an evolution problem are found.

Contents

1   *Introduction*

For a lower semicontinuous functional of the form

$$\Phi(v) = \int_\Omega \varphi(\nabla v)\, dx, \quad v \in H^{1,p}(\Omega;\mathbb{R}^m),$$

the convergence property

$$\Phi(u^k) \to \Phi(u) \quad \text{and} \quad u^k \to u \text{ in } H^{1,p}(\Omega;\mathbb{R}^m) \text{ weakly}$$

for a particular sequence (u^k) does not by itself inform us of the behavior the sequence

$(\varphi(\nabla u^k))$[1]. Here we show that if φ is nonnegative and has polynomial growth, then $(\varphi(\nabla u^k))$ is weakly convergent in $L^1(\Omega)$ to $\varphi(\nabla u)$. A consequence is that the Young measure generated by (∇u^k) represents the weak limit of a sequence $(\psi(\nabla u^k))$ when ψ is dominated by φ. Our interest in this question arose in the attempt to estimate convergence properties of numerical methods for functionals which are not lower semicontinuous, where φ plays the role of the relaxed density. Validity of the Young measure representation is useful knowledge in the study of the microstructure of ordered solids, cf. Ball and James [5,6], Chipot and Kinderlehrer [10], Ericksen [18-29], Fonseca [31-34], James [35], James and Kinderlehrer [36], Kinderlehrer [37], Kinderlehrer and Pedregal [38], Matos [41], and Pedregal [45,46]. We do not give any explicit applications to the numerical analysis in this paper except to say that our results confirm the validity of the Young measure representation for the limits of the approximations generated by finite element methods when the energy density has appropriate polynomial growth at infinity. We refer to [9,11,12,13,14] for discussions of these developments.

The proof of this and related facts is based on a method of Acerbi and Fusco [1] and subsequent application of the Dunford and Pettis criterion for weak convergence in L^1. Weak convergence of a sequence (f^k) in L^1 is sufficient but not necessary to give sense to the Young measure representation. Ball and Zhang [8] use the Chacon biting lemma to study this question under hypotheses weaker than ours.

The proofs of our results are in §1 - 3. Three applications are given in §4,5, and 6. The example of constraint management in §4 is a generalization of a result of S. Müller [44], cf. also K. Zhang [51]. In §5 a discussion of the Young measure representation when surface energies are present in the system, cf. [39]. Both of these use the convergence property above, or (1.3) below, without assuming that the functional is being driven to a minimum. An application to an evolution problem is given in §6, where it is shown how Young measure solutions may be found. This builds on some recent work of Slemrod [47]. Useful discussions of Young measures are given by Young [50] and Tartar [48,49], and more recently by Ball [3] and Evans [30]. One consequence of our considerations is that they lead to a notion of Young measures generated by functions whose gradients are in L^p for finite p, [45]. We begin with a description of our principal results.

THEOREM 1.1   *Let φ be continuous and quasiconvex and satisfy*

$$0 \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in M, \tag{1.1}$$

where $1 \leq p \leq \infty$. *Suppose that*

$$u^k \rightarrow u \quad in \ H^{1,p}(\Omega) \ weakly \ and \tag{1.2}$$

$$\int_{\Omega} \varphi(\nabla u) \, dx = \lim_{k \to \infty} \int_{\Omega} \varphi(\nabla u^k) \, dx \ . \tag{1.3}$$

Then there is a subsequence (not relabled) of the (u^k) *such that*

$$\varphi(\nabla u^k) \rightarrow \varphi(\nabla u) \quad in \ L^1(\Omega) \ weakly.$$

THEOREM 1.2   *Let* φ *be continuous and quasiconvex and satisfy*

$$0 \leq \varphi(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M},$$

where $1 \leq p \leq \infty$. *If* $u^k \rightarrow u$ *in* $H^{1,p}(\Omega)$ *weakly , then*

$$\int_E \varphi(\nabla u) \, dx \leq \lim \inf_{k \to \infty} \int_E \varphi(\nabla u^k) \, dx \tag{1.4}$$

for every (measurable) $E \subset \Omega$.

We wish to discuss THEOREM 1.2 a little prior to giving the proof. First note that according to the generalizations of Morrey's Theorem [43], for example Acerbi and Fusco [1], (1.4) holds whenever E is a domain with Lipschitz boundary. This information is insufficient to deduce (1.4) for more general sets, which is the crux of the problem.

The case of THEOREM 2 with $p = \infty$ is automatic since $\{ \varphi(\nabla u^k) \}$ are uniformly bounded in this case. Indeed, choose M with the property

$$\| \varphi(\nabla u^k) \|_{L^\infty(\Omega)} \leq M \quad \text{for all k.}$$

Given E, let U be an open neighborhood of E with $|U - E| < \varepsilon$. Now U is the union of countably many cubes $\{ D_j \}$ with disjoint interiors and for each D_j, (1.4) holds. Hence

$$\int_U \varphi(\nabla u) \, dx \leq \Sigma \int_{D_j} \varphi(\nabla u) \, dx$$

$$\leq \; \sum \; \lim\inf \; \int_{D_j} \varphi(\nabla u^k) \, dx$$

$$\leq \; \lim\inf \int_U \varphi(\nabla u^k) \, dx.$$

Finally, we have that

$$\int_E \varphi(\nabla u) \, dx \; \leq \; \lim\inf \int_E \varphi(\nabla u^k) \, dx \; + \; 2M\varepsilon \,.$$

Thus, if $u^k \to u$ in $H^{1,\infty}(\Omega)$ weak*, then

$$\int_E \varphi(\nabla u) \, dx \; \leq \; \lim\inf \int_E \varphi(\nabla u^k) \, dx \tag{1.5}$$

for any measurable $E \subset \Omega$.

The case $p = 1$ for Theorems 1 and 2 is easy and will not be discussed.

To illustrate how the preceding results apply to the Young measure representation, let us introduce the Banach space, for $p > 1$ fixed,

$$E \; = \; \{ \, \psi \in C(M) \colon \; \sup_M \frac{|\psi(A)|}{|A|^p + 1} \; < \; \infty \, \} \,. \tag{1.6}$$

THEOREM 1.3 *Let φ be quasiconvex and satisfy, for some constants $C \geq c > 0$,*

$$\max \{ \, c|A|^p - 1, 0 \, \} \; \leq \; \varphi(A) \; \leq \; C(1 + |A|^p), \quad A \in M, \tag{1.7}$$

where $1 \leq p \leq \infty$. Suppose that

$$u^k \; \to \; u \qquad in \; H^{1,p}(\Omega) \; weakly \; and \tag{1.8}$$

$$\int_\Omega \varphi(\nabla u) \, dx \; = \; \lim_{k \to \infty} \int_\Omega \varphi(\nabla u^k) \, dx \,. \tag{1.9}$$

Let $v = (v_x)_{x \in \Omega}$ be a Young measure generated by (u^k). Then for any $\psi \in E$, the sequence

$$\psi(\nabla u^k) \rightarrow \overline{\psi} \quad in \ \sigma(L^1(\Omega), L^\infty(\Omega)) \ where$$

$$\overline{\psi}(x) = \int_M \psi(A) \, dv_x(A) \quad in \ \Omega \ a.e. \tag{1.10}$$

Further, consider $W \in C(M)$ satisfying

$$W(A) \geq 0$$

and $\tag{1.11}$

$$c \, (|A|^p - 1) \leq W(A) \leq C \, (|A|^p + 1)$$

for some $p > 1$ and $0 < c \leq C$. Let

$$A_\Omega(y_o) = \{ \, v \in H^{1,p}(\Omega): \ v = y_o \ on \ \partial\Omega \, \} \ where \ y_o \in H^{1,p}(\Omega).$$

COROLLARY 1.4   *Let* W *satisfy* (1.11). *Suppose that* $(u^k) \subset A_\Omega(y_o)$ *satisfies*

$$\lim_{k \to \infty} \int_\Omega W(\nabla u^k) \, dx = \inf_{A_\Omega(y_o)} \int_\Omega W(\nabla v) \, dx \quad . \tag{1.12}$$

and

$$u^k \rightarrow u \quad in \ H^{1,p}(\Omega) \ weakly.$$

Let $v = (v_x)_{x \in \Omega}$ *be a Young measure generated by* (u^k). *Then for any* $\psi \in E$, *the sequence*

$$\psi(\nabla u^k) \rightarrow \overline{\psi} \quad in \ \sigma(L^1(\Omega), L^\infty(\Omega)) \ where$$

$$\overline{\psi}(x) = \int_M \psi(A) \, dv_x(A) \quad in \ \Omega \ a.e. \tag{1.13}$$

In particular, the $(W(\nabla u^k))$ *converges to a limit energy density* $\overline{W}$ *in* $\sigma(L^1(\Omega), L^\infty(\Omega))$ *where*

$$\overline{W}(x) = \int_M W(A) \, dv_x(A) \quad in \ \Omega \ a.e. \tag{1.14}$$

A version of COROLLARY 1.4 has also been proved independently by Matos [42] who obtains an improved class E by combining Ekeland's Lemma with the reverse Hölder inequality, although the convergence is then restricted to $\sigma(L^1(\Omega'), L^\infty(\Omega'))$ for $\Omega' \subset\subset \Omega$.

Note that a particular consequence of THEOREM 1.3 is that the sequence $\{ \, | M \cdot \nabla u^k |^p \, \}$, for a constant matrix M, converges weakly in $L^1(\Omega)$, although not to $| M \cdot \nabla u |^p$. Another consequence concerns the relaxation of W, or its quasiconvexification, cf. [7,15,16] for example. Assume that $p > 1$. The integrand

$$W^\#(F) \; = \; \inf_{H_o^{1,\infty}(\Omega)} \frac{1}{|\,\Omega\,|} \int_\Omega W(F + \nabla \zeta) \; dx \qquad\qquad (1.15)$$

is quasiconvex and relaxes the variational principle (1.8) in the sense that

$$\inf_{A_\Omega(y_o)} \int_\Omega W(\nabla v) \, dx \; = \; \inf_{A_\Omega(y_o)} \int_\Omega W^\#(\nabla v) \, dx \; .$$

Obviously a minimizing sequence for (1.12) is also a minimizing sequence for the functional with the integrand $W^\#$. For a given F, the infimum in (1.15) may or may not be realized, but given a minimizing sequence $u^k(x) \; = \; Fx + \zeta^k(x) \in H^{1,p}(\Omega;\mathbb{R}^m)$,

$$W^\#(F) \, | \, \Omega \, | \; = \; \lim_{k \to \infty} \int_\Omega W(\nabla u^k) \, dx \; .$$

Let $\mu \; = \; (\mu_x)_{x \in \Omega}$ be a Young measure generated by (u^k). We may assume that μ_x is independent of $x \in \Omega$, although we pass over the details of that here. Applying COROLLARY 1.4, we obtain in particular that

$$W^\#(F) \; = \; \int_M W(A) \, d\mu(A) \; , \qquad\qquad (1.16)$$

so the infimum is attained in a Young measure fashion. Moreover, the inequality $W^\# \leq W$ insures that

$$\mathrm{supp} \; \mu \; \subset \; \{A: W(A) \; = \; W^\#(A) \, \}.$$

Of course, if σ is any other Young measure generated by some sequence of the form $(v^k) \subset H^{1,p}(\Omega,\mathbb{R}^m)$ with $v^k \; = \; Fx$ on $\partial\Omega$, then

$$\int_M W(A) \, d\mu(A) \; \leq \; \int_M W(A) \, d\sigma(A) \; ,$$

so μ satisfies an ersatz minimizing principle as well.

2   *Proof of* Theorem 1.2

Our aim is to give a proof of the second result. THEOREM 1.1 will be a corollary of it. For this we adopt a technique of Acerbi and Fusco which has an important ingredient from a paper of F.-C. Liu [40]. The technique uses these facts from Acerbi and Fusco:

LEMMA 2.1   *Let* $G \subset \mathbb{R}^n$ *have* $|G| < \infty$. *Assume that* $\{M_k\}$ *is a sequence of subsets of* G *such that for some* $\varepsilon > 0$

$$|M_k| > \varepsilon \qquad \textit{for all } k.$$

Then there is a subsequence k_j *for which*

$$\bigcap M_{k_j} \neq \emptyset .$$

LEMMA 2.2   *Let* $\{f_k\}$ *be a sequence bounded in* $L^1(\Omega)$. *Then for each* $\varepsilon > 0$, *there is a triple* $(A_\varepsilon, \delta, S)$, *where* $A_\varepsilon \subset \Omega$ *with* $|A_\varepsilon| < \varepsilon$, $\delta > 0$, *and* S *is an infinite subset of the natural numbers, such that*

$$\int_D | f_k | \, dx \; < \; \varepsilon$$

whenever $D \cap A_\varepsilon = \emptyset$ *and* $|D| < \delta$.

For any $v \in C_0^\infty(\mathbb{R}^n)$, we set

$$M^*v(x) \; = \; M(|v(x)|) + M(|\nabla v(x)|)$$

where

$$Mf(x) \; = \; \sup_{r > 0} \frac{1}{|B_r|} \int_{B_r(x)} | f(z) | \, dz$$

is the maximal funtion of f. It is well known that if $v \in C_0^\infty(\mathbb{R}^n)$, then $M^*v \in C(\mathbb{R}^n)$ and

$$\| M^* v \|_{L^p(\mathbb{R}^n)} \le C(n,p) \| v \|_{H^{1,p}(\mathbb{R}^n)}, \ 1 < p \le \infty, \tag{2.1}$$

and in particular, for any $\lambda > 0$,

$$|\{ M^* v \ge \lambda \}| \le C(n,p) \lambda^{-p} \| v \|_{H^{1,p}(\mathbb{R}^n)}, \ 1 < p \le \infty. \tag{2.2}$$

LEMMA 2.3   *Let* $v \in C_0^\infty(\mathbb{R}^n)$ *and* $\lambda > 0$. *Set* $H^\lambda = \{ M^* v < \lambda \}$. *Then*

$$\frac{|v(x) - v(y)|}{|x - y|} \le C(n) \lambda, \ x, y \in H^\lambda, \tag{2.3}$$

where $C(n)$ *depends only on* n.

   We shall also make use of the well known fact that a Lipschitz function defined on a subset of $\mathbb{R}^n$ may be extended to all of $\mathbb{R}^n$ without increasing its Lipschitz constant.

PROOF OF THEOREM 1.2   We regard u^k and u as extended to functions in $H^{1,p}(\mathbb{R}^n)$ with norms controlled by their $H^{1,p}(\Omega)$ norms. Let $\varepsilon > 0$.

Step 1.   Since the functional of (1.4) is continuous in $H^{1,p}(\mathbb{R}^n)$ in the norm topology, because of the upper bound on φ, we may find $z, z^k \in C_0^\infty(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |\varphi(\nabla u) - \varphi(\nabla z)| \, dx \ < \ \varepsilon \tag{2.4}$$

$$\int_{\mathbb{R}^n} |\varphi(\nabla u^k) - \varphi(\nabla z + \nabla z^k)| \, dx \ < \ \varepsilon \tag{2.5}$$

and

$$\| u - u^k - z^k \|_{H^{1,p}(\mathbb{R}^n)} \ < \ \frac{1}{k}.$$

Thus $z^k \to 0$ in $H^{1,p}(\mathbb{R}^n)$ weakly and

$$\| z^k \|_{H^{1,p}(\mathbb{R}^n)} \ \le \ M \ < \ \infty. \tag{2.6}$$

Set

$$H^\lambda = \{ M^* z < \lambda \} \quad \text{and} \quad H^\lambda_k = \{ M^* z^k < \lambda \} .$$

According to LEMMA 2.3, we may find $\zeta^k, \eta \in H^{1,\infty}(\mathbb{R}^n)$ such that $\zeta^k = z^k$ on H^λ_k and $\eta = z$ on H^λ with

$$\| \zeta^k \|_{L^\infty(\mathbb{R}^n)} \leq \| z^k \|_{L^\infty(H^\lambda_k)} \leq \lambda$$

and

$$\| \zeta^k \|_{H^{1,\infty}(\mathbb{R}^n)} \leq C(n) \lambda,$$

and the same for η. After extraction of a subsequence we may suppose that

$$\zeta^k \rightarrow \zeta \quad \text{in } H^{1,\infty}(\mathbb{R}^n) \text{ weak*}.$$

We apply LEMMA 2.2 to the sequence $\{ M^*(z^k)^p \}$. By (1.2) and (2.1) these functions are bounded in $L^1(\Omega)$. So given $\varepsilon' > 0$, there is a triple $(A_{\varepsilon'}, \delta, S)$ with $| A_{\varepsilon'} | < \varepsilon'$ and

$$\int_D M^*(z^k)^p \, dx < \varepsilon'$$

whenever $D \cap A_{\varepsilon'} = \emptyset$ and $k \in S$.

Now let $G = \{ \zeta \neq 0 \}$. Since the $z^k \rightarrow 0$ in $L^p(\mathbb{R}^n)$ in norm, we may assume that $z^k \rightarrow 0$ pointwise a.e. in Ω. Thus if we set $G_0 = G \cap \{ x \in \Omega : z^k(x) \rightarrow 0 \}$, then $| G_0 | = | G |$. We write G_0 as a union,

$$G_0 = (G_0 \cap H^\lambda_k) \cup (G_0 \cap (\mathbb{R}^n - H^\lambda_k)) .$$

By (2.2),

$$| G_0 \cap (\mathbb{R}^n - H^\lambda_k) | \leq C \lambda^{-p} M \quad \text{for all } k. \tag{2.7}$$

This implies that

$$| G_0 | = | G | \leq 2C \lambda^{-p} M . \tag{2.8}$$

Otherwise, namely if

$$| G_o | \ > \ 2C \lambda^{-p} M, \quad \text{then} \quad | G_o \cap H_k^\lambda | \ > \ C \lambda^{-p} M,$$

by (2.7). Applying Lemma 2.1, there would be a subsequence k_j such that

$$G_o \cap \left(\cap H_{k_j}^\lambda \right) \ \neq \ \varnothing,$$

and for x in this intersection,

$$\zeta(x) \ = \ \lim \zeta^k(x) \ = \ \lim z^k(x) \ = \ 0,$$

which contradicts the definition of the set G. Hence (2.8) holds.

Step 2   Since $\varphi(\nabla u) \in L^1(\Omega)$, we may find $\sigma, 0 < \sigma < \varepsilon,$ and λ large enough that

$$\int_{A_\sigma \cup (\Omega - H^\lambda) \cup G} \varphi(\nabla u) \, dx \ < \ \varepsilon, \tag{2.9}$$

cf. (2.8) above. Let $E \subset \Omega$ be measurable and assume a subsequence of the u^k chosen (but not relabled) so that

$$\lim \int_E \varphi(\nabla u^k) \, dx \ = \ \liminf \int_E \varphi(\nabla u^k) \, dx .$$

Put

$$\alpha_k \ = \ \int_E \varphi(\nabla u^k) \, dx .$$

Since $\varphi \geq 0,$ by (2.5)

$$\alpha_k \ \geq \ \int_{E \cap H^\lambda \cap H^{\lambda,k} \cap (\Omega - A_\sigma)} \varphi(\nabla u^k) \, dx$$

$$\geq \; -\varepsilon \; + \int\limits_{E\cap H^{\lambda}\cap H^{\lambda,k}\cap(\Omega-A_{\sigma})} \varphi(\nabla z + \nabla z^k)\,dx \quad .$$

But $\nabla z = \nabla \eta$ and $\nabla z^k = \nabla \zeta^k$ in $H^{\lambda}\cap H^{\lambda,k}$ so that

$$\alpha_k \; \geq \; -\varepsilon \; + \int\limits_{E\cap H^{\lambda}\cap H^{\lambda,k}\cap(\Omega-A_{\sigma})} \varphi(\nabla\eta + \nabla\zeta^k)\,dx$$

$$= \; -\varepsilon \; + \int\limits_{E\cap H^{\lambda}\cap(\Omega-A_{\sigma})} \varphi(\nabla\eta + \nabla\zeta^k)\,dx \; - \int\limits_{E\cap H^{\lambda}\cap(\Omega-H^{\lambda,k})\cap(\Omega-A_{\sigma})} \varphi(\nabla\eta + \nabla\zeta^k)\,dx$$

$$= \; -\varepsilon \; + \beta_k \; - \gamma_k \; .$$

Since $\nabla(\eta + \zeta^k)$ is uniformly bounded and φ is quasiconvex, by the remark (1.5), we have that for K sufficiently large

$$\beta_k \; + \; \varepsilon \; \geq \int\limits_{E\cap H^{\lambda}\cap(\Omega-A_{\sigma})} \varphi(\nabla\eta + \nabla\zeta)\,dx \quad .$$

We now inspect γ_k. Using the bounds on $\nabla\eta$ and $\nabla\zeta^k$, and choosing λ large enough,

$$\gamma_k \; \leq \; C(1+\lambda^p)\,|\,(\Omega - H_k^{\lambda})\cap(\Omega - A_{\sigma})\,|$$

$$\leq \; C\,|\,\Omega - H_k^{\lambda}\,| \; + \int\limits_{(\Omega-H^{\lambda,k})\cap(\Omega-A_{\sigma})} CM^*(z^k)^p\,dx$$

$$\leq \; C\varepsilon \; + \; C\sigma \; \leq \; 2C\varepsilon.$$

Consequently, for k sufficiently large,

$$\alpha_k \; \geq \; -C\varepsilon \; + \int\limits_{E\cap H^{\lambda}\cap(\Omega-A_{\sigma})} \varphi(\nabla\eta + \nabla\zeta)\,dx \quad . \tag{2.10}$$

Step 3.   Again using the positivity of φ, from (2.10)

$$\alpha_k \; \geq \; -C\varepsilon \; + \int\limits_{E\cap H^{\lambda}\cap(\Omega-A_{\sigma})\cap(\Omega-G)} \varphi(\nabla\eta + \nabla\zeta)\,dx \quad .$$

Since $\zeta = 0$ in $\Omega - G$, we have that $\nabla\zeta = 0$ in $\Omega - G$, so, since $\eta = z$ in H^λ, we deduce that

$$\alpha_k \geq -C\varepsilon + \int_{E \cap H^\lambda \cap (\Omega - A_\sigma) \cap (\Omega - G)} \varphi(\nabla\eta)\,dx$$

$$\geq -C\varepsilon + \int_{E \cap H^\lambda \cap (\Omega - A_\sigma) \cap (\Omega - G)} \varphi(\nabla z)\,dx \quad .$$

By (2.4) and (2.9),

$$\alpha_k \geq -(1+C)\varepsilon + \int_{E \cap H^\lambda \cap (\Omega - A_\sigma) \cap (\Omega - G)} \varphi(\nabla u)\,dx$$

$$\geq -(1+C)\varepsilon + \int_E \varphi(\nabla u)\,dx - \int_{E \cap [A_\sigma \cup (\Omega - H^\lambda) \cup G]} \varphi(\nabla u)\,dx$$

$$\geq -(2+C)\varepsilon + \int_E \varphi(\nabla u)\,dx \quad .$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved.

3   Proofs of the other results

PROOF OF THEOREM 1.1   This follows from the Dunford-Pettis criterion. Assume that the sequence $(\varphi(\nabla u^k))$ is not $\sigma(L^1, L^\infty)$ relatively compact. Then for some $\varepsilon > 0$ and every $\delta > 0$, there is an $A_\delta \subset \Omega$ and an integer k_δ such that $|A_\delta| < \delta$ and

$$\int_{A_\delta} \varphi(\nabla u^{k_\delta})\,dx > \varepsilon \quad .$$

Since $\varphi(\nabla u) \in L^1(\Omega)$, there is a $\delta_0 > 0$ such that if $|E| < \delta_0$, then

$$\int_E \varphi(\nabla u) \, dx \; < \; \varepsilon. \tag{3.1}$$

Let us choose in particular $\delta_j = 2^{-j} \delta_0$. Then there is a sequence A_j, $|A_j| < \delta_j$, and k_j such that

$$\int_{A_j} \varphi(\nabla u^{k_j}) \, dx \; > \; \varepsilon \quad \text{for all } j.$$

Let $E = \cup A_j$, so $|E| \leq \delta_0$ and (3.1) holds. Thus

$$\varepsilon \; \leq \; \int_E \varphi(\nabla u^{k_j}) \, dx \; \leq \; \int_\Omega \varphi(\nabla u^{k_j}) \, dx \; - \; \int_{\Omega-E} \varphi(\nabla u^{k_j}) \, dx \quad .$$

Letting $k_j \to \infty$, we have by THEOREM 1.2 and the hypothesis (1.3) that

$$\varepsilon \; \leq \; \int_\Omega \varphi(\nabla u) \, dx) \; - \; \int_{\Omega-E} \varphi(\nabla u) \, dx$$

$$= \; \int_E \varphi(\nabla u) \, dx \; < \; \varepsilon,$$

a contradiction.

PROOF OF THEOREM 1.3: This also follows by the Dunford-Pettis criterion, using THEOREM 1.1.

4   *Constraint management in a limit case*

Certain variational principles in elasticity constrain the admissible variations $v \in H^{1,s}(\Omega;\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$, to satisfy

$$\det \nabla v \ > \ 0 \quad \text{in } \Omega \text{ a.e.}$$

In the limit case $p = n$, $\det \nabla v \in L^1(\Omega)$ for $v \in H^{1,n}(\Omega;\mathbb{R}^n)$ but it is not necessarily integrable to any higher power. Thus it is not automatic that if $u^k \to u$ in $H^{1,n}(\Omega;\mathbb{R}^n)$ weakly, that $\det \nabla u^k \to \det \nabla u$ in $L^1(\Omega)$ weakly. In fact, without additional requirements, this condition does not hold. One may refer to the counterexamples in Ball and Murat [7]. However, much is known about this situation, as we summarize below.

First of all, the determinant is a null lagrangian, namely, if $u,v \in H^{1,n}(\Omega;\mathbb{R}^n)$ and $u \Big|_{\partial\Omega}$

$= v \Big|_{\partial\Omega}$, then

$$\int_\Omega \det \nabla u \ dx \ = \ \int_\Omega \det \nabla v \ dx \ . \tag{4.1}$$

Assume that $u^k, u \in H^{1,n}(\Omega;\mathbb{R}^n)$ and

$$u^k \ \to \ u \ \text{in } H^{1,n}(\Omega;\mathbb{R}^n) \text{ weakly.} \tag{4.2}$$

Then for a subsequence of the (u^k), not relabeled, cf. eg. [2] ,

$$\det \nabla u^k \ \to \ \det \nabla u \quad \text{in } D'(\Omega). \tag{4.3}$$

Very recently, S. Müller [44] has shown that if (4.2) holds and $\det \nabla u^k \geq 0$, then

$$\det \nabla u^k \ \to \ \det \nabla u \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ weakly.} \tag{4.4}$$

We give a slight generalization of Müller's result. With it, alternate proofs of some results in elasticity may be given, for example, some of those in Zhang [51].

THEOREM 4.1   *Let* $u^k, u \in H^{1,n}(\Omega;\mathbb{R}^n)$ *satisfy*

$$u^k \ \to \ u \quad \textit{in } H^{1,n}(\Omega;\mathbb{R}^n) \textit{ weakly,} \tag{4.5}$$
$$\det \nabla u^k \ \geq \ 0 \quad \textit{in } \Omega \textit{ a.e., and} \tag{4.6}$$
$$u^k \Big|_{\partial\Omega} = u_o \Big|_{\partial\Omega}, \tag{4.7}$$

where $u_o \in H^{1,n}(\Omega;\mathbb{R}^n)$ *is fixed. Then*

$$\det \nabla u^k \quad \rightarrow \quad \det \nabla u \quad \text{in } L^1(\Omega) \text{ weakly.} \tag{4.8}$$

PROOF   First of all, $u = u_o$ on $\partial\Omega$. From Müller's result (4.4), we deduce that $\det \nabla u \geq 0$ in Ω a.e. By (4.1),

$$\int_\Omega \det \nabla u \, dx \quad = \quad \int_\Omega \det \nabla u^k \, dx \quad = \quad \int_\Omega \det \nabla u_o \, dx \quad, \text{for all } k. \tag{4.9}$$

Now let

$$\varphi(A) \quad = \quad \max \{ \det A, 0 \}, \quad A \in \mathbb{M},$$

which is continuous, quasiconvex, and satisfies

$$0 \leq \varphi(A) \leq C(1 + |A|)^n, \quad A \in \mathbb{M}.$$

Then $\varphi(\nabla u^k) = \det \nabla u^k$ and $\varphi(\nabla u) = \det \nabla u$, so, trivially, by (4.9),

$$\int_\Omega \varphi(\nabla u) \, dx \quad = \quad \lim_{k \rightarrow \infty} \int_\Omega \varphi(\nabla u^k) \, dx \ .$$

Consequently, by THEOREM 1.1, possibly for a subsequence which we do not relable,

$$\det \nabla u^k \quad \rightarrow \quad \det \nabla u \quad \text{in } L^1(\Omega) \text{ weakly.} \qquad\qquad \text{QED}$$

We wish to remark that we used Müller's result to conclude that $\det \nabla u \geq 0$ in Ω a.e. We could also have used the biting convergence theorem of Zhang [51] for this purpose. The idea of THEOREM 4.1 is that the sequence (u^k) may arise as a minimizing sequence for some variational principle subject to (4.6). Additional information then follows from the theorem.

5   *Application to functionals with surface energies*

We consider a simple situation where cooperative bulk and surface energies are minimized. Let $\Omega \subset \mathbb{R}^n$ have smooth boundary Γ and set

$$E(v) \quad = \quad \int_\Omega W(\nabla v) \, dx \quad + \quad \int_\Gamma \tau(\nabla v, v) \, dS \ , \quad v \in C^1(\bar{\Omega}; \mathbb{R}^m), \tag{5.1}$$

where ν denotes the exterior normal to Γ. The infimum of E over $C^1(\bar{\Omega};\mathbb{R}^m)$ is not necessarily the sum of the infima of its two summands, so we envision an application of our results when (1.3) will hold for each of the two terms but where these quantities will not be the unrestricted infima of their portions of the functional.

Assume that W is continuous and satisfies, for some $p > 1$ and $C \geq c > 0$,

$$\max\{ c|A|^p - 1, 0 \} \leq W(A) \leq C(1 + |A|^p), \quad A \in \mathbb{M}. \tag{5.2}$$

About τ we assume that it is continuous and, for some $s > 1$,

$$0 \leq \tau(A,\nu) \quad \text{and}$$
$$A \in \mathbb{M}, \tag{5.3}$$
$$c(|A_{tan}|^s - 1) \leq \tau(A,\nu) \leq C(|A|^s + 1),$$

where $A_{tan} = A(1 - \nu \otimes \nu)$ is the tangential part of A.

For a fixed $\nu \in \mathbb{S}^{n-1}$, let $D' \subset \{x \cdot \nu = 0\}$ be a domain and let dx' denote the $(n-1)$-Lebesgue measure on D'. By $D' \times (-r,r)$, $r > 0$, we abbreviate the name of the set

$$\{x \in \mathbb{R}^n: x' = (1 - \nu \otimes \nu)x \in D' \text{ and } |x \cdot \nu| < r \}.$$

Let $[E]$ denote the $n - 1$ dimensional Lebesgue measure of E. We define

$$\tau^\#(F,\nu) = \inf_{C'} \frac{1}{[D']} \int_{D'} \tau(F + \nabla\zeta,\nu)\,dx', \quad (F,\nu) \in \mathbb{M} \times \mathbb{S}^{n-1},$$
$$\tag{5.4}$$
$$C' = C_0^1(D' \times (-r,r)).$$

We always suppose that $[\partial D'] = 0$. Clearly $\tau^\# \geq 0$ and is independent of r. The relaxation of the functional E is given by

$$E^\#(v) = \int_\Omega W^\#(\nabla v)\,dx + \int_\Gamma \tau^\#(\nabla v,\nu)\,dS, \quad v \in C^1(\bar{\Omega};\mathbb{R}^m), \tag{5.5}$$

where $W^\#(A)$ is the ordinary quasiconvexification of W and $\tau^\#$ is defined by (5.4). A special property of $\tau^\#$ is that

$$\tau^{\#}(A,v) = \tau^{\#}(A_{tan},v), \quad A \in M,$$

which implies that

$$c(|A_{tan}|^s - 1) \leq \tau^{\#}(A,v) \leq C(|A|^s + 1), \quad A \in M, \tag{5.6}$$

and that $\tau^{\#}$ is well defined on $H^{1,s}(\Gamma;\mathbb{R}^m)$. An easy generalization of [39] tells us that

$$\inf_{C^1(\bar{\Omega})} E(v) = \inf_V E^{\#}(v), \quad V = H^{1,p}(\Omega;\mathbb{R}^m) \times H^{1,s}(\Gamma;\mathbb{R}^m). \tag{5.7}$$

Let $(u^k) \subset V$ be a minimizing sequence for E. Then (u^k) is a minimizing sequence for $E^{\#}$, which is bounded in V. Suppose that $u \in V$ and $u^k \to u$ in V weakly. By lower semicontinuity,

$$E^{\#}(u) = \lim_{k \to \infty} E^{\#}(v) = \inf_{C^1(\bar{\Omega})} E(v) = \inf_V E^{\#}(v) \quad \text{and}$$

$$\int_\Omega W^{\#}(\nabla u)\, dx = \lim_{k \to \infty} \int_\Omega W^{\#}(\nabla u^k)\, dx$$

$$\tag{5.8}$$

$$\int_\Gamma \tau^{\#}(\nabla_{tan}u,v)\, dS = \lim_{k \to \infty} \int_\Gamma \tau^{\#}(\nabla u^k,v)\, dS .$$

We may apply THEOREM 1.3, or a slight generalization of it in the case of $(\tau^{\#}(\nabla u^k,v))$, to deduce that

$$W^{\#}(\nabla u^k) \to W^{\#}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly and}$$
$$\tau^{\#}(\nabla u^k,v) \to \tau^{\#}(\nabla_{tan}u,v) \quad \text{in } L^1(\Gamma) \text{ weakly} .$$

If $\mu = (\mu_x)_{x \in \Omega}$ denotes a Young measure generated by (∇u^k), we have the limit energy representations

$$\bar{W}(x) = W^{\#}(\nabla u(x)) = \int_M W(A)\, d\mu_x(A), \quad x \in \Omega,$$

$$\bar{\tau}(x) = \tau^{\#}(\nabla_{tan}u(x),v(x)) = \int_\Gamma \tau^{\#}(A,v(x))\, d\mu_x(A), \quad x \in \Gamma,$$

and

$$\int_\Omega \bar{W}(x)\, dx + \int_\Gamma \bar{\tau}(x)\, dS = \inf_{C^1(\bar{B})} E(v) .$$

6   *Measure valued solutions of an evolution problem*

Some of our methods may be employed to study measure valued solutions of evolution problems. A more extensive treatment is given by Slemrod [47]; here we wish to explain merely how such solutions may come about. For further developments we refer to Demoulini [17]. To fix the ideas, we consider a scalar case. Suppose that $\varphi \in C^1(\mathbb{R}^n)$ satisfies

$$\max (c| a |^2 - 1, 0) \leq \varphi(a) \leq C(| a |^2 + 1)$$

$$| \nabla\varphi(a) | \leq C| a |$$

$$a \in \mathbb{R}^n, \qquad (6.1)$$

where $0 < c \leq C$. Let $q(a) = \nabla\varphi(a)$. Our interest is in solutions, possibly Young measures, which in some sense satisfy

$$-\text{div } \overline{q} + \frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \qquad (6.2)$$

$\mathbb{R}^+ = (0,\infty)$, subject to appropriate boundary conditions.

To render this more precise, let us agree that $\nu = (\nu_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+}$ is a Young measure solution of (6.2) provided that

ν *is a family of probability measures and*
$u \in L^\infty(\mathbb{R}^+; H^1_0(\Omega))$ *with* $\frac{\partial u}{\partial t} \in L^2(\Omega \times \mathbb{R}^+)$ *which satisfy*

$$-\text{div } \overline{q} + \frac{\partial u}{\partial t} = 0 \quad in \; H^1_0(\Omega \times \mathbb{R}^+), \qquad (6.3)$$

$$u \Big|_{\partial\Omega} = u_0 \Big|_{\partial\Omega}, \quad where \qquad (6.4)$$

$$\overline{q}(x,t) = \int_{\mathbb{R}^n} q(a) \, d\nu_{x,t}(a) \quad and$$

$$\nabla u(x,t) = \int_{\mathbb{R}^n} a \, d\nu_{x,t}(a) \qquad in \; \Omega \times \mathbb{R}^+ \; a.e. \qquad (6.5)$$

Above, $u_0 \in H^1_0(\Omega)$ is given. Moreover, we shall impose the condition that

v *is generated by a sequence* (∇u^h), $h > 0$, *where* $u^h \in L^\infty(\mathbb{R}^+; H_0^1(\Omega))$.   (6.6)

The equation (6.5) means that

$$\int_0^\infty \int_\Omega (\overline{q} \cdot \nabla \zeta + \frac{\partial u}{\partial t} \zeta) \, dxdt = 0 \quad \text{for } \zeta \in H_0^1(\Omega \times \mathbb{R}^+).$$ (6.7)

We shall give an outline of the proof of

THEOREM 6.1   *Assume* (6.1) *about* φ. *Then there exists a Young measure solution* $v = (v_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+}$ *of*

$$-\operatorname{div} \overline{q} + \frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

satisfying (6.3) - (6.6). *In addition*

$$\operatorname{supp} v_{x,t} \subset \{ a \in \mathbb{R}^n : \varphi(a) = \varphi^{**}(a) \}, \quad \text{in } \Omega \times \mathbb{R}^+ \text{ a.e.,}$$ (6.8)

where φ^{**} *is the convexification of* φ .

Recall that if $\varphi \in C^1(\mathbb{R}^n)$, then $\varphi^{**} \in C^1(\mathbb{R}^n)$, whence

$$q(a) = q^{**}(a) \quad \text{in} \quad \{ a \in \mathbb{R}^n : \varphi(a) = \varphi^{**}(a) \},$$

where $q^{**}(a) = \nabla \varphi^{**}(a)$. Note also that φ^{**} satisfies (6.1). Hence the

COROLLARY 6.2   *Assume* (6.1) *about* φ *and let* $v = (v_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+}$ *be a Young measure solution satisfying* (6.8). *Then* v *is a solution of the relaxed problem*

$$-\operatorname{div} \overline{q}^{**} + \frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega \times \mathbb{R}^+.$$ (6.9)

The constructed solution has some additional properties which we shall describe in the sequel.

Step1 An equilibrium problem. Let $w \in H_0^1(\Omega)$ and $h > 0$ and consider

$$\Phi(v) \ = \ \Phi_h(v) \ = \ \int_\Omega (\phi(\nabla v) + \tfrac{1}{2h} |v - w|^2)\, dx, \quad v \in H_0^1(\Omega), \text{ and } \quad (6.10)$$

$$\Phi^{**}(v) \ = \ \int_\Omega (\phi^{**}(\nabla v) + \tfrac{1}{2h} |v - w|^2)\, dx, \quad v \in H_0^1(\Omega), \qquad\qquad (6.11)$$

where ϕ^{**} is the convexification of ϕ. By a known relaxation theorem, cf. [16],

$$I \ = \ \inf_{H_0^1(\Omega)} \Phi(v) \ = \ \inf_{H_0^1(\Omega)} \Phi^{**}(v). \qquad\qquad (6.12)$$

Now let (v^k) be a minimizing sequence for $\Phi(v)$. We may assume there is a $u \in H_0^1(\Omega)$ such that

$$v^k \ \to \ u \quad \text{in } H_0^1(\Omega) \text{ weakly as } k \to \infty.$$

By lower semicontinuity,

$$\Phi(v^k) \ \to \ \Phi^{**}(u) \quad \text{as } k \to \infty,$$

and by the Rellich Theorem,

$$\int_\Omega |v^k - w|^2\, dx \ \to \ \int_\Omega |u - w|^2\, dx \quad \text{as } k \to \infty.$$

Hence

$$\int_\Omega \phi^{**}(\nabla u)\, dx \ = \ \lim_{k \to \infty} \int_\Omega \phi^{**}(\nabla v^k)\, dx \ = \ \lim_{k \to \infty} \int_\Omega \phi(\nabla v^k)\, dx \ .$$

Hence by THEOREM 1.1,

$$\phi^{**}(\nabla v^k) \ \to \ \phi^{**}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly and}$$
$$\phi(\nabla v^k) \ \to \ \phi^{**}(\nabla u) \quad \text{in } L^1(\Omega) \text{ weakly.}$$

Denoting by $\nu = (\nu_x)_{x \in \Omega}$ the Young measure generated by (∇v^k),

$$\text{supp } v \ \subset \ \{ \ a \in \mathbb{R}^n \colon \ \varphi(a) = \varphi^{**}(a) \ \} \ ,$$

$$\varphi^{**}(\nabla u) \ = \ \overline{\varphi} \ = \ \overline{\varphi}^{**} \quad \text{and} \quad \overline{q} \ = \ \overline{q}^{**} \quad \text{in } \Omega \text{ a.e.,} \tag{6.13}$$

where

$$\overline{\psi}(x) \ = \ \int_{\mathbb{R}^n} \psi(a) \, dv_x(a) \quad \text{in } \Omega \text{ a.e.}$$

In fact, the Young measure representation holds for any $\psi \in E$, where

$$E \ = \ \{ \ \psi \in C(M) \colon \ \sup_M \frac{|\psi(A)|}{|A|^2 + 1} < \infty \ \} \ .$$

We may now apply the technique developed in [10] to discuss stable Young measure minimizers of variational principles, cf. §5. As a consequence of this, we may write an equilibrium equation

$$\int_\Omega (\ \overline{q} \cdot \nabla \zeta + \tfrac{1}{h}(u - w) \ \zeta) \ dx \ = \ 0 \quad \text{for } \zeta \in H_0^1(\Omega). \tag{6.14}$$

Finally, the Young measure representation provides us with an elementary estimate for $\overline{q}$. Indeed, using the estimates of (6.1) and (6.13),

$$\int_\Omega | \ \overline{q} \ |^2 \, dx \ \leq \ \int_\Omega \int_{\mathbb{R}^n} | \ q(a) \ |^2 \, dv_x(a) dx$$

$$\leq \ C \int_\Omega \int_{\mathbb{R}^n} | \ a \ |^2 \, dv_x(a) dx$$

$$\leq \ C \int_\Omega \int_{\mathbb{R}^n} (\varphi(a) + 1) \, dv_x(a) dx$$

$$= \ C \int_\Omega (\varphi^{**}(\nabla u) + 1) \, dx \tag{6.15}$$

Step 2 Approximate solution   Let $u_o \in H_o^1(\Omega)$ be given and $h > 0$. We define a sequence of Young measure solutions $v^{h,j}$ and underlying functions $u^{h,j}$ by setting

$$v^{h,o} = \delta_{\nabla u_o} \quad \text{and} \quad u^{h,o} = u_o$$

and $v^{h,j+1}$ the solution of (6.12) with $w = u^{h,j}$ and $u^{h,j+1}$ its underlying function. We then are in possession of the energy densities

$$\varphi^{**}(\nabla u^{h,j}) = \langle v^{h,j}, \varphi \rangle = \langle v^{h,j}, \varphi^{**} \rangle \tag{6.16}$$

and the flux densities

$$\overline{q}^{h,j} = \langle v^{h,j}, q \rangle = \langle v^{h,j}, q^{**} \rangle . \tag{6.17}$$

Let $I^{h,j} = [hj, h(j+1))$, $\chi^{h,j} = \chi_I{}^{h,j}$, and

$$\lambda^{h,j}(t) = \begin{cases} 0 & t < hj \\ \dfrac{t}{hj} - 1 & hj \le t < h(j+1) \\ 1 & h(j+1) \le t \end{cases} .$$

Set

$$u^h(x,t) = \sum_j \{(1 - \lambda^{h,j}(t)) u^{h,j}(x) + \lambda^{h,j}(t) u^{h,j+1}(x)\} \in L^\infty(\mathbb{R}^+; H_o^1(\Omega)) \tag{6.18}$$

and

$$v_{x,t}^h = \sum_j \chi^{h,j}(t) v_x^{h,j} \quad \in E'. \tag{6.19}$$

Now from (6.18),

$$\frac{\partial u^h}{\partial t} = \frac{1}{h}(u^{h,j+1} - u^{h,j}) \quad \text{and} \quad \overline{q}^h = \langle v^h, q \rangle = \sum_j \overline{q}^{h,j} \chi^{h,j} \tag{6.20}$$

comprise a solution of

$$-\operatorname{div} \overline{q}^h + \frac{\partial u^h}{\partial t} = 0 \quad \text{in } H^{-1}(\Omega), \text{ for each } t,$$

from which it is elementary to check that

$$\int_0^\infty \int_\Omega (\, \overline{q}^h \cdot \nabla\zeta + \frac{\partial u^h}{\partial t} \, \zeta \,) \, dxdt \; = \; 0 \quad \text{for } \zeta \in H_o^1(\Omega \times \mathbb{R}^+). \tag{6.21}$$

Step 3 Estimates   Uniform estimates are available for $u^h \in L^\infty(\mathbb{R}^+; H_o^1(\Omega))$ and $\frac{\partial u^h}{\partial t} \in$

$L^2(\Omega \times \mathbb{R}^+)$. To begin, $u^{h,j}$ is admissible in the variational principle for $u^{h,j+1}$, so

$$\int_\Omega (\varphi^{**}(\nabla u^{h,j+1}) + \frac{1}{2h} \, | \, u^{h,j+1} - u^{h,j} \, |^2) \, dx \; \le \; \int_\Omega \varphi^{**}(\nabla u^{h,j}) \, dx \; .$$

Hence

$$\int_\Omega \varphi^{**}(\nabla u^{h,j}) \, dx \; \le \; \int_\Omega \varphi^{**}(\nabla u_o) \, dx \; = \; M^2 \tag{6.22}$$

and

$$\frac{1}{2h} \sum_j | \, u^{h,j+1} - u^{h,j} \, |^2 \; \le \; \int_\Omega \varphi^{**}(\nabla u_o) \, dx \; = \; M^2. \tag{6.23}$$

Since φ^{**} satisfies (6.1), the inequality (6.22) tells us that

$$\| \nabla u^{h,j} \|_{L^2(\Omega)} \; \le \; M \, . \tag{6.24}$$

By convexity of the L^2 norm and (6.24) we have that

$$\| u^h \|_{L^\infty(\mathbb{R}^+; H_o^1(\Omega))} \; \le \; M \, . \tag{6.25}$$

Rearranging a little in (6.23) and noting (6.20),

$$\int_0^\infty \int_\Omega | \frac{\partial u^h}{\partial t} |^2 \, dxdt \; \le \; M^2 \, . \tag{6.26}$$

Introduce the function

$$w^h(x,t) \; = \; \sum_j u^{h,j}(x) \, \chi^{h,j}(t) \; \in \; L^\infty(\mathbb{R}^+; H_o^1(\Omega)) \, . \tag{6.27}$$

Then (6.24) implies that

$$\| w^h \|_{L^\infty(\mathbb{R}^+;H^1_o(\Omega))} \leq M .$$ (6.28)

Finally, we wish to estimate $\overline{q}^h$ using (6.15), which provides the estimate

$$\| \overline{q}^h \|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} \leq C \int_\Omega (\varphi^{**}(\nabla u^{h,j}) + 1)dx \leq C(M^2 + 1).$$ (6.29)

Step 4 Passage to the limit   We let $h \to 0$. From the estimates (6.25), (6.26), (6.28), and (6.29), we may extract a subsequence of h as $h \to 0$ and

$$\nu = (\nu_{x,t})_{(x,t) \in \Omega \times \mathbb{R}^+} \in E' \quad \text{with} \quad \text{supp } \nu \subset \{ \varphi(a) = \varphi^{**}(a) \}$$
and ν is a Young measure,

$$w \in L^\infty(\mathbb{R}^+;H^1_o(\Omega)) \quad \text{with} \quad \nabla w = \langle \nu,a \rangle ,$$

$$\overline{q} \in L^\infty(\mathbb{R}^+;L^2(\Omega)) \quad \text{with} \quad \overline{q} = \langle \nu,q \rangle = \langle \nu,q^{**} \rangle, \text{ and}$$
$$u \in L^\infty(\mathbb{R}^+;H^1_o(\Omega)) \quad \text{with} \quad \frac{\partial u}{\partial t} \in L^2(\Omega \times \mathbb{R}^+)$$

which satisfy

$$\int_0^\infty \int_\Omega (\overline{q} \cdot \nabla \zeta + \frac{\partial u}{\partial t} \zeta) \, dxdt = 0 \quad \text{for } \zeta \in H^1_o(\Omega \times \mathbb{R}^+).$$ (6.30)

In fact, (6.30) above holds for $\zeta \in L^\infty(\mathbb{R}^+;H^1_o(\Omega))$. We remark that ν is a Young measure but it is not generated by the sequence (∇u^h) of (6.18), but rather by a diagonal subsequence of the functions which generate the (ν^h) of (6.19).

It remains to show that the Young measure ν and the limit function u are connected. We claim that $u = w$. In fact, we shall show that $\nabla u = \nabla w$ by means of an easy lemma.

LEMMA 6.3   Let $(f^{h,j}) \subset$ bounded set of $L^2(\Omega)$ for $h > 0$ and $j = 1,2,3,...,$ and set

$$f^h(x,t) = \sum_j f^{h,j}(x) \chi^{h,j}(t) \quad and$$
$$g^h(x,t) = \sum_j \{ (1 - \lambda^{h,j}(t)) f^{h,j}(x) + \lambda^{h,j}(t) f^{h,j+1}(x) \},$$

where $\chi^{h,j}$ *is the characteristic function of* $[hj, h(j+1))$ *and*

$$\lambda^{h,j}(t) \;=\; \begin{cases} \quad 0 & t \;<\; hj \\ \dfrac{t}{h} - j & hj \;\leq\; t \;<\; h(j+1) \\ \quad 1 & h(j+1) \;\leq\; t \end{cases} \quad .$$

Suppose that

$$f^h \;\rightarrow\; f \quad and \quad g^h \;\rightarrow\; g \quad in \;\; L^2_{loc}(\Omega \times \mathbb{R}^+) \;\; weakly.$$

Then $f = g.$

PROOF   It suffices to show that

$$\int_0^\infty \int_\Omega f \, \zeta \, dxdt \;=\; \int_0^\infty \int_\Omega g \, \zeta \, dxdt$$

for $\zeta \in C_0^\infty(\Omega)$ of the form $\zeta(x,t) = w(x)z(t)$. Let $z^{h,j} = z(hj)$ and

$$\zeta^h(x,t) \;=\; w(x) \sum_j z^{h,j} \, \chi^{h,j}(t) \, ,$$

$$\xi^h(x,t) \;=\; w(x) \sum_j \left\{ (1 - \lambda^{h,j}(t)) \, z^{h,j} + \lambda^{h,j}(t) \, z^{h,j-1} \right\} \, .$$

It is elementary to check that $\zeta^h \rightarrow \zeta$ and $\xi^h \rightarrow \zeta$ uniformly since z is smooth. Since

$$\int_0^\infty \int_\Omega f^h \, \xi^h \, dxdt \;=\; \int_0^\infty \int_\Omega g^h \, \zeta^h \, dxdt \, ,$$

the lemma follows.   QED

References

1.   Acerbi, E. and Fusco, N. 1984   Semicontinuity problems in the calculus of variations, Arch. Rat.
     Mech. Anal., 86, 125 - 145

2.   Ball, J. M.   1977   Constitutive inequalities and existence theorems in nonlinear elastostatics,
     Nonlinear analysis and mechanics: Heriot Watt Symposium, Vol I (Knops, R., ed.) Pitman Res.
     Notes in Math. 17, 187-241

3.   Ball, J. M.   1989   A version of the fundamental theorem for Young measures, *PDE's and
     continuum models of phase transitions,* Lecture Notes in Physics, 344,(Rascle, M., Serre, D., and
     Slemrod, M., eds.) Springer, 207-215

4.   Ball, J. M.   1989   Sets of gradients with no rank-one connections (to appear)

5.   Ball, J. M. and James, R. 1987   Fine phase mixtures as minimizers of energy, Arch. Rat. Mech.
     Anal.,100, 15-52

6.   Ball, J. M. and James, R. 1989   Proposed experimental tests of a theory of fine microstructure and the
     two well problem

7.   Ball, J. M. and Murat, F. 1984 $W^{1,p}$ - quasiconvexity and variational problems for multiple integrals,
     J. Fnal Anal, 58,225-253

8.   Ball, J. M. and Zhang, K.   1990   Lower semicontinuity of multiple integrals and the biting
     lemma, Proc. Royal Soc. Edinburgh, 114A, 367-379

9.   Chipot, M. and Collins, C.   Numerical approximation in variational problems with
     potential wells, (to appear)

10. Chipot, M. and Kinderlehrer, D.   1988   Equilibrium configurations of crystals, Arch. Rat. Mech.
     Anal. 103, 237-277

11. Chipot, M.   Numerical analysis of oscillations in nonconvex problems

12. Collins, C. and Luskin, M.   1989   The computation of the austenitic-martensitic phase
     transition,*PDE's and continuum models of phase transitions,*Lecture Notes in Physics, 344,
     (Rascle, M., Serre, D., and Slemrod, M., eds.) Springer, 34-50

13. Collins, C. and Luskin, M.   Numerical modeling of the microstructure of crystals with
     symmetry-related variants, *Proc. ARO US-Japan Workshop on Smart/Intelligent Materials and
     Systems,* Technomic

14. Collins, C., Kinderlehrer, D., and Luskin, M.   Numerical approximation of the solution of
     a variational problem with a double well potential, SIAM J. Numer. Anal.

15. Dacorogna, B. 1982   Weak continuity and weak lower semicontinuity of nonlinear functionals,
     Springer Lecture Notes 922

16. Dacorogna, B. 1989   *Direct methods in the Calculus of Variations,* Springer

17. Demoulini, S.   Thesis, University of Minnesota

18. Ericksen, J. L. 1979   On the symmetry of deformable crystals, Arch. Rat. Mech. Anal. 72, 1-13

19. Ericksen, J. L. 1980   Some phase transitions in crystals, Arch. Rat. Mech. Anal. 73, 99-124

20. Ericksen, J. L. 1981   Changes in symmetry in elastic crystals, IUTAM Symp. Finite Elasticity
     (Carlson, D.E. and Shield R.T., eds.) M. Nijhoff, 167-177

21.   Ericksen, J. L.   1981   Some simpler cases of the Gibbs phenomenon for thermoelastic solids, J.of
       thermal stresses, 4 , 13-30

22.   Ericksen, J. L.   1982   Crystal lattices and sublattices, Rend. Sem. Mat. Padova, 68, 1-9

23.   Ericksen, J. L.   1983   Ill posed problems in thermoelasticity theory, Systems of Nonlinear Partial
       Differential Equations, (Ball, J., ed) D. Reidel, 71-95

24.   Ericksen, J. L.   1984   The Cauchy and Born hypotheses for crystals, *Phase Transformations and
       Material Instabilities in Solids*, (Gurtin, M., ed) Academic Press, 61-78

25.   Ericksen, J. L.   1986   Constitutive theory for some constrained elastic crystals, Int. J. Solids
       Structures, 22, 951 - 964

26.   Ericksen, J. L.   1986   Stable equilibrium configurations of elastic crystals, Arch. Rat. Mech. Anal. 94,
       1-14

27.   Ericksen, J. L.   1987   Twinning of crystals I, *Metastability and Incompletely Posed Problems*, IMA
       Vol. Math. Appl. 3,(Antman, S., Ericksen, J.L., Kinderlehrer, D., Müller, I.,eds) Springer, 77-96

28.   Ericksen, J. L.   1988   Some constrained elastic crystals, *Material Instabilities in Continuum
       Mechanics*, (Ball, J. ed.) Oxford, 119 - 136

29.   Ericksen, J. L.   1989   Weak martensitic transformations in Bravais lattices, Arch. Rat. Mech. Anal,
       107, 23 - 36

30.   Evans, L. C.   1990   *Weak convergence methods for nonlinear partial differential equations*, C B M S
       74, Amer. Math. Soc.

31.   Fonseca, I.   1985   Variational methods for elastic crystals, Arch. Rat. Mech. Anal., 97, 189-220

32.   Fonseca, I.   1988   The lower quasiconvex envelope of the stored energy function for an elastic
       crystal, J. Math. pures et appl, 67, 175-195

33.   Fonseca, I.   Lower semicontinuity of surface energies (to appear)

34.   Fonseca, I.   The Wulff Theorem revisited (to appear)

35.   James, R. D.   1988   Microstructure and weak convergence, *Proc. Symp. Material Instabilities in
       Continuum Mechanics*, Heriot-Watt, (Ball, J. M., ed.), Oxford, 175-196

36.   James, R. D. and Kinderlehrer, D.   1989   Theory of diffusionless phase transitions,*PDE's and
       continuum models of phase transitions*, Lecture Notes in Physics, 344,(Rascle, M., Serre, D., and
       Slemrod, M., eds.) Springer, 51-84

37.   Kinderlehrer, D.   1988   Remarks about the equilibrium configurations of crystals, *Proc. Symp. Material
       instabilities in continuum mechanics*, Heriot-Watt (Ball, J. M. ed.) Oxford, 217-242

38.   Kinderlehrer, D. and Pedregal, P.   Remarks about Young measures supported on two wells

39.   Kinderlehrer, D. and Vergara-Caffarelli, G.   1989   The relaxation of functionals with surface energies,
       Asymptotic Analysis 2, 279-298

40.   Liu, F.-C.   1977   A Luzin tyoe property of Sobelov functions, Ind. U. Math. J., 26, 645-651

41.   Matos, J.   The absence of fine microstructure in α-β quartz

42.   Matos, J.   Thesis, University of Minnesota

43.   Morrey, C. B., Jr.   1966   *Multiple Integrals in the Calculus of Variations*, Springer

44.   Müller, S.   Higher integrability of determinants and weak convergence in L^1 (to appear)

45.   Pedregal, P.   1989   Thesis, University of Minnesota

46.   Pedregal, P.   1989   Weak continuity and weak lower semicontinuity for some compensation
            operators, Proc. Royal Soc. Edin. 113, 267 - 279

47.   Slemrod, M.   Dynamics of measure valued solutions to a backward-forward heat equation

48.   Tartar, L.   1983   The compensated compactness method applied to systems of conservation laws,
            Systems of nonlinear partial differential equations (Ball, J. M., ed) Riedel

49.   Tartar, L.   1984   Étude des oscillations dans les équations aux dérivées partielles nonlinéaires,
            Springer Lect. Notes Physics, 195, 384-412

50.   Young, L. C.   1969   *Lectures on calculus of variations and optimal control theory*, W.B. Saunders

51.   Zhang, K.   Biting theorems for Jacobians and their applications, Analyse nonlineare, (to
            appear)

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Validity of the Young measure representation is useful in the study of microstructure of ordered solids. Such a Young measure, generated by a minimizing sequence of gradients converging weakly in L^p, often needs to be evaluated on functions of pth power polynomial growth. We give a sufficient condition for this in terms of the variational principle. The principal result concerns lower semicontinuity of functionals integrated over arbitrary sets, THEOREM 1.2. The question arose in the numerical analysis of configurations. Several applications are given. Of particular note, Young measure solutions of an evolution problem are found.

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