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REGULARITY OF MINIMIZERS FOR A CLASS OF MEMBRANE ENERGIES

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Abstract Regularity properties for (local) minimizers of elastic energies have been challenging mathematical techniques for many years. Recently the interest has resurfaced due in part to the fact that existing partial regularity results do not suffice to ensure existence of (classical) solutions to problems involving free discontinuity sets. The analysis of such questions was started with the fundamental work of De Giorgi in the early 80's in connection with the Mumford-Shah model for image segmentation in computer vision, and later applied to some models for fracture mechanics, thin films, and membranes ([1], [18], [20]). In this paper it is shown that local minimizers in $W^{1,2}(\Omega;\mathbb{R}^d)$ of the functional

$$\mathcal{F}_0(u,\Omega):=\int_\Omega\left[\tfrac{1}{2}|Du|^2+f(|\nu(u)|)\right]dx$$

are Hölder continuous of any exponent $\gamma\in(0,1)$, where $\Omega\subset\mathbb{R}^2$ is an open, bounded set, f is a (not necessarily convex) function growing linearly at infinity, and $\nu(u)$ stands for the vector of all 2×2 minors of Du. As a consequence, it is possible to obtain existence of "classical" minimizers in $SBV(\Omega;\mathbb{R}^d)$ of

$$\mathcal{F}(u,\Omega):=\int_\Omega\left[\tfrac{1}{2}|\nabla u|^2+f(|\nu(u)|)\right]dx+\beta\int_\Omega|u-g|^q\,dx+\gamma H^{N-1}(S_u\cap\Omega)$$

where $g\in L^\infty(\Omega;\mathbb{R}^d)$, $q>1$, $\beta,\gamma>0$. These minimizers are "classical minimizers" in the sense that $H^1\left((\overline{S_u}\setminus S_u)\cap\Omega\right)=0$ and $u\in W^{1,2}(\Omega\setminus\overline{S_u})$.

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It is not restrictive to assume that

$$0 < \alpha < 1$$

and in what follows we will work under this assumption. Also, in order to simplify the notation the value of the constant C may change from one line to the next, and B_R, $R > 0$, will denote a generic open ball of radius R, centered at $x \in \Omega$, and such that $B_R \subset \Omega$.

Given $u \in SBV(\Omega; \mathbb{R}^d)$ we define

$$\nu(u) := \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2},$$

the 2-covector whose components are the 2×2 subdeterminants of ∇u.

Consider the energies

$$\mathcal{G}(K, u) := \int_{\Omega \setminus K} \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta \int_{\Omega \setminus K} |u - g|^q \, dx + \gamma H^1(\Omega \cap K),$$

$$\mathcal{F}(u, \Omega) := \int_{\Omega} \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta \int_{\Omega} |u - g|^q \, dx + \gamma H^1(S_u \cap \Omega),$$

and

$$\mathcal{F}_0(u, \Omega) := \int_{\Omega} \left[\frac{1}{2}|Du|^2 + f(|\nu(u)|)\right] dx.$$

Definition 2.1. *We say that $u \in W^{1,2}(\Omega; \mathbb{R}^d)$ is a $W^{1,2}$-*local minimizer *of*

$$I(v, \Omega) := \int_{\Omega} F(\nabla v) \, dx \,, \qquad v \in W^{1,2}(\Omega; \mathbb{R}^d)$$

if

$$I(u, B_R(x_0)) = \min\left\{I(v, B_R(x_0)) \ : \ v \in u + W_0^{1,2}(B_R(x_0); \mathbb{R}^d)\right\}$$

for all balls $B_R(x_0) \subset \Omega$.

The main result of this paper is the following theorem.

Theorem 2.2. *If $u \in W^{1,2}(\Omega, \mathbb{R}^d)$ is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then $u \in C^{0,\gamma}_{\mathrm{loc}}$ for all $\gamma \in (0,1)$.*

In the proof of Theorem 2.2 we will use classical arguments of regularity theory within the framework of the *Morrey spaces* $L^{p,\lambda}$; for a detailed study of these methods we refer the reader to [21], [24].

Definition 2.3. *Given $\lambda \geq 0$ we say that $f \in L^{p,\lambda}(\Omega; \mathbb{R})$ if there exists a constant $C > 0$ such that*

$$\int_{B_\rho(x) \cap \Omega} |f|^p \, dx \leq C \rho^\lambda$$

for all $x \in \Omega$ *and* $0 < \rho < \operatorname{diam}\Omega$. *The function* f *is said to be in* $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ *if* $f \in L^{p,\lambda}(\Omega')$ *for all* $\Omega' \subset\subset \Omega$.

It can be shown that, with $\Omega \subset \mathbb{R}^2$,

$$L^{p,0}(\Omega) = L^p(\Omega),\ L^{p,2}(\Omega) = L^\infty(\Omega),\ L^{p,\lambda}(\Omega) = \{0\} \quad \text{if } \lambda > 2.$$

and that $L^{p,\lambda}(\Omega)$ is a Banach space endowed with the norm

$$||f||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \int_{B_\rho(x)\cap\Omega} |f|^p\, dx \right\}^{\frac{1}{p}}.$$

Morrey proved that (see Theorem 3.5.2, [24])

Lemma 2.4. *If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *and* $Du \in L^{2,\lambda}_{loc}(\Omega)$ *for some* $0 < \lambda < 2$ *then* $u \in C^{0,\lambda/2}_{\mathrm{loc}}(\Omega)$.

In light of Lemma 2.4, we will prove Theorem 2.2 by showing that if u is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then for all $0 \le \lambda < 2$

$$\int_{B_\rho} |Du|^2\, dx \le C\left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Du|^2\, dx + C\rho^\lambda \tag{2.1}$$

for all $0 < \rho < R$ with $B_R \subset\subset \Omega$.
As a corollary we obtain,

Corollary 2.5. *Let* $u \in SBV(\Omega;\mathbb{R}^d)$ *be a minimizer for* $\mathcal{F}$. *Then* $(\overline{S_u}, u)$ *is a minimizer for* $\mathcal{G}$ *among all pairs* (K, v) *with* $K \subset \Omega$ *closed and* $v \in W^{1,2}(\Omega \setminus K;\mathbb{R}^d)$. *Moreover,*

$$H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

Following the argument introduced by De Giorgi, Carriero and Leaci [16], and outlined in [1], the corollary holds provided we can show that $W^{1,2}$–local minimizers of

$$v \in W^{1,2}(B_1;\mathbb{R}^d) \mapsto \int_{B_1} \left[\frac{1}{2}|Dv|^2 + M|\nu(v)|\right] dx$$

satisfy an estimate of the type

$$\int_{B_\rho} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx \le C\rho^\lambda \int_{B_1} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx + C\rho^\lambda,$$

for some $0 < \lambda < 2$ and $0 < \rho \le 1$ or, equivalently,

$$\int_{B_\rho} |Du|^2\, dx \le C\rho^\lambda \int_{B_1} |Du|^2\, dx + C\rho^\lambda.$$

We conclude that the assertion of the corollary holds true provided we prove (2.1).

The following two lemmas may be found in [21] (see Chapter 3, Theorem 3.1, page 87, and Lemma 2.1, respectively).

Lemma 2.6. *Let $\lambda < 2$, let $f \in L^{2,\lambda}(B_R;\mathbb{R}^2)$, and let $v \in W^{1,2}(B_R;\mathbb{R})$ satisfy*

$$\Delta v = \operatorname{div} f \quad \text{in } B_R.$$

Then $Dv \in L^{2,\lambda}_{\mathrm{loc}}(B_R;\mathbb{R}^2)$, and for every $\rho \le R$

$$\int_{B_\rho} |Dv|^2\,dx \le C\left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Dv|^2\,dx + C\rho^\lambda \,||f||^2_{L^{2,\lambda}(B_R)}.$$

Lemma 2.7. *Let $\phi : [0,+\infty) \to [0,+\infty)$ be a nonnegative, nondecreasing function, such that*

$$\phi(\rho) \le H\left[\left(\frac{\rho}{R}\right)^\gamma + \varepsilon\right]\phi(R) + KR^\beta$$

for all $0 < \rho < R \le R_0$ and for some constants $H, K \ge 0$ and $0 < \beta < \gamma$. Then there exist constants $\varepsilon_0 = \varepsilon_0(H,\gamma,\beta), C = C(H,\gamma,\beta)$ such that

$$\phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^\beta \phi(R) + K\rho^\beta\right]$$

for all $0 < \rho < R \le R_0$.

Lemma 2.8. *Let $p > 1$ and $0 \le \lambda < 2$. If $f_{ij} \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ for $i,j \in \{1,2\}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is a distributional solution of*

$$\Delta u = \sum D_{ij} f_{ij}$$

then $u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$.

Proof. Let $B_R \subset\subset \Omega$ and for every i,j let v_{ij} be the solution of (see Theorem 9.15 and Lemma 9.17, [22])

$$\begin{cases} \Delta v_{ij} = f_{ij} \\ v_{ij} \in W^{1,p}_0(B_R) \cap W^{2,p}(B_R), \end{cases}$$

and we set

$$w := \sum D_{ij} v_{ij}.$$

Then $w \in L^p(B_R)$ and $||w||_{L^p(B_R)} \le C \sum ||f_{ij}||_{L^p(B_R)}$. In addition, $\Delta w = \sum D_{ij} f_{ij}$ in $\mathcal{D}'$, so that the function

$$v := u - w$$

is harmonic, i.e. $\Delta v = 0$. Hence

$$\sup_{B_{R/2}} |v| \le C(p) \left(\frac{1}{|B_R|} \int_{B_R} |v|^p \, dx \right)^{1/p},$$

from which we deduce that for every $\rho \le R/2$ (thus, for all $0 < \rho \le R$)

$$\int_{B_\rho} |v|^p \, dx \le C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |v|^p \, dx.$$

We have

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \int_{B_\rho} (|v|^p + |w|^p) \, dx \\ &\le C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |v|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |u|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |u|^p \, dx + CR^\lambda. \end{aligned}$$

By Lemma 2.7 we deduce that for all $0 < \rho \le R$

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |u|^p \, dx + C\rho^\lambda \\ &\le \rho^\lambda \left[\frac{C}{R^\lambda} \int_{B_R} |u|^p \, dx + C \right], \end{aligned}$$

and so $u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$. □

We end this section with a list of algebraic inequalities, following an argument introduced in [8] (see also [17]).

Let $P, Q \in \mathbb{R}^d$ and set

$$A := \frac{|P|^2 - |Q|^2}{2}, \quad B := P \cdot Q, \quad \nu := P \wedge Q.$$

Lemma 2.9. *We have*

i) $2\sqrt{A^2 + B^2} \le |P|^2 + |Q|^2$;
ii) $0 \le |P|^2 + |Q|^2 - 2|\nu| \le 2\sqrt{A^2 + B^2}$;
iii) if $\nu = 0$ *then* $|P|^2 + |Q|^2 = 2\sqrt{A^2 + B^2}$;
iv) if $\alpha, \beta \in \mathbb{R}^d$ *and* $\nu \ne 0$ *then*

$$\left| \frac{1}{|\nu|} \nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \right|^2 \le 4\sqrt{A^2 + B^2} \, (|\alpha|^2 + |\beta|^2).$$

Proof. Since

$$|\nu|^2 = \sum_{i<j} |P_i Q_j - P_j Q_i|^2 = \frac{1}{2} \sum_{i,j} |P_i Q_j - P_j Q_i|^2 = |P|^2 |Q|^2 - (P \cdot Q)^2,$$

we have

$$|P|^2|Q|^2 = B^2 + |\nu|^2,$$

and so

$$4A^2 = (|P|^2 + |Q|^2)^2 - 4|P|^2|Q|^2 = (|P|^2 + |Q|^2)^2 - 4(B^2 + |\nu|^2),$$

and

$$4(A^2 + B^2) = (|P|^2 + |Q|^2)^2 - 4|\nu|^2.$$

Clearly i) and iii) follow. In addition, we have that

$$(|P|^2 + |Q|^2)^2 - 4|\nu|^2 \geq 0$$

hence

$$0 \leq |P|^2 + |Q|^2 - 2|\nu| \leq \sqrt{(|P|^2 + |Q|^2)^2 - 4|\nu|^2} = 2\sqrt{A^2 + B^2},$$

which yields assertion ii).

Now remark that if $\nu \neq 0$ then $P \neq 0$ and, setting

$$Q' := Q - \frac{P \cdot Q}{|P|^2} P,$$

then also $Q' \neq 0$. Define the orthonormal vectors

$$P_1 := \frac{P}{|P|}, \quad Q_1 := \frac{Q'}{|Q'|}.$$

We write

$$P = p\, P_1, \quad Q = s\, P_1 + q\, Q_1$$

with

$$p := |P|, \quad q := |Q'|, \quad s := \frac{P.Q}{|P|}.$$

Note that

$$\nu = pq\, P_1 \wedge Q_1, \quad |\nu| = pq,$$

and that if $v \in \mathbb{R}^d$ then

$$(P_1 \wedge Q_1) \cdot (P_1 \wedge v) = v \cdot Q_1, \quad (P_1 \wedge Q_1) \cdot (v \wedge Q_1) = v \cdot P_1.$$

We have

$$\begin{aligned}
&\frac{1}{|\nu|} \nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \\
&= (P_1 \wedge Q_1) \cdot (p\, P_1 \wedge \beta - s\, P_1 \wedge \alpha + q\, \alpha \wedge Q_1) - (p\, P_1 \cdot \alpha + s\, P_1 \cdot \beta + q\, Q_1 \cdot \beta) \\
&= [(q - p)\, P_1 - s\, Q_1] \cdot \alpha + [-s\, P_1 + (p - q)\, Q_1] \cdot \beta \\
&= v_1 \cdot \alpha + v_2 \cdot \beta,
\end{aligned}$$

with

$$v_1 := (q - p)\, P_1 - s\, Q_1 \qquad \text{and} \quad v_2 := -s\, P_1 + (p - q)\, Q_1.$$

We have

$$|v_1 \cdot \alpha + v_2 \cdot \beta|^2 \leq (|v_1|^2 + |v_2|^2)(|\alpha|^2 + |\beta|^2) = 2(|P|^2 + |Q|^2 - 2|\nu|)(|\alpha|^2 + |\beta|^2),$$

which, together with ii), concludes the proof of iv). □

3. Proof of the Regularity Theorem

In this section we assume that $u \in W^{1,2}(\Omega;\mathbb{R}^d)$ is a local minimizer of $\mathcal{F}_0$.

Proposition 3.1. *If $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ for some $0 \le \lambda < 2$ then $Du \in L^{2,q_0(\lambda)}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, where $q_0(\lambda) := \alpha + \lambda(1-\alpha/2)$.*

Before proceeding with the proof of this result, we remark that using an iterative scheme where

$$\lambda_0 := 0, \quad \lambda_{k+1} := q_0(\lambda_k),$$

then

$$\lim_{k\to+\infty} \lambda_k = \lim_{k\to+\infty} \alpha \sum_{i=0}^{k} \left(1 - \frac{\alpha}{2}\right)^i = 2,$$

hence (2.1) will follow for all $0 \le \lambda < 2$ and, as justified in Section 2, this suffices to assert Theorem 2.2.

The proof of Proposition 3.1 uses higher integrability properties of the functions

$$A := \frac{|D_1 u|^2 - |D_2 u|^2}{2}, \quad B := (D_1 u)\cdot(D_2 u),$$

where $D_1 u$ and $D_2 u$ stand for the column vectors in $\mathbb{R}^d$ of the derivatives of u with respect to x_1 and to x_2, respectively.

Proposition 3.2. *The functions A and B solve the system*

$$\begin{cases} \Delta A = D^2_{11} g - D^2_{22} g \\ \Delta B = 2 D^2_{12} g, \end{cases}$$

where

$$g := f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|).$$

In addition, if $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega;\mathbb{R}^{2d})$ for some $0 \le \lambda < 2$ then $\sqrt{|A| + |B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\mathrm{loc}}(\Omega;\mathbb{R})$.

Proof. Consider $\Phi := (\varphi, \psi) \in C^1_0(\Omega;\mathbb{R}^2)$, and let $\varepsilon > 0$ be small enough so that with $\Phi_\varepsilon(x) := x + \varepsilon\Phi(x)$, then $\Phi_\varepsilon : \Omega \to \Omega$ is a smooth diffeomorphism satisfying

$$\det D\Phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div}\Phi(x) + \omega_1(x,\varepsilon),$$
$$\det D\Phi_\varepsilon^{-1}(y) = 1 - \varepsilon \operatorname{div}\Phi(\Phi_\varepsilon^{-1}(y)) + \omega_2(y,\varepsilon),$$

where $\omega_i(\cdot,\varepsilon)/\varepsilon \to 0$, as $\varepsilon \to 0$, uniformly in Ω. Set

$$u_\varepsilon(y) := u\left(\Phi_\varepsilon^{-1}(y)\right), \quad y \in \Omega.$$

We have

$$\begin{aligned}\int_\Omega |Du_\varepsilon(y)|^2 dy &= \int_\Omega |Du(\mathbb{I} - \varepsilon D\Phi)|^2(\Phi_\varepsilon^{-1}(y))\, dy + o(\varepsilon)\\ &= \int_\Omega |Du(\mathbb{I} - \varepsilon D\Phi)|^2(1 + \varepsilon \operatorname{div}\Phi)\, dx + o(\varepsilon)\\ &= \int_\Omega |Du|^2\, dx + \varepsilon \int_\Omega \left[|Du|^2 \operatorname{div}\Phi - 2DuD\Phi \cdot Du\right] dx + o(\varepsilon),\end{aligned}$$

where the inner product of two $d \times 2$ matrices ξ and η is defined as $\xi \cdot \eta := \operatorname{trace}(\xi^T \eta)$.

On the other hand, since

$$\nu(u_\varepsilon(y)) = [\det D\Phi_\varepsilon^{-1}(y)]\, \nu(u)(\Phi_\varepsilon^{-1}(y)),$$

we also have that, setting $\Omega_\varepsilon := \{x \in \Omega : |\varepsilon \operatorname{div}\Phi - \omega_2||\nu(u)| \neq 0\}$,

$$\begin{aligned}&\int_\Omega f(|\nu(u_\varepsilon(y))|)\, dy = \int_\Omega f\left((1 - \varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|\right) \det D\Phi_\varepsilon\, dx\\ &= \int_{\Omega_\varepsilon} [f(|\nu(u)|) + (-\varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|)] \det D\Phi_\varepsilon\, dx\\ &\quad + \int_{\Omega_\varepsilon} \left[\frac{f((1 - \varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|) - f(|\nu(u)|)}{(-\varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|} - f'(|\nu(u)|)\right]\\ &\qquad\qquad (-\varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|) \det D\Phi_\varepsilon\, dx\\ &\quad + \int_{\Omega\setminus\Omega_\varepsilon} f(|\nu(u)|) \det D\Phi_\varepsilon\, dx\\ &= \int_\Omega f(|\nu(u)|) \det D\Phi_\varepsilon\, dx\\ &\quad + \int_\Omega (-\varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|) \det D\Phi_\varepsilon\, dx + o(\varepsilon),\\ &= \int_\Omega f(|\nu(u)|)\, dx + \varepsilon \int_\Omega [f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|)] \operatorname{div}\Phi\, dx + o(\varepsilon),\end{aligned}$$

because by Lebesgue's dominated convergence, by (H1), and due to the boundedness of f',

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} &\left|\frac{f((1 - \varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|) - f(|\nu(u)|)}{(-\varepsilon \operatorname{div}\Phi + \omega_2)|\nu(u)|} - f'(|\nu(u)|)\right|\\ &\qquad |\nu(u)| \left|\operatorname{div}\Phi - \frac{\omega_2}{\varepsilon}\right| |1 + \varepsilon \operatorname{div}\Phi + \omega_1|\, dx = 0.\end{aligned}$$

By the local minimality of u we have $\mathcal{F}_0(u_\varepsilon) - \mathcal{F}_0(u) \geq 0$, from which the Euler-Lagrange equation can be easily obtained,

$$\int_\Omega \left[\frac{1}{2}|Du|^2 \operatorname{div}\Phi - DuD\Phi\cdot Du\right] dx = \int_\Omega [\,|\nu(u)|f'(|\nu(u)|) - f(|\nu(u)|)\,] \operatorname{div}\Phi\, dx$$

for every $\Phi = (\varphi, \psi) \in C_0^1(\Omega; \mathbb{R}^2)$. This equation may be rewritten as

$$\int_{\Omega} [A(D_2\psi - D_1\varphi) - B(D_1\psi + D_2\varphi)]\, dx = \int_{\Omega} -g(D_1\varphi + D_2\psi)\, dx,$$

that is,

$$\begin{cases} D_1 A + D_2 B = D_1 g \\ D_2 A - D_1 B = -D_2 g, \end{cases}$$

and the first assertion follows. By (H3)

$$|g| \leq C(1 + |\nu(u)|^{1-\alpha})$$

and so, assuming that $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2d})$ we have that $|\nu(u)| \in L^{1,\lambda}_{\mathrm{loc}}(\Omega; \mathbb{R})$ and

$$g \in L^{\frac{1}{1-\alpha},\lambda}_{\mathrm{loc}}(\Omega).$$

We may now use Lemma 2.8 to obtain that

$$A, B \in L^{\frac{1}{1-\alpha},\lambda}_{\mathrm{loc}}(\Omega),$$

and by Hölder inequality we conclude that

$$\sqrt{|A| + |B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\mathrm{loc}}(\Omega).$$

□

Finally, in order to prove Proposition 3.1 we introduce the following notation:

$$q(\lambda) := 2\alpha + \lambda(1-\alpha),$$

$$\Omega_0 := \{x \in \Omega : |\nu(u)| = 0\},$$

$$\Omega_0' := \{x \in \Omega : |\nu(u)| > 0\},$$

$$\Omega_K := \{x \in \Omega : 0 < |\nu(u)| \leq K\},$$

$$\Omega_K' := \{x \in \Omega : |\nu(u)| > K\}.$$

Proof of Proposition 3.1. Fix $\phi \in W^{1,2}_0(\Omega; \mathbb{R}^d)$ and assume that $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2d})$ for some $0 \leq \lambda < 2$. For $\varepsilon \in \mathbb{R}$ set $u_\varepsilon(x) := u(x) + \varepsilon\phi(x)$. Define

$$P := D_1 u, \quad Q := D_2 u, \quad \alpha := D_1\phi, \quad \beta = D_2\phi, \quad \nu := \nu(u).$$

Since

$$\nu(u_\varepsilon) = \nu(u) + \varepsilon P \wedge \beta + \varepsilon\alpha \wedge Q + \varepsilon^2 \alpha \wedge \beta,$$

we have

$$\int_\Omega f(|\nu(u_\varepsilon)|)\,dx - \int_\Omega f(|\nu|)\,dx = \varepsilon \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx$$
$$+ |\varepsilon| \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx + o(\varepsilon).$$

Local minimality of u entails

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{F}_0(u_\varepsilon,\Omega)-\mathcal{F}_0(u,\Omega)}{\varepsilon} \le 0,$$

and so

$$\int_\Omega Du\cdot D\phi\,dx + \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx \le \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx.$$

We have

$$(M+1)\int_\Omega Du\cdot D\phi\,dx + M\int_{\Omega_0'}\left[\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)-(P\cdot\alpha+Q\cdot\beta)\right]dx$$
$$+\int_{\Omega_0'\cap\Omega_K}(f'(|\nu|)-M)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx$$
$$\le C\int_{\Omega_0}|Du||D\phi|\,dx + \omega_K\int_{\Omega_K'}|Du||D\phi|\,dx,$$

where

$$\omega_K := \sup_{t\ge K}|M-f'(t)|.$$

We recall that by (H2)

$$\omega_K \to 0 \qquad \text{as } K\to+\infty.$$

By Lemma 2.9 iii), iv), we deduce that

$$(M+1)\int_\Omega Du\cdot D\phi\,dx + \int_\Omega G\cdot D\phi\,dx$$
$$\le C\int_\Omega \sqrt{|A|+|B|}|D\phi|\,dx + \omega_K\int_\Omega |Du||D\phi|\,dx \tag{3.1}$$

with $G=(G_1,G_2)$ and

$$G_1 := \chi_{\Omega_0'\cap\Omega_K}(M-f'(|\nu|))\frac{\nu}{|\nu|}\wedge Q$$
$$G_2 := \chi_{\Omega_0'\cap\Omega_K}(f'(|\nu|)-M)\frac{\nu}{|\nu|}\wedge P,$$

and where χ_A stands for the characteristic function of the set A. By Lemma 2.9 ii), iii), and recalling that on Ω_K we have $|\nu|\le K$, we have

$$|G| \le C(K)(1+\sqrt{|A|+|B|}), \qquad \text{a.e. in } \Omega,$$

and by Proposition 3.2 we deduce that $G \in L^{2,q(\lambda)}(\Omega;\mathbb{R}^d)$. Next, for a fixed ball $B_R \subset\subset \Omega$ we compare u with the solution of the Dirichlet problem

$$\begin{cases} (M+1)\Delta v = \operatorname{div} G & \text{in } B_R \\ v - u \in W_0^{1,2}(B_R;\mathbb{R}). \end{cases} \tag{3.2}$$

By Lemma 2.6 $Dv \in L^{2,q(\lambda)}_{\rm loc}(B_R;\mathbb{R}^2)$ and for all $0 < \rho \le R$

$$\int_{B_\rho} |Dv|^2\,dx \le C\left(\frac{\rho}{R}\right)^{q(\lambda)} \int_{B_R} |Dv|^2\,dx + C(K)\rho^{q(\lambda)}. \tag{3.3}$$

From (3.1) and (3.2) we have for all $\phi \in W_0^{1,2}(B_R;\mathbb{R}^d)$

$$(M+1)\int_{B_R} (Du-Dv)\cdot D\phi dx \le C\int_{\Omega\cap B_R} \sqrt{|A|+|B|}|D\phi|dx + \omega_K \int_{B_R} |Du||D\phi|dx.$$

Therefore, taking $\phi := u - v$, and using the fact that by the definition of G and by (3.2)

$$|G| \le C|Du|, \qquad \int_{B_R} |Dv|^2 \le C\int_{B_R} |Du|^2,$$

we have

$$\int_{B_R} |Du - Dv|^2\,dx \le C\int_{B_R} (|A| + |B|)\,dx + C\omega_K \int_{B_R} |Du|^2\,dx.$$

Using (3.3) we now obtain

$$\int_{B_\rho} |Du|^2\,dx \le C\left[\left(\frac{\rho}{R}\right)^{q(\lambda)} + \omega_K\right] \int_{B_R} |Du|^2\,dx + C(K)R^{q(\lambda)},$$

and if K is large enough, so that ω_K is small, from Lemma 2.7 we conclude that for all $0 < \lambda' < q(\lambda)$

$$\int_{B_\rho} |Du|^2\,dx \le C\left(\frac{\rho}{R}\right)^{\lambda'} \int_{B_R} |Du|^2\,dx + C\rho^{\lambda'}, \tag{3.4}$$

and thus (3.4) holds true for $\lambda' = q_0(\lambda)$. □

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$$
(3.30)\qquad \begin{aligned}
\frac{\partial}{\partial t}(Dp_1) &= \frac{\tau_1}{\epsilon\sqrt{\tau_1^2+\tau_2^2}} H\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)\left(\frac{\tau_1 D\tau_1+\tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}}-\tau_y'(d)\,Dd\right)\\
&+ \frac{\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)_+}{\epsilon\left(\tau_1^2+\tau_2^2\right)^{3/2}}\left(\tau_2^2 D\tau_1-\tau_1\tau_2 D\tau_2\right),
\end{aligned}
$$

$$
(3.31)\qquad \begin{aligned}
\frac{\partial}{\partial t}(Dp_2) &= \frac{\tau_2}{\epsilon\sqrt{\tau_1^2+\tau_2^2}} H\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)\left(\frac{\tau_1 D\tau_1+\tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}}-\tau_y'(d)\,Dd\right)\\
&+ \frac{\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)_+}{\epsilon\left(\tau_1^2+\tau_2^2\right)^{3/2}}\left(\tau_1^2 D\tau_2-\tau_1\tau_2 D\tau_1\right),
\end{aligned}
$$

and

$$
(3.32)\qquad \frac{\partial}{\partial t}(Dd) = \frac{H\left(\sqrt{\tau_1^2+\tau_2^2}-\tau_y(d)\right)}{\epsilon}\left(\frac{\tau_1 D\tau_1+\tau_2 D\tau_2}{\sqrt{(\tau_1^2)+(\tau_2)^2}}-\tau_y'(d)\,Dd\right).
$$

In equations (3.30) - (3.32)

$$
(3.33)\qquad H(x) = \begin{cases} 0, & x<0 \\ 1, & x\ge 0 \end{cases}.
$$

If we now multiply (3.27) by $D\tau_1$, (3.28) by $D\tau_2$, (3.29) by Du, (3.32) by $\frac{\mu}{\mu+\beta}\tau_y'(d)\,Dd$ and add the resulting expressions we find that (3.1) and (3.2) hold where now

(3.34)
$$
f=\frac{1}{2}\left(\frac{\mu\left((D\tau_1)^2+(D\tau_2)^2\right)}{\mu+\beta}+\frac{\beta\left((D(\tau_1+p_1))^2+(D(\tau_2+p_2))^2\right)}{\mu+\beta}+(Du)^2+\frac{\mu}{\mu+\beta}\tau_y'(d)\,(Dd)^2\right),
$$

$$
(3.35)\qquad q_1 = Du\left(\frac{\mu}{\mu+\beta}D\tau_1+\frac{\beta}{\mu+\beta}D(\tau_1+p_1)\right),
$$

$$
(3.36)\qquad q_2 = Du\left(\frac{\mu}{\mu+\beta}D\tau_2+\frac{\beta}{\mu+\beta}D(\tau_2+p_2)\right),
$$

and

$$g = \frac{\mu \tau_y''(d)}{2(\mu+\beta)} d_t (Dd)^2$$

$$(3.37) \qquad - \frac{\mu}{\epsilon(\mu+\beta)} H\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)\left(\frac{\tau_1 D\tau_1 + \tau_2 D\tau_2}{\sqrt{\tau_1^2+\tau_2^2}} - \tau_y'(d)\, Dd\right)^2$$

$$- \frac{\mu}{\epsilon(\mu+\beta)} \frac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y(d)\right)_+}{(\tau_1^2+\tau_2^2)^{3/2}} (\tau_2 D\tau_1 - \tau_1 D\tau_2)^2 .$$

The hypothesis that $\tau_y'' \leq 0$ and $d_t \geq 0$ then guarantee that $g \leq 0$. Moreover, the boundary density satisfies

$$(3.38) \qquad q_1 \cos\theta + q_2 \sin\theta - f \leq -\frac{\mu\left(|Du| - \sqrt{(D\tau_1)^2 + (D\tau_2)^2}\right)^2}{2(\mu+\beta)}$$

$$-\frac{\beta\left(|Du| - \sqrt{(D(\tau_1+p_1))^2 + (D(\tau_2+p_2))^2}\right)^2}{2(\mu+\beta)} - \frac{\tau_y'(d)(Dd)^2}{2} \leq 0$$

and the latter two inequalities along with (3.2) yield the desired derivative bounds.

We conclude this section with an examination of the behavior of our system as the small parameter ϵ approaches zero from above. In what follows we let

$$(3.39) \qquad \Omega(x_0, y_0, R, t-s) = \left\{(x,\ y)\,|\sqrt{(x-x_0)^2 + (y-y_0)^2} \leq R + t - s\right\}$$

when $0 \leq s \leq t$ and

$$(3.40) \qquad B(x_0,\ y_0,\ R,\ t) = \left\{(x,y,s)\,|\sqrt{(x-x_0)^2 + (y-y_0)^2} \leq R + t - s, 0 \leq s \leq t\right\}$$

The a-priori estimates associated with (3.2) when f is given by (3.3) or (3.34) and D is one of the operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial y}$ guarantee that if the initial values for a family $(\tau_1^\epsilon,\ \tau_2^\epsilon,\ p_1^\epsilon,\ p_2^\epsilon,\ d^\epsilon, u^\epsilon)$ of distributional solutions are in L_2^{loc} independently of ϵ, then we may choose a sequence ϵ_i, $i = 1,\ 2, \ldots$, with the ϵ_i's decreasing to zero and limit functions $(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0)$ with the following properties:

(i) For any (x_0, y_0), $R > 0$, and $t > 0$ the sequence $(\tau_1^{\epsilon_i},\ \tau_2^{\epsilon_i},\ p_1^{\epsilon_i},\ p_2^{\epsilon_i},\ d^{\epsilon_i},\ u^{\epsilon_i})$ converge strongly in $L_2(B(x_0, y_0, R, t))$ to $(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0)$. Moreover, the limit functions have weak t, x, and y derivatives and the sequences $D(\tau_1^{\epsilon_i}\tau_2^{\epsilon_i}, p_1^{\epsilon_i},\ p_2^{\epsilon_i},\ d^{\epsilon_i},\ u^{\epsilon_i})$ converge weakly in $L_2(B(x_0, y_0, R, t))$ to $D(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0,\ d^0,\ u^0)$ where again $D = \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$.

(ii) The hypotheses (2.28) and (2.35) on the yield stress further guarantee that $\tau_y(d^{\epsilon_i})$ converges strongly in $L_2(B(x_0,\ y_0, R,\ t))$ to $\tau_y(d^0)$ and that $D\tau_y(d^{\epsilon_i}) = \tau_y'(d^{\epsilon_i})\, Dd^{\epsilon_i}$ converges weakly in $L_1(B(x_0,\ y_0, R,\ t))$

to $D\tau_y(d^0) = \tau_y'(d^0) Dd^0$ where once again $D = \frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$. Moreover, equations (2.30) -(2.41) and the convergence results (i) above imply that $\tau_m^{\epsilon_i}\frac{\partial p_n^{\epsilon_i}}{\partial t}$, m and $n = 1$ and 2, and $\sqrt{(\tau_1^{\epsilon_i})^2 + (\tau_2^{\epsilon_i})^2}\frac{\partial d^{\epsilon_i}}{\partial t}$ converge weakly in $L_1(B(x_0, y_0, R, t))$ to $\tau_m^0\frac{\partial p_n^0}{\partial t}$, m and $n = 1$ and 2, and $\sqrt{(\tau_1^0)^2 + (\tau_2^0)^2}\frac{\partial d^0}{\partial t}$ respectively.

Finally, the identities

$$\tau_1^{\epsilon_i}\frac{\partial p_1^{\epsilon_i}}{\partial t} + \tau_2^{\epsilon_i}\frac{\partial p_2^{\epsilon_i}}{\partial t} = \sqrt{(\tau_1^{\epsilon_i})^2 + (\tau_2^{\epsilon_i})^2}\frac{\partial d^{\epsilon_i}}{\partial t} \tag{3.41}$$

$$\tau_2^{\epsilon_i}\frac{\partial p_1^{\epsilon_i}}{\partial t} - \tau_1^{\epsilon_i}\frac{\partial p_2^{\epsilon_i}}{\partial t} = 0 \tag{3.42}$$

guarantee that the limit functions satisfy

$$\tau_1^0\frac{\partial p_1^0}{\partial t} + \tau_2^0\frac{\partial p_1^0}{\partial t} = \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2}\frac{\partial d^0}{\partial t} \tag{3.43}$$

and

$$\tau_2^0\frac{\partial p_1^0}{\partial t} - \tau_1^0\frac{\partial p_2^0}{\partial t} = 0. \tag{3.44}$$

(iii) For any $\delta > 0$, the measure

$$m\left(\left\{(x, y, s) \in B(x_0, y_0, R, t) \,|\, \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2} - \tau_y\left(d^0\right) \geq \delta\right\}\right) = 0. \tag{3.45}$$

The last identity implies that

$$\begin{aligned} & m\left(\left\{(x, y, s) \in B(x_0, y_0, R, t) \,|\, \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2} - \tau_y(d^0) \geq 0\right\}\right) \\ = \; & m\left(\left\{(x, y, s) \in B(x_0, y_0, R, t) \,|\, \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2} - \tau_y(d^0) = 0\right\}\right). \end{aligned} \tag{3.46}$$

In what follows we shall refer to

$$\left\{(x, y, s) \in B(x_0, y_0, R, t) \,|\, \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2} - \tau_y\left(d^0\right) = 0\right\}$$

as the yield set. On open subsets $\mathcal{U}$ of the yield set we introduce Θ by

$$\tau_1^0 = \tau_y\left(d^0\right)\cos\Theta \text{ and } \tau_2^0 = \tau_y\left(d^0\right)\sin\Theta. \tag{3.47}$$

The relationships (3.41) and (3.42) then imply that

$$\cos\Theta\frac{\partial p_1^0}{\partial t} + \sin\Theta\frac{\partial p_2^0}{\partial t} = \frac{\partial d^0}{\partial t} \tag{3.48}$$

and

$$\sin\Theta\frac{\partial p_1^0}{\partial t} - \cos\Theta\frac{\partial p_2^0}{\partial t} = 0. \tag{3.49}$$

Since the weak limits $(\tau_1^0,\ \tau_2^0,\ p_1^0,\ p_2^0, u^0)$ also satisfy the conservation laws (2.36) - (2.38) we find that in $\mathcal{U}$ the following equations are satisfied in the distributional sense

$$\cos\Theta\tau_y'\left(d^0\right)\frac{\partial d^0}{\partial t}-\tau_y\left(d^0\right)\sin\Theta\frac{\partial\Theta}{\partial t}-\frac{\partial u^0}{\partial x}=-\frac{\partial p_1^0}{\partial t}\tag{3.50}$$

$$\sin\Theta\tau_y'\left(d^0\right)\frac{\partial d^0}{\partial t}+\tau_y\left(d^0\right)\cos\Theta\frac{\partial\Theta}{\partial t}-\frac{\partial u^0}{\partial y}=-\frac{\partial p_2^0}{\partial t}\tag{3.51}$$

and

$$\frac{\partial u^0}{\partial t}-\frac{\partial}{\partial x}\left(\tau_y\left(d^0\right)\cos\Theta+\frac{\beta}{\mu+\beta}p_1^0\right)-\frac{\partial}{\partial y}\left(\tau_y\left(d^0\right)\sin\Theta+\frac{\beta}{\mu+\beta}p_2^0\right)=0.\tag{3.52}$$

These equations represent a closed system for $(p_1^0,\ p_2^0,\ d^0,\ u^0,\ \Theta)$ on the yield surface. They imply that (3.52) holds and that

$$\left(1+\tau_y'\left(d^0\right)\right)\frac{\partial d^0}{\partial t}=\cos\Theta\frac{\partial u^0}{\partial x}+\sin\Theta\frac{\partial u^0}{\partial y},\tag{3.53}$$

$$\frac{\partial\Theta}{\partial t}=\frac{1}{\tau_y\left(d^0\right)}\left(-\sin\Theta\frac{\partial u^0}{\partial x}+\cos\Theta\frac{\partial u^0}{\partial y}\right),\tag{3.54}$$

$$\frac{\partial p_1^0}{\partial t}=\frac{\cos\Theta}{1+\tau_y'\left(d^0\right)}\left(\cos\Theta\frac{\partial u^0}{\partial x}+\sin\Theta\frac{\partial u^0}{\partial y}\right),\tag{3.55}$$

and

$$\frac{\partial p_2^0}{\partial t}=\frac{\sin\Theta}{1+\tau_y'\left(d^0\right)}\left(\cos\Theta\frac{\partial u^0}{\partial x}+\sin\Theta\frac{\partial u^0}{\partial y}\right).\tag{3.56}$$

Not surprisingly, we find that in open sets $\mathcal{E}$ of $\left\{(x,\ y,s)\ \in\ B\left(x_0,y_0,R,t\right)|0<\sqrt{(\tau_1^0)^2+(\tau_2^0)^2}<\tau_y\left(d^0\right)\right\}$ the weak limits satisfy the elasticity equations

$$\frac{\partial\tau_1^0}{\partial t}-\frac{\partial u^0}{\partial x}=0,\tag{3.57}$$

$$\frac{\partial\tau_2^0}{\partial t}-\frac{\partial u^0}{\partial y}=0.\tag{3.58}$$

$$\frac{\partial u^0}{\partial t}-\frac{\partial\tau_1^0}{\partial x}-\frac{\partial\tau_2^0}{\partial y}=\frac{\beta}{\mu+\beta}\left(\frac{\partial p_1^0}{\partial x}+\frac{\partial p_2^0}{\partial y}\right)\tag{3.59}$$

and

$$\frac{\partial p_1^0}{\partial t} = \frac{\partial p_2^0}{\partial t} = \frac{\partial d^0}{\partial t} = 0. \tag{3.60}$$

The assertions in part (i) which pertain to $(\tau_1^{\epsilon_i}, \tau_2^{\epsilon_i}, p_1^{\epsilon_i}, p_2^{\epsilon_i}, u^{\epsilon_i})$ follow from (3.2), (3.3), and (3.34). Equations (2.28), (2.35), (3.2), and $\frac{\partial d^{\epsilon_i}}{\partial t} \geq 0$ imply that the d^{ϵ_i}'s are bounded in $L_1(B(x_0, y_0, R, t))$. Their L_2 boundedness follows from the inequality

$$\left(d^{\epsilon_i}(x\ y, s)\right)^2 \leq 2\left(d(x, y, 0)\right)^2 + 2s\int_0^s \left(\tfrac{\partial d^{\epsilon_i}}{\partial \eta}\right)^2 (x, y, \eta)\, d\eta \tag{3.61}$$

which in turn implies that

$$\int_0^t \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t-s\}} \left(d^{\epsilon_i}(x, y, s)\right)^2 dxdy \right) ds$$

$$\leq 2t \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t\}} d^2(x, y, 0)\, dxdy$$

$$+2\int_0^t s \left(\int_0^s \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t-\eta\}} \left(\tfrac{\partial d^{\epsilon_i}}{\partial \eta}\right)^2 (x, y, \eta)\, dxdy \right) d\eta \right) ds \tag{3.62}$$

$$\leq 2t \iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t\}} d^2(x, y, 0)\, dxdy$$

$$+2t^2 \int_0^t \left(\iint_{\{\sqrt{(x-x_0)^2+(y-y_0)^2} \leq R+t-\eta\}} \left(\tfrac{\partial d^{\epsilon_i}}{\partial \eta}\right)^2 (x, y, \eta)\, dxdy \right) d\eta$$

As noted previously, the assertions of (ii) follow directly from those of (i) and the governing equations (2.39) - (2.41).

The veracity of (iii) follows from the inequality

(3.63)

$$
\begin{aligned}
0 \;\le\;& \delta m\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^o) \ge \delta\} \\
\le\;& \iiint_{\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0) \ge \delta\}} \left(\sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0)\right) dxdyds \\
=\;& \iiint_{\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0) \ge \delta\}} \left(\sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2}\right) dxdyds \\
+\;& \iiint_{\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0) \ge \delta\}} \left(\tau_y(d^{\epsilon_i}) - \tau_y(d^0)\right) dxdyds \\
+\;& \iiint_{\{(x,y,s) \in B(x_0,y_0,R,t) | \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} - \tau_y(d^0) \ge \delta\}} \left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y(d^{\epsilon_i})\right) dxdyds,
\end{aligned}
$$

the strong convergence results of part (i) which guarantee that the first two integrals on the right-hand side of (3.63) converge to zero as the ϵ_i's tend to zero, and from the observation that the third integral is bounded from above by

$$
\iiint_{B\,(x_0,y_0,R,t)} \left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+ dxdyds
$$

which in turn is bounded by

(3.64)

$$
\left(m\left(B\left(x_0,y_0,R,t\right)\right)\right)^{1/2} \left(\iiint_{B\,(x_0,y_0,R,t)} \left(\left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+\right)^2 dxdyds \right)^{1/2}.
$$

The identity (3.2) with f given by (3.3) and g by (3.6) guarantees that

(3.65)

$$
\iiint_{B\,(x_0,y_0,R,t)} \left(\left(\sqrt{(\tau_1^{\epsilon_i})^2+(\tau_2^{\epsilon_i})^2} - \tau_y\left(d^{\epsilon_i}\right)\right)_+\right)^2 dxdyds
$$

$$
\le \epsilon_i \tfrac{(\mu+\beta)}{\mu} \iint_{\left\{\sqrt{(x-x_0)^2+(y-y_0)^2} \le R+t\right\}} \left(\tfrac{\mu\left(\tau_1^2+\tau_2^2\right)}{2(\mu+\beta)} + \tfrac{\beta\left((\tau_1+p_1)^2+(\tau_2+p_2)^2\right)^2}{2(\mu+\beta)} + \tfrac{u^2}{2} + \tfrac{\mu}{\mu+\beta}\int_0^d \tau_y\left(\eta\right) d\eta \right)(x,y,0)\, dxdy
$$

and (3.64) and (3.65) imply that the third integral on the right-hand side of (3.63) tends to zero as ϵ_i tends to zero.

To establish (3.57)-(3.60) in open subsets $\mathcal{E}$ of $\left\{(x,y,s) \in B\left(x_0,y_0,R\ t\right) | 0 < \sqrt{(\tau_1^0)^2+(\tau_2^0)^2} < \tau_y\left(d^0\right)\right\}$ it suffices to show that $\frac{\partial d^0}{\partial t} = 0$ on $\mathcal{E}$. Equations (3.43) and (3.44) will then guarantee that $\frac{\partial p_1^0}{\partial t} = \frac{\partial p_2^0}{\partial t} = 0$ and these identities, along with (2.36)-(2.38) will guarantee that (3.57)-(3.60) hold.

In what follows we let $\delta > 0$,

$$E_\delta = \left\{ (x,\ y,\ s) \in B\,(x_0,\ y_0, R,\ t)\,|\sqrt{(\tau_1^0)^2 + (\tau_2^0)^2} - \tau_y\left(d^0\right) \leq -\delta < 0 \right\}\ . \tag{3.66}$$

and observe that

$$\begin{aligned} \left| \iint\limits_{E_\delta} \frac{\partial d^0}{\partial s} dxdyds \right| \leq & \left| \iiint\limits_{E_\delta} \left(\frac{\partial d^0}{\partial s} - \frac{\partial d^{\epsilon_i}}{\partial s} \right) dxdyds \right| \\ & + \left| \iiint\limits_{E_\delta \cap \left\{ \frac{\partial d^{\epsilon_i}}{\partial s} > 0 \right\}} \frac{\partial d^{\epsilon_i}}{\partial s}\, dxdyds \right| \end{aligned} \tag{3.67}$$

The weak convergence of $\frac{\partial d^{\epsilon_i}}{\partial s}$ to $\frac{\partial d^0}{\partial s}$ guarantees that the first integral on the right-hand side of (3.67) may be made arbitrarily small. We estimate the second integral by

$$\left(\iiint\limits_{B\,(x_0, y_0, R, t)} \left(\tfrac{\partial d^{\epsilon_i}}{\partial s}\right)^2 dxdyds \right)^{1/2} m\Big(E_\delta \cap \left\{\tfrac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}\Big)^{1/2}.$$

That the first factor is bounded follows from (3.2) with f given by (3.34) and $D = \frac{\partial}{\partial t}$. Thus, it suffices to show that $\lim\limits_{i\to\infty} m\left(E_\delta \cap \left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}\right) = 0$. To establish this assertion we observe that $\left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\} = \left\{\sqrt{(\tau_1^{\epsilon_i})^2 + (\tau_2^{\epsilon_i})^2} - \tau_y\,(d^{\epsilon_i}) > 0\right\}$, and that

$$\begin{aligned} \delta m\Big(E_\delta \cap \left\{\tfrac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}\Big) \leq & \iiint\limits_{E_\delta \cap \left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}} \left(\tau_y\,(d^0)\ - \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2}\right) dxdyds \\ \leq & \iiint\limits_{E_\delta \cap \left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}} \left(\tau_y\,(d^0) - \tau_y\,(d^{\epsilon_i})\right) dxdyds \\ & + \iiint\limits_{E_\delta \cap \left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}} \left(\sqrt{(\tau_1^{\epsilon_i})^2 + (\tau_2^{\epsilon_i})^2} - \sqrt{(\tau_1^0)^2 + (\tau_2^0)^2}\right) dxdyds. \end{aligned} \tag{3.68}$$

The strong convergence results of (i) imply that the latter two integrals tend to zero as the ϵ_i's tend to zero thereby yielding $\lim\limits_{i\to\infty} m\left(E_\delta \cap \left\{\frac{\partial d^{\epsilon_i}}{\partial s} > 0\right\}\right) = 0$.

4 Computational Experiments

In this section we present some computational experiments for the dimensionless system (2.36) - (2.41) when the normalized yield stress is given by

$$\tau_y = 1 + c_1 + (c_1 - c_2)\, d - \frac{c_1}{1+d} \tag{4.1}$$

and

$$0 < c_2 < c_1. \tag{4.2}$$

Since the flows associated with this system may be quite complicated we restrict our attention to problems with Riemann type data where

$$(\tau_1, \tau_2, p_1, p_2, d)\left(x, y, 0^+\right) \equiv (0,0,0,0,0) \tag{4.3}$$

and

$$u\left(x, y, 0^+\right) = \begin{cases} u_0, & \text{if } \ xy > 0 \\ -u_0, & \text{if } \ xy < 0 \end{cases} \tag{4.4}$$

where u_0 is a constant. The solutions generated by this data exhibit a high degree of symmetry and thus when visualizing them we may confine our attention to one of the four quadrants $(k-1)\frac{\pi}{2} \le \theta \le \frac{k\pi}{2}$, $k = 1, \ \ldots \ , 4$. The data for $u(x, y, 0^+)$ is not H_1^{loc} but the functions

$$u^h(x,y,0^+) = \begin{cases} u_0, \text{ if } x > \dfrac{h}{2} \text{ and } y > \dfrac{h}{2} \text{ or } x < -\dfrac{h}{2} \text{ and } y < -\dfrac{h}{2}, \\[2ex] -\ u_0, \text{ if } x < -\dfrac{h}{2} \text{ and } y > \dfrac{h}{2} \text{ or } x > \dfrac{h}{2} \text{ and } y < -\dfrac{h}{2}, \\[2ex] -\ u_0 + \dfrac{2u_0}{h}\left(x + \dfrac{h}{2}\right), \text{ if } -\dfrac{h}{2} \le x \le h/2 \text{ and } y \ge \dfrac{h}{2}, \\[2ex] u_0 - \dfrac{2u_0}{h}\left(x + \dfrac{h}{2}\right), \text{ if } -\dfrac{h}{2} \le x \le \dfrac{h}{2} \text{ and } y \le -\dfrac{h}{2}, \\[2ex] u_0 - \dfrac{2u_0}{h}\left(y + \dfrac{h}{2}\right), \text{ if } -\dfrac{h}{2} \le y \le \dfrac{h}{2} \text{ and } x \le -\dfrac{h}{2}, \\[2ex] -\ u_0 + \dfrac{2u_0}{h}\left(y + \dfrac{h}{2}\right), \text{ if } -\dfrac{h}{2} \le y \le \dfrac{h}{2} \text{ and } x \ge \dfrac{h}{2}, \\[2ex] u_0 - \dfrac{2u_0}{h}\left(x + \dfrac{h}{2}\right) - \dfrac{2u_0}{h}\left(y + \dfrac{h}{2}\right) + \dfrac{3u_0}{h^2}\left(x + \dfrac{h}{2}\right)\left(y + \dfrac{h}{2}\right), \\ \quad \text{if } -\dfrac{h}{2} \le x \le \dfrac{h}{2} \text{ and } -\dfrac{h}{2} \le y \le \dfrac{h}{2}. \end{cases} \tag{4.5}$$

are H_1^{loc} and this, together with our L_2^{loc} contractivity estimate of the previous section, is sufficient to guarantee that the solution to (2.36) - (2.41) taking on the data (4.3) and (4.5) has a strong L_2^{loc} limit as $h \rightarrow 0^+$ which satisfies (2.36) - (2.41), (4.3), and (4.4). This limiting behavior is true when $\epsilon > 0$ is fixed and also in the $\epsilon = 0^+$ limit when the rate independent equations (3.52) - (3.60) govern.

Our updating algorithm is as follows. We assume we are given $(\tau_1, \tau_2, p_1, p_2, d, u)^N (x, y)$ on the $x - y$ plane. These represent the approximate solution at time $t = (N - 1/2)\,\delta$ where δ is our time step and $N \geq 1$. To advance these data we successively solve the following systems:

$$
(4.6) \qquad \begin{cases} \dfrac{\partial \tau_1}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \quad \dfrac{\partial \tau_2}{\partial t} = 0, \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(\tau_1 + \dfrac{\beta}{\mu + \beta} p_1\right) = 0, \\ \text{and} \quad \dfrac{\partial p_1}{\partial t} = \dfrac{\partial p_2}{\partial t} = \dfrac{\partial d}{\partial t} = 0, \quad 0 \leq t \leq \delta, \end{cases}
$$

$$
(4.7) \qquad \begin{cases} \dfrac{\partial \tau_1}{\partial t} = 0, \quad \dfrac{\partial \tau_2}{\partial t} - \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial y}\left(\tau_2 + \dfrac{\beta}{\mu + \beta} p_2\right) = 0, \\ \\ \text{and} \quad \dfrac{\partial p_1}{\partial t} = \dfrac{\partial p_2}{\partial t} = \dfrac{\partial d}{\partial t} = 0, \quad 0 \leq t \leq \delta, \end{cases}
$$

and

$$
(4.8) \qquad \begin{cases} \dfrac{\partial}{\partial t}(\tau_1 + p_1) = \dfrac{\partial}{\partial t}(\tau_2 + p_2) = \dfrac{\partial u}{\partial t} = 0, \\ \\ \dfrac{\partial p_1}{\partial t} = \dfrac{\tau_1 \left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon \sqrt{\tau_1^2 + \tau_2^2}}, \qquad \dfrac{\partial p_2}{\partial t} = \dfrac{\tau_2 \left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon \sqrt{\tau_1^2 + \tau_2^2}} \\ \text{and} \quad \dfrac{\partial d}{\partial t} = \dfrac{\left(\sqrt{\tau_1^2 + \tau_2^2} - \tau_y(d)\right)_+}{\epsilon}, \quad 0 \leq t \leq \delta. \end{cases}
$$

Our principal reason for this splitting is that the systems (4.6) and (4.7) may be updated exactly by elementary characteristic methods and (4.8) may be easily integrated to any desired order of accuracy via Runge Kutta methods.

For (4.6) we use $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ as initial data and let $(\tau_1^1, \tau_2^1, p_1^1, p_2^1, d^1, u^1)$ denote the solution to (4.6) with these data at time $t = \delta$. We then solve (4.7) using $(\tau_1^1, \tau_2^1, p_1^1, p_2^1, d, u^1)$ as initial data and let $(\tau_1^2, \tau_2^2, p_1^2, p_2^2, d^2, u^2)$ denote the solution at $t = \delta$. We next repeat the process but first solve (4.7) with the data $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ and let $(\tau_1^3, \tau_2^3, p_1^3, p_2^3, d^3, u^3)$ denote the solution at $t = \delta$. We then use $(\tau_1^3, \tau_2^3, p_1^3, p_2^3, d^3, u^3)$ as data for (4.6) and let $(\tau_1^4, \tau_2^4, p_1^4, p_2^4, d^4, u^4)$ denote the solution at $t = \delta$. Finally we average the approximate solutions indexed by (2) and (4) and denote the result as $(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5)$; that is

$$
(4.9) \qquad \left(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5\right) = \frac{1}{2}\left(\tau_1^2 + \tau_1^4, \tau_2^2 + \tau_2^4, p_1^2 + p_1^4, p_2^2 + p_2^4, d^2 + d^4, u^2 + u^4\right).
$$

We note that this particular approximation represents a second order update to the "elastic" wave equation:

(4.10)
$$\begin{cases} \dfrac{\partial \tau_1}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \quad \dfrac{\partial \tau_2}{\partial t} - \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(\tau_1 + \dfrac{\beta\, p_1}{\mu+\beta}\right) - \dfrac{\partial}{\partial y}\left(\tau_2 + \dfrac{\beta\, p_2}{\mu+\beta}\right) = 0, \\[2ex] \dfrac{\partial p_1}{\partial t} = \dfrac{\partial p_2}{\partial t} = \dfrac{\partial d}{\partial t} = 0, \quad 0 \le t \le \delta \end{cases}$$

taking on the data $(\tau_1, \tau_2, p_1, p_2, d, u)^N$ at $t = 0$ and does better than either of the approximates labeled 2 or 4; in particular solution symmetries are preserved via the averaging algorithm.

The final step in our algorithm involves solving (4.8) with the data $(\tau_1^5, \tau_2^5, p_1^5, p_2^5, d^5, u^5)$. Over the interval $0 \le t \le \delta$ we have

$$\tau_1 + p_1 \equiv \tau_1^5 + p_1^5, \ \tau_2 + p_2 \equiv \tau_2^5 + p_2^5, \text{ and } u \equiv u^5 \tag{4.11}$$

and

$$\begin{cases} \dfrac{\partial \tau_1}{\partial t} = \dfrac{\tau_1\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y\,(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}}, \quad \dfrac{\partial \tau_2}{\partial t} = \dfrac{\tau_2\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y\,(d)\right)_+}{\epsilon\sqrt{\tau_1^2+\tau_2^2}} \\[3ex] \text{and} \quad \dfrac{\partial d}{\partial t} = \dfrac{\left(\sqrt{\tau_1^2+\tau_2^2} - \tau_y\,(d)\right)_+}{\epsilon}. \end{cases} \tag{4.12}$$

If we let

$$\tau_1 = J\cos\Theta \text{ and } \tau_2 = J\sin\Theta, \tag{4.13}$$

then equation (4.12) implies

$$J + d \equiv J^5 + d^5 \quad \text{and} \quad \Theta \equiv \Theta^5, \quad 0 \le t \le \delta \tag{4.14}$$

and

$$\frac{\partial d}{\partial t} = \frac{\left(J^5 + d^5 - d - \tau_y\,(d)\right)_+}{\epsilon}, \quad 0 \le t \le \delta\,. \tag{4.15}$$

In (4.14), $J^5 = \sqrt{(\tau_1^5)^2 + (\tau_2^5)^2}$ and $0 \le \Theta^5 < 2\pi$ satisfies

$$\cos\Theta^5 = \frac{\tau_1^5}{J^5} \text{ and } \sin\Theta^5 = \frac{\tau_2^5}{J^5}. \tag{4.16}$$

In what follows we let d^6 denote our update of (4.15) taking on the data d^5 at $t = 0$. Equation (4.14) then implies that

$$\begin{cases} J^6 = J^5 + d^5 - d^6, \quad \tau_1^6 = J^6\dfrac{\tau_1^5}{J^5}, \quad \tau_2^6 = J^6\dfrac{\tau_2^5}{J^5}, \\[2ex] u^6 = u^5, \quad p_1^6 = p_1^5 + \tau_1^5 - \tau_1^6, \quad \text{and} \quad p_2^6 = p_2^5 + \tau_2^5 - \tau_2^6. \end{cases} \tag{4.17}$$

Our approximate solution at $t = (N + 1/2)\,\delta$ is given by the update labeled 6. To obtain the approximate solution at $t = \delta/2$ we merely solve (4.8) over the interval $0 \le t \le \delta/2$ with the prescribed initial data and take the value of their update at $t = \delta/2$ to be $(\tau_1, \tau_2, p_1, p_2, d, u)^1$.

The snapshots shown in Figure 1-18 were run with the normalized yield stress given by (4.1) when $c_1 = 1$ and $c_2 = .5$. The parameter u_0 defining the initial data was set to 1.5 and we chose $\delta = h = .01$. The parameter ϵ was set to 0.1. Surface renderings of $J = \sqrt{\tau_1^2 + \tau_2^2}, d$, and u are shown at times .3, .4, and .5.

The purely one dimensional nature of the solutions away from the corner where strong interactions take place is evident from these simulations and it is clear from these calculations that our algorithm captures the sharp contact discontinuities in J and u correctly. Our algorithm is easy to implement and avoids a number of thorny issues we would have to contend with if we tried to integrate the reduced $\epsilon = 0^+$ equations directly.

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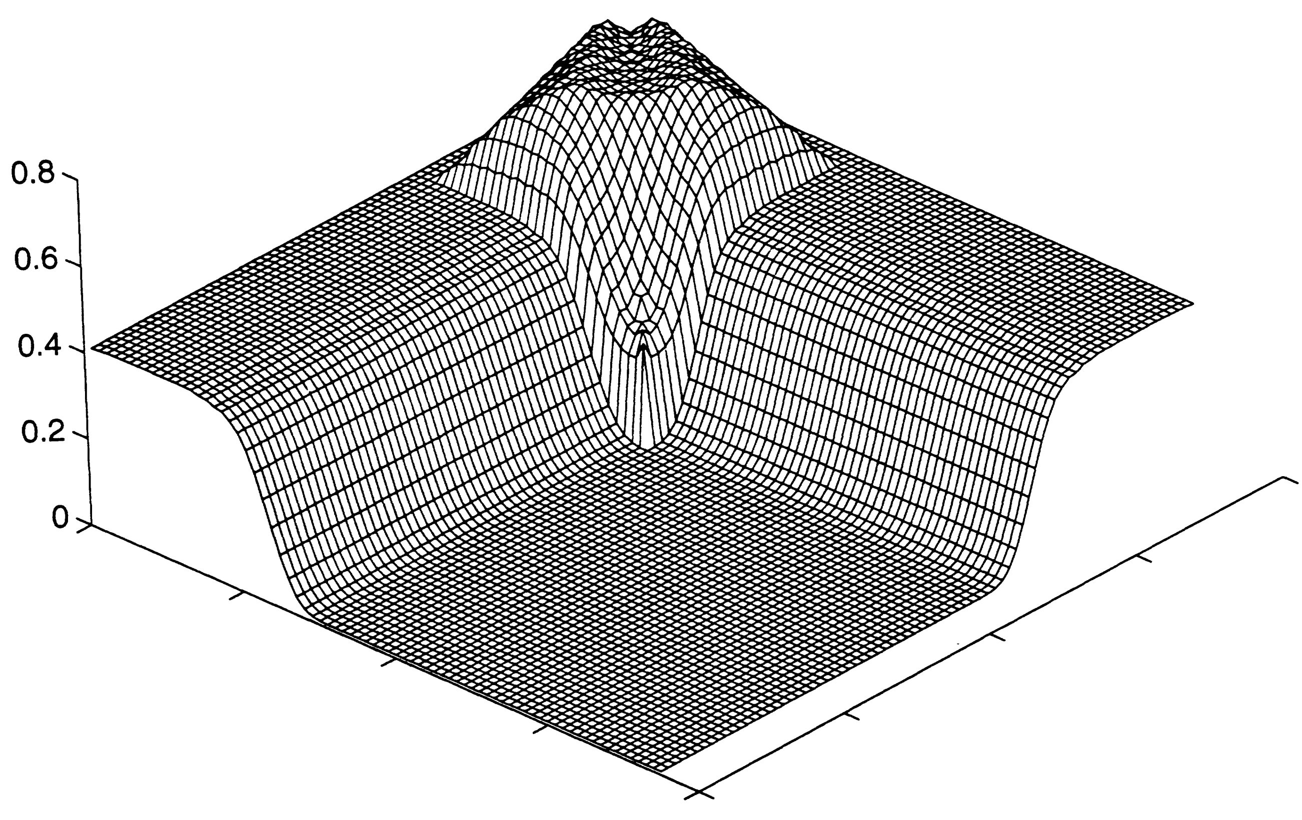
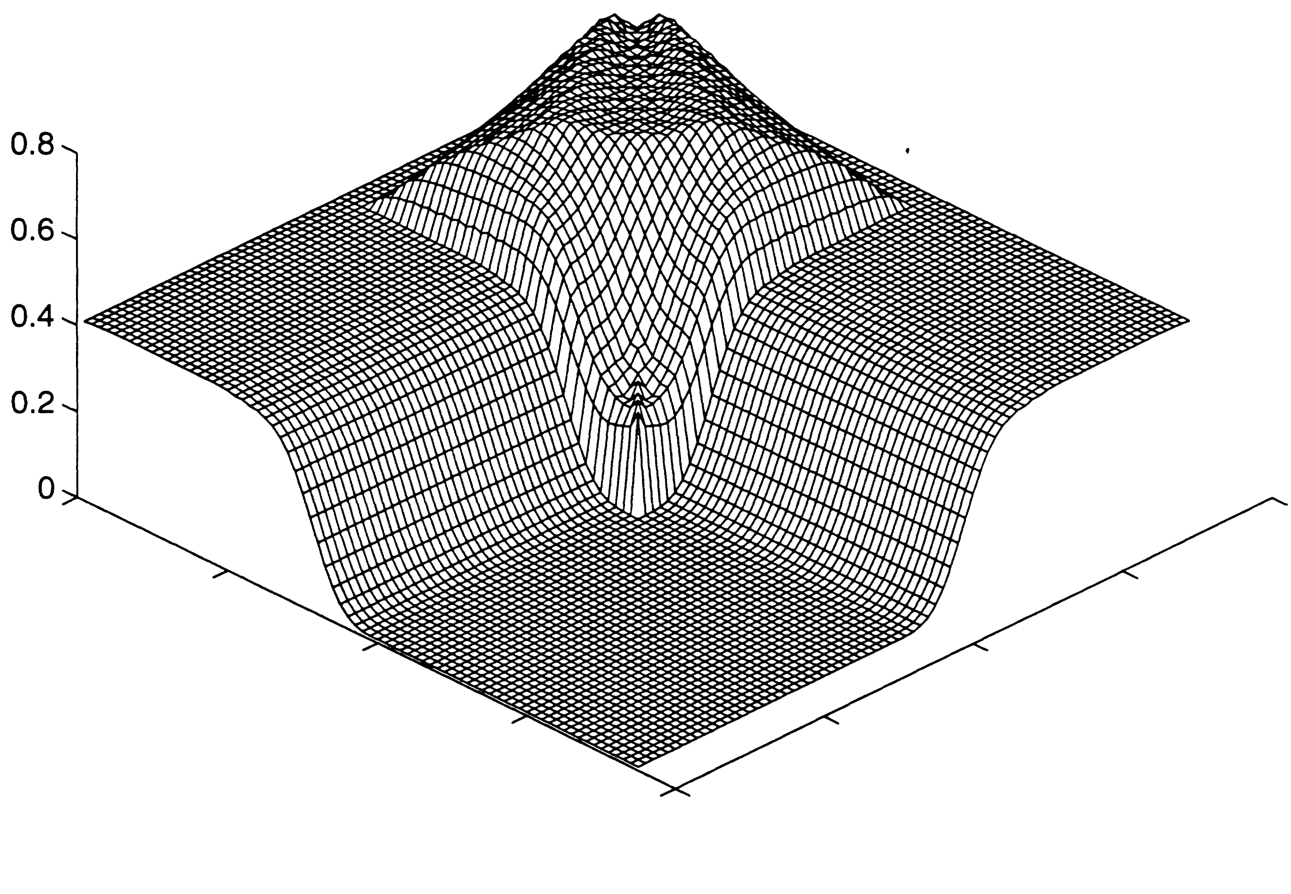
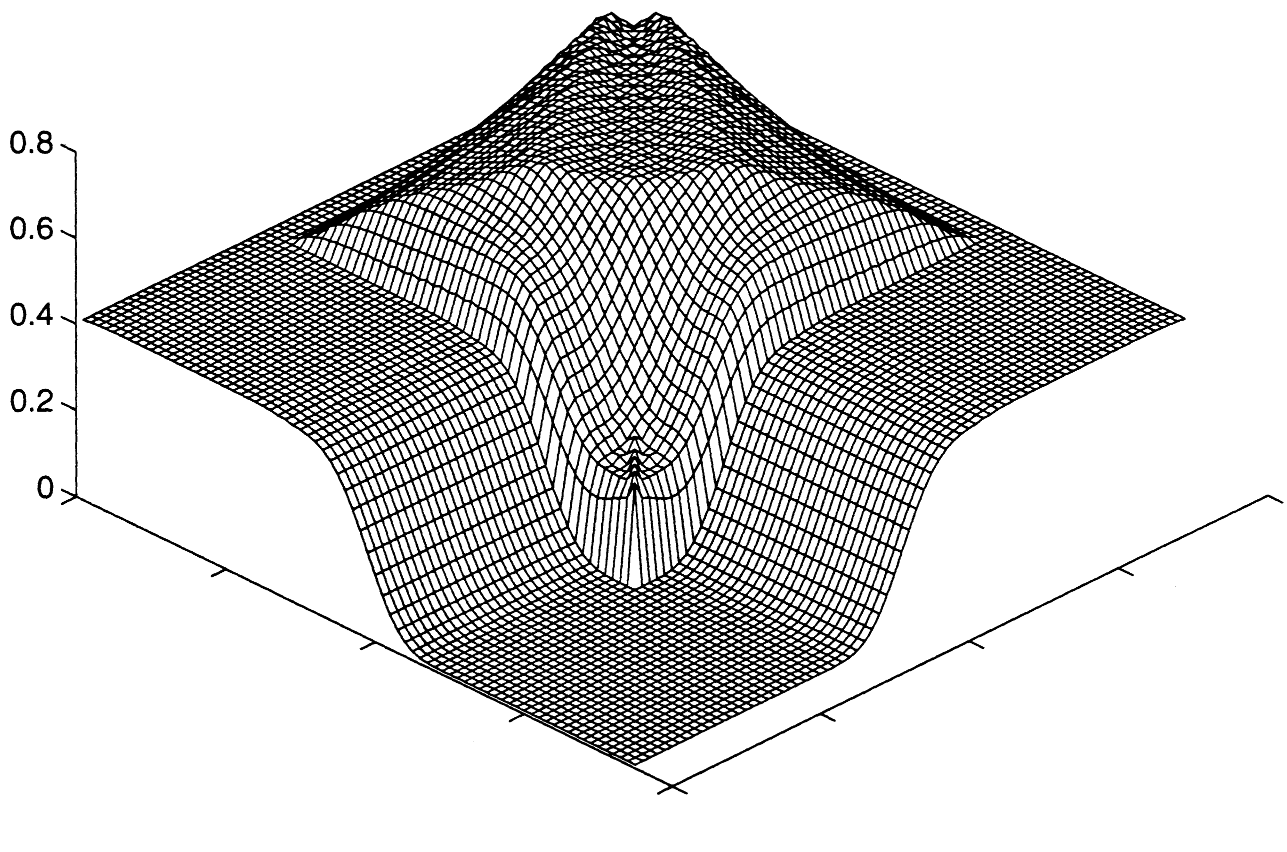
time = 0.3 , accumulated plastic strain viewed head on



time = 0.4 , accumulated plastic strain viewed head on



time = 0.5 , accumulated plastic strain viewed head on



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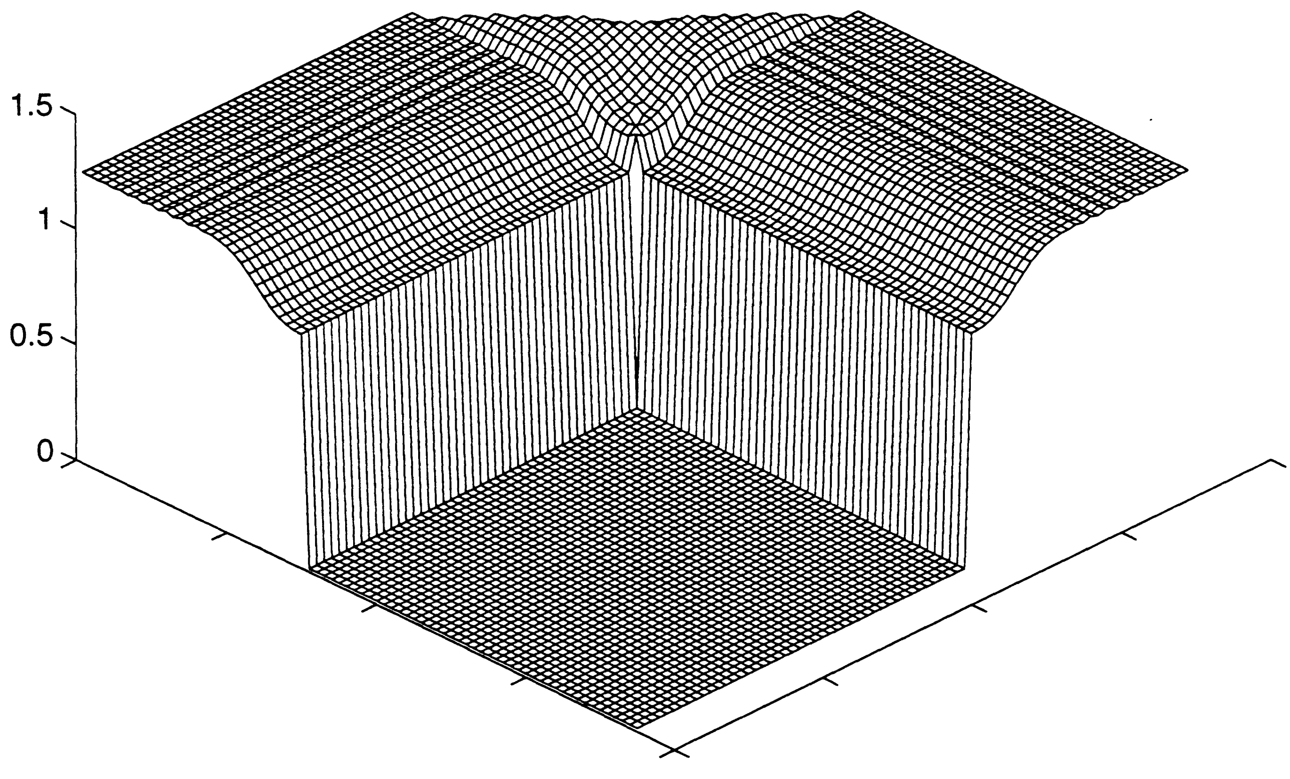

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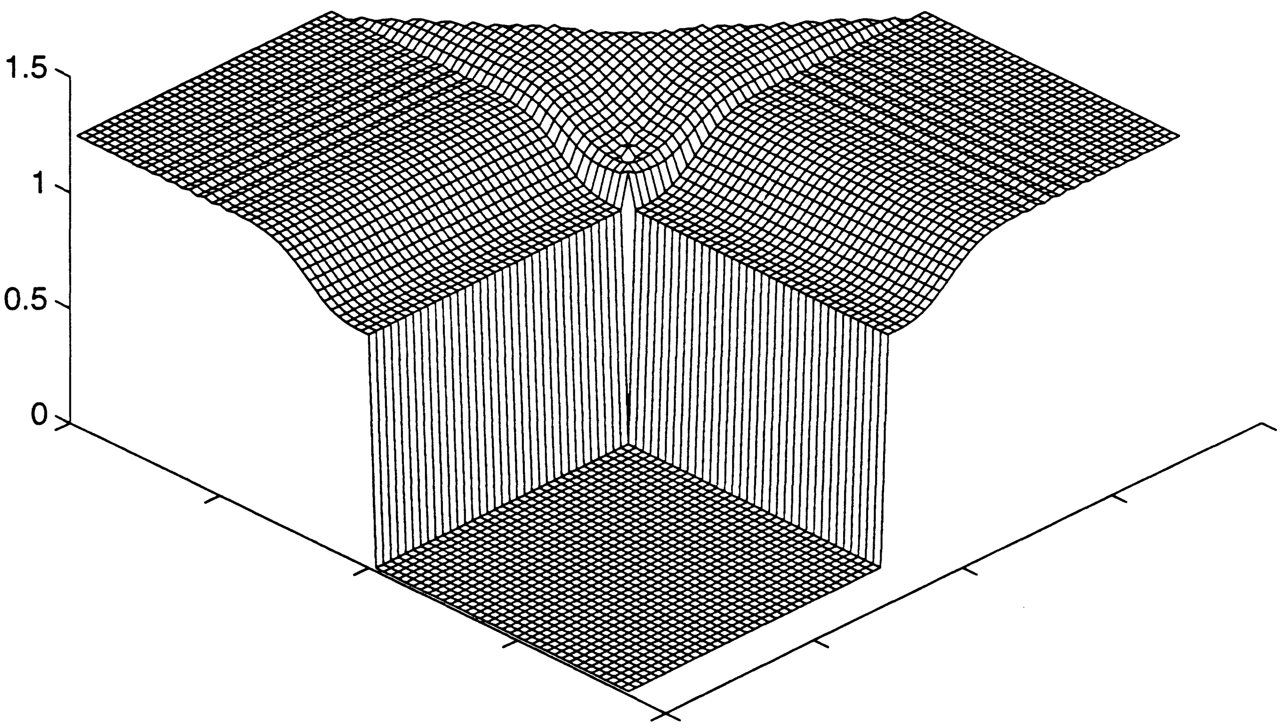

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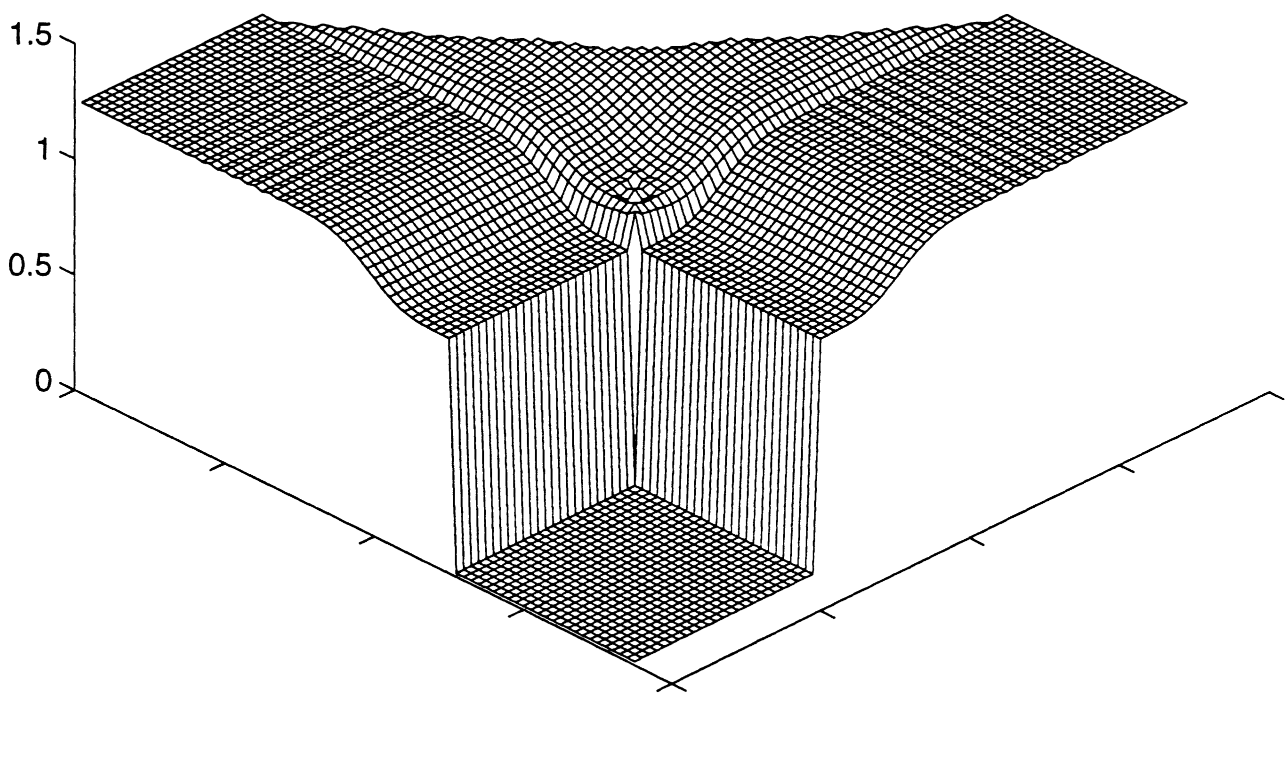
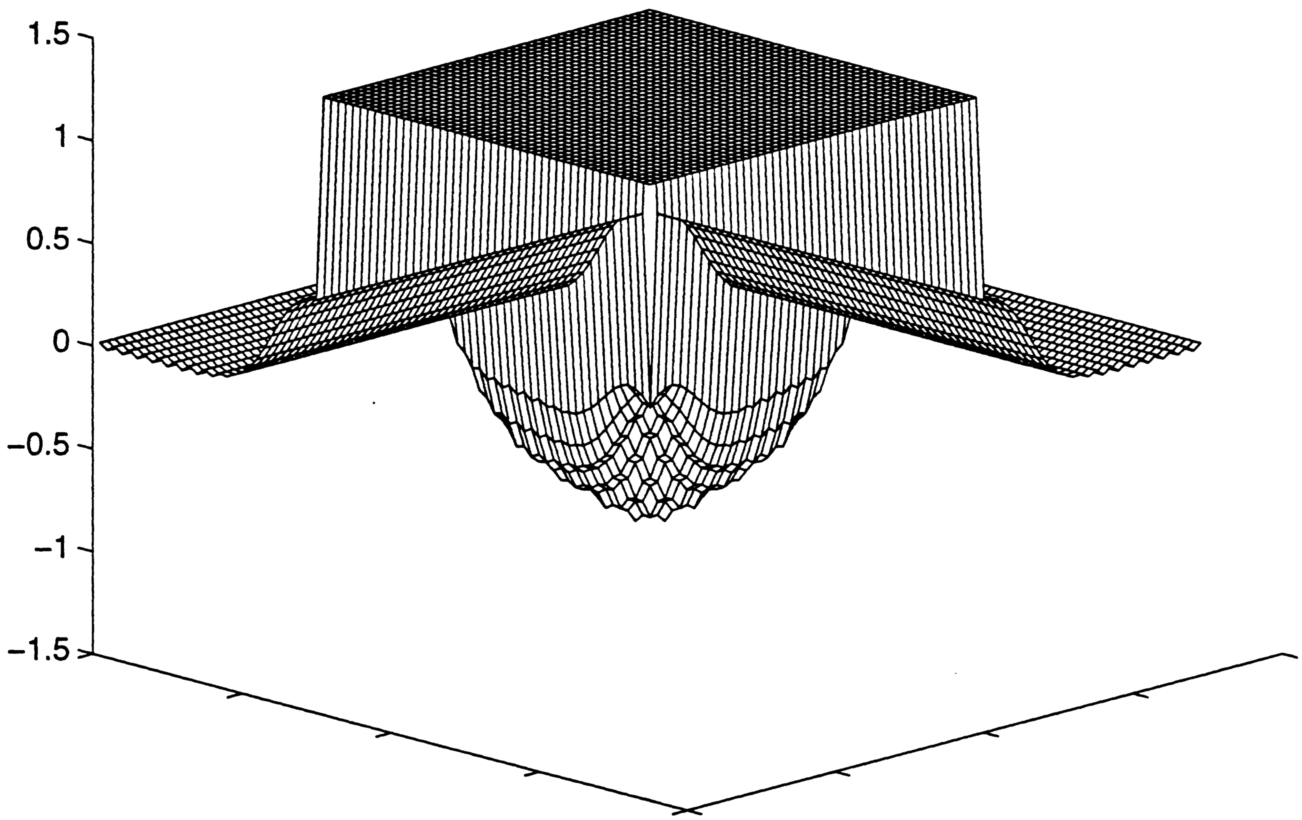


time = 0.3 , velocity field viewed head on



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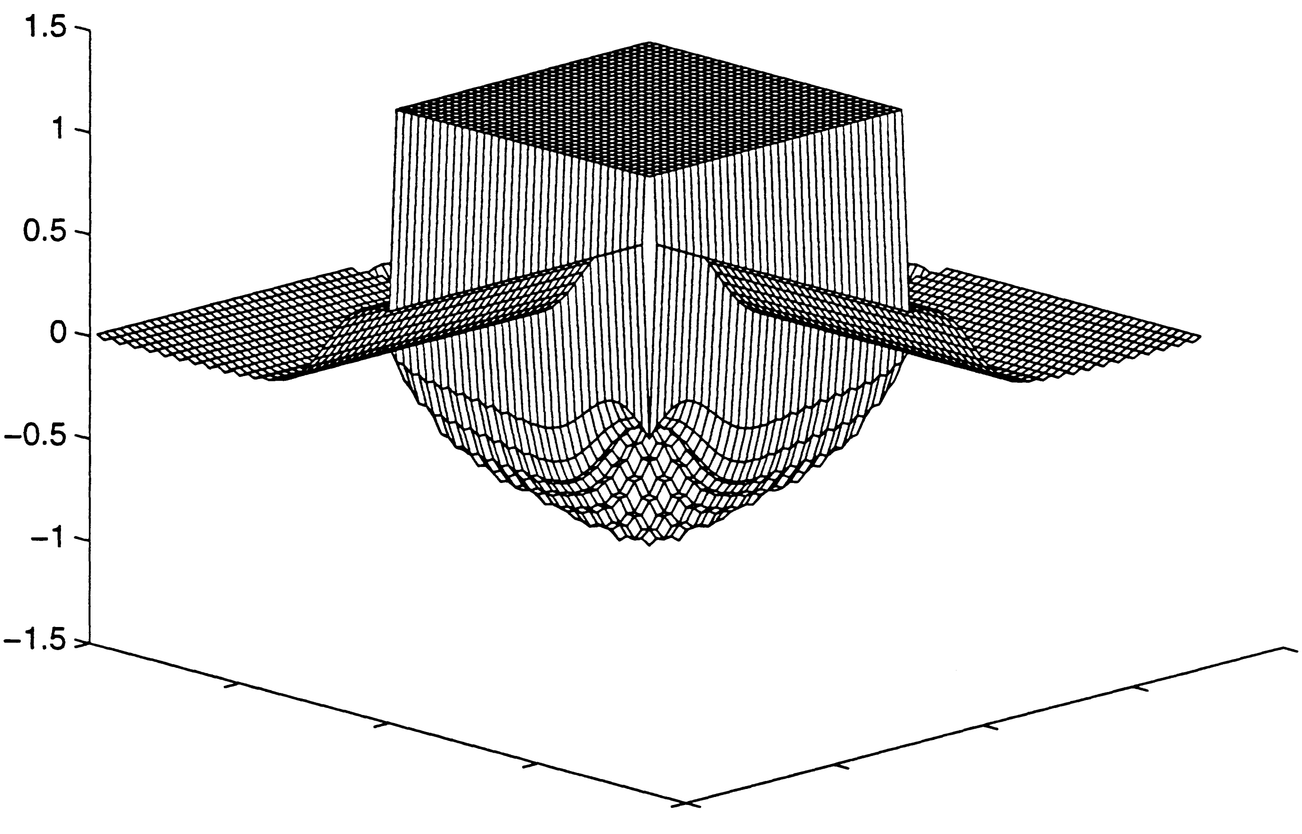

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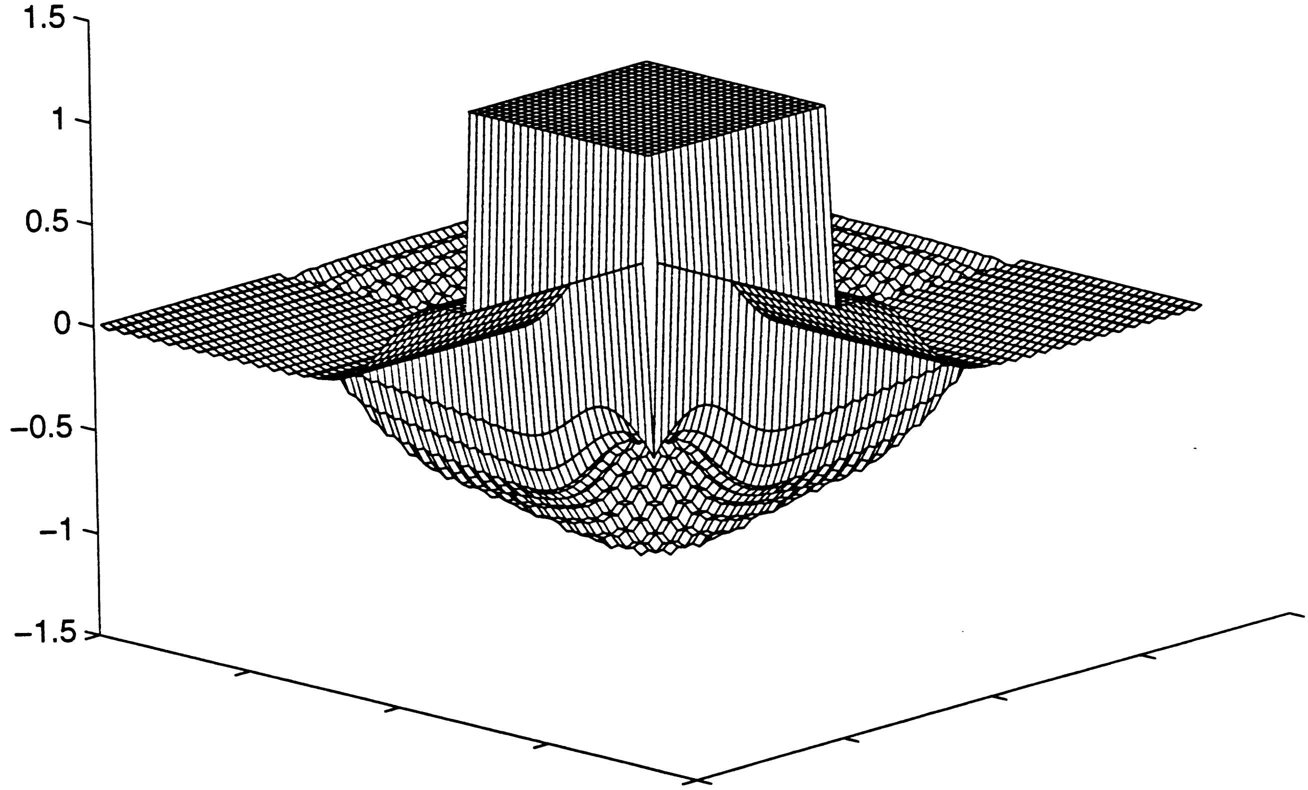
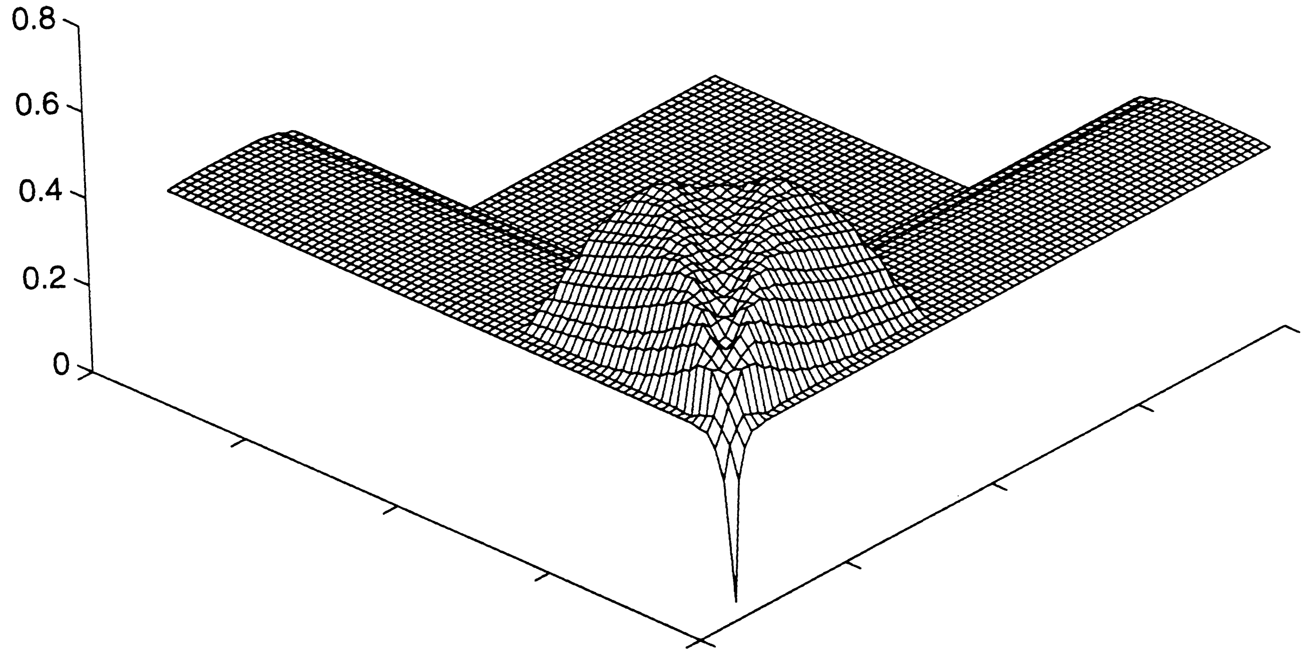


time = 0.3 , accumulated plastic strain viewed from behind



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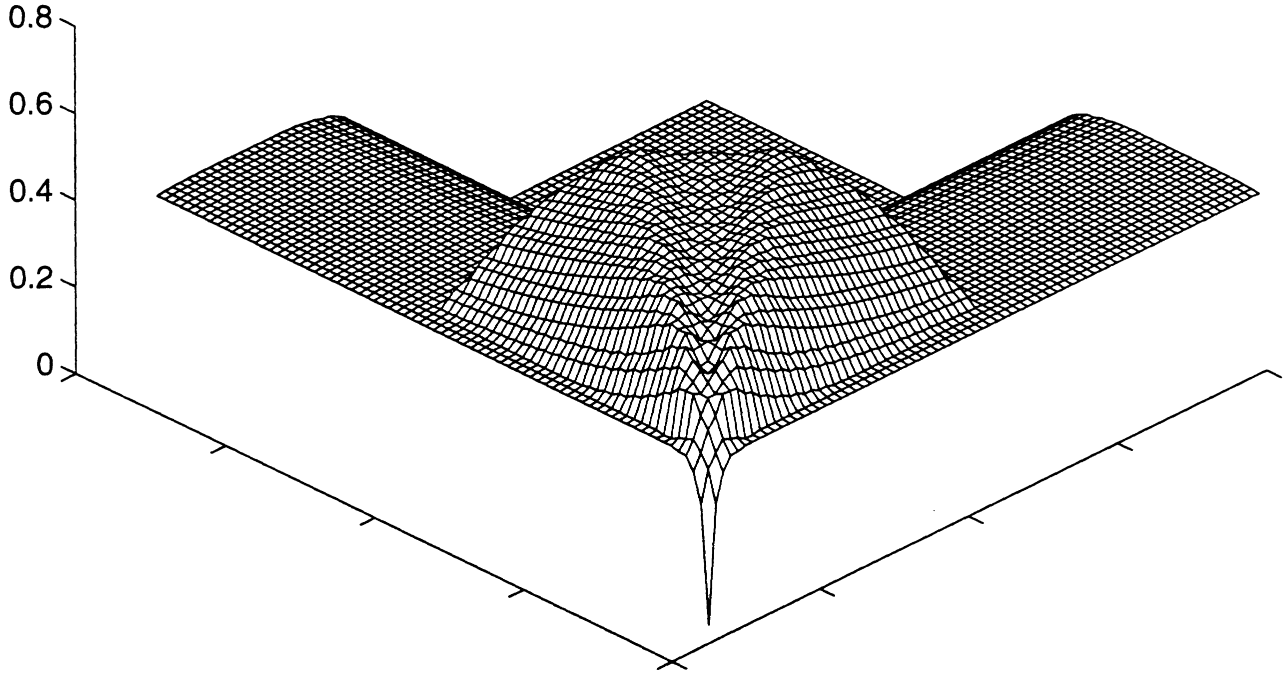

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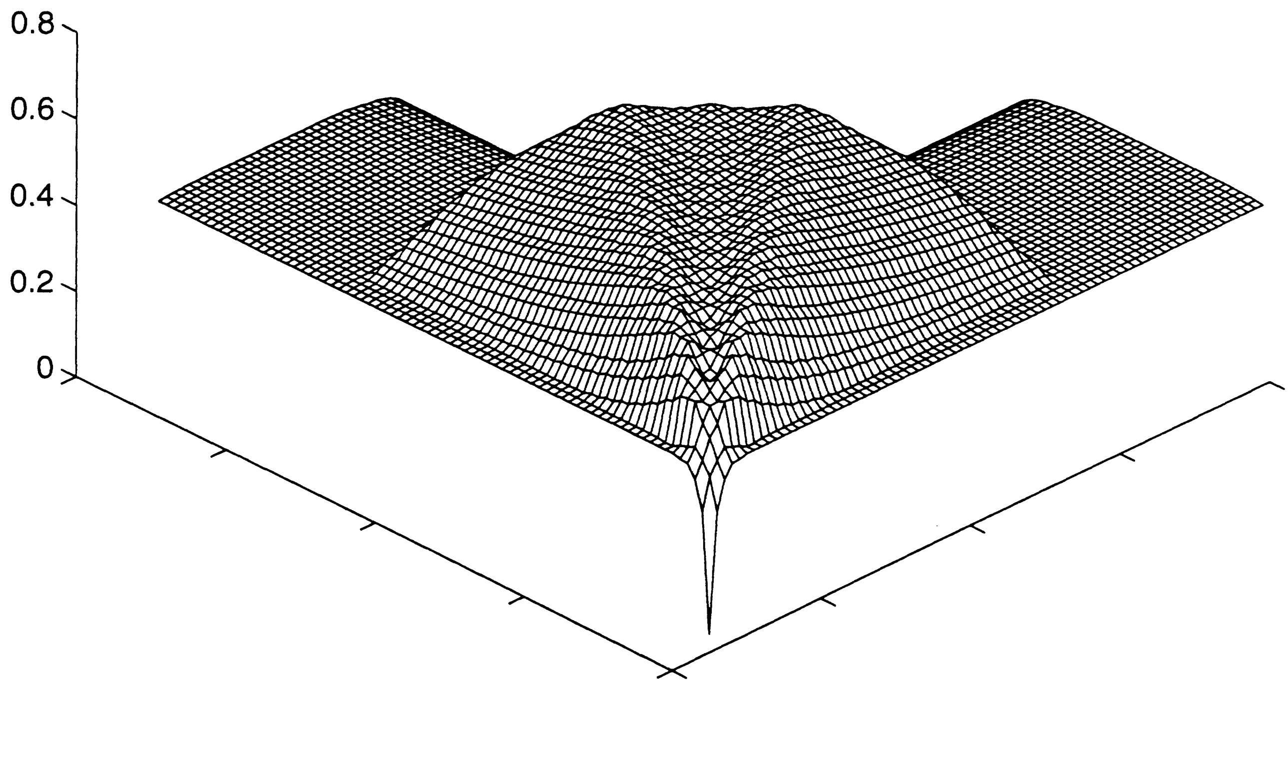
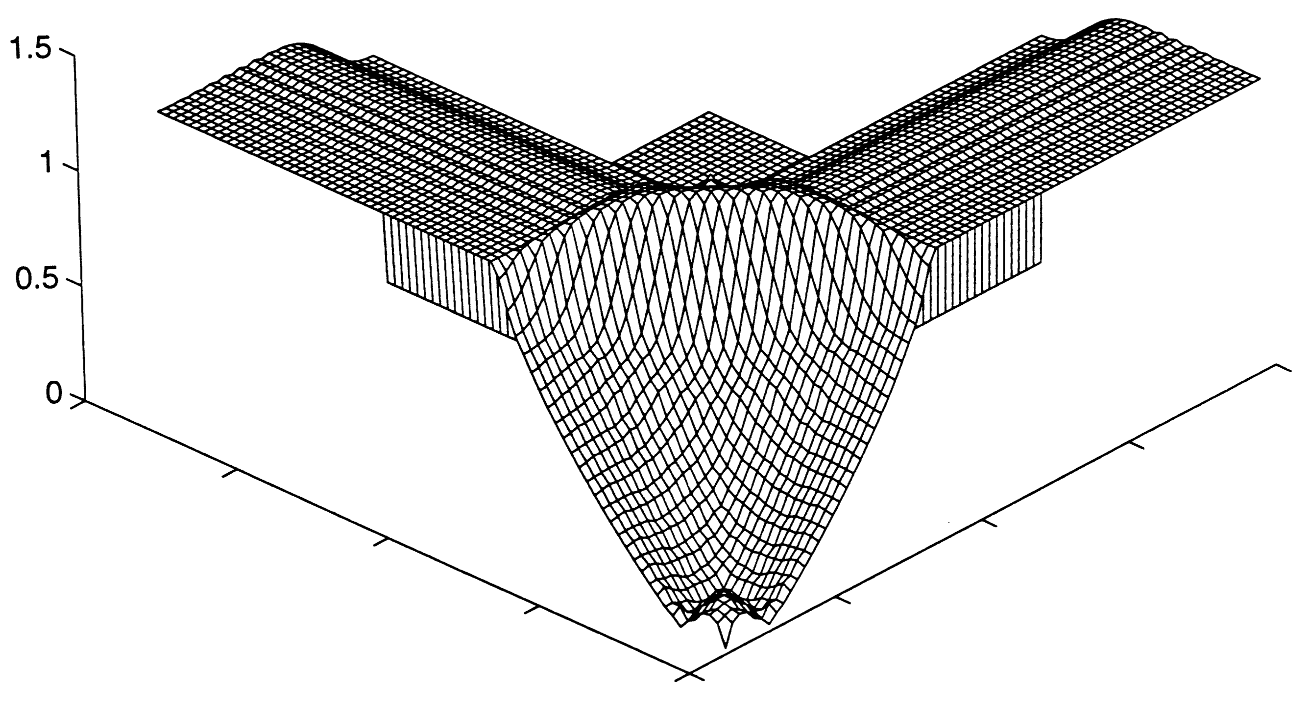


time = 0.3 , total shear stress viewed from behind



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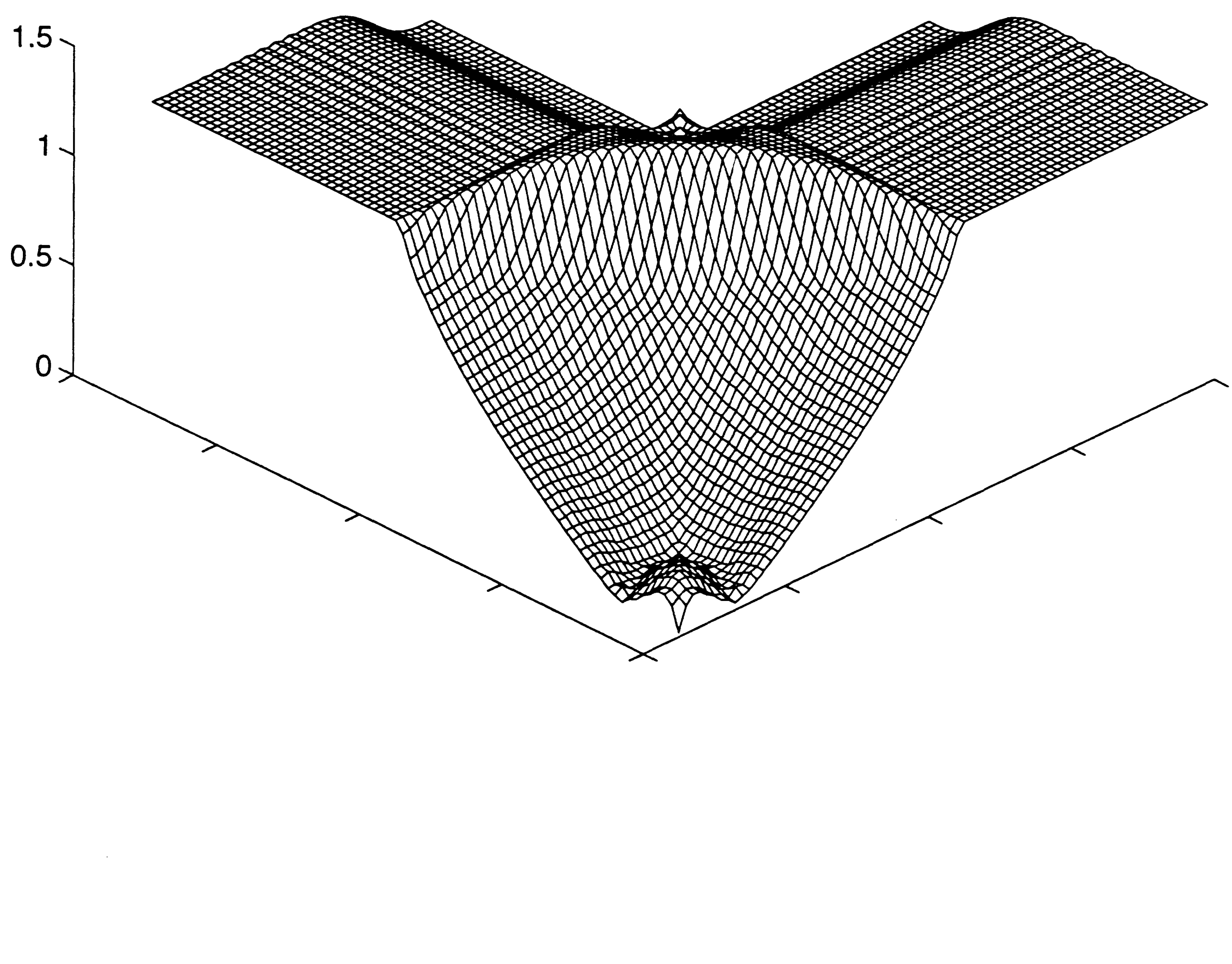

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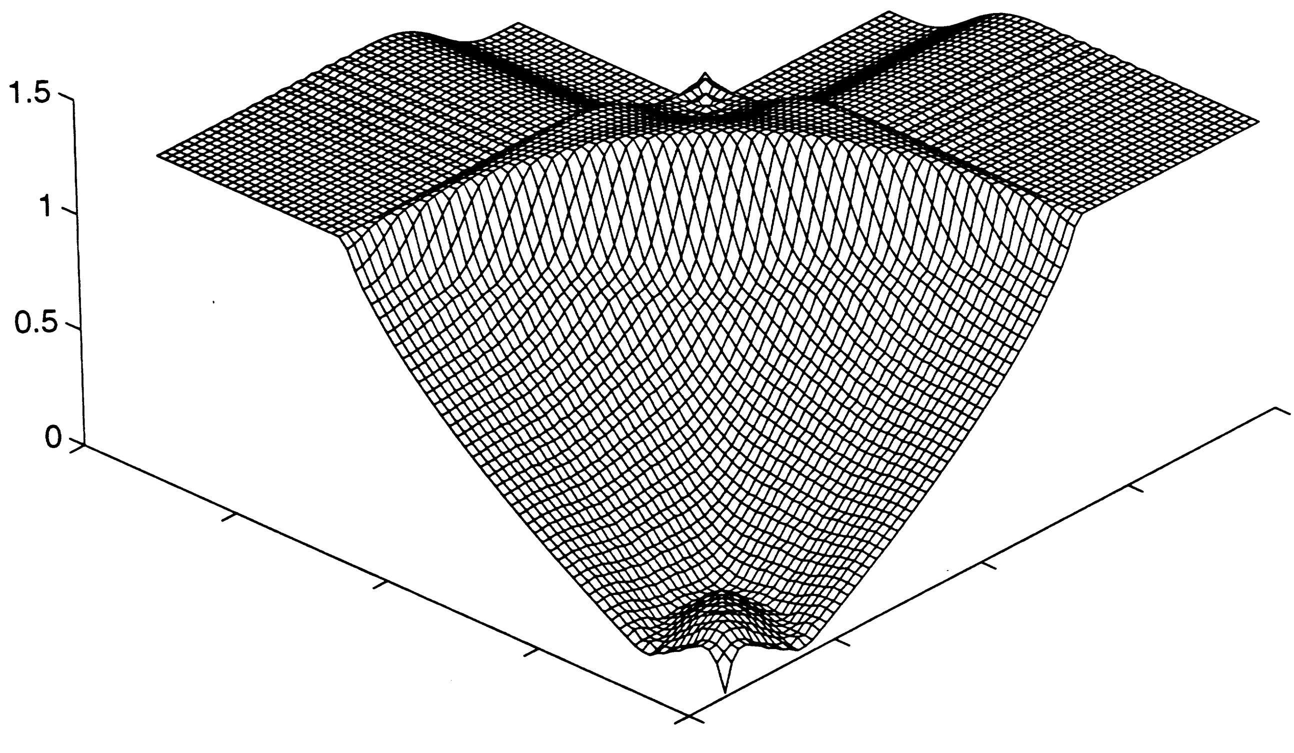
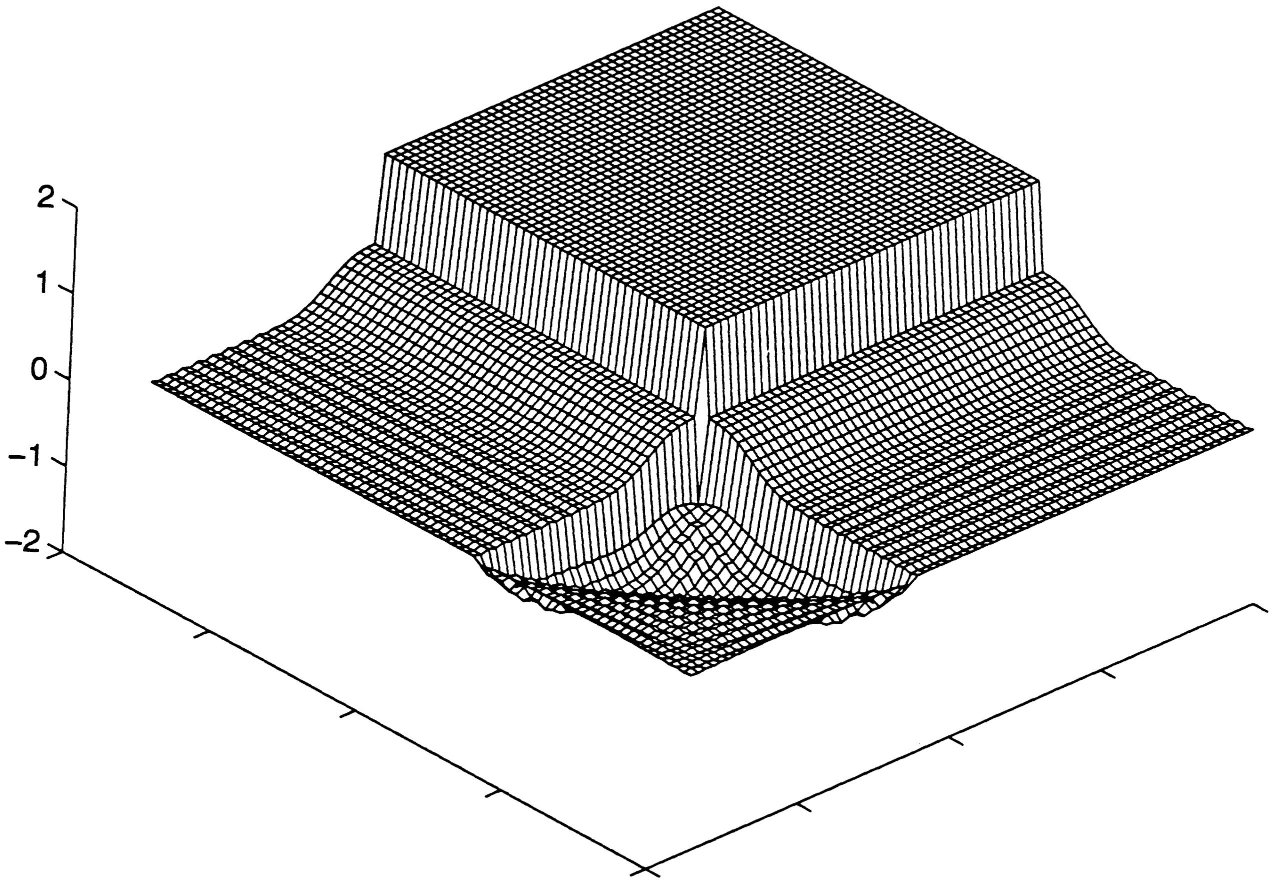


time = 0.3 , velocity field viewed from behind



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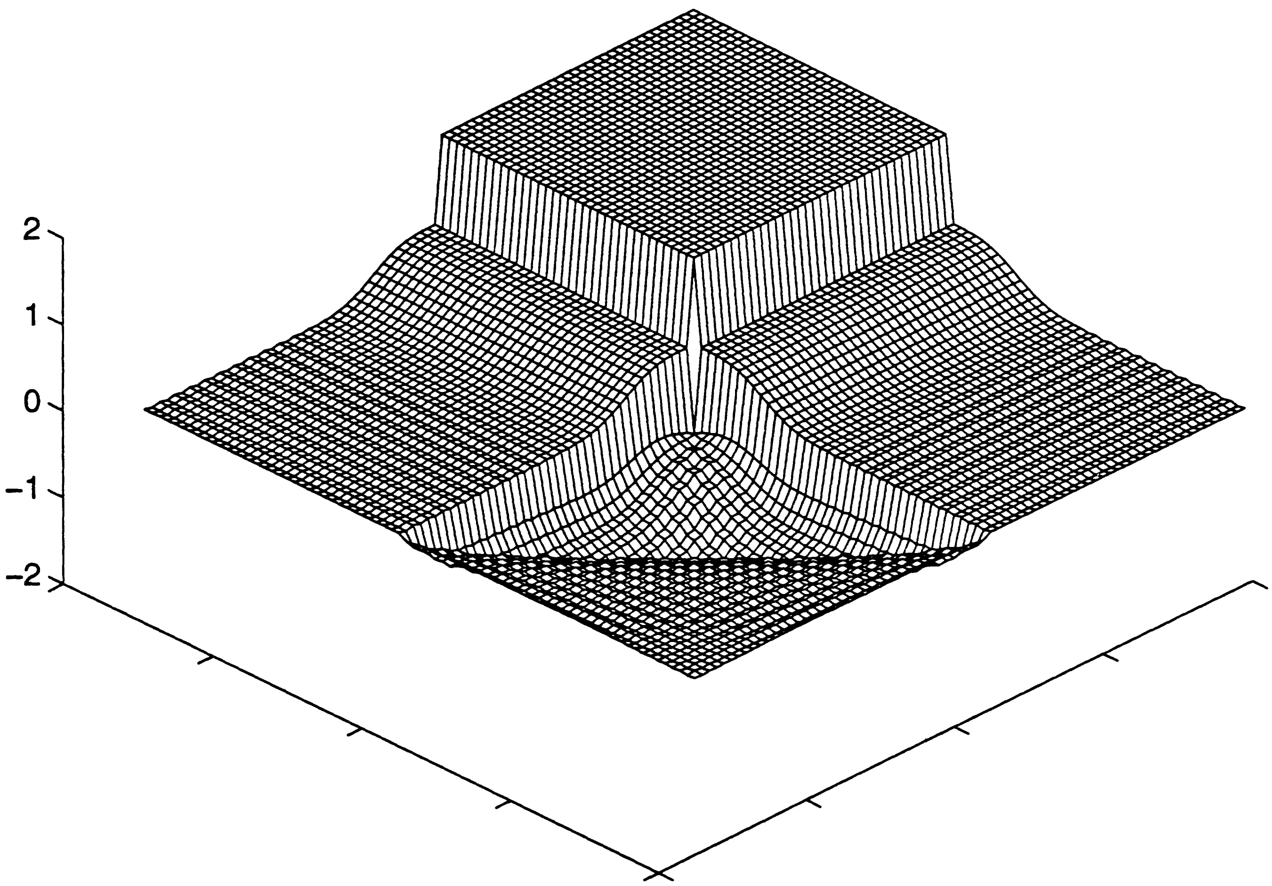

time = 0.5 , velocity field viewed from behind