NAMS
90-2

MOTION OF A SET BY THE CURVATURE OF ITS BOUNDARY

by

Halil Mete Soner
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 90-82-NAMS-2

May 1990

MOTION OF A SET BY THE CURVATURE OF ITS BOUNDARY

HALIL METE SONER

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213
U.S.A.

March 1990

revised: May 1990

ABSTRACT

The study of a crystal shrinking or growing in a melt gives rise to equations relating the normal velocity of the motion to the curvature of the crystal boundary. Often these equations are anisotropic, indicating the prefered directions of the crystal structure. In the isotropic case this equation is called the mean curvature flow or the curve shortening equation, and has been studied by differential geometric tools. In particular, it is known that there are no classical solutions to these equations. In this paper we develop a weak theory for the "generalized mean curvature" equation using the newly developed theory of viscosity solutions. Our approach is closely related to that of Osher & Sethian, Chen, Giga & Goto, and Evans & Spruck, who view the boundary of the crystal as the level set of a solution to a nonlinear parabolic equation. Altough we use their results in an essential way, we do not require that the boundary is a level set. Our main results are the existence of a solution, large time asymptotics of this solution, and its connection to the level set solution of Osher & Sethian, Chen, Giga & Goto, and Evans & Spruck. In general there is no uniqueness, even for classical solutions, but we prove a uniqueness result under restrictive assumptions. We also construct a class of explicit solutions which are dialations of Wullf crystals.

Key words: Viscosity solutions, mean curvature flow, phase transitions, Wullf crystal

AMS Subject Classification: 35A05, 53A10

TABLE OF CONTENTS

1. INTRODUCTION

In this note we consider a collection of subsets $\{C(t)\}_{t\in[0,T]}$ of R^d evolving according to the equation

$$(E) \qquad \beta(\theta)\,\mathrm{V} + \mathrm{trace}\,(G(\theta)\,\mathrm{R}) - \upsilon = 0,$$

where θ,V,R are the outward unit normal vector, the normal velocity, the curvature tensor of the hypersurfarce $\partial C(t)$, respectively, and $\beta > 0$, $G \geq 0$, υ are given quantities. The isotropic case, $\beta\equiv 1$, $\upsilon=0$, G = identity/$(d-1)$, is the mean curvature flow or cuve shortening equation;

$$(MCE) \qquad \mathrm{V} = -\kappa,$$

where κ is the mean curvature. This particular equation is studied extensively by differential geometric tools. In a series of papers Gage[G 1983] [G 1984], Gage & Hamilton [GH 1986], Grayson[Gr 1987] analysed the flow of an embedded plane curve under the equation (MCE). They showed that a smooth embedded plane curve first becomes convex and then shrinks to a point in finite time. Also the limiting shape is a circle. Huisken [H 1984] generalized this result showing that any convex set, in any space dimension, shrinks to a point smoothly. The flow of a smooth curve embedded in a smooth Riemannian surface is pursued by Grayson [Gr 1989a]. Recently, Angenent generalized some of the two dimensional results [A 1989a,b,c] to the nonlinear case.

The behavior of non-embedded curves is the subject of Abresch & Langer [AL 1986] and Epstein & Weinstein [EW 1987]. In particular, they show that non-embedded curves develop singularities before they shrink to a point. In higher dimensions even the smooth embedded hypersurfaces develop singularities. Grayson [Gr 1989b] gives the example of a "dumbbell" shape in R^3. This is a region obtained by connecting two spheres by a thin long pipe. Grayson argues that under the mean curvature flow the boundary of this region will pinch off, leaving two bubbles. Also numerical studies of Sethian [S 1989] supports this observation. So a weaker formulation of this equation is necessary for to obtain a general theory. To our knowledge Brakke[Br 1978] was first to reformulate the above evolution problem using geometric measure theory. Then he constructed generalized solutions, global in time, for a large class of initial conditions. Recently an alternative weak formulation of (E), which provides uniqueness in addition to existence is given by Osher & Sethian [OS 1988](also see Sethian [S 1985]), Evans & Spruck [ES 1989a,b] and Chen, Giga & Goto[CGG 1989] (Chen, Giga & Goto considers a more general class of equations than (E), which they call geometric. A discussion of this class of equations is given by Giga & Goto[GG 1990].). Also Barles [B 1985] studied a similar problem related to a flame propagation model. Their approach is to consider $\partial C(t)$ as the level curve of a continuous function Φ, i.e.,

$$\partial C(t) = \{ x \in R^d \ : \ \Phi(x,t) = 0 \ \} \tag{1.1}$$

with

$$C(t) = \{ x \in R^d \ : \ \Phi(x,t) > 0 \ \}.$$

Then, Φ formally satisfies the parabolic equation

$$\beta\Big(-\frac{\nabla\Phi(x,t)}{|\nabla\Phi(x,t)|}\Big)\frac{\frac{\partial}{\partial t}\Phi(x,t)}{|\nabla\Phi(x,t)|} = \mathrm{trace}\Big[G\Big(-\frac{\nabla\Phi(x,t)}{|\nabla\Phi(x,t)|}\Big)\nabla\Big(\frac{\nabla\Phi(x,t)}{|\nabla\Phi(x,t)|}\Big)\Big] + \upsilon\,, \quad , \forall\ t > 0,\ \Phi(x,t) = 0\,. \tag{1.2}$$

The above equation is nonlinear, degenerate and undefined when $\nabla\Phi(x,t) = 0$. Evans & Spruck and Chen *et al* circumvent these rather subtle technical problems using the newly developed theory of viscosity solutions of nonlinear partial differential equations as in Crandall & Lions[CL 1983], Crandall, Evans & Lions[CEL 1984], Jensen[J 1988], Jensen Lions & Souganidis[JLS 1988], Lions[L 1983 a,b], and Ishii[I 1989].

In this paper, we define a notion of viscosity solutions of (E) which is closely related to the one given in [CGG 1989] and [ES 1989a]. We observe that the "signed" distance function

$$d_C(x,t) = \begin{cases} \text{distance } (x,\partial C(t)) & \text{if } x \in C(t) \\ -\text{ distance } (x,\partial C(t)) & \text{if } x \notin C(t) \end{cases} \tag{1.3}$$

of $C(\cdot)$ satisfies

$$\beta(-\nabla d_C(x,t))\,\frac{\partial}{\partial t}d_C(x,t) = \mathrm{trace}[G(-\nabla d_C(x,t))\,D^2 d_C(x,t)] + \upsilon \qquad \forall\ x \in \partial C(t), \tag{1.4}$$

as long as $\partial C(t)$ is smooth. Using the viscosity formulations of first and second derivatives of semi-continuous functions, we give a "direct" definition of viscosity solutions by requiring that (1.4) should hold in the viscosity sense (see Definitions 5.1 and Section 4 for the precise definition). Using the results and the techniques of [CGG 1989] and [ES 1989a], we obtain an existence result and prove that any supersolution $\{C(t)\}_{t\geq 0}$ of (E) includes any subsolution $\{\Gamma(t)\}_{t\geq 0}$ of (E) provided that closure of $\Gamma(0)$ is a compact subset of $C(0)$. In general there are more than one solutions of the initial value problem. In fact this is the case whenever the level set of the solution of Evans & Spruck and Chen *et al* develops a non-empty interior (see Section 8). However, the comparison result enables us to define two solutions one of which contains all the subsolutions and the second is included by all the supersolutions. These solutions are given by

$$U(t) = \{\ x \in R^d \ :\ \Phi(x,t) \geq 0\ \},$$

and

$$L(t) = \{\ x \in R^d \ :\ \Phi(x,t) > 0\ \}$$

where Φ is a solution of (1.2) which satisfies the initial condition (see Section 10). We also prove uniqueness of solutions to the initial value problem if $\upsilon \leq 0$ and initialy the solution has the property which we call strictly sharshaped (see Section 11).

Equation (E) arises in the study of nonequilibrium thermomechanics of two-phase continua; see Gurtin[Gu 1988a,b], Angenent & Gurtin[AG 1989], and the references therein for a systematic development of the subject. Under several simplifying assumptions the evolution of the interface is described by (E). In this context υ is the energy difference between two phases which is assumed to be constant, $\beta(\theta)$ measures the drag opposing the motion in the θ direction, and the coefficient $G(\theta)$ is a linear function of the interfacial energy (or surface tension) and its second derivatives. The anisotropy of the equation is essential in this theory and is related to the geometry of the underlying crystal structure. However, the isotropic case was derived as a simple model of the motion of the interphase by Mullins [M 1960] and by Allen & Cahn [AC 1979]. For a more detailed discussion, we refer the reader to Sekerka [Se 1973].

The stationary version of (E) is formally related to a variational problem and its celebrated solution is known as the Wullf crystal of the interfacial energy; see for example [W1901], [D 1944], [J 1974], [J 1975], [F 1990] for the properties and the definition of it and see Section 6.1 of Angenent & Gurtin [AG 1989] for the connection between the equation (E) and the Wulff crystal. It is also known that any solution of (E) shrinks to an empty set in finite time if $\upsilon \leq 0$ or if initially the solution is small (see [AG 1989], [ES 1990a], and [CGG 1990]). But if $\upsilon > 0$ and initially the solution is large enough, then the solution grows for all time (see [AG 1989]). In Sections 12 and 13 we construct a class of explicit solutions which are dialations of the Wullf crystal of $(1/\beta)$ and then use them to obtain asymptotic results. In addition we show that asymptotically the solution looks like the Wullf crystal of $(1/\beta)$ if it is growing. This result was conjectured by Angenent & Gurtin[AG 1989].

Acknowledgment. I greatly acknowledge valuable discussions with **M. E. Gurtin**. In particular, he showed me the correct form of the equation, explained the physical model and gave me the example discussed in Section 8. This work was supported by the National Science Foundation, and by the NSF-ARO Grant for Nonlinear analysis and continuum mechanics in the science of materials.

2. PRELIMINARIES

In this section we make several definitions which will be used throughout the paper. We will use the notation cl$A$, int$A$, andA^c to denote the closure of A, interior A and the complement of A , respectively. Let S^{d-1} be the set of all unit vector in R^d. We assume

(A) β and G are continuous on S^{d-1}, and $\beta(\theta) > 0$, $G(\theta) \geq 0$, $G(\theta)\eta\cdot\theta = 0$, $\forall\ \eta,\theta$.

Definition 2.1

(a) A subset of R^d is called ***proper*** if the interior of itself and its complement are non-empty.

(b) Let $\{C(t)\}_{t\in[0,T]}$ be a collection of proper subsets of R^d. The ***signed distance function*** of $\{C(t)\}_{t\in[0,T]}$ is

$$d_C(x,t) = \begin{cases} \text{dist}(x,\partial C(t)) & \text{if } x\in C(t) \\ -\ \text{dist}(x,\partial C(t)) & \text{if } x\notin C(t) \end{cases} \quad ,\forall\ t\in[0,T],$$

where dist(x,A) denotes the distance between the point x and the set A.

Finally we define the notion of a classical solution of (E). Basically we require that the signed distance function is smooth and satisfies (1.4). Also an additional "causality" condition is needed (see example 5.5), but we do not require the global continuity of the distance function.

Definition 2.2 We say that an open collection of smooth subsets$\{C(t)\}_{t\geq 0}$ of R^d is a ***classical solution*** of (E) if;

(a) There is $0 < T \leq \infty$ such that

$$C(t) = \varnothing \qquad \forall\ t\geq T,$$

or

$$C(t) = R^d \qquad \forall\ t\geq T,$$

and for $t<T$, $C(t)$ is proper, d_C is smooth in a (space-time) neighbourhood of every boundary point $x\in\partial C(t)$, and d_C satisfies (1.4).

(b) For every $t<T$ and $x\in R^d$,

$$\limsup_{(z,s)\to(x,t)} [d_C(z,s)\wedge 0] = \limsup_{(z,s)\uparrow(x,t)} [d_C(z,s)\wedge 0], \tag{2.1}$$

$$\liminf_{(z,s)\to(x,t)} [d_C(z,s)\vee 0] = \liminf_{(z,s)\uparrow(x,t)} [d_C(z,s)\vee 0]. \tag{2.2}$$

3. SEMI-CONTINUOUS ENVELOPES

In this section we define the upper and lower semi-continuous envelopes of a collection of subsets of R^d. We also prove several elementary properties of them which will be used in later sections.

For a given collection of proper subsets $\{C(t)\}_{t\in[0,T)}$ of R^d, define

(3.1) $$C^*(t) = \bigcap_{\varepsilon>0} \quad \bigcup_{|t-s|\le\varepsilon \; s<T} C(s),$$

(3.2) $$C_*(t) = \bigcup_{\varepsilon>0} \quad \bigcap_{|t-s|\le\varepsilon \; s<T} C(s)\ .$$

Lemma 3.1 *Let* $\{C(t)\}_{t\in[0,T)}$ *be a collection of proper subsets of* R^d.

(a) $R^d \setminus C_*(t) = (R^d \setminus C(\cdot))^*(t), \quad \forall\, t\in[0,T),$

(b) $R^d \setminus C^*(t) = (R^d \setminus C(\cdot))_*(t), \quad \forall\, t\in[0,T),$

(c) *If* $(x_n,t_n) \to (x,t)$ *and* $x_n \in \mathrm{cl}C^*(t_n)$, then $x\in \mathrm{cl}C^*(t)$,

(d) *If* $(x_n,t_n) \to (x,t)$ *and* $x_n \notin \mathrm{int}C_*(t_n)$, then $x \notin \mathrm{int}C_*(t)$,

(e) *The map* $(x,t) \to d_{C^*}(x,t)$ *is upper semi continuous,*

(f) *The map* $(x,t) \to d_{C_*}(x,t)$ *is lower-semi continuous.*

Proof :

(a) (b) follows from the De Morgan's rule.

(c) Since $x_n \in \mathrm{cl}C^*(t_n)$ and $(x_n,t_n) \to (x,t)$, there is a sequence $(y_n, s_n) \to (x,t)$ as n tends to infinity, and satisfying $y_n \in C(s_n)$ for all n, m. Set $\varepsilon_1 = 1$ and

$$\varepsilon_n = \sup\{|s_k - t| \ : \kappa \ge n\ \}.$$

Observe that

$$y_n \in C(s_n) \subset \bigcup_{|t-s|\le\varepsilon_k \; s\ge 0} C(s), \quad \text{if } n \ge k.$$

Hence,

$$x \in \text{cl}\,\{ \bigcup_{|t-s|\le\varepsilon_k \;\; s\ge 0} C(s)\}, \quad \forall\, k\,.$$

(d) follows from (a) and (c).

(e) Set $d_{C^*}(x,t) = \beta$. Suppose that $\beta \ge 0$. Then, there is a sequence $y_n \notin C^*(t)$ such that

$$|x - y_n| \to \beta\,.$$

Also, the definition of $C^*(t)$ yields that there are $\varepsilon_n > 0$ satisfying

$$y_n \notin \bigcup_{|t-s|\le\varepsilon_n \;\; s\ge 0} C(s).$$

Hence $y_n \notin C^*(s)$ for all $s \in ((t - \varepsilon_n) \vee 0, t + \varepsilon_n)$ and consequently

$$d_{C^*}(z,s) \le |z - y_n|\;, \;\forall\; z \in R^d, \; s \in ((t - \varepsilon_n) \vee 0, t + \varepsilon_n).$$

Therefore,

$$\lim_{\varepsilon\to 0}\; \sup\{\, d_{C^*}(z,s) \;:\; |z - x| + |t - s| < \varepsilon\;\} \le \lim_{n\to\infty\; z\to x} |z - y_n| = \beta\;.$$

Now suppose that $\beta < 0$. We claim that

$$\alpha = \lim_{(z,s)\to(x,t)} d_{C^*}(x,t) < 0.$$

Indeed if it is not the case, there exists a subsequence $(\omega_n,t_n) \to (x,t)$ such that $\omega_n \in C^*(t_n)$. Then, part (c) implies that $x \in \text{cl}C^*(t)$ which contradicts the assumption $\beta < 0$.

Choose $(z_n,t_n) \to (x,t)$ such that

$$\alpha = \lim d_{C^*}(z_n,t_n)\,.$$

Choose another sequence $y_n \in \text{cl}C^*(t_n)$ satisfying

$$\alpha = \lim - |z_n - y_n|.$$

Since $|x - y_n| \leq |z_n - y_n| + |z_n - x|$, $|y_n|$ is uniformly bounded in n. Hence we may assume that y_n is convergent. Let $y = \lim y_n$. Then, $\alpha = -|x - y|$, and part (c) yields that $y \in \mathrm{cl}C^*(t)$. So

$$\alpha = -|x - y| \leq d_{C^*}(x,t) = \beta.$$

(f) follows from (b) and (e). ▭

We will use $(a \wedge b)$ and $(a \vee b)$ to denote $\min\{a,b\}$ and $\max\{a,b\}$, respectively.

Lemma 3.2 *Let $\{C(t)\}_{t \in [0,T]}$ be a collection of proper subsets of R^d. Then, $[d_{C^*}(x,t) \wedge 0]$ and $[d_{C_*}(x,t) \vee 0]$ are upper and lower semi-continuous envelopes of the functions $[d_C(x,t) \wedge 0]$ and $[d_C(x,t) \vee 0]$, respectively, i.e., for $(x,t) \in R^d \mathrm{x}[0,T)$,*

(3.3) $$d_{C^*}(x,t) \wedge 0 = \limsup_{(z,s) \to (x,t)} [d_C(z,s) \wedge 0] ,$$

(3.4) $$d_{C_*}(x,t) \vee 0 = \liminf_{(z,s) \to (x,t)} [d_C(z,s) \vee 0] .$$

Proof: Since $C(t)$ is included in $C^*(t)$, $d_C(x,t) \leq d_{C^*}(x,t)$ for all $t \geq 0$, and x. Then, the upper semi-continuity of $d_{C^*}(x,t)$ yields

$$d_{C^*}(x,t) \wedge 0 \geq \limsup_{(z,s) \to (x,t)} [d_C(z,s) \wedge 0].$$

Suppose that $x \in \mathrm{cl}C^*(t)$. Then there are $(z_n,t_n) \to (x,t)$ such that $z_n \in C(t_n)$. Hence,

$$\begin{aligned} 0 &= d_{C^*}(x,t) \wedge 0 \\ &= \lim [d_C(z_n,t_n) \wedge 0] \\ &\leq \limsup_{(z,s) \to (x,t)} [d_C(z,s) \wedge 0]. \end{aligned}$$

Suppose $x \notin \mathrm{cl}C^*(t)$. Then, there is $y \in \partial C^*(t)$ such that

$$d_{C^*}(x,t) = -|x - y|.$$

Since $y \in \partial C^*(t)$, there is a sequence $(\omega_n, t_n) \to (y,t)$ such that $\omega_n \in C(t_n)$. Then,

$$d_C(x,t_n) \geq - |x - \omega_n| \,,$$

and consequently

$$\begin{aligned}
- |x - y| &= [d_C{}^*(x,t) \wedge 0] \\
&= d_{C^*}(x,t) \\
&\geq \limsup_{(z,s)\to(x,t)} \; [d_C(z,s) \wedge 0] \\
&\geq \lim d_C(x,t_n) \\
&\geq \lim - |x - \omega_n| \\
&= - |x - y| \,.
\end{aligned}$$

This completes the proof of (3.3), and (3.4) is proved after observing that

$$\begin{aligned}
d_{C_*}(x,t) &= - d_{(R^d \setminus C(\cdot))^*}(x,t) \\
&= - \limsup_{(z,s)\to(x,t)} \; [\, d_{(R^d \setminus C(\cdot))}(z,s)] \\
&= \liminf_{(z,s)\to(x,t)} \; [d_C(z,s)] \,. \qquad \square
\end{aligned} \tag{3.5}$$

4. SUB & SUPER DIFFERENTIALS

We first recall the definition of sub and superdifferentials of semi-continuous functions as in [CL 1983] and [CEL 1984]. We then define the sub and superdifferentials of a set-valued map.

Let $S(d)$ be the collection of all dxd symmetric matrices.

Definition 4.1 Let $T > 0$, Φ be a function on R^d x $[0,T)$, Φ^* and Φ_* be the upper semi-continuous and lower semi-continuous envelope of Φ, respectively (see (3.3) and (3.4)).

(a) The set of ***superdifferentials*** of Φ at $(x,t) \in R^d$x$(0,T)$ is,

$$D^+\Phi(x,t) = \{ (n,p) \in R^d \times R \ : \ \limsup_{(y,h)\to 0} \frac{\Phi^*(x+y,t+h) - \Phi^*(x,t) - ph - n\cdot y}{|(y,h)|} \leq 0 \ \}.$$

(b) The set of ***second superdifferentials*** of Φ at $(x,t) \in R^d$x$(0,T)$ is,

$$D_x^{+2}\,_t^{+1}\Phi(x,t) = \{ (n,A,p) \in R^d \times S(d) \times R \ :$$

$$\limsup_{(y,h)\to 0} \frac{\Phi^*(x+y,t+h) - \Phi^*(x,t) - ph - n\cdot y - (1/2)Ay\cdot y}{|y|^2 + |h|} \leq 0 \ \}.$$

(c) The set of ***subdifferentials*** of Φ at $(x,t) \in R^d$x$(0,T)$ is,

$$D^-\Phi(x,t) = \{ (n,p) \in R^d \times R \ : \ \liminf_{(y,h)\to 0} \frac{\Phi_*(x+y,t+h) - \Phi_*(x,t) - ph - n\cdot y}{|(y,h)|} \geq 0 \ \}.$$

(d) The set of ***second subdifferentials*** of Φ at $(x,t) \in R^d$x$(0,T)$ is,

$$D_x^{-2}\,_t^{-1}\Phi(x,t) = \{ (n,A,p) \in R^d \times S(d) \times R \ :$$

$$\liminf_{(y,h)\to 0} \frac{\Phi_*(x+y,t+h) - \Phi_*(x,t) - ph - n\cdot y - (1/2)Ay\cdot y}{|y|^2 + |h|} \geq 0 \ \}.$$

See Appendix A for several well-known properties of these sets. We continue with the definition of sub and superdifferentials of a set valed map.

Definition 4.2 For a given collection of subsets $\{C(t)\}_{t\geq 0}$ of R^d,

$$\mathbf{D}^+ C(t) = \bigcup_{x \in R^d} \mathbf{D}^{+2}_{x}{}^{+1}_{t}[d_{C^*} \wedge 0]\,(x,t) \qquad \forall\ t < T(C(\cdot)),$$

$$\mathbf{D}^- C(t) = \bigcap_{x \in R^d} \mathbf{D}^{-2}_{x}{}^{-1}_{t}[d_{C_*} \vee 0)\,(x,t) \qquad \forall\ t < T(C(\cdot)),$$

where $T(C(\cdot))$ is called the ***extinction time*** and is given by

$$T(C(\cdot)) = \begin{cases} \inf\{\ t \geq 0\ :\ \mathrm{cl}C^*(t) = R^d \ \text{or}\ \mathrm{int}C_*(t)\ \} \\ \infty \ \text{if}\ C(t) \text{ is a proper subset of } R^d \text{ forall } t \geq 0. \end{cases}$$

Remark 4.3

(a) For $x \notin C^*(t)$ $d_{C^*}(x,t) = -\inf\{\ |x - y|\ :\ y \in \partial C^*(t)\ \}$. Hence,

(4.1) $$\mathbf{D}^+ C(t) = \bigcup_{x \in \mathrm{cl}C^*(t)} \mathbf{D}^{+2}_{x}{}^{+1}_{t}[d_{C^*} \wedge 0]\,(x,t).$$

(b) Similarly

(4.2) $$\mathbf{D}^- C(t) = \bigcup_{x \notin \mathrm{int}C_*(t)} \mathbf{D}^{-2}_{x}{}^{-1}_{t}[d_{C_*} \vee 0]\,(x,t).$$

Other properties of the above sets are gathered in Appendix A. Also see Example 5.5 for a discusion of the particular definition.

5 VISCOSITY SOLUTIONS; DEFINITION

We start with rewritting the equation (1.2) as

$$(1.2) \qquad \frac{1}{|\nabla \Phi(x,t)|} F\left(\nabla \Phi(x,t)\,,\, D^2\Phi(x,t),\, \frac{\partial}{\partial t}\Phi(x,t)\right) = 0\,,$$

where ∇ and D^2 denote the gradient and the Hessian matrix with respect to the x variable alone and for $(n,A,p) \in [R^d \setminus \{0\}] \mathrm{x} S(d) \mathrm{x} R$,

$$(5.1) \qquad F(n,A,p) = \beta(-\frac{n}{|n|})\,p - \sum_{ij=1}^{d} [G_{ij}(-\frac{n}{|n|})\,A_{ij}] - \upsilon\,|n|.$$

Let F^* and F_* be the upper and lower semi-continuous envelopes of F. Note that F^* and F_* are both defined on $R^d \mathrm{x} S(d) \mathrm{x} R$ and are given by

$$F^*(n,A,p) = \limsup_{\substack{(m,B,q)\to(n,A,p) \\ m \neq 0}} F(m,B,q)\,,$$

$$F_*(n,A,p) = \liminf_{\substack{(m,B,q)\to(n,A,p) \\ m \neq 0}} F(m,B,q)\,.$$

Since $\beta > 0$, and $G \geq 0$, the function F and its upper and lower semi-continuous envelopes satisfy,

$$F(n,A,p) < F(n,A-B,p+q) \qquad \forall q > 0,\, B \geq 0.$$

Hence the equation (1.4) is degenerate parabolic. Also $G(\theta)\theta = 0$ implies that,

$$(5.2) \qquad F(\lambda(n,A,p)) = \lambda F(n,(\mathrm{I} - \frac{n\otimes n}{|n|^2})A,p) \qquad \forall\ |n| \neq 0, \lambda > 0.$$

Definition 5.1 A collection $\{C(t)\}_{t\geq 0}$ of subsets of R^d is

(a) a ***viscosity subsolution*** of (E) if

$$F_*(n,A,p) \leq 0\ , \quad \forall\ (n,A,p) \in \mathbf{D}^+C(t),\ t \in [0,T(C(\cdot)))\ ,$$

(b) a ***viscosity supersolution*** of (E) if

$$F^*(n,A,p) \geq 0 \ , \quad \forall \ (n,A,p) \in \mathbf{D}^- C(t), \ t \in [0,T(C(\cdot))) \ ,$$

(c) a ***viscosity solution*** of (E) if it is both viscosity subsolution and viscosity supersolution of (E).

See Appendix B for an equivalent definition. Also a stability theorem is stated in Appendix C.

Remark 5.2

(a) Since $[d_{C^*} \wedge 0](x,t)$ is the upper semi-continuous envelope of $[d_C \wedge 0](x,t)$, $\{C(t)\}_{t \geq 0}$ is a subsolution of (E) if and only if $[d_C \wedge 0](x,t)$ is a viscosity subsolution of (1.2) on $R^d \times [0,T(C(\cdot)))$. The viscosity solutions of equations like (1.2) defined in [CGG 1989] and [ES 1989a]. Similarly $\{C(t)\}_{t \geq 0}$ is a supersolution of (E) if and only if $[d_C \wedge 0](x,t)$ is a viscosity supersolution of (1.2) on $R^d \times [0,T(C(\cdot)))$.

(b) $\{C(t)\}_{t \geq 0}$ is a supersolution of (E) if and only if it is a subsolution of

$$\beta(-\theta)\, \mathbf{V} = -\, \mathrm{trace} G(-\theta)\, \mathbf{R} - \upsilon.$$

(c) Suppose that $\{C(t)\}_{t \geq 0}$ is a viscosity subsolution of (E). For $T \leq T(C(\cdot))$, define

$$\Gamma(t) = \begin{cases} C(t) & \text{if } t \leq T \\ \varnothing & \text{if } t > T. \end{cases}$$

Then, $T(\Gamma(\cdot)) = T$ and $\{\Gamma(t)\}_{t \geq 0}$ is also a subsolution.

We make the following definition to distinguish between a "maximal" sub or supersolution and others constructed like $\{\Gamma(t)\}_{t \geq 0}$.

Definition 5.3

We say that a collection of subsets $\{C(t)\}_{t\geq 0}$ of R^d is ***maximal*** if whenever $T(C(\cdot))$ is finite either

$$\mathrm{int}C_*(t) = \varnothing \qquad \forall\ t \geq T(C(\cdot)),$$

or

$$\mathrm{cl}C^*(t) = R^d \qquad \forall\ t \geq T(C(\cdot)).$$

We continue by showing that any classical solution $\{C(t)\}_{t\geq 0}$ of (E) is a maximal viscosity solution. For simplicity we make a simplifying assumption which rules out several pathological cases. We assume that there is $I = \{t_1, t_2, \ldots, t_N\}$ such that

(5.3) $\qquad d_C$ is continuous on $R^d \times ([0,T(C(\cdot))) \setminus I)$.

Lemma 5.4 *Any classical subsolution (or supersolution) of* (E) satisfying (5.3) *is a maximal viscosity subsolution (or viscosity supersolution) of* (E).

Proof: The maximality of classical sub or supersolutions follow from Definition 2.2. Let $\{C(t)\}_{t\geq 0}$ be a classical subsolution (E), and $(n,A,p) \in \mathbf{D}^+C(t)$ with $t < T(C(\cdot))$. Then there is x such that

$$(n,A,p) \in \mathbf{D}^{+2}_{x}{}^{+1}_{t}[d_{C^*}\wedge 0](x,t).$$

First assume that $[d_{C^*}\wedge 0](x,t) = [d_C \wedge 0](x,t)$. Using (4.1) we may assume that $x \in \mathrm{cl}C(t)$. If $x \in \mathrm{int}C(t)$ the smoothness of $\{C(t)\}_{t\geq 0}$ yields that $(x,t) \in \mathrm{int}\,\{\, (y,s) \in R^d \mathrm{x} [0,\infty) : y \in C(s)\,\}$. Hence,

$$(n,p) = (0,0),\ A \geq 0\ ,$$

and

$$F_*(n,A,p) \leq F_*(0,0,0) = 0.$$

So suppose that $x \in \partial C(t)$. Set $\eta = \nabla d_C(x,t)$. Then, $-\eta$ is the outward unit normal vector at $x \in \partial C(t)$. Let $\xi \in R^d$ be such that $\xi \cdot \eta > 0$. Then, the smoothness of $\partial C(t)$ yields that

$$(x - \tau\xi) \notin C(t) \qquad \forall \text{ sufficiently small } \tau > 0.$$

Using the definition of the subdifferential and Corollary 14.3, we obtain

$$0 \geq \lim_{\tau \downarrow 0} \frac{[d_C \wedge 0](x-\tau\xi,t) - [d_C \wedge 0](x,t) + \tau\xi \cdot n}{\tau|\xi|}$$

$$\geq \lim_{\tau \downarrow 0} \frac{d_C(x-\tau\xi,t) - d_C(x,t) + \tau\xi \cdot n}{\tau|\xi|}$$

$$= [-\eta \cdot \xi + n \cdot \xi]/|\xi|.$$

Hence,

(5.4) $$n \cdot \xi \leq \eta \cdot \xi \qquad \forall\ \xi \cdot \eta \geq 0\,.$$

Also,

$$(x+\tau\eta) \in C(t) \qquad \forall \text{ sufficiently small } \tau > 0,$$

and a similar argument based on $[d_C \wedge 0](x+\tau\eta, t) = 0$ yields

(5.5) $$n \cdot \eta \geq 0\,.$$

Inequalities (5.4) and (5.5) imply that

(5.6) $$n = \rho\eta$$

for some $\rho \in [0,1]$. Set

$$V = \frac{\partial}{\partial t} d_C(x,t)\,.$$

Then, for every $\varepsilon > 0$,

$$x + \tau(1-\varepsilon)V\eta \notin C(t-\tau) \qquad \forall \text{ sufficiently small } \tau > 0\,.$$

Since $(n,p) \in \mathbf{D}^+[d_C \wedge 0](x,t)$,

$$0 \geq \lim_{\tau \downarrow 0} \frac{[d_C \wedge 0](x + \tau(1 - \varepsilon)V\eta, t - \tau) - [d_C \wedge 0](x,t) - \tau(1 - \varepsilon)V\eta \cdot n + p\tau}{\tau}$$

$$= \lim_{\tau \downarrow 0} \frac{d_C(x + \tau(1 - \varepsilon)V\eta, t - \tau) - d_C(x,t) - \tau(1 - \varepsilon)V\eta \cdot n + p\tau}{\tau}$$

$$= V(1 - \varepsilon) - V - (1 - \varepsilon)V\eta \cdot n + p$$

$$= -\varepsilon V - (1 - \varepsilon)V\rho + p.$$

Hence,

$$p \leq \rho V.$$

A similar argument based on

$$x - \tau(1 + \varepsilon)V\eta \in C(t + \tau) \qquad \forall \text{ sufficiently small } \tau > 0,$$

yields that $p \geq \rho V$ and therefore

$$p = \rho V. \tag{5.7}$$

Let $\zeta \in R^d$ be a unit vector orthogonal to η. Since the boundary of $C(t)$ is smooth, there is a sequence $z_m \in \partial C(t)$ converging to x and

$$\lim \frac{z_m - x}{|z_m - x|} = \zeta.$$

Set $\tau_m = |z_m - x|$ and $w_m = (z_m - x)/\tau_m$. Then, w_m converges to ζ and

$$0 = d_C(z_m, t)$$

$$= d_C(x,t) + \int_0^1 \nabla d_C(x + \tau\tau_m w_m, t) \cdot \tau_m w_m \, d\tau$$

$$= \int_0^1 [\nabla d_C(x,t) + \int_0^1 D^2 d_C(x + r\tau\tau_m w_m, t) \, \tau\tau_m w_m \, dr] \cdot \tau_m w_m \, d\tau \, .$$

Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$\lim \left[\frac{\eta \cdot w_m}{\tau_m}\right] = \lim \left[\frac{\nabla d_C(x,t) \cdot w_m}{\tau_m}\right] = -(1/2)\, D^2 d_C(x,t)\zeta\cdot\zeta \ .$$

Also,

$$0 \geq \lim_{m\to\infty} \frac{[d_C \wedge 0](z_m,t) - [d_C \wedge 0](x,t) - \tau_m w_m \cdot n - (\tau_m)^2 (1/2)\, A w_m \cdot w_m}{(\tau_m)^2}$$

$$= \lim_{m\to\infty} \frac{d_C(z_m,t) - d_C(x,t) - \tau_m w_m \cdot n - (\tau_m)^2 (1/2)\, A w_m \cdot w_m}{(\tau_m)^2}$$

$$= \lim_{m\to\infty} \left[-\rho \frac{\eta \cdot w_m}{\tau_m} - (1/2)\, A w_m \cdot w_m \right]$$

$$= \rho\, (1/2)\, D^2 d_C(x,t)\zeta\cdot\zeta - (1/2)\, A\zeta\cdot\zeta \ .$$

Hence,

$$[A - \rho\, D^2 d_C(x,t)]\zeta\cdot\zeta \geq 0 \qquad \forall\ \zeta\cdot\eta = 0,$$

or equivalently

$$(5.8) \qquad (I - \eta\otimes\eta)A \geq \rho\, (I - \nabla d_C(x,t)\otimes\nabla d_C(x,t))\, D^2 d_C(x,t)$$

$$= \rho\, D^2 d_C(x,t) \ .$$

Suppose that $\rho > 0$. Then, (5.6), (5.7) and (5.8) imply that

$$F_*(n,A,p) = F(n,A,p)$$

$$= F(\rho(n,A,p))$$

$$\leq F\left(\nabla d_C(x,t)\ ,\ D^2 d_C(x,t),\ \frac{\partial}{\partial t} d_C(x,t)\right)$$

$$\leq 0 \ .$$

If $\rho = 0$, then $(n,p) = (0,0)$. Also, $(I - \eta\otimes\eta)A \geq 0$. Hence

$$F_*(n,A,p) = F_*(0,A,0) \leq 0.$$

Recall that we assumed $[d_C\wedge 0](x,t) = [d_{C^*}\wedge 0](x,t)$. If $[d_C\wedge 0](x,t) \neq [d_{C^*}\wedge 0](x,t)$, then using (2.1) we can construct a sequence x_m,t_m,n_m,A_m,p_m such that, $[d_C\wedge 0](x_m,t_m) = [d_{C^*}\wedge 0](x_m,t_m)$,

$$\lim\ (x_m,t_m,n_m,A_m,p_m) = (x,t,n,A,p),$$

and

$$(n_m,A_m,p_m + K_m) \in \mathrm{D}_x^{+2}\,{}_t^{+1}[d_C\wedge 0](x_m,t_m)$$

for some $K_m \geq 0$. Such a sequence is constructed by considering the local maxima of the map $[d_C\wedge 0](y,s) - \Psi(y,s) - [m(t- s)]^{-1}$ on the region $R^d\mathrm{x}(0,t)$, where Ψ is a smooth function as in Theorem 14.1(b) with $\Phi = [d_C\wedge 0]$. Using the previous argument we conclude that

$$F_*(n_m,A_m,p_m) \leq F_*(n_m,A_m,p_m + K_m) \leq 0.$$

Passing to the limit as m tends to infinity yields that $\{C(t)\}_{t\geq 0}$ is a viscosity subsolution of of (E). The assertion about the supersolutions is proved by using the proved result and Remark 5.2(b). □

We give a simple example to clarify the definition.

Example 5.5

Define

$$C(t) = \begin{cases} \{\ (x,y) \in R^2\ :\ 2(1 - t) < |x|^2 + |y|^2 < 4 - 2t\ \} & \text{if } t < 1 \\ \{\ (x,y) \in R^2\ :\ |x|^2 + |y|^2 < 4 - 2t\ \} & \text{if } 1 \leq t < 2 \\ \varnothing & \text{if } t \geq 2. \end{cases}$$

Then, $C(\cdot)$ is a classical solution of (MCE) with $d = 2$. In fact it is the unique viscosity solution of (MCE) with initial condition

(5.9) $$C(0) = \{ (x,y) \in R^2 \ : \ 2 < |x|^2 + |y|^2 < 4 \}.$$

Also define

$$\Gamma(t) = \begin{cases} C(t) & \text{if } t < (1/2) \\ \{ (x,y) \in R^2 \ : \ |x|^2 + |y|^2 < 4 - 2t \} & \text{if } (1/2) \le t < 2 \\ \varnothing & \text{if } t \ge 2 . \end{cases}$$

Then for any $x \in \partial\Gamma^*(t)$,

$$F(n,A,p) \le 0 \qquad \forall \ (n,A,p) \in \mathbf{D}_{x}^{+2} \, {}_{t}^{+1} d_{\Gamma^*}(x,t).$$

Observe that $\{\Gamma(t)\}_{t\ge 0}$ does not satisfy (2.1) and thus is not a classical solution. Also this example indicates why we need a subdifferential which is larger than the set

$$\bigcup_{x \in \partial C^*(t)} \mathbf{D}_{x}^{+2} \, {}_{t}^{+1} d_{C^*} (x,t) \, .$$

However if $\{C(t)\}_{t\ge 0}$ is "continuous" in the time variable, there may be an equivalent definition which only uses the above set.

6 EXISTENCE by PERRON'S METHOD

In this section we obtain an existence result by assuming the existence of a viscosity subsolution and a viscosity supersolution of (E). Existence for the initial value problem is discussed in Section 10. We follow the approach of Ishii who was first to use the Perron's method to obtain existence of viscosity solutions[I 1987]. Our proof is very closely related to the proof of Proposition 2.3 in [CGG 1989].

Lemma 6.1

(a) *Let C be a collection of subsolutions of* (E). *Define*

$$C(t) = \cup \{ \Gamma(t) \ : \Gamma(\cdot) \in C \text{ and } t < T(\Gamma(\cdot))\} \cup \varnothing.$$

Then $\{C(t)\}_{t\geq 0}$ *is a subsolution of* (E).

(b) *Let A be a collection of supersolutions of* (E). *Define*

$$C(t) = \cap \{ \Gamma(t) \ : \Gamma(\cdot) \in A \text{ and } t < T(\Gamma(\cdot))\} \cap R^d.$$

Then $\{C(t)\}_{t\geq 0}$ *is a supersolution of* (E).

Proof:(a) Remark 5.2 (a) and Proposition 2.2 of [CGG 1990] implies the result provided that for all $t \in [0,T(C(\cdot))\ )$ and x,

$$[d_C \wedge 0](x,t) = \sup \{[d_\Gamma \wedge 0](x,t) : \Gamma(\cdot) \in C \text{ and } t < T(\Gamma(\cdot))\}. \tag{6.1}$$

Indeed if $x \in C(t)$ with $t < T(C(\cdot))$, then $x \in \Gamma(t)$ for some $\Gamma(\cdot) \in C$ which is proper at time t, and (6.1) follows easily. Suppose $x \notin C(t)$ with $t < T(C(\cdot))$. Then, there are $y_n \in C(t)$ such that

$$d_C(x,t) = \lim - |x - y_n|.$$

The definition of $C(t)$ implies that for each n, $y_n \in \Gamma_n(t)$ for some $\Gamma_n \in C$ which is proper at time t. Hence,

$$- |x - y_n| \leq d_{\Gamma_n}(x,t) \qquad \forall\ n,$$

and

$$[d_C \wedge 0](x,t) \le \sup \{[d_\Gamma \wedge 0](x,t) : \Gamma(\cdot) \in C \text{ and } t < T(\Gamma(\cdot))\} .$$

To prove the reverse inequality, observe that for every $t < T(C(\cdot))$ and positive integer n there is $\Gamma_n(\cdot) \in C$ which is proper at time t and

$$[d_{\Gamma_n} \wedge 0](x,t) \ge \sup \{[d_\Gamma \wedge 0](x,t) : \Gamma(\cdot) \in C \text{ and } t < T(\Gamma(\cdot))\} - (1/n).$$

Also choose $z_n \in \Gamma_n(t)$ satisfying

$$- |x - z_n| \ge [d_{\Gamma_n} \wedge 0](x,t) - (1/n).$$

Then by the definition of $C(t)$, $z_n \in C(t)$ and

$$[d_C \wedge 0](x,t) \ge - |x - z_n|.$$

Combining above inequalities yield (6.1).

(b) Follows from part (a) and Remark 5.2(b). □

We need the following technical lemma in our main existence result.

Lemma 6.2 *Suppose that there are* $\delta > 0$, $x_0 \in R^d$, $t_0 > 0$ *and smooth functions* f *and* g *satisfying*

(6.2) $$F^*(\nabla f(x_0,t_0), D^2 f(x_0,t_0), \frac{\partial}{\partial t} f(x_0,t_0))) < 0,$$

(6.3) $$f(x_0,t_0) = g(x_0,t_0),$$

and

(6.4) $$f(x,t) > 0 \;\Rightarrow\; g(x,t) \ge 0, \qquad \forall \;\; |x - x_0| + |t - t_0| < \delta.$$

Then,

(6.5) $$F_*(\nabla g(x_0,t_0), D^2 g(x_0,t_0), \frac{\partial}{\partial t} g(x_0,t_0))) \le 0 .$$

Proof: Once again we analyse three cases seperately;

(1) $(\nabla f(x_0,t_0), \frac{\partial}{\partial t} f(x_0,t_0)) \neq 0$

Set

$$\eta = (\nabla f(x_0,t_0), \frac{\partial}{\partial t} f(x_0,t_0)).$$

Then, for any $\upsilon \in R^{d+1}$ satisfying $\eta \cdot \upsilon > 0$, there is $\rho(\upsilon) > 0$ such that

$$f((x_0,t_0) + \rho\upsilon) > 0 \qquad \forall \ \rho \in (0,\rho(\upsilon)] .$$

Assumption (6.4) yields

$$g((x_0,t_0) + \rho\upsilon) \geq 0 \qquad \forall \ \rho \in [0,\rho(\upsilon)],\ \upsilon \in R^{d+1} \text{ and } \eta \cdot \upsilon > 0,$$

and therefore there is $\alpha \geq 0$ such that

(6.6)
$$\begin{aligned}(\nabla g(x_0,t_0), \tfrac{\partial}{\partial t} g(x_0,t_0)) &= \alpha\, \eta \\ &= \alpha \ (\nabla f(x_0,t_0), \tfrac{\partial}{\partial t} f(x_0,t_0)) .\end{aligned}$$

Let $\zeta \in R^{d+1}$ be a unit vector orthogonal to η. By the implicit function theorem, there is a sequence (z_m,t_m) converging to (x_0,t_0) such that $f(z_m,t_m) = 0$ and

$$\lim \frac{z_m - x}{|z_m - x|} = \zeta .$$

Set $\tau_m = |(z_m - x, t_m - t)|$ and $w_m = (z_m - x, t_m - t)/\tau_m$. Then, w_m converges to ζ and

$$\begin{aligned}0 &= f(z_m,t_m) \\ &= f(x_0,t_0) + \int_0^1 (\nabla f((x_0,t_0)+\tau\tau_m w_m), \tfrac{\partial}{\partial t} f((x_0,t_0)+ \tau\tau_m w_m))\cdot \tau_m w_m \ d\tau \\ &= \int_0^1 [\eta + \int_0^1 D_{x,t}^2 f((x_0,t_0)+ r\tau\tau_m w_m) \ \tau\tau_m w_m \ dr]\cdot \tau_m w_m \ d\tau,\end{aligned}$$

where $D^2_{x\,t}f$ is the Hessian matrix of f with respect to all of its variables. Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$\lim \left[\frac{\eta\cdot w_m}{\tau_m}\right] = -\,(1/2)\, D^2_{x\,t}f(x_0,t_0)\zeta\cdot\zeta\ .$$

Also, an approximation argument based on (6.4) and $(\nabla f(x_0,t_0), \frac{\partial}{\partial t}f(x_0,t_0)) \neq 0$ yields $0 \leq g(z_m,t_m)$. Hence,

$$0 \leq g(z_m,t_m)$$

$$= g(x_0,t_0) + \int_0^1 (\nabla g((x_0,t_0)+\tau\tau_m w_m), \frac{\partial}{\partial t} g((x_0,t_0)+\ \tau\tau_m w_m))\cdot\tau_m w_m\ \ d\tau$$

$$= \int_0^1 [(\nabla g(x_0,t_0), \frac{\partial}{\partial t} g(x_0,t_0))$$

$$+ \int_0^1 D^2_{x\,t}g((x_0,t_0)+\ r\tau\tau_m w_m)\ \tau\tau_m w_m\ \ dr]\cdot\tau_m w_m\ \ d\tau.$$

Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$0 \leq \lim \left[\frac{(\nabla g(x_0,t_0), \frac{\partial}{\partial t} g(x_0,t_0))\cdot w_m}{\tau_m}\right] + (1/2)\, D^2_{x\,t}g(x_0,t_0)\zeta\cdot\zeta$$

$$= \lim \left[\frac{\alpha\eta\cdot w_m}{\tau_m}\right] + (1/2)\, D^2_{x\,t}g(x_0,t_0)\zeta\cdot\zeta$$

$$= -\,\alpha\,(1/2)\, D^2_{x\,t}f(x_0,t_0)\zeta\cdot\zeta + (1/2)\, D^2_{x\,t}g(x_0,t_0)\zeta\cdot\zeta.$$

Hence,

$$[\alpha\, D^2_{x\,t}f(x_0,t_0) - D^2_{x\,t}g(x_0,t_0)]\zeta\cdot\zeta \leq 0 \qquad \forall\ \zeta\cdot\eta = 0.$$

Combinning (6.2), (6.6), and the above inequality yield (6.5).

(2) $(\nabla g(x_0,t_0),\frac{\partial}{\partial t}g(x_0,t_0)) \neq 0$

Using the negation of (6.5), proceed exactly as in the previous case to obtain

$$\text{(6.7)}\quad [D_x^2 f(x_0,t_0) - \alpha D_x^2 g(x_0,t_0)]\zeta\cdot\zeta \leq 0 \quad \forall\ \zeta\cdot(\nabla g(x_0,t_0),\frac{\partial}{\partial t}g(x_0,t_0)) = 0,$$

$$\text{(6.8)}\quad (\nabla f(x_0,t_0),\frac{\partial}{\partial t}f(x_0,t_0)) = \alpha\,(\nabla g(x_0,t_0),\frac{\partial}{\partial t}g(x_0,t_0)),$$

for some $\alpha \geq 0$. If $\alpha > 0$, then (6.5) follows from (6.2) (6.7) and (6.8). Now suppose that $\alpha = 0$. Then, (6.2), (6.7), and the definition F^* imply that

$$0 > F^*(\nabla f(x_0,t_0),D^2 f(x_0,t_0),\frac{\partial}{\partial t}f(x_0,t_0))$$

$$= F^*(0,D^2 f(x_0,t_0),0)$$

$$\geq \limsup_{\rho\downarrow 0} F(\rho\nabla g(x_0,t_0),D^2 f(x_0,t_0),0).$$

However, (6.7) with $\alpha = 0$ yields

$$F(\rho\nabla g(x_0,t_0),D^2 f(x_0,t_0),0) \geq 0$$

for every $\rho > 0$.

(3) $(\nabla f(x_0,t_0),\frac{\partial}{\partial t}f(x_0,t_0)) = (\nabla g(x_0,t_0),\frac{\partial}{\partial t}g(x_0,t_0)) = 0$

Set

$$A = D^2 f(x_0,t_0),$$

$$B = D^2 g(x_0,t_0).$$

Then, (6.4) and the fact that $(\nabla f(x_0,t_0),\frac{\partial}{\partial t}f(x_0,t_0)) = (\nabla g(x_0,t_0),\frac{\partial}{\partial t}g(x_0,t_0)) = 0$ yield

(6.9) $\qquad Ae\cdot e > 0 \quad \Rightarrow \quad Be\cdot e \geq 0 \qquad \forall\ e\in R^d.$

Let $e,f \in R^d$ be given. We claim that

$$Ae\cdot e + Af\cdot f > 0 \qquad \Rightarrow \qquad Be\cdot e + Bf\cdot f \geq 0.$$

Indeed if $Ae\cdot e$ and $Af\cdot f$ are both strictly positive, then (6.9) yields the result. If they are both negative, then there is nothing to prove. So we may assume that

$$Ae\cdot e > 0 \geq Af\cdot f, \text{ and } \quad Be\cdot e > 0 > Bf\cdot f.$$

Define two second order polynomials by

$$P_1(r) = A(f+re)\cdot(f+re), \text{ and} \qquad P_2(r) = B(f+re)\cdot(f+re).$$

Let λ_1, λ_2 be the roots of P_1 and μ_1, μ_2 be the roots of P_2. Observe that

$$\lambda_1\lambda_2 = Af\cdot f \,/\, Ae\cdot e\,, \qquad \mu_1\mu_2 = Bf\cdot f \,/\, Be\cdot e\,.$$

Hence to prove the claim it suffices to show that

$$\lambda_1\lambda_2 + 1 > 0 \qquad \Rightarrow \qquad \mu_1\mu_2 + 1 \geq 0.$$

Using (6.9) we conclude that whenever $P_1(r) > 0$, then $P_2(r) \geq 0$. We also know that $Ae\cdot e > 0 \geq Af\cdot f$, and $Be\cdot e > 0 > Bf\cdot f$. Hence

$$\lambda_1 \leq \mu_1 < 0 < \mu_2 \leq \lambda_2\,,$$

and consequently $\lambda_1\lambda_2 \leq \mu_1\mu_2$.

The hypothesis (6.2) and the assumption (A) yield that trace $G(\theta^*)A > 0$ for some $\theta^* \in S^{d-1}$. Using this inequality and the non-negativity of $G(\theta^*)$ we may represent $G(\theta^*)$ as

$$G(\theta^*) = \sum_{i=1}^{2M} e_i \otimes e_i$$

for some vectors e_i in R^d satisfying

$$Ae_{2k-1}\cdot e_{2k-1} + Ae_{2k}\cdot e_{2k} > 0 \qquad \forall k=1,..,M.$$

Then we have

$$Be_{2k-1}\cdot e_{2k-1} + Be_{2k}\cdot e_{2k} \geq 0.$$

Sum this inequality over k to obtain that trace $G(\theta^*)B \geq 0$, and (6.5) follows.

Theorem 6.3 *Suppose that* $\{L(t)\}_{t\geq 0}$, $\{U(t)\}_{t\geq 0}$ *are viscosity subsolution and viscosity supersolution of* (E), *satisfying*

(6.10) $$L(t) \subset U(t), \qquad \forall\ t < T(L(\cdot))\wedge T(U(\cdot)).$$

Then there exist a viscosity solution $\{C(t)\}_{t\geq 0}$ *of* (E) *satisfying*

(6.11) $$L(t) \subset C(t) \subset U(t) \qquad \forall\ t < T(L(\cdot))\wedge T(U(\cdot)).$$

Proof: Set $T_0 = T(L(\cdot))\wedge T(U(\cdot))$, $S_0 = T(U(\cdot))$ and

$$C = \left\{ \{\Gamma(t)\}_{t\geq 0} \ :\ \begin{matrix} \{\Gamma(t)\}_{t\geq 0} \text{ is a subsolution of (E)} \\ \Gamma(t) \subset U(t) \quad \forall\ t < S_0\wedge T(\Gamma(\cdot)) \end{matrix} \right\}.$$

Then Lemma 6.1 (a) yields that

$$C(t) = \begin{cases} \cup\ \{\ \Gamma(t)\ :\ \Gamma(\cdot)\in C \text{ and } t < T(\Gamma(\cdot))\}\ \cup\ \varnothing & \text{if } t < S_0 \\ \varnothing & \text{if } t \geq S_0 \end{cases}$$

is a subsolution of (E). Also (6.10) implies that $\{L(t)\}_{t\geq 0}\in C$. Hence $T_0 \leq T(C(\cdot)) \leq S_0$. Suppose that $\{C(t)\}_{t\geq 0}$ is not a supersolution of (E). Using Lemma 14.6 we conclude that there exists a smooth function $\tilde{\Phi}$ and $(x_0,t_0)\in R^d\times(0,T(C(\cdot)))$ such that

$$\begin{aligned} 0 &= [d_{C_*}\vee 0](x_0,t_0) - \tilde{\Phi}(x_0,t_0) \\ &= \min\{\ [d_{C_*}\vee 0](x,t) - \tilde{\Phi}(x,t)\ :\ (x,t)\in R^d\times[0,\infty)\ \} \end{aligned}$$

and

$$F^*(\nabla\tilde{\Phi}(x_0,t_0), D^2\tilde{\Phi}(x_0,t_0), \frac{\partial}{\partial t}\tilde{\Phi}(x_0,t_0)) < 0\ .$$

In view of (4.2) we may assume that

(6.12) $$x_0 \notin \mathrm{int}C_*(t_0).$$

Set

$$\Phi(x,t) = \tilde{\Phi}(x,t) - [\,|x - x_0|^2 + |t - t_0|^2]^2.$$

Then,

$$(6.13)\qquad \begin{cases} \text{(a)}\ \ [d_{C_*} \vee 0](x,t) - \Phi(x,t) \geq [\,|x - x_0|^2 + |t - t_0|^2]^2 \quad \forall\ (x,t) \\ \text{(b)}\ \ \Phi(x_0,t_0) = [d_{C_*} \vee 0](x_0,t_0) \\ \text{(c)}\ \ F^*(\nabla \Phi(x_0,t_0), D^2\Phi(x_0,t_0), \frac{\partial}{\partial t}\Phi(x_0,t_0)) < 0. \end{cases}$$

Since $C(t) \subset U(t)$ for all $t < S_0$ and $\{U(t)\}_{t\geq 0}$ is a supersolution of (E), (6.13) (c) implies that

$$[d_{C_*} \vee 0](x_0,t_0) = \Phi(x_0,t_0) < [d_{U_*} \vee 0](x_0,t_0)\,.$$

Therefore there is δ_1 such that

$$(6.14)\qquad B_{\delta_1}(x_0,t_0) = \{\,(x,t) \in R^d \mathrm{x}[0,\infty) \ :\ |x - x_0|^2 + |t - t_0|^2 < (\delta_1)^2\,\} \subset \bigcup_{t\geq 0} [U(t) \times \{t\}]\,.$$

Also due to the smoothness of Φ and the upper semi-continuity of F^* there is δ_2 satisfying

$$(6.15)\qquad F^*(\nabla \Phi(x,t), D^2\Phi(x,t), \frac{\partial}{\partial t}\Phi(x,t)) < 0 \qquad \forall\ (x,t) \in B_{\delta_2}(x_0,t_0).$$

For $t < T(C(\cdot))$, let

$$\Psi(x,t) = \max\,\{\Phi(x,t) + (\delta_0)^4/2\,,\,[d_C \vee 0](x,t)\}\,,$$

where

$$\delta_0 = \min\,\{\delta_1, \delta_2\}.$$

Suppose $(x,t) \notin B_{\delta_0}(x_0,t_0)$, then (6.13) (a) yields that

$$\Phi(x,t) + (\delta_0)^4/2 < \Phi(x,t) + [|x - x_0|^2 + |t - t_0|^2]^2$$

$$\leq [d_{C_*} \vee 0](x,t)$$

$$\leq [d_C \vee 0](x,t) .$$

Hence, for $t < T(C(\cdot))$,

(6.16) $$\Psi(x,t) = [d_C \vee 0](x,t), \qquad \forall \ (x,t) \notin B_{\delta_0}(x_0,t_0).$$

Finally define

$$S(t) = \{ (x,t) \in R^d x[0,\infty) : \ \Psi(x,t) > 0 \} \quad \text{if } t < T(C(\cdot)),$$

and $S(t)$ is defined to be the empty set for $t \geq T(C(\cdot))$. The definition of $S(t)$ and (6.16) imply that

$$\bigcup_{t\geq 0} [S(t) \times \{t\}] \subset \bigcup_{t\geq 0} [C(t) \times \{t\}] \cup B_{\delta_0}(x_0,t_0).$$

Since $C(t)$ is included in $U(t)$ for all $t < S_0$ and $\delta_0 \leq \delta_1$, (6.14) and (6.16) yield that

(6.17) $$S(t) \subset U(t) \quad \forall \ t \in [0,S_0).$$

Using the smoothness of Ψ and the inequality

$$\Psi(x_0,t_0) = \Phi(x_0,t_0) + (\delta_0)^4/2$$

$$= [d_{C_*} \vee 0](x_0,t_0) + (\delta_0)^4/2$$

$$\geq (\delta_0)^4/2 ,$$

we conclude that

(6.18) $$B_{\delta}(x_0,t_0) \subset \bigcup_{t\geq 0} [S(t) \times \{t\}]$$

for some $\delta > 0$. Suppose that

$$(6.19) \qquad B_{\delta}(x_0,t_0) \subset \bigcup_{t\geq 0} [C(t) \times \{t\}]$$

for some $\delta > 0$. Then, $x_0 \in \mathrm{int}C^*(t_0)$. Recall that x_0 is chosen from the complement of $\mathrm{int}C_*(t_0)$ (see (6.12)). Hence, (6.19) does not hold for any positive δ and (6.18) yields

$$(6.20) \qquad \bigcup_{t<T(C(\cdot))} [C(t) \times \{t\}] \neq \bigcup_{t<T(C(\cdot))} [S(t) \times \{t\}].$$

Since $\{C(t)\}_{t\geq 0}$ is the largest subsolution of (E) included in $\{U(t)\}_{t\geq 0}$, (6.17), (6.20) and the fact that $C(t)$ is included in $S(t)$ imply that $\{S(t)\}_{t\geq 0}$ is not a subsolution of (E). Hence to complete the proof of this theorem it suffices to prove that $\{S(t)\}_{t\geq 0}$ is a subsolution of (E).

First note that $T(S(\cdot)) = T(C(\cdot))$. Suppose that a smooth function γ and$(y_0,s_0) \in R^d \times (0,T(S(\cdot)))$ satisfy

$$0 = [d_{S^*} \wedge 0](y_0,s_0) - \gamma(y_0,s_0) > [d_{S^*} \wedge 0](y,s) - \gamma(y,s) \quad \forall \ (y,s) \neq (y_0,s_0).$$

We need to show that

$$(6.21) \qquad F_*(\nabla\gamma(y_0,s_0), D^2\gamma(y_0,s_0), \frac{\partial}{\partial t}\gamma(y_0,s_0))) \leq 0.$$

Since $[d_{C^*} \wedge 0] \leq [d_{S^*} \wedge 0]$, if

$$[d_{C^*} \wedge 0](y_0,s_0) = [d_{S^*} \wedge 0](y_0,s_0)$$

(6.21) follows from the the fact that $\{C(t)\}_{t\geq 0}$ is a subsolution of (E). So we may assume that

$$(6.22) \qquad [d_{C^*} \wedge 0](y_0,s_0) < [d_{S^*} \wedge 0](y_0,s_0).$$

We analyse three cases seperately;

(1) $\Phi(y_0,s_0) + (\delta_0)^4/2 > 0$.

Then,$[d_{S^*} \wedge 0]$ is equal to zero on a neighborhood of (y_0,s_0). Hence, $\nabla\gamma(y_0,s_0) = 0$, $\frac{\partial}{\partial t}\gamma(y_0,s_0) = 0$, $D^2\gamma(y_0,s_0) \geq 0$ and (6.21) follows easily.

(2) $\Phi(y_0,s_0) + (\delta_0)^4/2 = [d_{S^*}\wedge 0](y_0,s_0) = 0$

We have

$$\gamma(y_0,s_0) = \Phi(y_0,s_0) + (\delta_0)^4/2 = 0.$$

Suppose that $\Phi(y,s) + (\delta_0)^4/2 > 0$ for some (y,s). Since $\Psi \geq \Phi + (\delta_0)^4/2$, we have $[d_{S^*}\wedge 0](y,s) = 0$. We also know that, $\gamma \geq [d_{S^*}\wedge 0]$. Hence,

$$\gamma(y,s) \geq 0 \qquad \text{whenever } \Phi(y,s) + (\delta_0)^4/2 > 0.$$

Now use the previous lemma with $f = \Phi + (\delta_0)^4/2$, $g = \gamma$ and $(x_0,t_0) = (y_0,s_0)$.

(3) $[d_{S^*}\wedge 0](y_0,s_0) < 0$.

Hence $y_0 \notin \mathrm{cl}S^*(s_0)$. Choose $z_0 \in \partial S^*(s_0)$ such that

$$[d_{S^*}\wedge 0](y_0,s_0) = -|y_0 - z_0|.$$

Define,

$$\xi(y,s) = \gamma(y + y_0 - z_0,s) + |y_0 - z_0|.$$

Then,

$$\begin{aligned}[d_{S^*}\wedge 0](z_0,s_0) - \xi(z_0,s_0) &= [d_{S^*}\wedge 0](y_0,s_0) - \gamma(y_0,s_0) \\ &\geq [d_{S^*}\wedge 0](y,s) - \gamma(y,s) \qquad \forall\ y,s, \\ &\geq -|z - y| + [d_{S^*}\wedge 0](z,s) - \gamma(y,s) \qquad \forall\ y,s,z.\end{aligned}$$

Let $y = z + y_0 - z_0$,

$$\begin{aligned}[d_{S^*}\wedge 0](z_0,s_0) - \xi(z_0,s_0) &\geq -|y_0 - z_0| + [d_{S^*}\wedge 0](z,s) - \gamma(z + y_0 - z_0,s) \\ &= [d_{S^*}\wedge 0](z,s) - \xi(z,s) \qquad \forall\ y,s.\end{aligned}$$

Since $z_0 \in \partial S^*(s_0)$, $\Psi(z_0,s_0) \geq 0$. Also (6.22) yields that $z_0 \notin \mathrm{cl}C^*(s_0)$. Hence $\Psi(z_0,s_0) = \Phi(z_0,s_0) + (\delta_0)^4/2$. Using the previous two cases at the point z_0,s_0 we obtain,

$$F_*(\nabla \xi(z_0,s_0), D^2\xi(z_0,s_0), \frac{\partial}{\partial t}\xi(z_0,s_0))) \leq 0.$$

Now (6.21) follows after observing that

$$(\nabla \xi(z_0,s_0), D^2\xi(z_0,s_0), \frac{\partial}{\partial t}\xi(z_0,s_0)) = (\nabla \gamma(y_0,s_0), D^2\gamma(y_0,s_0), \frac{\partial}{\partial t}\gamma(y_0,s_0)).$$

(4) $\Phi(y_0,s_0) + (\delta_0)^4/2 < 0$, $[d_{S^*} \wedge 0](y_0,s_0) = 0$

Using (6.22) we conclude that $\Psi = \Phi + (\delta_0)^4/2$ in a neighbourhood of (y_0,s_0). This contradicts with $[d_{S^*} \wedge 0](y_0,s_0) = 0$.

□

7.COMPARISON

Theorem 7.1 *Suppose that* $\{L(t)\}_{t\geq 0}$ *and* $\{U(t)\}_{t\geq 0}$ *are viscosity subsolution and viscosity supersolution of* (E), *respectively. Assume that for each* $T > 0$, *there is a positive constant* $R(T)$ *satisfying*

(7.1) $$L(t)\,,\, U(t) \subset B_{R(T)} \qquad \forall\ t \leq T\,.$$

Also, assume that there is $\alpha > 0$, *such that*

(7.2) $$[d_{L^*} \wedge 0](x,0) \leq [d_{U_*} \vee 0](x,0) - \alpha \qquad \forall x \in R^d\,.$$

Then,

(7.3) $$L^*(t) \subset U_*(t) \qquad \forall\ 0 \leq t < T(L(\cdot)) \wedge T(U(\cdot)).$$

Proof : Set $T_0 = T(L(\cdot)) \wedge T(U(\cdot))$, for $(x,t) \in R^d \mathrm{x}[0,T_0)$ define

$$u(x,t) = [d_{L^*} \wedge 0](x,t),$$

and

$$v(x,t) = [d_{U_*} \vee 0](x,t) - \alpha\,.$$

Remark 5.2(a) yields that u and v are viscosity subsolution and viscosity supersolution of (1.2) on $R^d \mathrm{x}[0,T_0)$, respectively. Clearly, (7.2) yields that

$$u(x,0) \leq v(x,0) \qquad \forall\, x \in R^d\,.$$

Also, using (7.1) we obtain that for any $t < T \wedge T_0$ and $|x| = R(T) + \alpha$,

$$\begin{aligned} u(x,t) &= d_{L^*}(x,t) \\ &\leq -\,\mathrm{distance}(x, \partial B_{R(T)}) \\ &= -\,\alpha \\ &\leq [d_{U_*} \vee 0](x,t) - \alpha \\ &= v(x,t)\,. \end{aligned}$$

Hence, Theorem 4.1 of [CGG 1989] implies that

(7.4) $$u(x,t) \leq v(x,t) \qquad \forall\, t < T \wedge T_0 \text{ and } |x| \leq R(T) + \alpha\,.$$

Suppose $x \in L^*(t)$ for some $t < T \wedge T_0$, then $|x| \leq R(T)$ and

$$u(x,t) = 0 \leq v(x,t) \leq [d_{U_*} \vee 0](x,t) - \alpha.$$

Therefore $d_{U_*}(x,t) \geq \alpha$, in particular $x \in U_*(t)$. □

For $\delta > 0$, define

$$L^{\delta}(t) = \{\ (x,t) \in R^d \text{x}[0,\infty)\ :\ d_{L^*}(x,t) > -\ \delta\ \},$$

$$U_{\delta}(t) = \{\ (x,t) \in R^d \text{x}[0,\infty)\ :\ d_{U_*}(x,t) > \delta\ \}.$$

Remark 7.2 Condition (7.2) is equivalent to

(7.5) $$L^*(0) \subset (U_*)_{\alpha}(0).$$

The following is a weak regularity result in the time variable and it will be used in Sections 9 and 10.

Lemma 7.3 *Suppose that* $\{C(t)\}_{t \geq 0}$ *and* $\{U(t)\}_{t \geq 0}$ *are viscosity subsolution and viscosity supersolution of* (E), *respectively. Then,*

(a) $$\limsup_{s \uparrow t} [d_{C^*} \wedge 0](x,s) = [d_{C^*} \wedge 0](x,t) \qquad \forall\ (x,t) \in R^d \text{x}(0, T(C(\cdot))\ ),$$

(b) $$\liminf_{s \uparrow t} [d_{U_*} \vee 0](x,s) = [d_{U_*} \vee 0](x,t) \qquad \forall\ (x,t) \in R^d \text{x}(0, T(U(\cdot))\ ).$$

Proof: (a) We analyse two cases seperately;

(1) $x \in \text{cl}C^*(t)$.

Suppose to the contrary. Then

$$-\ \alpha = \limsup_{s \uparrow t} [d_{C^*} \wedge 0](x,s) < 0.$$

and consequently there is $\delta > 0$ such that

$$[d_{C^*}\wedge 0](y,s) \le -(\alpha/2), \quad \forall\ |x-y| \le \delta,\ s \in [t-\delta, t).$$

Hence, for any positive p

$$[d_{C^*}\wedge 0](y,s) - [d_{C^*}\wedge 0](x,t) - (s-t)p$$

$$\le -(\alpha/2) - 0 - (s-t)p$$

$$\le 0 \qquad \forall\ |x-y| \le \delta,\ s \in [t - ((\alpha/2)p\wedge\delta), \infty).$$

Therefore, $(0,0,p) \in \mathbf{D}^+C(t)$. Since $\beta > 0$ (see assumption A),

$$F_*(0,0,p) \ge \min\{\ \beta(-n/|n|)\,p\ : n \in R^d\ \} > 0 \qquad \forall\ p > 0.$$

The above inequality contradicts the subsolution property of $\{C(t)\}_{t\ge 0}$.

(2) $x \notin \mathrm{cl}C^*(t)$.

Choose $z \in \partial C^*(t)$ such that

$$d_{C^*}(x,t) = -|x-z|.$$

Then, the upper semi-continuity and the sublinearity of $[d_{C^*}\wedge 0]$ imply that

$$d_{C^*}(x,t) \ge \limsup_{s\uparrow t}\ [d_{C^*}\wedge 0](x,s)$$

$$\ge \limsup_{s\uparrow t}\ [d_{C^*}\wedge 0](z,s) - |x-z|$$

$$= \limsup_{s\uparrow t}\ [d_{C^*}\wedge 0](z,s) + d_{C^*}(x,t).$$

Since $z \in \partial C^*(t)$, we apply case (1) to obtain

$$\limsup_{s\uparrow t}\ [d_{C^*}\wedge 0](z,s) = 0.$$

(b) Follows from part(a) and Remark 5.2(b). ▭

8.NON-UNIQUENESS, EXAMPLES

We give two planar isotropic examples to show that there is no general uniqueness result. Non-uniqueness of solutions is related to the development of an interior of the level sets of viscosity solutions to (1.2). Similar examples are also discussed in Section 8.2 of [ES 1989a].

Example 8.1

Let $h(z,t)$ be the solution of

(8.1)(a) $$\frac{\partial}{\partial t} h(z,t) = \frac{\partial^2}{\partial^2 z} h(z,t)\, [1 + (\frac{\partial}{\partial z} h(z,t))^2]^{-1} \qquad \forall\ t>0,\ z>0,$$

(8.1)(b) $$\frac{\partial}{\partial z}\ h(0,t) = 0 \qquad \forall\ t>0,$$

(8.1)(c) $$\lim_{\xi \to \infty} \frac{\partial}{\partial z}\ h(\xi,t) = 1 \qquad \forall\ t>0,$$

(8.1)(d) $$h(z,0) = z \qquad \forall\ z \geq 0.$$

The existence of such a solution can be proved by an approximation argument. Define

$$D(t) = \{\ (x,y) \in R^2\ \ : |x| > h(|y|,t)\ \},$$

and

$$C(t) = \{\ (x,y) \in R^2\ \ : |y| < h(|x|,t)\ \}\ .$$

A straightforward calculation shows that both $\{C(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$ are classical solution of (MCE) with initial condition,

$$C(0) = D(0) = \{\ (x,y) \in R^2\ \ : |x| > |y|\ \}.$$

Example 8.2

Let $(h(z,t),A(t))$ and T be a solution of

(8.2)(a) $$\frac{\partial}{\partial t} h(z,t) = \frac{\partial^2}{\partial^2 z} h(z,t)\, [1 + (\frac{\partial}{\partial z} h(z,t))^2]^{-1} \qquad \forall\ t \in [0,T),\ z \in (0,A(t)),$$

(8.2)(b)	$\frac{\partial}{\partial z} h(0,t) = h(A(t),t) = 0$	$\forall\ t \in [0,T)$,
(8.2)(c)	$\lim_{\xi \uparrow A(t)} \frac{\partial}{\partial z} h(\xi,t) = -\infty$	$\forall\ t \in [0,T)$,
(8.2)(d)	$h(z,0) = z\sqrt{1 - z^2}$	$\forall\ z \in (0,1)$,
(8.2)(e)	$A(0) = 1.$	

For $t \leq T$ define

$$D(t) = \{\ (x,y) \in [-A(t),A(t)] \times R \ : |y| < h(|x|,t)\ \}.$$

Let $(p(z,t),b(t),B(t))$ and T be a solution of

(8.3)(a)	$\frac{\partial}{\partial t} p(z,t) = \frac{\partial^2}{\partial^2 z} p(z,t)\ [1 + (\frac{\partial}{\partial z} p(z,t))^2]^{-1}$	$\forall\ t \in [0,T),\ z \in (b(t),B(t))$,
(8.3)(b)	$p(b(t),t) = p(B(t),t) = 0$	$\forall\ t \in [0,T)$,
(8.4)(c)	$\lim_{\xi \downarrow b(t)} \frac{\partial}{\partial z} p(\xi,t) = -\lim_{\xi \uparrow B(t)} \frac{\partial}{\partial z} p(\xi,t) = \infty$	$\forall\ t \in [0,T)$,
(8.3)(d)	$p(z,0) = z\sqrt{1 - z^2}$	$\forall\ z \in (0,1)$,
(8.3)(e)	$b(0) = 0, \qquad B(0) = 1.$	

For $t \leq T$ define

$$C(t) = \{\ (x,y) \in [[-B(t),-b(t)] \cup [b(t),B(t)]] \times R \ : |y| < p(|x|,t)\ \}.$$

If there are solutions to (8.2) and (8.3), it is easy to show that both $\{C(t)\}_{t \geq 0}$ and $\{D(t)\}_{t \geq 0}$ are classical solution of (E) with initial condition,

$$C(0) = D(0) = \Gamma = \{\ (x,y) \in [0,1] \times [0,1] \ : \ |y| < |x|\sqrt{1 - x^2}\ \}$$

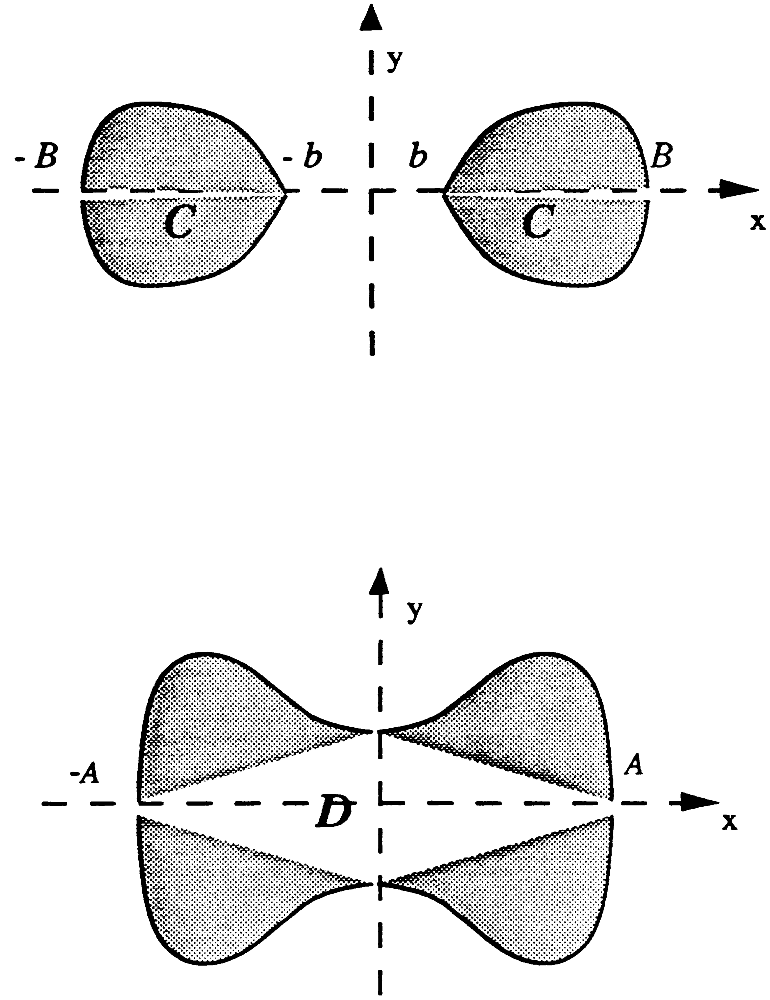

Figure 1 Two solutions with initial data $C(0)$.

9.UNIQUENESS FOR NONPOSITIVE υ

In this section we prove a uniqueness result for $\upsilon \leq 0$ and a class of initial conditions. These initial conditions along with other properties do not have self-intersections. We start with the description of the condition we impose on the initial conditions.

For $\rho \in (0,1)$, $\Gamma \subset R^d$ and $x \in R^d$ define

$$\begin{aligned} \Gamma(x,\rho) &= \rho(\Gamma \oplus \{-x\}) \oplus \{x\} \\ &= \{\ \rho y + (1-\rho)x\ : y \in \Gamma\ \}. \end{aligned}$$

Definition 9.1 We say that a bounded open subset Γ of R^d is ***strictly starshaped*** around a point $x \in R^d$ if there is $\rho_0 \in (0,1)$ such that

$$\min\{\ \mathrm{dist}(z,\Gamma^c)\ :\ z \in \mathrm{cl}\Gamma(x,\rho)\ \} > 0, \qquad \forall\ \rho \in [\rho_0,1).$$

The compactness of Γ implies the following.

Lemma 9.2 *A bounded open set Γ is strictly starshaped around x if and only if there is ρ_0 such that*

$$\text{(9.1)} \quad \sup\{-\ \mathrm{dist}(z,\Gamma(x,\rho)) - \mathrm{dist}(z,\Gamma^c)\ :\ z \in R^d\ \} = -\ \alpha(\rho) < 0 \qquad \forall \rho \in [\rho_0,1).$$

Theorem 9.3 *Suppose that $\{L(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$ are viscosity subsolution and viscosity supersolution of* (E), *respectively. Assume that $\upsilon \leq 0$ and there is a bounded open set Γ satisfying*

$$\text{(9.2)} \qquad \mathrm{cl}L^*(0) \subset \mathrm{cl}\Gamma \subset \mathrm{cl}U_*(0)$$

Further assume that Γ is strictly starshaped around a point x and for each $T > 0$, there is $R(T)$ such that

$$L(t), U(t) \subset B_{R(T)} \qquad \forall\ t \leq T.$$

Then,

$$L^*(t) \subset \mathrm{cl}U_*(t) \qquad \forall\ \ 0 \leq t < T(L(\cdot)) \wedge T(U(\cdot)).$$

Proof: Since the equation (E) is invariant under translation without loss of generality we may assume that Γ *is* strictly starshaped around the origin. Fix $\rho < 1$, and define

$$C(t) = \rho\, L(\frac{t}{\rho^2}), \qquad \forall \;\; t \geq 0.$$

We claim that $\{C(t)\}_{t\geq 0}$ is a subsolution of (E). Indeed suppose

$$(n, A, p) \in \mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{C^*} \wedge 0](x,t)$$

at some $(x,t) \in R^d \mathrm{x}[0, T(L(\cdot)))$. Since $C^*(t) = \rho\, L^*(\frac{t}{\rho^2})$ and

$$d_{C^*}(x,t) = \rho d_{L^*}(\frac{x}{\rho}, \frac{t}{\rho^2}), \qquad \forall \;\; (x,t),$$

the definition of the set of second superdifferentials yields

$$0 \geq \limsup_{(y,h)\to 0} \;\; \{ \frac{[d_{C^*} \wedge 0](x+y, t+h) - [d_{C^*} \wedge 0](x,t) - n\cdot y - ph - (1/2)Ay\cdot y}{|h| + |y|^2} \}$$

$$= \limsup_{(y,h)\to 0} \;\; \{ \frac{\rho[d_{L^*} \wedge 0]((x+y)/\rho, (t+h)/\rho^2) - \rho[d_{L^*} \wedge 0](x/\rho, t/\rho^2)}{|h| + |y|^2} + \frac{-n\cdot y - ph - (1/2)Ay\cdot y}{|h| + |y|^2} \}$$

$$= \frac{1}{\rho} \;\; \{ \limsup_{(z,s)\to 0} \frac{[d_{L^*} \wedge 0]((x/\rho) + z, (t/\rho^2) + s) - [d_{L^*} \wedge 0](x/\rho, t/\rho^2)}{|s| + |z|^2} + \frac{-n\cdot z - \rho ps - (1/2)\,\rho Az\cdot z}{|s| + |z|^2} \}.$$

Hence,

$$(n, \rho A, \rho p) \in \mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{L^*} \wedge 0](x/\rho, t/\rho^2).$$

Since $\{L(t)\}_{t\geq 0}$ is a subsolution of (E),

$$
\begin{aligned}
0 &\geq F_*(n,\rho A,\rho p)\\
&= \rho F_*(n,A,p) - \upsilon\,|n|(1-\rho)\\
&\geq \rho F_*(n,A,p)\;.
\end{aligned}
$$

Combine (9.1) and (9.2) to obtain

$$[d_{C^*}\wedge 0](x,0) \leq [d_{U^*}\vee 0](x,0) - \alpha(\rho)\;.$$

Hence the hypothessis of Theorem 7.1 are satisfyied by $\{C(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$. Thus

(9.4) $$\rho\, L^*(\frac{t}{\rho^2}) = C^*(t) \subset U_*(t), \qquad \forall\; t \in [0,T(L(\cdot))\wedge T(U(\cdot))\,)$$

and ρ sufficiently close to 1. Suppose $x \in L^*(t)$ with $t < T(L(\cdot))\wedge T(U(\cdot))$, using Lemma 7.3 (a) we obtain a sequence $(x_n\,,t_n) \rightarrow (x,t)$ satisfying

$$t_n < t \quad \text{and} \quad x_n \in L^*(t_n) \qquad \forall\; n\,.$$

Set $\rho_n = \sqrt{t/t_n}$. Then

(9.5) $$\rho_n\, x_n \in \rho_n\, L^*(t_n) = C^*(t).$$

Since $\rho_n < 1$ and $\rho_n \rightarrow 1$, (9.4) and (9.5) yield that for sufficiently large n

$$\rho_n\, x_n \in U_*(t).$$

Now let n tend to infinity to conclude that $x \in \mathrm{cl}U_*(t)$. □

10. EXISTENCE; INITIAL VALUE PROBLEM

In this Section we construct a maximal viscosity solution to (E) with a given initial data. Our construction is closely related to Section 6 of [CGG 1989]. In view of Section 6, to obtain an existence result it suffices to construct a viscosity subsolution and a viscosity supersolution satisfying the given initial data

(10.1)(a) $\quad \mathrm{int}C^*(0) = \Gamma,$

(10.1)(b) $\quad \mathrm{cl}C_*(0) = \mathrm{cl}\Gamma.$

For $r \geq 0$, define $w(r)$ by

$$w(r) = \frac{rR - \ln(rR+1)}{KR^2}$$

where

$$K = \max\{\ \mathrm{trace}G(n) / \beta(n)\ :\ n\in R^d \text{ and } |n| = 1\ \}\vee 1\ ,$$

$$R = (\max\{\ |\upsilon|/\beta(n)\ :\ n\in R^d \text{ and } |n| = 1\ \}/K)\vee 1.$$

For a given $x_0\in R^d$ and $\rho>0$, define

$$L(x_0,\rho)(t) = \{\ x\in R^d\ :\ \rho - t - w(|x - x_0|) > 0\ \},$$

and

$$U(x_0,\rho)(t) = \{\ x\in R^d\ :\ -\rho + t + w(|x - x_0|) > 0\ \}.$$

Then,

$$L(x_0,\rho)(t) = \mathrm{B}_{\rho(t)}(x_0),$$

and

$$U(x_0,\rho)(t) = R^d \setminus \mathrm{B}_{\rho(t)}(x_0),$$

where $\mathrm{B}_\rho(x)$ denotes the d-dimensional sphere with radius ρ and center x and $\rho(t)$ is the unique solution of

$$\rho = t + w(\rho(t)).$$

Since the curvature tensor and the normal velocity of $B_{\rho(t)}(x_0)$ is

$$(\text{ identity}/\rho(t), \frac{d}{dt}\rho(t)),$$

a straightforward calculation shows that $\{L(x_0,\rho)(t)\}_{t\geq 0}$ is a classical subsolution of (E). Similarly the curvature tensor and the normal velocity of $R^d \setminus B_{\rho(t)}(x_0)$ is

$$(\text{ - identity}/\rho(t), - \frac{d}{dt}\rho(t)),$$

and $\{U(x_0,\rho)(t)\}_{t\geq 0}$ is a classical supersolution of (E). Finally, define

(10.2) $\qquad L(t) = \cup \{L(x_0,\rho)(t) : L(x_0,\rho)(0) \subset \Gamma\},$

and

(10.3) $\qquad U(t) = \cap \{ U(x_0,\rho)(t) : \Gamma \subset U(x_0,\rho)(t) \}.$

Then, Lemma 6.1 implies that $\{L(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$ are viscosity subsolution and viscosity supersolution of (E), respectively. Also, $\{L(t)\}_{t\geq 0}$ satisfies (10.1) (a) and $\{U(t)\}_{t\geq 0}$ satisfies (10.1) (b). Moreover,

$$L(t) \subset U(t) \qquad \forall \ t < T(L(\cdot)) \wedge T(U(\cdot)).$$

Theorem 10.1 *For any given proper subset Γ of R^d there is a viscosity solution $\{C(t)\}_{t\geq 0}$ of (E) satisfying (10.1). Moreover, the extinction time $T(C(\cdot))$ is strictly positive.*

Proof: The existence of $\{C(t)\}_{t\geq 0}$ follows from the preceeding calculations and Theorem 6.3. Since Γ is a proper subset of R^d, there are $R_1>0$, $R_2>0$ and $x_0 \in R^d$ and $y_0 \in R^d$ such that

$$L(x_0,R_1)(0) \subset \Gamma \subset U(y_0,R_2)(0).$$

Then, the definitions of $L(x_0,R_1)(t)$ and $U(y_0,R_2)(t)$ imply that

$$\varnothing \neq L(x_0,R_1)(t) \subset L(t) \subset C(t) \quad \forall\ t < R_1,$$

and

$$C(t) \subset U(t) \subset U(y_0,R_2)(t) \neq R^d \quad \forall\ t < R_2.$$

Therefore,

$$T(C(\cdot)) \geq R_1 \wedge R_2\,. \qquad \square$$

Remark 10.2 Let $\{V(t)\}_{t\geq 0}$ be a viscosity supersolution of (E) satisfying (10.1) (b). Then,

$$Y(t) = U(t) \cap V(t)$$

is again a viscosity supersolution of (E) with initial condition (10.1). Hence, by using $\{Y(t)\}_{t\geq 0}$ instead of $\{U(t)\}_{t\geq 0}$, in the proof of the above theorem we obtain a viscosity solution included by the given viscosity supersolution $\{V(t)\}_{t\geq 0}$.

We need a technical lemma to prove that the viscosity solution constructed in the proofs of Theorem 6.3 and Theorem 10.1 is indeed a maximal one. Suppose that $\{C(t)\}_{t\geq 0}$ and $\{\Gamma(t)\}_{t\geq 0}$ are viscosity subsolutions of (E) satisfying,

$$[d_{\Gamma^*} \wedge 0](x,0) \leq [d_{C^*} \wedge 0](x,t_0), \qquad \forall\ x, \tag{10.4}$$

at some point t_0. Set

$$S(t) = \begin{cases} C(t) & \text{if } t \leq t_0 \\ \Gamma(t - t_0) & \text{if } t > t_0. \end{cases}$$

Lemma 10.3. *$\{S(t)\}_{t\geq 0}$ is a viscosity subsolution of* (E).

Proof: Let $(n,A,p) \in \mathrm{D}_x^{+2}{}_t^{+1}[d_{S^*} \wedge 0](x_0,s_0)$. We need to show that

(10.5) $$F_*(n,A,p) \leq 0.$$

If $s_0 \neq t_0$, then (10.5) follows easily from the subsolution properties of $\{C(t)\}_{t \geq 0}$ and $\{\Gamma(t)\}_{t \geq 0}$. So assume that $s_0 = t_0$. Using (10.4) and Lemma 7.3 (a) we obtain

$$[d_{S^*} \wedge 0](x,t) = \begin{cases} [d_{C^*} \wedge 0](x,t) & \text{if } t \leq t_0 \\ [d_{\Gamma^*} \wedge 0](x,t - t_0) & \text{if } t > t_0. \end{cases}$$

Let Ψ be as in Theorem 14.1 (b) with $\Phi = [d_{S^*} \wedge 0]$ and $(x,t) = (x_0,t_0)$. For $\varepsilon > 0$, define

$$\Psi_\varepsilon(x,t) = \Psi(x,t) + [|x - x_0|^4 + |t - t_0|^2] + \varepsilon \frac{1}{t_0 - t}.$$

Choose $(x_\varepsilon,t_\varepsilon)$ such that $t_\varepsilon < t_0$, and

$$[d_{C^*} \wedge 0](x_\varepsilon,t_\varepsilon) - \Psi_\varepsilon(x_\varepsilon,t_\varepsilon) \geq [d_{C^*} \wedge 0](x,t) - \Psi_\varepsilon(x,t) \qquad \forall\, x \in R^d,\ t \in [0,t_0).$$

It is easy to show that $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ as ε tends to zero. Also the subsolution property of $\{C(t)\}_{t \geq 0}$ implies that

$$F_*(\nabla \Psi_\varepsilon(x_\varepsilon,t_\varepsilon), \mathrm{D}^2 \Psi_\varepsilon(x_\varepsilon,t_\varepsilon), \frac{\partial}{\partial t} \Psi_\varepsilon(x_\varepsilon,t_\varepsilon)) \leq 0.$$

Since F_* is non-decreasing in the p-variable and

$$\liminf_{\varepsilon \downarrow 0} \frac{\partial}{\partial t} \Psi_\varepsilon(x_\varepsilon,t_\varepsilon) \geq \frac{\partial}{\partial t} \Psi(x_0,t_0),$$

we obtain (10.5) by letting ε go to zero. □

Theorem 10.4 *For any given non-empty bounded open subset Γ of R^d there is a maximal viscosity solution* $\{C(t)\}_{t \geq 0}$ *of* (E) *satisfying* (10.1).

11. CONNECTION BETWEEN (E) & (1.2)

Let u be the unique viscosity solution of

(1.2) $$F(\nabla u(x,t), D^2u(x,t), \frac{\partial}{\partial t}u(x,t)) = 0, \qquad \forall\, x \in R^d, t > 0,$$

with initial condition

$$u(x,0) = \begin{cases} \text{dist}(x,\partial\Gamma)\wedge 1 & \text{if } x \in \Gamma \\ -[\text{dist}(x,\partial\Gamma)\wedge 1] & \text{if } x \notin \Gamma \end{cases}$$

where Γ is a given bounded open subset of R^d. The existence, the uniqueness, and the continuity of u are proved in Theorem 6.8 of [CGG 1989]. Set

(11.1) $$L(t) = \{\, x \in R^d \;;\; u(x,t) > 0 \,\},$$

(11.2) $$U(t) = \{\, x \in R^d \;;\; u(x,t) \geq 0 \,\}.$$

Theorem 11.1 *For any bounded non-empty open subset Γ of R^d, $\{L(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$ are maximal viscosity solutions of* (E) *satisfying* (10.1). *Moreover*

(11.3) $$L^*(t) \subset C(t) \subset \text{cl}U_*(t),$$

for all $t \in [0, T(L(\cdot))\wedge T(U(\cdot))\,)$, and any other maximal viscosity solution $\{C(t)\}_{t\geq 0}$ of (E) *with initial condition* (10.1).

Proof: Let A be the collection of all viscosity solutions of (E) with initial conditions which are compact in Γ. Set

$$J(t) = \cup\, \{\, C(t) \;:\; \{C(t)\}_{t\geq 0} \in A\,,\; t < T(C(\cdot))\,\} \cup \emptyset.$$

Since Γ is bounded, if $\{C(t)\}_{t\geq 0} \in A$ then there is $\alpha > 0$ such that

(11.4) $$[d_{C^*}\wedge 0](x,0) \leq u(x,0) - \alpha.$$

Also $\{C(t)\}_{t\geq 0}$ is included in the viscosity supersolution defined by (10.3). In particular $\{C(t)\}_{t\geq 0}$ is bounded. Therefore (11.4) and Theorem 4.1 of [CGG 1989] yield that

11. CONNECTION BETWEEN (E) & (1.2)

Let u be the unique viscosity solution of

$$F(\nabla u(x,t), D^2u(x,t), \frac{\partial}{\partial t}u(x,t)) = 0, \qquad \forall\, x \in R^d,\, t > 0, \tag{1.2}$$

with initial condition

$$u(x,0) = \begin{cases} \text{dist}(x,\partial\Gamma)\wedge 1 & \text{if } x \in \Gamma \\ -[\text{dist}(x,\partial\Gamma)\wedge 1] & \text{if } x \notin \Gamma \end{cases}$$

where Γ is a given bounded open subset of R^d. The existence, the uniqueness, and the continuity of u are proved in Theorem 6.8 of [CGG 1989]. Set

$$L(t) = \{\, x \in R^d \;;\; u(x,t) > 0 \,\}, \tag{11.1}$$

$$U(t) = \{\, x \in R^d \;;\; u(x,t) \geq 0 \,\}. \tag{11.2}$$

Theorem 11.1 *For any bounded non-empty open subset Γ of R^d, $\{L(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$ are maximal viscosity solutions of* (E) *satisfying* (10.1). *Moreover*

$$L^*(t) \subset C(t) \subset \text{cl}U_*(t), \tag{11.3}$$

for all $t \in [0, T(L(\cdot))\wedge T(U(\cdot))\,)$, and any other maximal viscosity solution $\{C(t)\}_{t\geq 0}$ of (E) *with initial condition* (10.1).

Proof: Let A be the collection of all viscosity solutions of (E) with initial conditions which are compact in Γ. Set

$$J(t) = \cup\, \{\, C(t) \;:\; \{C(t)\}_{t\geq 0} \in A\,,\; t < T(C(\cdot)) \,\} \cup \varnothing.$$

Since Γ is bounded, if $\{C(t)\}_{t\geq 0} \in A$ then there is $\alpha > 0$ such that

$$[d_{C^*}\wedge 0](x,0) \leq u(x,0) - \alpha. \tag{11.4}$$

Also $\{C(t)\}_{t\geq 0}$ is included in the viscosity supersolution defined by (10.3). In particular $\{C(t)\}_{t\geq 0}$ is bounded. Therefore (11.4) and Theorem 4.1 of [CGG 1989] yield that

$$[d_{C^*}\wedge 0](x,t) \le u(x,t) - \alpha.$$

Suppose that $x\in C(t)$ with $t \le T(C(\cdot))$. Then, $u(x,t) \ge \alpha > 0$. Hence, $J(t)$ is included in $L(t)$ forall $t \ge 0$.

Suppose that $x_0\in L(t_0)$ with $t_0 < T(L(\cdot))$, then $u(x_0,t_0) > \gamma$ for some $\gamma > 0$. By Theorem 10.4 there is a maximal viscosity solution $\{C(t)\}_{t\ge 0}\in \boldsymbol{A}$ such that

(11.4) $$[d_{C^*}\vee 0](x,0) = u(x,0) - \gamma.$$

Then, theorem 4.1 of [CGG 1989] implies that $[d_{C^*}\vee 0](x,t) \ge u(x,t) - \gamma$ for all $(x,t)\in R^d \times [0,T(C(\cdot))]$. Since $\{C(t)\}_{t\ge 0}$ is maximal and Γ is bounded

$$C_*(T(C(\cdot))) = \varnothing.$$

Therefore the continuity of u and (11.4) yield that

$$T(C(\cdot)) \ge \inf \{\, t \ge 0 \;:\; \text{there is } x \text{ such that } u(x,t) > \gamma \,\}.$$

Hence $x_0\in C(t_0)$, $t_0 < T(C(\cdot))$ and consequently $L(t) = J(t)$ forall $t \ge 0$.

Now let $\boldsymbol{B}$ be the collection of all viscosity solutions of (E) with initial data which compactly includes Γ. A similar argument yields

$$U(t) = \cap \{\, C(t) \;:\; \{C(t)\}_{t\ge 0}\in \boldsymbol{B} \,,\; t < T(C(\cdot)) \,\} \cap R^d.$$

Let $\{C(t)\}_{t\ge 0}$ be a viscosity solution of (E) and (10.1). Theorem 7.1 implies that

$$K^*(t) \subset C(t) \subset V_*(t),$$

forall $\{K(t)\}_{t\ge 0}\in \boldsymbol{A}$, $\{V(t)\}_{t\ge 0}\in \boldsymbol{B}$, and $t \in [0,T(L(\cdot))\wedge T(U(\cdot)) \,)$. ▭

Corollary 11.2 *The level set*

$$\Gamma(t) = \{\, x\in R^d \;;\; u(x,t) = 0 \,\}$$

has non-empty interior if and only if there are more than one viscosity solutions to (E). *When there is a unique solution* $\{C(t)\}_{t\ge 0}$ *to* (E),

$$\Gamma(t) = \partial C(t) \qquad \forall\; t < T(C(\cdot)).$$

12. A CLASS OF EXPLICIT SOLUTIONS

In this Section we construct a class of explicit solutions of (E) which are related to Wulff crystals [W 1901](also see Dinghas[D 1944], Taylor[T 1974], [T 1975],and Fonseca[F 1990]). We will use these solutions in the asymptotic analysis of (E). Let

$$B(\theta) = \frac{1}{\beta(\theta/|\theta|)}\,|\theta|$$

and we assume

$$\sum_{j=1}^{d}\sum_{i=1}^{d} \frac{\partial^2}{\partial\theta_i\partial\theta_j}B(\theta)\xi_i\,\xi_j > 0 \qquad \forall\ \ \theta\cdot\xi = 0. \tag{12.1}$$

For $x \in R^d$ and $x \neq 0$, set $\hat{x} = x/|x|$. Then define

$$R(x) = \min\left\{ \frac{B(\theta)}{\theta\cdot\hat{x}}\ :\ \theta\in R^d \text{ and } \theta\cdot\hat{x} > 0 \right\} \qquad \forall\ \ x \neq 0, \tag{12.2}$$

and

$$\Theta(x) = \left\{ \theta\in S^{d-1}:\ \theta\cdot\hat{x} > 0 \text{ and } \frac{B(\theta)}{\theta\cdot\hat{x}} = R(x) \right\} \qquad \forall\ \ x \neq 0, \tag{12.3}$$

where $S^{d-1} = \{\ \theta\in R^d\ :\ |\theta| = 1\ \}$. We gather several elementary properties of the above functions into a Lemma.

Lemma 12.1

(a) *There is a continuously differentiable function $\theta(x)$ such that*

$$\Theta(x) = \{\theta(x)\} \qquad \forall\ \ x \neq 0.$$

In particular,

$$\nabla B(\theta(x))[\theta(x)\cdot\hat{x}] - B(\theta(x))\,\hat{x} = 0. \tag{12.4}$$

(b) *For all $x \neq 0$,*

$$B(\theta(x)) = R(x)[\theta(x)\cdot\hat{x}] = \max\{\ R(y)\ [\theta(x)\cdot\hat{y}]\ :\ \ y \neq 0\ \}.$$

In particular,

(12.5) $$\nabla R(x)[\theta(x)\cdot\hat{x}] + R(x)\frac{\theta(x)}{|x|} - R(x)[\theta(x)\cdot\hat{x}]\frac{x}{|x|^2} = 0.$$

Proof:

(a) For $i = 1,..,d$ set

(12.6) $$H_i(\theta,x) = [\theta\cdot\hat{x}]\frac{\partial}{\partial\theta_i}B(\theta) - B(\theta)\,\hat{x}_i\,.$$

Then,

(12.7) $$H(\theta,x) = (H_1(\theta,x),...,H_d(\theta,x)) = 0 \qquad \forall\ \theta\in\Theta(x).$$

We calculate directly that

$$\frac{\partial}{\partial\theta_j}H_i(\theta,x) = [\theta\cdot\hat{x}]\frac{\partial^2}{\partial\theta_i\partial\theta_j}B(\theta) + \hat{x}_j\frac{\partial}{\partial\theta_i}B(\theta) - \hat{x}_i\frac{\partial}{\partial\theta_j}B(\theta).$$

Let ξ be a vector orthogonal to θ. Using (12.1) we conclude that

$$\sum_{j=1}^{d}\sum_{i=1}^{d}[\frac{\partial}{\partial\theta_j}H_i(\theta,x)]\xi_i\,\xi_j = [\theta\cdot\hat{x}]\sum_{j=1}^{d}\sum_{i=1}^{d}[\frac{\partial^2}{\partial\theta_i\partial\theta_j}B(\theta)\,]\xi_i\,\xi_j > 0,$$

$$\forall\ \theta\cdot\xi = 0,\ \theta\cdot\hat{x} > 0.$$

Hence for every x, there is a unique solution $\theta(x)\in S^{d-1}$ of the equation (12.7). Using the implicit function theorem we conclude that $\theta(x)$ is continuously differentiable.

(b) Follows from straightforward calculations. □

Let h be a real-valued, continuously differentiable, strictly decreasing function on $[0,\infty)$. For $x \neq 0$, define

$$u(x) = h(|x|/R(x)).$$

Using the previous lemma, we calculate directly that

$$\nabla u(x) = h'(|x|/R(x))[\frac{x}{|x|R(x)} - \frac{|x|}{(R(x))^2}\nabla R(x)]$$

$$= h'(|x|/R(x))[\frac{x}{|x|R(x)} - \frac{|x|}{(R(x))^2}\{R(x)\frac{x}{|x|^2} - R(x)\frac{\theta(x)}{|x|[\theta(x)\cdot\hat{x}]}\}].$$

$$= h'(|x|/R(x))\frac{\theta(x)}{R(x)[\theta(x)\cdot\hat{x}]}$$

(12.8)
$$= h'(|x|/R(x))\frac{\theta(x)}{B(\theta(x))}.$$

Since h is decreasing and $\theta(x) \in S^{d-1}$,

(12.9)
$$\nabla(\frac{\nabla u(x)}{|\nabla u(x)|}) = -\nabla(\theta(x)).$$

Set

$$g(\theta) = D^2(B(\theta)) / (d-1).$$

Lemma 12.2 *For any non-zero $x \in R^d$,*

(12.11)
$$\mathrm{trace}[g(\theta(x))(I - \theta(x)\otimes\theta(x))\nabla(\theta(x))] = \frac{R(x)}{|x|}.$$

Proof: Recall that $H_i(\theta(x),x) = 0$, where H_i is as in (12.6). Differentiate this equation with respect to x_j and then use the same equation to obtain

(12.12)
$$[(\theta(x)\cdot x)D^2(B(\theta(x))]\nabla(\theta(x)) = B(\theta(x))[I - \frac{1}{\theta(x)\cdot x}x\otimes\theta(x)].$$

Since $B(\theta)$ is homogenous of degree one, $D^2(B(\theta))\theta = 0$ for every θ. Hence,

$$(d-1)g(\theta(x))(I - \theta(x)\otimes\theta(x)) = D^2(B(\theta(x))).$$

Using (12.12) we obtain,

$$g(\theta(x))(\mathrm{I} - \theta(x) \otimes \theta(x))\nabla(\theta(x)) =$$

$$= \frac{B(\theta(x))}{(d - 1)(\theta(x)\cdot x)}(\mathrm{I} - \frac{1}{\theta(x)\cdot x} x\otimes\theta(x))$$

$$= \frac{R(x)}{(d - 1)|x|}(\mathrm{I} - \frac{1}{\theta(x)\cdot x} x\otimes\theta(x)).$$

We prove (12.11) after observing that trace $(\mathrm{I} - \frac{1}{\theta(x)\cdot x} x\otimes\theta(x)) = (d - 1)$. ▭

For any λ define

$$C(t) = \{x\in R^d\backslash\{0\} \ : \ h(|x|/R(x)) > e^{\lambda t}\} \cup \{0\}.$$

Recall that for $\theta\in S^{d-1}$ $\beta(\theta) = (B(\theta))^{-1}$. Identities (12.8) and (12.11) imply that at $x\in \partial C(t)$,

$$\text{normal velocity} = \mathbf{V} = \frac{\lambda e^{\lambda t}}{h'(|x|/R(x)) \ \beta(\theta(x))}$$

$$= \frac{\lambda h(|x|/R(x))}{h'(|x|/R(x)) \ \beta(\theta(x))} ,$$

$$\text{outward unit normal} = \theta = \theta(x)$$

$$\text{trace} g(\theta(x))(\mathrm{I} - \theta(x) \otimes \theta(x))\nabla(\frac{\nabla u(x)}{|\nabla u(x)|}) = -\frac{R(x)}{|x|}.$$

Hence, $\{C(t)\}_{t\geq 0}$ is a classical solution of (E) with

$$G(\theta) = Cg(\theta)$$

provided that

$$\frac{\lambda h(\rho)}{h'(\rho)} = -\frac{C}{\rho} + \upsilon \qquad \forall\ \rho = |x|/R(x). \tag{12.13}$$

Example 12.3 Suppose $\upsilon > 0$. Then a solution to (12.13) with $\lambda = -1$ is

$$h(\rho) = \exp[-\frac{\rho}{\upsilon} - \frac{C}{\upsilon^2}\ln(\rho - \frac{C}{\upsilon})] \qquad \forall\ \rho > \frac{C}{\upsilon}.$$

Hence, for any $\alpha > 0$,

$$C_1(t) = \{x \in R^d \backslash \{0\} \ :\ h(|x|/R(x)) > \alpha\, e^{-t}\} \cup \{0\}$$

is a classical solution of

$$\beta(\theta)\,\mathbf{V} = -\operatorname{trace} Cg(\theta)\,\mathbf{R} + \upsilon\ .$$

The solution $\{C_1(t)\}_{t \geq 0}$ is increasing in time with extinction time infinity.

Example12.4 Suppose $\upsilon > 0$. Then a solution to (12.13) with $\lambda = 1$ is

$$h(\rho) = \exp[\frac{\rho}{\upsilon} + \frac{C}{\upsilon^2}\ln(-\rho + \frac{C}{\upsilon})] \qquad \forall\ \rho \in [0, \frac{C}{\upsilon}].$$

Hence, for any $\alpha > 0$,

$$C_2(t) = \{x \in R^d \backslash \{0\} \ :\ h(|x|/R(x)) > \alpha\, e^{t}\} \cup \{0\}$$

is a classical solution of

$$\beta(\theta)\,\mathbf{V} = -\operatorname{trace} Cg(\theta)\,\mathbf{R} + \upsilon\ .$$

In this case the solution is decreasing in time with extinction time

$$T(C_2(\cdot)) = \frac{C}{\upsilon^2}\ln(\frac{C}{\upsilon}) - \ln\alpha\,.$$

Example 12.5 Let $\upsilon = 0$. Then with $\lambda = 1$,

$$h(\rho) = \exp\left(-\frac{\rho^2}{2C}\right)$$

is a solution of (12.13). Hence, for any $\alpha > 0$,

$$C_3(t) = \{x \in R^d \setminus \{0\} \;:\; h(|x|/R(x)) > \alpha e^t\} \cup \{0\}$$

$$= \{x \in R^d \setminus \{0\} \;:\; [|x|/R(x)] < \sqrt{2C[-t - \ln\alpha]}\;\} \cup \{0\}$$

is a classical solution of

$$\beta(\theta)\, \mathbf{V} = -\, \mathrm{trace} C g(\theta)\, \mathbf{R}.$$

In this case the solution is decreasing in time with extinction time

$$T(C_3(\cdot)) = -\ln \alpha\,.$$

Definition 12.6 The open set

$$W(1/\beta) = \{\, x \in R^d \setminus \{0\} : \; |x| < R(x)\,\} \cup \{0\}$$

is called the ***Wulff crystal*** of the surface energy $(1/\beta)$.

All the solutions constructed in the above examples are dialations of the Wulff crystal $W(1/\beta)$. We collect the previous examples into the following lemma.

Lemma 12.7 *For* $i = 1,2,3$,

$$C_i(t) = \alpha_i(t)\, W(1/\beta)\,,$$

where $h(\alpha_i(t)) = \alpha e^{\lambda t}$. *In particular,* $\alpha_i(t)$ *is the unique solution of*

$$\text{(12.14)} \qquad \frac{d}{dt}\alpha_i(t) = \left[-\frac{C}{\alpha_i(t)} + \upsilon\right] \qquad \forall\ t > 0,$$

with initial condition

(12.15) $\alpha_i(0) = h^{-1}(\alpha).$

Remark 12.8 If β satisfies (12.1), then the Wulff crystal $\upsilon W(1/\beta)$ is a solution of the stationary problem,

$$0 = -\ \mathrm{trace}\, g(\theta)\, \mathbf{R} + \upsilon.$$

This fact was proved by Angenet & Gurtin in two dimensions (see Section 6.1 [AG 1989]).

13. LARGE TIME ASYMPTOTICS

In this section we show that any solution of (E), with bounded initial condition, has finite extinction time if $\upsilon \le 0$ or if it is initially small. We also show that if $\upsilon > 0$, then solutions of (E) with large enough but bounded initial condition has infinite extintion time. These results already proved by Angenent & Gurtin [AG 1989] for classical solutions in two dimensions. Also Chen, Giga & Goto [CGG 1989] proved the finite extinction when $\upsilon = 0$. We use the comparison result Theorem 7.1 together with the explicit solutions constructed in the previous section. Our techniques also show that when the solution is growing, asymptotically it has the shape of the Wulff region $W(1/\beta)$.

We employ the notation $A \subset\subset B$ if A is a compact subset of B. When B is bounded, A and B satisfies (7.5),i.e., $A \subset B_\alpha$ for a suitable α.

Lemma 13.1 *Suppose that* $\{C(t)\}_{t\ge 0}$ *is a viscosity subsolution of* (E) *and there is K such that*

$$C^*(0) \subset\subset B_K(0).$$

Further assume that

$$\upsilon \le 0, \textit{ and } \frac{\text{trace}G(\theta)}{\beta(\theta)} \ge g_0 > 0, \qquad \forall\ \theta.$$

Then,

$$T(C(\cdot)) \le \frac{K^2}{2g_0}.$$

Proof: Let

$$\Gamma(t) = \alpha(t)\, W(1)$$

where α is the solution of (12.14) with $C = g_0$ and initial condition $\alpha(0) = K$. Then, Lemma 12.5 yields that $\{\Gamma(t)\}_{t\ge 0}$ is a classical solution of

$$\mathbf{V} = -\, g_0 \kappa .$$

Since the mean curvature of $\partial\Gamma(t)$ is always negative, $\{\Gamma(t)\}_{t\ge 0}$ is a supersolution of (E). Moreover, $\Gamma(0) = B_K(0)$. Hence Theorem 7.1 yields that

$$C^*(t) \subset \Gamma(t) \quad \forall\, t < T(C(\cdot)) \wedge T(\Gamma(\cdot)).$$

We complete the proof after observing that $T(\Gamma(\cdot)) = \dfrac{K^2}{2g_0}$. □

Let $g(\theta)$ be as in the previous section.

Lemma 13.2 *Suppose that* $\{C(t)\}_{t\geq 0}$ *is a viscosity subsolution of* (E) *and there is* K_1 *such that*

$$C^*(0) \subset\subset K_1 W(1/\beta).$$

Further assume that

$$\upsilon > 0,\ G(\theta) \geq g(\theta) g_1 > 0,\ \forall \theta,\ \text{and}\ K_1 < \frac{g_1}{\upsilon}.$$

Then, $T(C(\cdot))$ *is finite.*

Proof: Let

$$\Gamma_1(t) = \alpha_1(t)\, W(1/\beta)$$

where α_1 is the solution of (12.14) with $C = g_1$ and initial condition $\alpha(0) = K_1$. Then, Lemma 12.5 yields that $\{\Gamma_1(t)\}_{t\geq 0}$ is a classical solution of

$$\beta(\theta)\mathbf{V} = -\operatorname{trace} g(\theta) g_1 \mathbf{R} + \upsilon.$$

Since the curvature tensor of $\partial\Gamma_1(t)$ is always negative definite and $G(\theta) \geq g_1 g(\theta)$, $\{\Gamma_1(t)\}_{t\geq 0}$ is a supersolution of (E). Therefore Theorem 7.1 yields

$$C^*(t) \subset \Gamma_1(t) \qquad \forall\, t < T(C(\cdot)) \wedge T(\Gamma_1(\cdot)).$$

Using the assumption $K_1 < g_1/\upsilon$, we obtain

$$\Gamma_1(t) = \{\, x \in R^d \ :\ |x| < R(x)\frac{g_1}{\upsilon} \text{ and } h(|x|/R(x)) > h(K_1)\, e^{-t} \,\}$$

where

$$h(\rho) = \exp[\frac{\rho}{\upsilon} + \frac{g_1}{\upsilon^2}\ln(-\rho + \frac{g_1}{\upsilon})] \qquad \forall\, \rho \in [0, \frac{g_1}{\upsilon}] .$$

Recall that the above function is the one computed in Example 12.4 with $C = g_1$. Hence, $T(\Gamma_1(\cdot))$ is finite and so is $T(C(\cdot))$. □

Lemma 13.3 *Suppose that* $\{C(t)\}_{t\geq 0}$ *is a maximal viscosity supersolution of* (E) *and there is* K_2 *such that*

$$K_2 W(1/\beta) \subset\subset C^*(0),$$

and $C^*(0)$ *is bounded. Further assume that*

$$\upsilon > 0,\; 0 < G(\theta) \leq g(\theta) g_2,\; \forall \theta \text{ ,and } K_2 > \frac{g_2}{\upsilon}.$$

Then, $T(C(\cdot))$ *is infinite.*

Proof: Let

$$\Gamma_2(t) = \alpha_2(t)\, W(1/\beta)$$

where α_2 is the solution of (12.14) with $C = g_2$ and initial condition $\alpha(0) = K_2$. Then, Lemma 12.5 yields that $\{\Gamma_2(t)\}_{t\geq 0}$ is a classical solution of

$$\beta(\theta)\mathbf{V} = -\,\text{trace} g(\theta) g_2\, \mathbf{R} + \upsilon\, .$$

Since the curvature tensor of $\partial\Gamma_2(t)$ is always negative definite, $\{\Gamma_2(t)\}_{t\geq 0}$ is a subsolution of (E). Therefore Theoerem 7.1 yields

$$\Gamma_2(t) \subset C_*(t) \qquad \forall\, t < T(C(\cdot))\wedge T(\Gamma_2(\cdot)).$$

Using the assumption $K_2 > g_2/\upsilon$, we proceed as in Example 12.3 to obtain

$$\Gamma_2(t) = \{\, x \in R^d \;:\; |x| \leq R(x)\frac{g_2}{\upsilon} \,\} \cup$$
$$\{\, x \in R^d \;:\; |x| > R(x)\frac{g_2}{\upsilon} \text{ and } h(|x|/R(x)) > h(K_2)\, e^{-t} \,\}$$

where

$$h(\rho) = \exp[-\frac{\rho}{\upsilon} - \frac{g_2}{\upsilon^2} \ln (\rho - \frac{g_2}{\upsilon})] \qquad \forall \rho > \frac{g_2}{\upsilon} .$$

We complete the proof after observing that $T(\Gamma_2(\cdot))$ is infinite, $\{C(t)\}_{t\geq 0}$ is maximal and $C(t)$ is bounded for each t. ▭

Proposition 13.4 *Suppose that* $\{C(t)\}_{t\geq 0}$ *is a maximal viscosity solution of* (E) *and there are* K_1 *and* K_2 *such that*

$$K_1 W(1/\beta) \subset\subset C^*(0),$$

and

$$C_*(0) \subset\subset K_2 W(1/\beta).$$

Further assume that

$$\upsilon > 0, \;\; 0 < g(\theta) g_1 < G(\theta) \leq g(\theta) g_2, \qquad \forall \theta ,$$

and

$$K_1 > \frac{g_1}{\upsilon}, \quad K_2 > \frac{g_2}{\upsilon}.$$

Then,

$$\alpha_1(t)\, W(1/\beta) \subset C(t) \subset \alpha_2(t)\, W(1/\beta) \;\; \forall t > 0, \tag{13.1}$$

where $\alpha_1(\cdot)$ *and* $\alpha_2(\cdot)$ *are solutions of* (12.14) *with* $C = g_i$ *and initial conditions* $\alpha_i(0) = K_i$ *for* $i = 1,2$.

Proof: Lemma 12.5 implies that $\alpha_i(t) W(1/\beta)$ is a solution of

$$\beta(\theta) \mathrm{V} = - \mathrm{trace}\, g_i g(\theta)\, \mathrm{K} + \upsilon , \;\; \text{for } \; \mathrm{i} = 1,2.$$

The negativity of the curvature of $W(1/\beta)$ and the comparison principle yield the result.

▭

Remark 13.5 Since for $i = 1,2$ $\alpha_i(t)$ tends to infinity, from (12.14) we obtain that

$$\lim_{t \to \infty} \frac{\alpha_i(t)}{t} = \upsilon.$$

Hence, in some sense

$$\lim_{t \to \infty} \frac{1}{t} C(t) = \upsilon W(1/\beta).$$

The above asymptotic result is conjectured by Angenent & Gurtin (see page 354 [AG 1989]).

14. APPENDIX

A. Properties of Sub & Superdifferentials.

In this Section we gather some properties of the set of sub and superdifferentials. The proof of the following Lemma is similar to the proof of Proposition 1.1 [CEL 1984] and Lemma I.4 [CL 1984] (Also see Lemma 2.15 in [J 1989]).

For an open set O, $C^{2,1}(O)$ denotes the set of functions which are twice continuously differentiable in x and once continuously differentiable in t.

Lemma 14.1

(a) $(n,p) \in D^+\Phi(x,t)$ *if and only if there is a continuously differentiable function* Ψ *such that*

(14.1)(a) $\quad \nabla\Psi(x,t) = n,$

(14.1)(b) $\quad \frac{\partial}{\partial t}\Psi(x,t) = p,$

(14.1)(c) $\quad \Phi^*(x,t) - \Psi(x,t) > \Phi^*(z,s) - \Psi(z,s), \qquad \forall\ (z,s) \neq (x,t).$

(b) $(n,A,p) \in D_x^{+2}{}_t^{+1}\Phi(x,t)$ *if and only if there is* $\Psi \in C^{2,1}(R^d \times R)$ *such that*

(14.2)(a) $\quad \nabla\Psi(x,t) = n,$

(14.2)(b) $\quad \frac{\partial^2}{\partial x_i x_j}\Psi(x,t) = A_{ij} \qquad i,j = 1,..,d,$

(14.2)(c) $\quad \frac{\partial}{\partial t}\Psi(x,t) = p,$

(14.2)(d) $\quad \Phi^*(x,t) - \Psi(x,t) > \Phi^*(z,s) - \Psi(z,s), \qquad \forall\ (z,s) \neq (x,t).$

(c) $(n,p) \in D^-\Phi(x,t)$ *if and only if there is a continuously differentiable function* Ψ *such that*

(14.3)(a) $\nabla \Psi(x,t) = n,$

(14.3)(b) $\frac{\partial}{\partial t} \Psi(x,t) = p,$

(14.3)(c) $\Phi_*(x,t) - \Psi(x,t) < \Phi_*(z,s) - \Psi(z,s), \qquad \forall\ (z,s) \neq (x,t).$

(d) $(n,A,p) \in \mathbf{D}_x^{-2} {}_t^{-1} \Phi(x,t)$ *if and only if there is* $\Psi \in C^{2,1}(R^d \mathrm{x} R)$ *such that*

(14.4)(a) $\nabla \Psi(x,t) = n,$

(14.4)(b) $\frac{\partial^2}{\partial x_i x_j} \Psi(x,t) = A_{ij} \qquad i,j = 1,..,d,$

(14.4)(c) $\frac{\partial}{\partial t} \Psi(x,t) = p,$

(14.4)(d) $\Phi_*(x,t) - \Psi(x,t) < \Phi_*(z,s) - \Psi(z,s), \qquad \forall\ (z,s) \neq (x,t).$

Proof:(a)(c) Proved in Proposition 1.1[CEL 1984].

(b) Without loss of generality we may assume that $A = 0$, $n = 0$, $p = 0$, and $(x,t) = 0$. For $\rho > 0$, let

$$h(\rho) = \sup \left\{ \frac{(\Phi^*(z,s)) \vee 0}{\sqrt{|z|^4 + |s|^2}} : \sqrt{|z|^4 + |s|^2} \leq \rho \right\}.$$

Then h is continuous on $[0,\infty)$, with $h(0) = 0$. Let

$$\Psi(z,s) = F(\sqrt{|z|^4 + |s|^2}),$$

where

$$F(r) = \frac{1}{r} \int_r^{2r} \int_\xi^{2\xi} h(\rho)\, d\rho\, d\xi.$$

Straightforward calculations show that Ψ satisfies (14.2).

(d) Similar to part (b). □

Remark 14.2 In the above lemma, Ψ can not be smoother than $C^{2,1}$ in parts (b), (d) and C^1 in parts (a) and (c). However by a simple approximation argument we obtain an equivalent definition of viscosity solutions which uses only smooth functions(see Appendix B). Also the global smoothness of these tests functions is not necessary.

Corollary 14.3

(a) *If* $(n,A,p) \in \mathbf{D}^{+2}_{x}{}^{+1}_{t}\Phi(x,t)$, *then* $(n,p) \in \mathbf{D}^{+}\Phi(x,t)$.

(b) *If* $(n,A,p) \in \mathbf{D}^{-2}_{x}{}^{-1}_{t}\Phi(x,t)$, *then* $(n,p) \in \mathbf{D}^{-}\Phi(x,t)$.

Lemma 14.4 *Suppose* $|n| > 0$, *and* $An = 0$. *Then, the following are equivalent*

(a) $(n,A,p) \in \mathbf{D}^{+}C(t_0)$,

(b) *there are* $x_0 \in \partial C^*(t_0)$ *and a collection of open sets* $\{S(t)\}_{t\geq 0}$ *satisfying (see figure 1 below)*

(14.5)(a) d_S *is* $C^{2,1}(N)$ *on a neighborhood* N *of* (x_0,t_0),

(14.5)(b) $\nabla d_S(x_0,t_0) = \frac{n}{|n|}$,

(14.5)(c) $\frac{\partial^2}{\partial x_i x_j} d_S(x_0,t_0) = \frac{A_{ij}}{|n|}$, $i,j = 1,..,d$,

(14.5)(d) $\frac{\partial}{\partial t} d_S(x_0,t_0) = \frac{p}{|n|}$,

(14.5)(e) $\bigcup_{s\geq 0} [[\, \mathrm{cl}C^*(s) \cap (S(s))^c] \times \{s\}] = \{(x_0,t_0)\}$.

(c) $\frac{1}{|n|}(n,A,p) \in \mathbf{D}^{+2}_{x}{}^{+1}_{t}\, d_{C^*}(x_0,t_0)$ *for some* $x_0 \in \partial C^*(t_0)$.

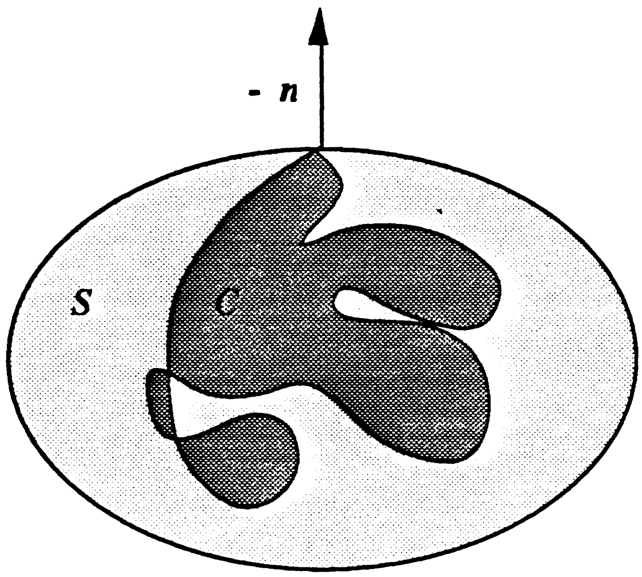

Figure 2

Proof :

(a) $\Rightarrow$ **(b)** The definition of $\mathbf{D}^{+}C(t_0)$ implies that

$$(n,A,p) \in \mathbf{D}_{x}^{+2}{}_{t}^{+1}\,[d_{C^*}\wedge 0](x_0,t_0)$$

at some $x_0 \in R^d$. Since $|n| \neq 0$, $x_0 \notin \mathrm{int}C^*(t_0)$. We analyse the remaining two cases seperately;

(1) $x_0 \in \partial C^*(t_0)$.

Let Ψ be as in Lemma 14.1 (b). Set

$$S(t) = \{\ (z,t) \in R^d \times [0,\infty):\ \ \Psi(z,t) > 0\ \}\ .$$

Then, (14.5) (a)-(d) is satisfied by $\{S(t)\}_{t\geq 0}$ due to the smoothness of Ψ and the positivity of $|n|$. Since $\Psi(x_0,t_0) = 0$,

$$x_0 \in [\mathrm{cl}C^*(t_0) \cap (S(t_0))^c].$$

Suppose that

(c) $\Rightarrow$ (a) The fact that

$$[d_{C^*} \wedge 0] \le d_{C^*} ,$$

implies that $D^{+2\ +1}_{x\ \ t}\, d_{C^*}(x,t)$ is included in $D^{+2\ +1}_{x\ \ t}\, [d_{C^*} \wedge 0](x,t)$ at every $t \ge 0$, and $x \in \partial C^*(t)$. Therefore

$$\frac{1}{|n|}(n,A,p) \in D^{+2\ +1}_{x\ \ t}\, [d_{C^*} \wedge 0](x_0,t_0) .$$

Also, $(0,0,0) \in D^{+2\ +1}_{x\ \ t}\, [d_{C^*} \wedge 0](x_0,t_0)$. Hence the convexity of $D^{+2\ +1}_{x\ \ t}\, [d_{C^*} \wedge 0](x_0,t_0)$ yields (a). □

Observe that

$$\begin{aligned} D^- C(t) &= \bigcup_{x \in R^d} D^{-2\ -1}_{x\ \ t}(- d_{(R^d \setminus C(\cdot))^*} \vee 0)\ (x,t) \\ &= \bigcup_{x \in R^d} - D^{+2\ +1}_{x\ \ t}(d_{(R^d \setminus C(\cdot))^*} \vee 0)\ (x,t) \\ &= D^+ (R^d \setminus C(\cdot))(t) \ . \end{aligned}$$

Hence we have the following analogue of the previous Lemma for the superdifferentials.

Lemma 14.5 *Suppose* $|n| > 0$, *and* $An = 0$. *Then, the following are equivalent*

(a) $(n,A,p) \in D^- C(t_0)$,

(b) *there are* $x_0 \in \partial C^*(t_0)$ *and a collection of open sets* $\{S(t)\}_{t \ge 0}$ *satisfying (see figure 2 below)*

(14.9)(a) d_S *is* $C^{2,1}(N)$ *on a neighborhood* N *of* (x_0,t_0),

(14.9)(b) $\nabla d_S(x_0,t_0) = \frac{n}{|n|}$, (14.9)(c) $$\frac{\partial^2}{\partial x_i x_j} d_S(x_0,t_0) = \frac{A_{ij}}{|n|},\ i,j = 1,..,d,$$

(14.9)(d) $$\frac{\partial}{\partial t} d_S(x_0,t_0) = \frac{p}{|n|},$$

(14.9)(e) $$\bigcup_{s\geq 0} [[(\mathrm{cl}C^*(s))^c \cap S(s)]\mathrm{x}\{s\}] = \{(x_0,t_0)\}.$$

(c) $\frac{1}{|n|}(n,A,p) \in \mathbf{D}_x^{-2}{}_t^{-1}\, d_{C^*}(x_0,t_0)$ *for some* $x_0 \in \partial C^*(t_0)$.

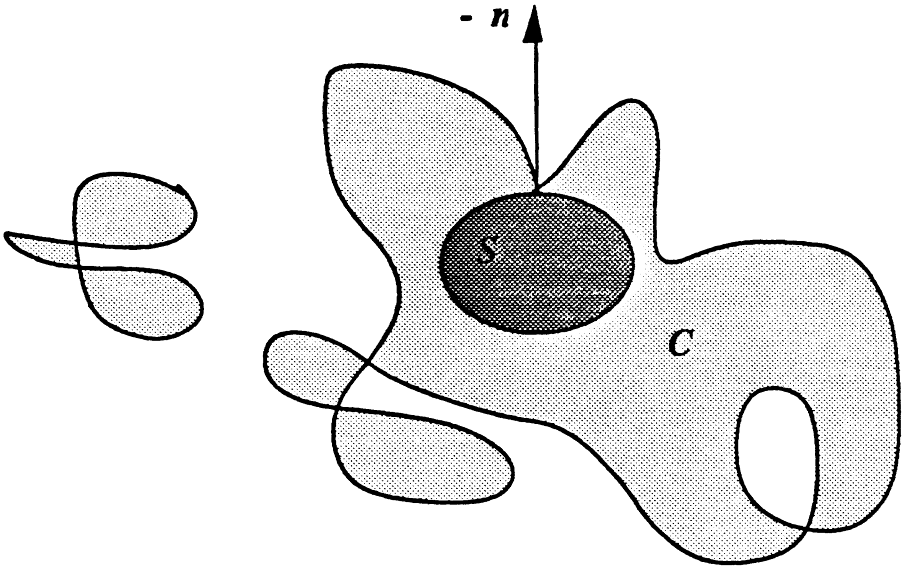

Figure 3

B. An equivalent definition

Following the proof of Theorem 1.1 of [CEL 1984], we obtain an equivalent definition (also see Lemma 2.5 in [J 1989]).

Lemma 14.6

(a) $\{C(t)\}_{t\geq 0}$ *is a viscosity subsolution of* (E) *if and only if for all smooth* Ψ,

$$F_*(\nabla \Psi(x,t), D^2\Psi(x,t), \frac{\partial}{\partial t}\Psi(x,t)) \leq 0,$$

whenever $[d_{C^*}\wedge 0] - \Psi$ *attains its maximum at* (x,t).

(b) $\{C(t)\}_{t\geq 0}$ *is a viscosity supersolution of* (E) *if and only if for all smooth* Ψ,

$$F^*(\nabla \Psi(x,t), D^2\Psi(x,t), \frac{\partial}{\partial t}\Psi(x,t)) \geq 0,$$

whenever $[d_{C^*}\wedge 0] - \Psi$ *attains its minimum at* (x,t).

C. Stability

The following is an analogue of the stability theorem of Barles & Perthame [BP 1988] for the first order Hamilton-Jacobi equations. We will state it only for subsolutions, an analogue result holds for supersolutions. Let $\{C_n(t)\}_{t\geq 0}$ be a sequence of viscosity subsolutions of (E). Set $C_n(t) = \varnothing$ for $t \geq T(C_n(\cdot))$ and then define

$$C(t) = \limsup_{s\to t \quad n\to\infty} C_n(s)$$

$$= \bigcap_{\substack{m=1\,2..\\ \varepsilon>0}} \bigcup_{\substack{n=m\,m+1..\\ |t-s|\leq\varepsilon}} C_n(s)\,.$$

Lemma 14.7 $\{C(t)\}_{t\geq 0}$ *is a viscosity subsolution of* (E).

References

[[AL 1986] U. Abresch and J. Langer, The normalized curve shortenning flow and homothetic solutions, *J. Differential. Geometry*, 23 (1986), 175-196

[AC 1979] S. M. Allen and J. W. Cahn, A macroscopic theory for antiphase boundary motion and its application to antiphase domain coaqrsing, *Act. Metall.*, 27 (1979), 1085-1095

[A 1989a] S. Angenent, Parabolic equations for curves on surfaces (I). Curves with p-integrable curvature, preprint

[A 1989b] S. Angenent, Parabolic equations for curves on surfaces (II). Intersections, blow-up and generalized solutions, preprint

[A 1989c] S. Angenent, On the formation of singularities in the curve shortening flow, preprint

[AG 1989], S. Angenent and M. E. Gurtin, Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface, *Arch. Rat. Mech. An.*, 108 (1989), 323-391

[B 1985] G. Barles, Remark on a flame propagation model, Rapport INRIA, #464, Dec. 1985

[BP 1988] G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, *SIAM Control & Opt.*, 26/5 (1988), 1133-1148

[Br 1978] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Princeton University Press, Princeton, 1978

[CGG 1989] Y.-G. Chen, Y. Giga and S. Gogo, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, preprint (Announcement: *Proc. Japan Acad. Ser. A*, 65(1989), 207-210)

[CEL 1984] M. G. Crandall, L. C. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. A.M.S.*,282 (1984), 487-502

[CL 1983] M. G. Crandall, and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. A.M.S.*,277 (1983),1-42

[D 1944] A. Dinghas, Uber einen geometrischen satzs von Wulff fur die gleichgewichtsform von kristallen, *Zeit. Kris.*, 105 (1944), 304-314

[ES 1989a] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I,preprint

[ES 1989b] L. C. Evans and J. Spruck, Motion of level sets by mean curvature II,preprint

[F 1990] I. Fonseca, The Wulff's Theorem revisited, preprint

[G 1983] M. Gage, An isotropic inequality with applications to curve shortening, *Duke Math. J.*, 50 (1983), 1225-1229

[G 1984] M. Gage, Curve shorthening makes convex curves circular, *Invent. Math.*, 76 (1984), 357-368

[GH 1986] M.Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, *J. Differential Geometry*, 23 (1986), 69-96

[GG 1990] Y. Giga and S. Goto, Motion of hypersurfaces and geometric equations, preprint

[Gr 1987] M. Grayson, The heat equation shrinks embedded plane curves to round points, *J. Differential Geometry*, 26 (1987), 285-314

[Gr 1989a] M. Grayson, Shorthening embedded curves, *Annals of Math.*, 129 (1989), 71-111

[Gr 1989b] M. Grayson, A short note on the evolution of surfaces via mean curvature, *Duke Math. J.*, 58 (1989), 555-558

[Gu 1988a], M. E. Gurtin, Multiphase thermomechanics with interfacial structure 1. Heat conduction and the capillary balance law, *Arch. Rat. Mech. An.*, 104 (1988), 195-221

[Gu 1988b], M. E. Gurtin, Multiphase thermomechanics with interfacial structure. Towards a nonequilibrium thermomechanics of two phase materials, *Arch. Rat. Mech. An.*, 104 (1988), 275-312

[H 1984] G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geometry*, 20 (1984), 237-266

[I 1987] H. Ishii, Perron's method for Hamilton-Jacobi equations, *Duke Math. J.*, 55 (1987), 369-384

[I 1989] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's, *C.P.A.M.*, 42 (1989), 15-45

[J 1988] R. Jensen, The maximum principle for viscosity solutions of second order fully nonlinear partial differential equations, *Arch. Rat. Mech. An.*, 101 (1988), 1-27

[JLS 1988], R. Jensen, P.-L. Lions and P. E. Souganidis, A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations, *Proc. A.M.S.*, 102 (1988), 975-978

[L 1983a] P.-L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations I, *Comm. PDE.*, 8 (1983) 1101-1134

[L 1983b] P.-L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part II: viscosity solutions and uniqueness, *Comm. PDE.*, 8 (1983) 1229-1276

[M 1960] W. W. Mullins, Grain boundary grooving by volume diffusion, *Trans. Metall. Soc. AIME,* 218 (1960), 354-361

[OS 1988] S. Osher and J. A. Sethian, Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations, *J. Computational Physics*, 79 (1988), 12-49

[Se 1973] R. F. Sekerka, *Morphological stability, Crystal growth: an Introduction*, (1973) Noth-Holland, Amsterdam

[S 1985] J. A. Sethian, Curvature and evolution of fronts, *Comm. Math. Physics*, 101 (1985), 487-499

[S 1989] J. A. Sethian, Recent numerical algorithms for hypersurfaces moving with curvature dependent speed: Hamilton-Jacobi equations and conservation laws, *J. Differential Geometry*, 31/1 (1990), 131-162

[T 1974] J. Taylor, Existence and structure of solutions to a class of nonelliptic variational problems, *Symposia Math.*, 14 (1974), 499-508

[T 1975] J. Taylor, Unique structure of solutions to a class of nonelliptic variational problems, *Proc.Symposia Pure Math. A.M.S.*,27 (1975), 419-427

[W 1901] G. Wullf, Frage der Geschwindigkeit des washsthums und der auflosung der krystallflachen, *Zeit. Krystall. Min.*, 34 (1901), 449-530

UNCLASSIFIED MASTER COPY - FOR REPRODUCTION PURPOSES

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE:

3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): NAMS-2

5. MONITORING ORGANIZATION REPORT NUMBER(S):

6a. NAME OF PERFORMING ORGANIZATION: Carnegie Mellon University

6b. OFFICE SYMBOL (If applicable):

7a. NAME OF MONITORING ORGANIZATION: U. S. Army Research Office

6c. ADDRESS (City, State, and ZIP Code): Department of Mathematics, Pittsburgh, PA 15213

7b. ADDRESS (City, State, and ZIP Code): P. O. Box 12211, Research Triangle Park, NC 27709-2211

8a. NAME OF FUNDING / SPONSORING ORGANIZATION: U. S. Army Research Office

8b. OFFICE SYMBOL (If applicable):

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:

8c. ADDRESS (City, State, and ZIP Code): P. O. Box 12211, Research Triangle Park, NC 27709-2211

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification): Motion of a Set by the Curvature of its Boundary

12. PERSONAL AUTHOR(S): Halil Mete Soner

13a. TYPE OF REPORT: Technical

13b. TIME COVERED: FROM ______ TO ______

14. DATE OF REPORT (Year, Month, Day): May 1990

15. PAGE COUNT: 74

16. SUPPLEMENTARY NOTATION: The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): viscosity solutions, mean curvature flow, phase transitions, Wullf crystal

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The study of a crystal shrinking or growing in a melt gives rise to equations relating the normal velocity of the motion to the curvature of the crysta; boundary. Often these equations are anisotropic, indicating the prefered directions of the crystal structure. In the isotropic case this equation is called the mean curvature flow or the curve shortening equation, and has been sutdied by differential geometric tools. In particular, it is known that there are no classical solutions to these equations. In this paper we develop a weak theory for the "generalized mean curvature" equation using the newly developed theory of viscosity solutions. Our approach is closely related to that of Osher & Sethian, Chen, Giga & Goto, and Evans & Spruck, who view the boundary of the crystal as the level set of a solution to a nonlinear parabolic equation. Although we use their results in an essential way, we do not require that the boundary is a level set. Our main results are the existence of a solution, large time asymptotics of this solution, and its connection to the level set solution of Osher & Sethian, Chen, Giga & Goto, and Evans & Spruck. In general there is no uniqueness,

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL:

22b. TELEPHONE (Include Area Code):

22c. OFFICE SYMBOL:

even for classical solutions, but we prove a uniqueness result under restrictive assumptions. We also construct a class of explicit solutions which are dialations of Wullf cyrstals.