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On the Long-time Behavior of Ferroelectric and Ferromagnetic Systems

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1. Introduction

In this note we examine some new models for ferroelectric (FE) and ferromagnetic materials (FM). These models are analogous to ones used in [1] to describe the dynamics of elastic materials which can exhibit phase changes.

We begin with Maxwell's equations

$$\frac{\partial \mathbf{B}}{\partial t} = -\ \mathrm{curl}\ \mathbf{E} \quad \text{and} \quad \mathrm{curl}\ \mathbf{H} = \mathbf{J}. \tag{1.1}$$

Here, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields and $\mathbf{J}$ is the current. We assume that

$$\mathbf{J} = \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{E},\ \sigma \geq 0 \tag{1.2}$$

where $\mathbf{D}$ is the electric displacement, $\frac{\partial \mathbf{D}}{\partial t}$ is the displacement current, and $\sigma \mathbf{E}$ is the conduction current.

In usual dielectrics which are homogeneous and isotropic $\mathbf{D}$ and $\mathbf{E}$ and $\mathbf{B}$ and $\mathbf{H}$ are assumed to be related by the constitutive equations

$$\mathbf{D} = \epsilon \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H} \tag{1.3}$$

where ϵ and μ are constants. Insertion of (1.2) and (1.3) into (1.1) yields a linear hyperbolic system which can be solved in a region $\mathcal{D}$ subject to initial values for $\mathbf{E}$ and $\mathbf{H}$ and the specification of the tangential components of either $\mathbf{E}$ or $\mathbf{H}$ on $\partial\mathcal{D}$. Our primary focus will be on the case where $(\mathbf{n} \times \mathbf{E})_- = 0$ on ∂D but we will describe briefly what obtains when $(\mathbf{n} \times \mathbf{H})_- = 0$ on ∂D.

In the case of ferroelectric materials one replaces $(1.3)_1$ by

$$\mathbf{D} = \epsilon(\mathbf{E} + \mathbf{P})$$

where $\mathbf{P}$ is the electric polarization while in the case of ferromagnetic materials the relation $(1.3)_2$ is replaced by

$$\mathbf{B} = \mu(\mathbf{H} + \mathbf{M})$$

where $\mathbf{M}$ is the magnetic polarization. The polarizations are intended to reflect the crystalline structure of the underlying material. At equilibrium these fields will be spatially distributed in a time independent

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fashion which describes the equilibrium microstructure of the material. The stability of these equilibria is very much an open question and will not be resolved here. Carr and Pego [2] have demonstrated that for the Landau-Ginzburg equation, one which gives rise to the same equilibra as our more complicated system, the equilibrium solutions are metastable with exponentially long lifetimes. Greenberg[1] has also shown that similar phenomena is true for elasticity equations which exhibit phase transitions.

In what follows we shall treat in detail the case of ferroelectric materials. We start with some geometric simplifications. We assume the region is a cylinder with generators parallel to the z-axis and a uniform simply connected cross section Ω in the $x-y$ plane. We consider only fields which are independent of z, assume that the electric field $\mathbf{E}$ and electric polarization $\mathbf{P}$ are of the form:

$$\mathbf{E} = ae\mathbf{e}_3 \quad \text{and} \quad \mathbf{P} = ap\mathbf{e}_3,$$

and assume that the magnetic field $\mathbf{H}$ is given by

$$\mathbf{H} = b(h_1\mathbf{e}_1 + h_2\mathbf{e}_2)$$

where $a > 0$ and $b > 0$ are constants. Additionally, we assume that

$$\mathbf{D} = \epsilon(\mathbf{E} + \mathbf{P}) \quad \text{and} \quad \mathbf{B} = \mu\mathbf{H} \tag{1.4}$$

and that $a > 0$ and $b > 0$ satisfy

$$\frac{b}{a} = \sqrt{\frac{\epsilon}{\mu}}. \tag{1.5}$$

In this situation, Maxwell's equations (1.1) and (1.2) reduce to the following systems for e, p, h_1, and h_2:

$$\left.\begin{aligned} e_t + p_t + \sigma_1 e &= c(h_{2x} - h_{1y}) \\ h_{1t} &= -ce_y \\ h_{2t} &= ce_x. \end{aligned}\right\} \tag{1.6}$$

Here $\sigma_1 = \dfrac{\sigma}{\epsilon} \geq 0$ and $c = \dfrac{1}{\sqrt{\epsilon\mu}}$ in the speed of light. We close the system by assuming that the polarization, p, evolves as

$$\delta^2\left(p_{tt} - \lambda^2\Delta p\right) + \alpha\delta p_t = \beta\left(e - g(p)\right) \tag{1.7}$$

where α, β, λ, and δ are positive constants, Δ is the two dimensional laplacian, and $g(p) = \Phi^{(1)}(p)$ is the derivative of a symmetric, double-well potential Φ with equally valued minima at $p = \mp 1$ and a single local maxima at $p = 0$. For technical reasons we also assume that

$$\Phi(p) \sim \frac{kp^2}{2}, \ \Phi^{(1)}(p) \sim kp, \quad \Phi^{(2)}(p) \sim k, \ \text{and} \ \Phi^{(3)} \sim 0 \tag{1.8}$$

as $|p| \to \infty$ for some $0 < k < \infty$.

Equation (1.7) is similar to the better studied Landau-Ginzburg equation

$$-\delta^2\lambda^2\Delta p + \alpha\delta p_t = \beta(e - g(p)) \tag{1.9}$$

which may also be used to model the evolution of p. The latter equation propagates information at infinite speeds whereas (1.7) transmits information at speeds s satisfying $|s| = \lambda < \infty$; a desirable property.

Typically, initial data are prescribed for (e, p, p_t, h_1, h_2) at points $(x, y) \in \Omega$ and boundary conditions are given on $\partial\Omega$ for times $t > 0$. The boundary conditions we impose are that

$$e\left(x^B, y^B, t\right) = 0 \quad \text{and} \quad \frac{\partial p}{\partial n}(x^B, y^B, t) = 0. \tag{1.10}$$

Equation $(1.10)_1$ is the implementation of $(\mathbf{n} \times \mathbf{E})_- = 0$ for this special geometry. In $(1.10)_2$, $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ is the unit exterior normal to $\partial\Omega$ at $\left(x^B, y^B\right)$ and $\mathbf{t} = \mathbf{n}^\perp = -n_2\mathbf{e}_1 + n_1\mathbf{e}_2$ in the unit tangent to $\partial\Omega$ at $\left(x^B, y^B\right)$.

We cast the system (1.6) and (1.7) in a slightly more symmetric looking fashion. To affect this reduction, we assume that the initial data

$$e^0(x, y) = \lim_{t \to 0^+} e(x, y, t) \ , \ (x, y) \in \Omega \tag{1.11}$$

and

$$\left(h_1^0, h_2^0\right)(x, y) = \lim_{t \to 0^+} \left(h_1, h_2\right)(x, y, t) \ , \ (x, y) \in \Omega \tag{1.12}$$

satisfy

$$\frac{\partial h_1^0}{\partial x} + \frac{\partial h_2^0}{\partial y} = 0 \ , \ (x, y) \in \Omega, \tag{1.13}$$

$$\left(h_1^0, h_2^0\right) \cdot (n_1, n_2) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega, \tag{1.14}$$

and

$$e^0\left(x^B, y^B\right) = 0 \ , \ \left(x^B, y^B\right) \in \partial\Omega. \tag{1.15}$$

Equations $(1.6)_2$ and $(1.6)_3$, when combined with (1.13) and (1.14) imply that

$$\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} = 0 \ , \ (x, y) \in \Omega \text{ and } t > 0 \tag{1.16}$$

and

$$(h_1, h_2) \cdot (n_1, n_2) = 0 \ , \ \left(x^B, y^B\right) \in \partial\Omega \text{ and } t > 0 \tag{1.17}$$

and these latter two identities imply the existence of a potential ψ such that

$$h_1 = -c\frac{\partial \psi}{\partial y} \quad \text{and} \quad h_2 = c\frac{\partial \psi}{\partial x} \ , \ (x, y) \in \Omega \quad \text{and} \quad t \geq 0, \tag{1.18}$$

and

$$\psi\left(x^B, y^B, t\right) = g(t) \ , \ \left(x^B, y^B\right) \in \partial\Omega \quad \text{and} \quad t \geq 0. \tag{1.19}$$

Additionally, $(1.6)_2$ and $(1.6)_3$ imply the existence of a function $t \longrightarrow H(t)$ such that

$$e(x,y,t) = \frac{\partial \psi}{\partial t}(x,y,t) - \frac{dH}{dt}(t) \tag{1.20}$$

for all $(x,y) \in \Omega$ and $t \geq 0$. Moreover, (1.9), (1.19), and (1.20) imply that $\frac{dg}{dt} = \frac{dH}{dt}$. If we now let

$$\phi \stackrel{def}{=} \psi - g \ , \ (x,y) \in \Omega \text{ and } t \geq 0, \tag{1.21}$$

we find that

$$e = \phi_t \quad , \quad h_1 = -c\frac{\partial \phi}{\partial y} \quad , \quad \text{and } \ h_2 = c\frac{\partial \phi}{\partial x}, \tag{1.22}$$

and that ϕ satisfies

$$\phi_{tt} + \sigma_1 \phi_t + p_t = c^2 \Delta \phi \ , \ (x,y) \in \Omega \text{ and } t > 0, \tag{1.23}$$

and

$$\phi\left(x^B, y^B, t\right) \equiv 0 \ , \ \left(x^B, y^B\right) \in \partial\Omega \text{ and } t > 0. \tag{1.24}$$

Moreover, ϕ is coupled to p through

$$\delta^2 p_{tt} + \alpha \delta p_t - \beta\left(\phi_t - g(p)\right) = \delta^2 \lambda^2 \Delta p, \ (x,y) \in \Omega \text{ and } t > 0, \tag{1.25}$$

and

$$\frac{\partial p}{\partial n}\left(x^B, y^B, t\right) = 0 \ , \ \left(x^B, y^B\right) \in \partial\Omega \text{ and } t > 0. \tag{1.26}$$

In section 2 we analyze the long time behavior of solutions of (1.23) - (1.26). This is done through a succession of "energy" type estimates and our basic results are that as $t \to \infty$ the function ϕ converges to zero in a suitably strong sense while p converges to a solution of the equilibrium problem

$$\delta^2 \lambda^2 \Delta p = \beta g(p), \ (x,y) \in \Omega \text{ and } \frac{\partial p}{\partial n} = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{1.27}$$

These results imply that e, h_1, and h_2 converge to zero as $t \to \infty$. In section 3 we analyze the equilibrium problem (1.27). Our principal result is that the nontrivial solutions of (1.27) may be obtained by finding the critical points of an even function, $\mathcal{J}$, defined on R^N. The dimension N is determined by the magnitude of the parameter $\frac{\beta}{2\delta^2\lambda^2}\left|\Phi^{(2)}(0)\right|$. In section 4 we discuss the ferromagnetic problem and show that similar results to the ferroelectric problem obtain.

2. Large Time Behavior of Solutions to (1.23)-(1.26).

In this section we focus on the large time behavior of solutions of the system:

$$\phi_{tt} + \sigma_1 \phi_t + p_t = c^2 \Delta \phi \quad , \quad (x,y) \in \Omega \text{ and } t > 0 \tag{2.1}$$

$$\delta^2 p_{tt} + \alpha \delta p_t - \beta \left(\phi_t - g(p) \right) = \delta^2 \lambda^2 \Delta p \quad , \quad (x,y) \in \Omega \text{ and } t > 0 \tag{2.2}$$

satisfying the boundary conditions

$$\phi \left(x^B, y^B, t \right) = \frac{\partial p}{\partial n} \left(x^B, y^B, t \right) = 0 \quad , \quad \left(x^B, y^B \right) \in \partial\Omega \text{ and } t > 0. \tag{2.3}$$

Once again the parameters $\alpha, \beta, \delta, \lambda$, and c are positive, $\sigma_1 \geq 0$, and e, h_1, and h_2 are related to ϕ by

$$e = \phi_t \quad , \quad h_1 = -c \frac{\partial \phi}{\partial y} \quad , \quad \text{and} \quad h_2 = c \frac{\partial \phi}{\partial x}. \tag{2.4}$$

Information about the large time behavior of the system (2.1) - (2.3) will follow from a series of energy identities; the most basic of which is obtained by multiplying (2.1) by $\beta \phi_t$ and (2.2) by p_t and adding the resulting expressions. The identity is

$$\frac{\partial E}{\partial t} - \text{div}\mathbf{q} = -\sigma_1 \beta \phi_t^2 - \alpha \delta p_t^2 \leq 0 \tag{2.5}$$

where

$$E = \frac{\beta}{2} \left(\phi_t^2 + c^2 | \bigtriangledown \phi |^2 \right) + \frac{\delta^2}{2} \left(p_t^2 + \lambda^2 | \bigtriangledown p |^2 \right) + \beta \Phi(p), \tag{2.6}$$

$$\mathbf{q} = \beta c^2 \phi_t \bigtriangledown \phi + \delta^2 \lambda^2 p_t \bigtriangledown p, \tag{2.7}$$

and

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad , \quad \left(x^B, y^B \right) \in \partial\Omega \text{ and } t > 0, \tag{2.8}$$

and once again Φ is the double-well potential satisfying $\Phi^{(1)}(p) = g(p)$ and (1.8). The key point of this and succeeding estimates is the fact that the term $\beta p_t \phi_t$ which comes from the multiplication of (2.1) by $\beta \phi_t$ exactly cancels the term $-\beta p_t \phi_t$ which comes from multiplying (2.2) by p_t.

To obtain the higher order estimates we differentiate the system (2.1) - (2.3) with respect to time. One differentiation implies that the pair (ϕ_t, p_t) satisfies

$$\phi_{ttt} + \sigma_1 \phi_{tt} + p_{tt} = c^2 \Delta \phi_t \quad , \quad (x,y) \in \Omega \text{ and } t > 0, \tag{2.9}$$

$$\delta^2 p_{ttt} + \alpha \delta p_{tt} - \beta \left(\phi_{tt} - g^{(1)}(p) p_t \right) = \delta^2 \lambda^2 \Delta p_t, \; (x,y) \in \Omega \text{ and } t > 0, \tag{2.10}$$

and the boundary conditions

$$\phi_t \left(x^B, y^B, t \right) = \frac{\partial p_t}{\partial n} \left(x^B, y^B, t \right) = 0 \quad , \quad \left(x^B, y^B \right) \in \partial\Omega \text{ and } t > 0, \tag{2.11}$$

while two differentiations imply that the pair (ϕ_{tt}, p_{tt}) satisfies

$$\phi_{tttt} + \sigma_1 \phi_{ttt} + p_{ttt} = c^2 \Delta \phi_{tt} \quad , \quad (x,y) \in \Omega \text{ and } t > 0 \tag{2.12}$$

$$\delta^2 p_{tttt} + \alpha \delta p_{ttt} - \beta \left(\phi_{ttt} - g^{(1)}(p) p_{tt} - g^{(2)}(p) p_t^2 \right) = \delta^2 \lambda^2 \Delta p_{tt} \quad , \quad (x,y) \in \Omega \text{ and } t > 0 \tag{2.13}$$

and the boundary conditions

$$\phi_{tt} \left(x^B, y^B, t \right) = \frac{\partial p_{tt}}{\partial n} \left(x^B, y^B, t \right) = 0 \quad , \quad \left(x^B, y^B \right) \in \partial \Omega \text{ and } t > 0. \tag{2.14}$$

Associated with the identities (2.9) - (2.11) and (2.12) - (2.14) we obtain identities of the form:

$$\frac{\partial E}{\partial t} - \text{div}\mathbf{q} = G \quad , \quad (x,y) \in \Omega \text{ and } t > 0, \tag{2.15}$$

and

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad , \quad \left(x^B, y^B \right) \in \partial \Omega. \tag{2.16}$$

Equations (2.15) and (2.16) imply that

$$\frac{d}{dt} \iint\limits_{\Omega} E(x,y,t) dx dy = \iint\limits_{\Omega} G(x,y,t) dx dy \tag{2.17}$$

and

$$\iint\limits_{\Omega} E(x,y,t) dx dy = \iint\limits_{\Omega} E(x,y,0^+) dx dy + \int_0^t \left(\iint\limits_{\Omega} G(x,y,s) dx dy \right) ds. \tag{2.18}$$

In the case of (2.9) - (2.11)

$$E = \frac{\beta}{2} \left(\phi_{tt}^2 + c^2 | \bigtriangledown \phi_t |^2 \right) + \frac{\delta^2}{2} \left(p_{tt}^2 + \lambda^2 | \bigtriangledown p_t |^2 \right), \tag{2.19}$$

$$\mathbf{q} = \beta c^2 \phi_{tt} \bigtriangledown \phi_t + \delta^2 \lambda^2 p_{tt} \bigtriangledown p_t, \tag{2.20}$$

and

$$G = -\sigma_1 \beta \phi_{tt}^2 - \alpha \delta p_{tt}^2 - \beta g^{(1)}(p) p_t p_{tt}. \tag{2.21}$$

whereas in the case of (2.12) - (2.14)

$$E = \frac{\beta}{2} \left(\phi_{ttt}^2 + c^2 | \bigtriangledown \phi_{tt} |^2 \right) + \frac{\delta^2}{2} \left(p_{ttt}^2 + \lambda^2 | \bigtriangledown p_{tt} |^2 \right), \tag{2.22}$$

$$\mathbf{q} = \beta c^2 \left(\phi_{ttt} \bigtriangledown \phi_{tt} \right) + \delta^2 \lambda^2 p_{ttt} \bigtriangledown \phi_{tt}, \tag{2.23}$$

and

$$G = -\sigma_1 \beta \phi_{ttt}^2 - \alpha \delta p_{ttt}^2 - \beta \left(g^{(1)}(p) p_{tt} p_{ttt} + g^{(2)}(p) p_t^2 p_{ttt} \right). \tag{2.24}$$

Next, we record some immediate consequences of the identities (2.5) and (2.15) when E is given by (2.6), (2.19), and (2.22).

Lemma 1. The identities (2.5) and (2.6) imply that

(i) ϕ is in $H_0^1(\Omega)$ uniformly in t,
(ii) p is in $H^1(\Omega)$ uniformly in t,
(iii) ϕ_t and p_t are in $L_2(\Omega)$ uniformly in t,
(iv) $\alpha > 0$ and $\delta > 0$ imply that p_t is in $L_2(\Omega \times [0, \infty))$, and
(v) if $\sigma_1 > 0$, then ϕ_t is in $L_2(\Omega \times [0, \infty))$. ∎

Lemma 2. The identities (1.8), (2.15) and (2.19) and the result (iv) of Lemma 1 imply that

(i) ϕ_t is in $H_0^1(\Omega)$ uniformly in t,
(ii) p_t is in $H^1(\Omega)$ uniformly in t,
(iii) ϕ_{tt} and p_{tt} are in $L_2(\Omega)$ uniformly in t,
(iv) $\alpha > 0$ and $\delta > 0$ imply that p_{tt} is in $L_2(\Omega \times [0, \infty))$,
(v) if $\sigma_1 > 0$, then ϕ_{tt} is in $L_2(\Omega \times [0, \infty))$. ∎

Proof. The key step in establishing Lemma 2 is the observation that (1.8) implies that

$$-\beta g^{(1)}(p) p_t p_{tt} \leq \frac{\alpha\delta}{2} p_{tt}^2 + \frac{\beta^2 k_1^2}{2\alpha\delta} p_t^2$$

and this inequality, (2.21), and the fact that Lemma 1 guarantees that $p_t \in L_2(\Omega \times [0, \infty))$ implies the results claimed. Here, k_1 is an upper bound for $|g^{(1)}(p)|$.

The next set of estimates will be a simple consequence of (1.8), (2.15), and (2.22) and the following inequality which pertains to functions which are $H^1(\Omega)$ uniformly in t and in $L_2(\Omega \times [0, \infty))$, namely

$$\int_0^\infty \left(\iint_\Omega f^4(x,y,t)\,dx\,dy \right) dt \leq K_1 \times \left(\sup_{0 \leq t < \infty} \|f\|_{1,\Omega}^2(t) \right) \times \int_0^\infty \left(\iint_\Omega f^2(x,y,t)\,dx\,dy \right) dt, \qquad (2.25)$$

where K_1 is independent of f and depends only on Ω.[1]

Lemma 3. The identities (1.8), (2.15), (2.22) and (2.25) imply that

(i) ϕ_{tt} is in $H_0^1(\Omega)$ uniformly in t,
(ii) p_{tt} is in $H^1(\Omega)$ uniformly in t,
(iii) ϕ_{ttt} and p_{ttt} are in $L_2(\Omega)$ uniformly in t,
(iv) $\alpha > 0$ and $\delta > 0$ imply that p_{ttt} is in $L_2(\Omega \times [0, \infty))$,
(v) if $\sigma_1 > 0$, then ϕ_{ttt} is in $L_2(\Omega \times [0, \infty))$. ∎

Proof. The key step in establishing this result is the observation that (1.8) implies that

$$-\beta g^{(1)}(p) p_{tt} p_{ttt} \leq \frac{\alpha\delta}{4} p_{ttt}^2 + \frac{\beta^2 k_2^2}{\alpha\delta} p_{tt}^2$$

and

$$-\beta g^{(2)}(p) p_t^2 p_{ttt} \leq \frac{\alpha\delta}{4} p_{ttt}^2 + \frac{\beta^2 k_2^2}{\alpha\delta} p_t^4$$

where k_2 is an upper bound for $|\Phi^{(3)}(p)|$. These inequalities guarantee that the source term G defined in (2.24) satisfies

---

[1] Here $\|f\|_{1,\Omega}^2(t) = \iint_\Omega f^2(x,y,t)\,dx\,dy + \iint_\Omega \left(f_x^2 + f_y^2\right)(x,y,t)\,dx\,dy$

$$G \le -\sigma_1\beta\phi_{ttt}^2 - \frac{\alpha\delta}{2}p_{ttt}^2 + \frac{\beta^2 k_2^2}{\alpha\delta}\left(p_t^4 + p_{tt}^2\right)$$

and the last inequality along with the results of Lemmas 1 and 2 and the inequality (2.25) imply the results claimed.

The underlying equations (2.1) - (2.3), (2.9) - (2.11), and (2.12) - (2.14) together with the results of Lemmas 1-3 also yield

Lemma 4. (i) $\Delta\phi$ and Δp are in $H^1(\Omega)$ uniformly in t and (ii) $\Delta\phi_t$ and Δp_t are in $L_2(\Omega)$ uniformly in t.■

The a-priori estimates of Lemmas 1-4 imply that if the initial data is sufficiently smooth, then for each $T > 0$ the functions

$$(\phi^t, p^t)(x,y,s) = (\phi, p)(x,y,t+s) \quad , \quad (x,y) \in \Omega \quad \text{and} \quad 0 \le s \le T \tag{2.26}$$

are uniformly bounded in $H^3(\Omega \times [0,T])$ independently of t with bounds that depend on the size of the data and the number $T > 0$[2]. Moreover, the fact that p_t, p_{tt}, and p_{ttt} are in $L_2(\Omega \times [0,\infty))$ guarantees that p_t and p_{tt} converge to zero strongly in $L_2(\Omega)$ as t tends to infinity and additionally that the functions

$$\left(p_s^t, p_{ss}^t, p_{sss}^t\right)(x,y,s) = \left(p_s, p_{ss}, p_{sss}\right)(x,y,t+s) \quad , \quad 0 \le s \le T \tag{2.27}$$

converge to zero in $L_2(\Omega \times [0,T])$ as t tends to infinity. Our next task is to prove

Theorem 1. For each $T > 0$ the functions ϕ^t converge strongly to zero in $H^2(\Omega \times [0,T])$ as t tends to infinity.[3] ■

Proof. We assume the theorem is false. Then, we can find an increasing sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n = \infty$ and an $\epsilon > 0$ such that $||\phi^{t_n}||_{2,T} \ge \epsilon$. We note that the sequence $\{\phi^{t_n}\}_{n=1}^{\infty}$ is also bounded in $H^3(\Omega \times [0,T])$ and thus we can find a subsequence $\{\tau_{k(n)}\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \tau_{k(n)} = \infty$ of the original sequence and a function ϕ^∞ in $H^3(\Omega \times [0,T])$ such that

$$\lim_{n\to\infty} ||\phi^{\tau_{k(n)}} - \phi^\infty||_{2,T} = 0 \quad \text{and} \quad ||\phi^\infty||_{2,T} \ge \epsilon. \tag{2.28}$$

Moreover, we may assume, without loss of generality, that the sequence $\{p^{\tau_{k(n)}}\}_{n=1}^{\infty}$ converges strongly to p^∞ in $H^2(\Omega \times [0,T])$, that p^∞ is in $H^3(\Omega \times [0,T])$, and finally that ϕ^∞ and p^∞ satisfy the limit equations

$$\phi_{ss}^\infty + \sigma_1\phi_s^\infty = c^2\Delta\phi^\infty \quad , \quad (x,y) \in \Omega \ \text{ and } \ 0 \le s \le T, \tag{2.29}$$

$$\beta\phi_s^\infty = \beta g(p^\infty) - \delta^2\lambda^2\Delta p^\infty \quad , \quad (x,y) \in \Omega \ \text{ and } \ 0 \le s \le T, \tag{2.30}$$

$$p_s^\infty = 0 \quad , \quad (x,y) \in \Omega \ \text{ and } 0 \le s \le T, \tag{2.31}$$

and

[2] Recall that $H^k(\Omega \times [0,T])$ consists of all functions f on $\Omega \times [0,T]$ with partial derivatives $\partial_x^m \partial_y^n \partial_t^p f$ of order $m+n+p \le k$ which are in $L_2(\Omega \times [0,T])$. For such functions

$$||f||_{k,T}^2 = \sum_{m+n+p\le k} \int_0^T \left(\iint_\Omega \left(\partial_x^m \partial_y^n \partial_s^p f\right)^2 (x,y,s)\,dx\,dy \right) ds$$

[3] The implications of this result for the primary fields of interest; namely e, h_1, and h_2 follow directly from (2.4).

$$\phi^{\infty}\left(x^B,y^B,s\right)=\frac{\partial p^{\infty}}{\partial n}\left(x^B,y^B\right)=0 \quad , \quad \left(x^B,y^B\right)\in\partial\Omega \text{ and } 0\le s\le T. \tag{2.32}$$

If we now differentiate (2.30) with respect to s and exploit (2.31) we find that $\phi^{\infty}_{ss}\equiv 0$ and this in turn reduces (2.29) to $\sigma_1\phi^{\infty}_s=c^2\Delta\phi^{\infty}$. If we now differentiate the last relation with respect to s we find that $\Delta\phi^{\infty}_s\equiv 0$, $(x,y)\in\Omega$ and $0\le s\le T$. Differentiating $(2.32)_1$, also yields $\phi^{\infty}_s=0$ on $\partial\Omega$ and these two facts in turn imply that $\phi^{\infty}_s\equiv 0$ for $(x,y)\in\bar{\Omega}$ and $0\le s\le T$. Finally, equations (2.29) and (2.32) and the identities $\phi^{\infty}_s\equiv\phi^{\infty}_{ss}\equiv 0$ imply that $\phi^{\infty}\equiv 0$ in $\Omega\times[0,T]$ and this in turn yields $||\phi^{\infty}||_{2,T}\equiv 0$ which is a contradiction. ∎

We note that we have made no reference in the proof as to whether σ_1 is positive or zero. Had we assumed $\sigma_1>0$, then our basic a-priori estimates would have guaranteed that $(\phi^t_s,\phi^t_{ss},\phi^t_{sss})$ all converged to zero in $L_2(\Omega\times[0,T])$ as t went to infinity and thus the limit equations (2.29) and (2.30) would have directly taken the form $\Delta\phi^{\infty}\equiv 0$ and $\beta g(p^{\infty})-\delta^2\lambda^2\Delta p^{\infty}=0$. These relations would then have yielded the desired result.

The preceding proof gives us considerable information about the ω-limit set of solutions of (2.1) - (2.3). In particular we know that if $(\phi^{\infty},p^{\infty})$ is in the ω-limit set, then $\phi^{\infty}\equiv 0$ and p^{∞} is a solution of the equilibrium problem

$$\delta^2\lambda^2\Delta p^{\infty}-\beta g\left(p^{\infty}\right)=0 \quad , \quad (x,y)\in\Omega \tag{2.33}$$

and

$$\frac{\partial p^{\infty}}{\partial n}\left(x^B,y^B\right)=0 \quad , \quad \left(x^B,y^B\right)\in\partial\Omega. \tag{2.34}$$

We further note that if $\{t_n\}_{n=1}^{\infty}$ is an increasing sequence of times satisfying

$$\lim_{n\to\infty} t_n=\infty \quad \text{and} \quad \lim_{n\to\infty}||p^{t_n}-p^{\infty}||_{2,T}=0, \tag{2.35}$$

then the energy identity (2.5) implies that the averaged energy

$$\mathcal{E}\left(t,T\right)\stackrel{\text{def}}{=}\int_0^T\left(\iint_{\Omega}\left(\frac{\beta}{2}\left(\phi_s^2+c^2|\nabla\phi|^2\right)+\frac{\delta^2}{2}\left(p_s^2+\lambda^2|\nabla p|^2\right)+\beta\Phi(p)\right)(x,y,t+s)\,dxdy\right)ds \tag{2.36}$$

satisfies

$$\lim_{n\to\infty}\mathcal{E}\left(t_n,T\right)=T\iint_{\Omega}\left(\frac{\delta^2\lambda^2}{2}|\nabla p^{\infty}|^2+\beta\Phi\left(p^{\infty}\right)\right)(x,y)\,dxdy\stackrel{def}{=}T\mathcal{E}_{\infty}, \tag{2.37}$$

in fact (2.5) implies that $\mathcal{E}(t,T)$ is monotone decreasing in t and thus we obtain the stronger result

$$\lim_{t\to\infty}\mathcal{E}(t,T)=T\mathcal{E}_{\infty}. \tag{2.38}$$

We note that the constants $\mathcal{E}_{\infty}$ are not arbitrary, rather they must be one of the critical values of the functional

$$E(p)=\iint_{\Omega}\left(\frac{\delta^2\lambda^2}{2}|\nabla p|^2+\beta\Phi(p)\right)(x,y)dxdy$$

as p ranges over $H^1(\Omega)$. These critical values are the energies associated with nontrivial solutions of (2.33) and (2.34). In the next section we shall show that finding the critical points of the above functional (and the associated critical values) is equivalent to finding the critical points (and critical values) of a real valued function defined on a finite dimensional euclidean space. The dimension of this euclidean space is related to the size of the parameter $\frac{\beta}{2\lambda^2\delta^2}\left|\Phi^{(2)}(0)\right|$. In the general case we have not succeeded in showing that for a fixed set of parameters and domain Ω either problem has only a finite number critical points but we note that the results of [1] imply that in the one-dimensional case where Ω is an interval, say $(0,1)$, there are only a finite number of critical points of E and thus only a finite number of critical values. We note there are always multiple solutions to (2.33) and (2.34) giving rise to a given critical value $\mathcal{E}_\infty$ of E; the evenness of E implies that if p^∞ is a nontrivial solution with energy $\mathcal{E}_\infty = E\left(p^\infty\right)$, then so is $-p^\infty$. The above considerations lead us to

Theorem 2. Suppose the number of pairs $(p,-p)$ of solutions to (2.33) and (2.34) are finite and suppose further that (ϕ, p) is the solution to (2.1) - (2.3) corresponding to a fixed initial condition

$$\left(\phi, \phi_t, p, p_t\right)\left(x, y, 0^+\right) = \left(\phi^0, \phi^1, p^0, p^1\right)(x, y), (x, y) \in \Omega \tag{2.39}$$

which is smooth enough so that the estimates of Lemmas 1-4 obtain. Then, there exists a unique limit $(0, p^*)$ so that

$$\lim_{t\to\infty}\left(||\phi^t||_{2,T} + ||p^t - p^*||_{2,T}\right) = 0. \tag{2.40}$$

Additionally, p^* must be one of the solutions of (2.33) and (2.34). ∎

Proof. The results of Theorem 1 guarantee that ϕ has the appropriate limiting behavior. We now assume that p has no limit and we let $0, (p_1, -p_1), (p_2, -p_2)$, $(p_M, -p_M)$ be the finite set of equilibrium solutions to (2.33) and (2.34). The hypothesis that p has no limit guarantees that for each index $j = 0, 1, \ldots, M$ we can find an $\epsilon_j > 0$ and increasing sequence of times t_n^j with $\lim_{n\to\infty} t_n^j = \infty$ such that

$$||p^{t_n^j} \mp p_j||_{2,T} \geq \epsilon_j \tag{2.41}$$

for $j = 0, 1 \ldots, M$ and $n = 1, 2, \ldots$. But the uniform boundedness of the $p^{t_n^j}$'s in $H^3\left(\Omega \times [0,T]\right)$ guarantees we can find an increasing subsequence $\tau_{k(n,j)}$ of the times t_n^j which tends to infinity such that the function $p^{\tau_{k(n,j)}}$ converge strongly in $H^2\left(\Omega \times [0,T]\right)$ to some solution of (2.33) and (2.34) and this contradicts (2.41). ∎

We conclude this section with some remarks about the system (1.6) and (1.7) when the boundary condition $(1.10)_1$ is replaced by

$$\left(h_1, h_2\right) \cdot \left(-n_2, n_1\right)\left(x^B, y^B, t\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{2.42}$$

This latter condition when combined with $(1.6)_2$ and $(1.6)_3$ implies that e satisfies the Neumann condition

$$\frac{\partial e}{\partial n}\left(x^B, y^B, t\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{2.43}$$

We again insist that p satisfies $(1.10)_2$. To analyze the long time behavior in this situation we could again introduce a potential ϕ via (1.22) and ϕ would again satisfy (1.23) but (1.24) would be replaced by

$$\frac{\partial \phi}{\partial n}\left(x^B, y^B, t\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega.$$

Identical energy estimates obtain for this problem but in this situation we loose $L_2(\Omega)$ estimates for ϕ and thus cannot avail ourselves of standard compactness results to conclude that ϕ has the desired limiting properties as t tends to infinity. Thus, when the magnetic field satisfies (2.42) and p satisfies $(1.10)_2$ we find it preferable to work directly with the original system (1.6) and (1.7). We now assume that the initial data for the magnetic field, $\mathbf{h}^0$, is divergence free and thus satisfies the compatibility condition $\int_\Omega \mathbf{h}^0 \cdot \mathbf{n} ds = 0.$ This hypothesis guarantees that for all $t > 0, \mathbf{h}$ satisfies

$$\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} = 0 \text{ and } \int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} ds = 0. \tag{2.44}$$

For the new boundary condition our results depend upon whether $\sigma_1 > 0$ or $\sigma_1 = 0$. When $\sigma_1 > 0, e$ and $\mathbf{h}$ converge to zero as t tends to infinity and p converges to a solution of (2.33) and (2.34). When $\sigma_1 = 0, e$ converges to the constant e^∞ defined by

$$e^\infty = \frac{1}{\mathrm{A}(\Omega)} \iint_\Omega \left(e\left(x,y,0\right) + p\left(x,y,0\right) - p^\infty\left(x,y\right)\right) dxdy$$

where $\mathrm{A}(\Omega)$ is the area of Ω and p converges to p^∞ which now satisfies

$$\beta g\left(p^\infty\right) - \delta^2\lambda^2 \Delta p^\infty = \beta e^\infty \tag{2.45}$$

and the boundary conditions (2.34). In the case where $\sigma_1 = 0$ the magnetic field $\mathbf{h}$ also converges to zero as t tends to infinity.

To establish these results we use identities satisfied by solutions of (1.6), (1.7), $(1.10)_2$, and (2.42). These are obtained from our previous ones by making use of (2.4). We let

$$E_1 = \frac{\beta}{2}\left(e^2 + h_1^2 + h_2^2\right) + \frac{\delta^2}{2}\left(p_t^2 + \lambda^2 |\nabla p|^2\right) + \beta\Phi(p), \tag{2.46}$$

$$E_2 = \left\{\begin{array}{ccc} \frac{\beta}{2}\left(e_t^2 + h_{1t}^2 + h_{2t}^2\right) & + & \frac{\delta^2}{2}\left(p_{tt}^2 + \lambda^2 |\nabla p_t|^2\right. \\ \frac{\beta}{2}\left(e_t^2 + c^2 |\nabla e|^2\right) & + & \frac{\delta^2}{2}\left(p_{tt}^2 + \lambda^2 |\nabla p_t|^2\right) \end{array}\right\}, \tag{2.47}$$

$$E_3 = \left\{\begin{array}{ccc} \frac{\beta}{2}\left(e_{tt}^2 + h_{1tt}^2 + h_{2tt}^2\right) & + & \frac{\delta^2}{2}\left(p_{ttt}^2 + \lambda^2 |\nabla p_{tt}|^2\right) \\ \frac{\beta}{2}\left(e_{tt}^2 + c^2 |\nabla e_t|^2\right) & + & \frac{\delta^2}{2}\left(p_{ttt}^2 + \lambda^2 |\nabla p_{tt}|^2\right) \end{array}\right\}, \tag{2.48}$$

$$\mathbf{Q}_1 = \beta c e\left(h_2, -h_1\right) + \delta^2\lambda^2 p_t \nabla p, \tag{2.49}$$

$$\mathbf{Q}_2 = \beta c e_t\left(h_{2t}, -h_{1t}\right) + \delta^2\lambda^2 p_{tt} \nabla p_t = \beta c^2 e_t \nabla e + \delta^2\lambda^2 p_{tt} \nabla p_t, \tag{2.50}$$

$$\mathbf{Q}_3 = \beta c e_{tt}\left(h_{2tt}, - h_{1tt}\right) + \delta^2 p_{ttt} \nabla p_{tt} = \beta c^2 e_{tt} \nabla e_t + \delta^2\lambda^2 p_{ttt} \nabla p_{tt}, \tag{2.51}$$

$$G_1 = -\beta\sigma_1 e^2 - \alpha\delta p_t^2, \tag{2.52}$$

$$G_2 = -\beta\sigma_1 e_t^2 - \alpha\delta p_{tt}^2 - \beta g^{(1)}(p) p_t p_{tt}, \tag{2.53}$$

and

$$G_3 = -\beta\sigma_1 e_{tt}^2 - \alpha\delta p_{ttt}^2 - \beta\left(g^{(1)}(p) p_{tt} + g^{(2)}(p) p_t^2\right) p_{ttt}. \tag{2.54}$$

It is then easily checked that for indices $i = 1 - 3$ the following identities are satisfied by solutions of (1.6), (1.7), $(1.10)_2$, and (2.42):

$$\frac{\partial E_i}{\partial t} - \text{div}\mathbf{Q}_i = G_i \quad (x, y) \in \Omega \text{ and } t > 0 \tag{2.55}$$

and

$$\mathbf{Q}_i \cdot \mathbf{n}\left(x^B, y^B, t\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega \text{ and } t > 0. \tag{2.56}$$

Additionally, the electric field, e, satisfies

$$e_{tt} + p_{tt} = c^2 \Delta e \quad , \quad (x, y) \in \Omega \text{ and } \frac{\partial e}{\partial n} = 0, \ \left(x^B, y^B\right) \in \partial\Omega. \tag{2.57}$$

The implications of these identities are summarized in

Lemma 5.

(i) If the initial data for $e, \mathbf{h}$, and p are sufficiently smooth, then for each $t \geq 0$ and $T > 0$ the functions

$$e^t(x, y, s) = e(x, y, t + s) \quad , \quad (x, y) \in \Omega \text{ and } 0 \leq s \leq T \tag{2.58}$$

and

$$p^t(x, y, s) = p(x, y, t + s) \quad , \quad (x, y) \in \Omega \text{ and } 0 \leq s \leq T \tag{2.59}$$

are respectively in $H^2(\Omega \times [0, T])$ and $H^3(\Omega \times [0, T])$ with bounds which depend only on the initial data and T.

(ii) The derivatives p_t and p_{tt} converge strongly to zero in $L_2(\Omega)$ as $t \to \infty$ and p also satisfies the decay estimates (2.27).

(iii) If $\sigma_1 > 0$, then e and e_t converge strongly to zero in $L_2(\Omega)$ as $t \to \infty$ and the functions

$$\left(e^t, e_s^t, e_{ss}^t\right)(x, y, s) = (e, e_s, e_{ss})(x, y, t + s) \quad , \quad 0 \leq s \leq T \tag{2.60}$$

converge to zero strongly in $L_2(\Omega \times [0, T])$ as $t \to \infty$. ∎

Thus, if we exploit (ii) and (iii) of the preceding lemma we find that if $(e, \mathbf{h}, p)$ is a solution of (1.6), (1.7), (1.10), and (2.42), then

$$\lim_{t \to \infty} \iint_\Omega \left(h_1^2 + h_2^2\right)(x, y, t)\, dx\, dy = 0. \tag{2.61}$$

This latter result follows from the fact that **h** satisfies (2.42), (2.44), and

$$\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} = \omega \stackrel{def}{=} \frac{1}{c}\left(e_t + p_t\right) \quad , \quad (x,y) \in \Omega \text{ and } t > 0 \tag{2.62}$$

where

$$\iint_{\Omega} \omega(x,y,t)dxdy \equiv 0 \text{ and } \lim_{t\to\infty} \iint_{\Omega} \omega^2(x,y,t)dxdy = 0. \tag{2.63}$$

The fact that solutions of (2.42), (2.44), (2.62), and (2.63) satisfy the a-priori estimate

$$\iint_{\Omega} \left(h_1^2 + h_2^2\right)(x,y,t)dxdy \le \frac{1}{\wedge_2} \iint_{\Omega} \omega^2(x,y,t)dxdy, \tag{2.64}$$

guarantees that **h** satisfies (2.61). The constant $\wedge_2$ in (2.64) is the smallest positive eigenvalue of the laplacian on Ω with eigenfunctions satisfying a zero Neumann condition on $\partial\Omega$ which are orthogonal to constants.

The situation when $\sigma_1 = 0$ is more subtle. Here we use the arguments employed earlier to show that ϕ^∞ was zero to conclude that

$$\lim_{t\to\infty} \left(||e_s^t||_{0,T}^2 + ||h_1^t||_{0,T}^2 + ||h_2^t||_{0,T}^2\right) = 0. \tag{2.65}$$

Equations (2.27) and (2.65) then imply that the ω-limit set of solutions of (1.6), (1.7), $(1.10)_2$, and (2.42) consist of fields $(e^\infty, \mathbf{h}^\infty, p^\infty)$ where e^∞ is a constant on $\Omega, \mathbf{h}^\infty \equiv 0$, and e^∞ and p^∞ are related by

$$\beta e^\infty = \beta g\left(p^\infty\right) - \delta^2\lambda^2\Delta p^\infty \quad , \quad (x,y) \in \Omega \tag{2.66}$$

and

$$\frac{\partial p^\infty}{\partial n}\left(x^B, y^B\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{2.67}$$

The constant e^∞ is related to p^∞ by

$$e^\infty = \frac{1}{\mathrm{A}(\Omega)} \iint_{\Omega} \left(e(x,y,0) + p(x,y,0) - p^\infty(x,y)\right) dxdy \tag{2.68}$$

and again $\mathrm{A}(\Omega)$ is the area of Ω. Finally that the averaged energy satisfies

$$\begin{aligned} &\lim_{t\to\infty} \int_0^T \left(\iint_{\Omega} E_1\left(x,y,t+s\right) dxdy \right) ds \\ &= T\left(\frac{\beta}{2}\mathrm{A}(\Omega)\left(e^\infty\right)^2 + \iint_{\Omega} \left(\frac{\delta^2\lambda^2}{2}| \nabla p^\infty|^2 + \beta\Phi\left(p^\infty\right)\right)(x,y)dxdy\right) \end{aligned} \tag{2.69}$$

where E_1 is the energy density defined in (2.46). This concludes section 2.

3. The Equilibrium Problem (2.33) and (2.34).

In this section we examine the equilibrium problem (2.33) and (2.34), namely

$$-\Delta p + \wedge p\gamma\left(p^2\right) = \wedge p \quad , \quad (x,y) \in \Omega \tag{3.1}$$

and

$$\frac{\partial p}{\partial n}\left(x^B, y^B\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{3.2}$$

Here

$$\wedge = \frac{\beta}{\delta^2\lambda^2} \tag{3.3}$$

where $0 < \delta$, $0 < \lambda$, and $0 < \beta$ are the parameters introduced in section 1. This problem is equivalent to (2.33) and (2.34) with

$$g(p) = p\left(\gamma\left(p^2\right) - 1\right). \tag{3.4}$$

We assume that $\gamma(\cdot)$ is $C^2[0,\infty)$ and satisfies

$$\gamma(0) = 0 \text{ and } 0 < \gamma^{(1)}(s) \text{ for } 0 \leq s, \tag{3.5}$$

$$\gamma(1) = 1, \tag{3.6}$$

$$\gamma(s) \sim (k+1) \quad , \quad 0 < k < \infty, \text{ as } s \to \infty, \tag{3.7}$$

and

$$\lim_{s\to\infty} s^n \gamma^{(n)}(s) = 0 \quad , \quad n = 1 \text{ and } 2. \tag{3.8}$$

These assumptions guarantee the function $g(\cdot)$ defined in (3.4) satisfies the assumptions laid down in (1.8). For such g's the potential Φ is given by

$$\Phi = \frac{1}{2}\left(\Psi\left(p^2\right) - p^2\right) \text{ and } \Psi(s) = \int_0^s \gamma(y)dy. \tag{3.9}$$

Φ has the double-well character and as $|p| \to \infty$

$$\Phi \sim \frac{k}{2}p^2 \tag{3.10}$$

as desired.

The problem (3.1) and (3.2) has the trivial equilibria $p \equiv \pm 1$ and $p \equiv 0$ and the non constant equilibria p satisfy the a-priori bounds $-1 \leq p \leq 1$. These inequalities follow from (3.5) and (3.6) and the maximum (minimum) principle for the Laplace operator.

Our basic result is that finding the non trivial equilibrium solutions of (3.1) and (3.2) is equivalent to the finite dimensional problem finding the critical points of an even C^2 function,

$\mathcal{J}$, on R^N. The integer N is equal to the number of eigenvalues of $-\Delta$ (with eigenfunctions which satisfy (3.2)) which are less than $\wedge$.

The function $\mathcal{J}$ has critical values satisfying

$$\frac{\wedge \mathrm{A}(\Omega)\,(\Psi(1)-1)}{2} \leq \mathcal{J}_{\substack{\text{critical}\\ \text{value}}} \leq 0,$$

has an isolated local maxima at $\mathbf{u} = \mathbf{0}$ satisfying $\mathcal{J}(\mathbf{0}) = 0$, and satisfies the asymptotic estimate

$$\mathcal{J}\,(\mathbf{u}) \sim M\|\mathbf{u}\|^2 \quad , \quad M > 0$$

as $\|\mathbf{u}\|$ tends to infinity. These estimates guarantee that for δ large enough

$$\mathcal{C} \overset{def}{=} \left\{\mathbf{u} \in R^N \,|\nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}) = \mathbf{0}\right\} \subset \left\{\mathbf{u} \in R^N \middle|\, \|\mathbf{u}\| < \delta\right\}. \tag{3.11}$$

The critical points may be obtained by examining the limit points of the gradient flow

$$\frac{d\mathbf{u}}{dt} = -\,\nabla_{\mathbf{u}}\,\mathcal{J}(\mathbf{u});$$

specifically, if we let

$$\mathcal{S}(\epsilon, 0) = \left\{\mathbf{u} \in R^N | \mathcal{J}(\mathbf{u}) \equiv \epsilon\right\} \tag{3.12}$$

and

$$\mathcal{S}(\epsilon, \infty) = \left\{\mathbf{u}_\infty \in R^N | \mathbf{u}_\infty \in \omega - \text{limit set of } \mathcal{S}(\epsilon, 0)\right\}, \tag{3.13}$$

then

$$\mathcal{C} = \bigcup_{\epsilon \in I} \mathcal{S}(\epsilon, \infty). \tag{3.14}$$

where I is the internal $\dfrac{\wedge \mathrm{A}(\Omega)(\Psi(1)-1)}{2} \leq \epsilon \leq 0$. Though not a particularly effective computational algorithm these observations point out that solutions of (3.1) and (3.2) can be obtained by taking the limits of a finite dimensional system of differential equations rather than the infinite dimensional system described in sections 1 and 2.

Now, and in what follows, we assume Ω has a complete set of eigenfunctions, ϕ_i, with eigenvalues, $\wedge_i$, satisfying

$$-\Delta \phi_i = \wedge_i \phi_i \text{ in } \Omega \text{ and } \frac{\partial \phi_i}{\partial n} = 0 \text{ on } \partial\Omega. \tag{3.15}$$

The numbers $\wedge_i$ and $\wedge$ are ordered as indicated below

$$0 = \wedge_1 \leq \wedge_2 \leq \ldots \leq \wedge_N < \wedge < \wedge_{N+1} \leq \ldots \leq \wedge_{N+i} \tag{3.16}$$

and the eigenfunctions, ϕ_i, satisfy the normalization conditions

$$\phi_i = \frac{1}{A^{1/2}(\Omega)} \quad \text{and} \quad \iint_\Omega \phi_i \phi_j \; dxdy = \delta_{i,j}. \tag{3.17}$$

We let

$$\mathcal{M}_N = \left\{ u \in H^1(\Omega) \mid u = \sum_{i=1}^{N} u_i \phi_i \right\} \tag{3.18}$$

and

$$\mathcal{N}_N = \left\{ v \in H^1(\Omega) \mid v = \sum_{i=1}^{\infty} v_i \phi_{N+i} \right\} \tag{3.19}$$

and note that for functions $u \in \mathcal{M}_N$ the $L_2(\Omega)$ and $H^1(\Omega)$ norms generate equivalent topologies whereas for functions $v \in \mathcal{N}_N$

$$||v||_{1,\Omega}^2 \stackrel{def}{=} \iint_\Omega (\nabla v \cdot \nabla v + v^2)\, dxdy = \sum_{i=1}^{\infty} (\wedge_{N+i} + 1)\, v_i^2 \le \frac{(\wedge_{N+1} + 1)}{(\wedge_{N+1} - \wedge)}\, |||v|||_{1,\Omega}^2 \tag{3.20}$$

where

$$|||v|||_{1,\Omega}^2 \stackrel{def}{=} \sum_{i=1}^{\infty} (\wedge_{N+i} - \wedge)\, v_i^2 \le ||v||_{1,\Omega}^2. \tag{3.21}$$

Moreover, solving the equilibrium problem (3.1) and (3.2) is equivalent to finding $u \in \mathcal{M}_N$ and $v \in \mathcal{N}_N$ such that

$$(\wedge_i - \wedge)\, u_i + \wedge \iint_\Omega \phi_i\, (u+v)\, \gamma\left((u+v)^2\right) dxdy = 0 \quad , \quad 1 \le i \le N \tag{3.22}$$

and

$$(\wedge_{N+i} - \wedge)\, v_i + \wedge \iint_\Omega \phi_{N+i}\, (u+v)\, \gamma\left((u+v)^2\right) dxdy = 0 \quad , \quad 1 \le i, \tag{3.23}$$

and solving (3.22) and (3.23) is equivalent to finding the critical points $u \in \mathcal{M}_N$ and $v \in \mathcal{N}_N$ of

$$\begin{aligned} J(u,v) &\stackrel{def}{=} \frac{1}{2} \iint_\Omega \left(\nabla u \cdot \nabla u + \nabla v \cdot \nabla v - \wedge\, (u^2 + v^2) + \wedge \Psi\left((u+v)^2\right)\right) dxdy \\ &= \frac{1}{2} \sum_{i=1}^{N} (\wedge_i - \wedge)\, u_i^2 + \frac{1}{2} \sum_{i=1}^{\infty} (\wedge_{N+i} - \wedge)\, v_i^2 + \frac{1}{2} \iint_\Omega \Psi\left((u+v)^2\right) dxdy, \end{aligned} \tag{3.24}$$

that is solutions of

$$\frac{\partial J}{\partial u_i} = 0 \quad , \quad 1 \le i \le N \tag{3.25}$$

and

$$\frac{\partial J}{\partial v_i} = 0 \quad , \quad 1 \le i. \tag{3.26}$$

We observe that if $u \in \mathcal{M}_N$ and $v \in \mathcal{N}_N$ satisfy (3.22) and (3.23), then the following additional identities must hold

$$\sum_{i\in 1}^{N} (\wedge_i - \wedge)\, u_i^2 + \wedge \iint_{\Omega} u(u+v)\gamma\left((u+v)^2\right) dxdy = 0 \tag{3.27}$$

and

$$\sum_{i=1}^{\infty} (\wedge_{N+i} - \wedge)\, v_i^2 + \wedge \iint_{\Omega} v(u+v)\gamma\left((u+v)^2\right) dxdy = 0. \tag{3.28}$$

These last identities imply that if $u \in \mathcal{M}_N$ and $v \in N_N$ is a critical point of J, then

$$J_{\substack{\text{critical}\\\text{value}}}(u,v) = \frac{\wedge}{2} \iint_{\Omega} \left(\Psi\left((u+v)^2\right) - (u+v)^2\right) \gamma\left((u+v)^2\right) dxdy. \tag{3.29}$$

The fact the function $p \overset{def}{=} u+v$ satisfies (3.1) and (3.2) and the bounds $-1 \le p \le 1$ and the fact that $\gamma(\cdot)$ satisfies (3.5) and (3.6) guarantees that any critical value of J satisfies the bounds

$$\frac{\wedge \mathrm{A}(\Omega)}{2} \left(\Psi(1) - 1\right) \le J_{\substack{\text{critical}\\\text{value}}}(u,v) \le 0. \tag{3.30}$$

Moreover, the lower bound is achieved at the critical points $(u,v) \equiv (\pm 1, 0) = \left(\pm A^{1/2}(\Omega)\phi_1, 0\right)$. We are interested in the other critical points of J.

For fixed $u \in \mathcal{M}_N$ we first focus on the system (3.23) (equivalently (3.26)). We note that if $v \in \mathcal{N}_N$ is a solution of (3.23), then

$$\begin{aligned}
(\wedge_{N+1} - \wedge) &\iint_{\Omega} v^2 dxdy \le |||v|||_{1,\Omega}^2 \\
&\le \sum_{i=1}^{\infty} (\wedge_{N+i} - \wedge)\, v_i^2 + \wedge \iint_{\Omega} \gamma\left((u+v)^2\right) \left(v^2\right)^2 dxdy \\
&= -\wedge \iint_{\Omega} \gamma\left((u+v)^2\right) uv\, dxdy \\
&\le \wedge(k+1) \left(\iint_{\Omega} u^2 dxdy\right)^{1/2} \left(\iint_{\Omega} v^2 dxdy\right)^{1/2} \\
&\le \frac{\wedge(k+1)}{\wedge_{N+1} - \wedge} \left(\iint_{\Omega} u^2 dxdy\right)^{1/2} |||v|||_{1,\Omega}
\end{aligned} \tag{3.31}$$

where $k+1 = \lim_{s\to\infty} \gamma(s)$ is the upper bound for $\gamma(\cdot)$ on $s \ge 0$. The last inequality together with the strong monotonicity estimate

$$|||v^2 - v^1|||_{1,\Omega}^2 \le \sum_{i=1}^{\infty} \left(v_i^2 - v_i^1\right)\left(T_i\left(u, v^2\right) - T_i\left(u, v^1\right)\right)$$

(with $T_i(u,v)$ defined by the expression on the left-hand side of (3.23)) guarantees that for each $u \in \mathcal{M}_N$ there is a unique $v = \hat{v}(u) \in \mathcal{N}_N$ satisfying (3.23). Moreover, (3.5) - (3.9) imply that this mapping is C^2 on $\mathcal{M}_N$ and has the following additional properties:

$$\hat{v}(-u) = -\hat{v}(u), \tag{3.32}$$

$$\hat{v}_i(te) = \frac{-\wedge\gamma^{(1)}(0)t^3}{\wedge_{N+1} - \wedge} \iint_\Omega \phi_{N+1} e^3 dxdy \quad , \quad \text{as } t \to 0^+, \tag{3.33}$$

and

$$\hat{v}(te)/t = o(1) \quad , \quad \text{as } t \to \infty. \tag{3.34}$$

In (3.33) and (3.34), $e = \sum_{i \in 1}^{N} e_i \phi_i$ and $\iint_\Omega e^2 dxdy = \sum_{i=1}^{N} e_i^2 = 1$.

We now turn our attention to the system (3.22) where

$$v = \hat{v}(u) \overset{def}{=} \sum_{i=1}^{\infty} \hat{v}_i(u) \phi_{N+i} \tag{3.35}$$

and the $\hat{v}_i(u)$'s are the unique solution of (3.23). Once again the solutions $u = \sum_{i=1}^{N} u_i \phi_i$ of this system are critical points of

$$J\left(u, \hat{v}(u)\right) = \frac{1}{2} \sum_{i=1}^{N} \left(\wedge_i - \wedge\right) u_i^2 + \frac{1}{2} \sum_{i=1}^{\infty} \left(\wedge_{N+i} - \wedge\right) \hat{v}_i^2(u) + \frac{\wedge}{2} \iint_\Omega \Psi\left(\left(u + \hat{v}(u)\right)^2\right) dxdy, \tag{3.36}$$

that is the solutions of $\dfrac{\partial J}{\partial u_i} = 0 \quad , \quad i = 1, 2, \ldots N$. Moreover, the fact that $v = \hat{v}(u)$ satisfies (3.28) implies that $J\left(u, \hat{v}(u)\right)$ reduces to

$$\mathcal{J}(\mathbf{u}) \overset{\text{def}}{=} J\left(u, \hat{v}(u)\right) = \frac{1}{2} \sum_{i=1}^{N} \left(\wedge_i - \wedge\right) u_i^2 + \frac{\wedge}{2} \iint_\Omega \left(\Psi\left(\left(u + \hat{v}(u)\right)^2\right) - \hat{v}(u)\left(u + \hat{v}(u)\right)\gamma\left(\left(u + \hat{v}(u)\right)^2\right)\right) dxdy \tag{3.37}$$

and the inequality (3.30) implies that critical values of $\mathcal{J}(\cdot)$ also satisfy (3.30).

We now record some facts about $\mathcal{J}(\cdot)$. The first is that

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) = \frac{\partial J}{\partial u_i}\left(u, \hat{v}(u)\right) \quad , \quad 1 \le i \le N. \tag{3.38}$$

This identity follows from the fact that $v = \hat{v}(u)$ satisfies (3.26). An immediate consequence of (3.37) is the identity

$$
\begin{aligned}
\frac{\partial^2 \mathcal{J}}{\partial u_i \partial u_j} &= (\wedge_i - \wedge)\,\delta_{ij} + \iint\limits_{\Omega} \phi_i \phi_j \left(\gamma\left((u+\hat{v}(u))^2\right) + 2\left((u+\hat{v}(u))^2 \gamma^{(1)}\left((u+\hat{v}(u))^2\right)\right)\right) dxdy \\
&+ \wedge \iint\limits_{\Omega} \phi_i \left(\gamma\left((u+\hat{v}(u))^2\right) + 2\,(u+\hat{v}(u))^2 \gamma^{(1)}\left((u+\hat{v}(u))^2\right) \sum_{p=1}^{\infty} \frac{\partial \hat{v}_p}{\partial u_j}(u)\phi_p\right) dxdy.
\end{aligned} \tag{3.39}
$$

The asymptotic estimate (3.33) guarantees that $\hat{v}(0) = 0$ and $\frac{\partial \hat{v}}{\partial u_j}(0) = 0$ and these identities, along with (3.5) and (3.22), imply that

$$
\frac{\partial \mathcal{J}}{\partial u_i}(0) = 0 \text{ and } \frac{\partial^2 \mathcal{J}}{\partial u_i \partial u_j}(0) = \text{diag }\left(\wedge_1 - \wedge, \wedge_2 - \wedge, \ldots, \wedge_N - \wedge\right)
$$

and thus that $\mathbf{u} = 0$ is an isolated local maxima of $\mathcal{J}$. We note that $\mathcal{J}(\mathbf{0}) = 0$. The asymptotic estimates (3.7) and (3.35) guarantee that for e's satisfying

$$
e = \sum_{i=1}^{N} e_i \phi_i \text{ and } \iint\limits_{\Omega} e^2 dxdy = \sum_{i=1}^{N} e_i^2 = 1 \tag{3.40}
$$

$\mathcal{J}$ satisfies

$$
\mathcal{J}(t\mathbf{e}) \sim \frac{t^2}{2}\left(\sum_{i=1}^{n} \wedge_i\, e_i^2 + \wedge k\right) \text{ as } t \to \infty \tag{3.41}
$$

and the latter estimate, together with the fact that the critical values of $\mathcal{J}$ satisfy (3.30), guarantees that for δ large enough all critical points of $\mathcal{J}$ satisfy (3.11) - (3.14).

We can also apply the Lyusternik-Schnirelman theory (see e.g. [4], [5]) to the function $\mathcal{J}(\mathbf{u})$ on $\mathcal{M}_N$ to determine critical levels of this functional and corresponding non-trivial solutions of

$$
\nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}) = 0. \tag{3.42}
$$

With the exception of the constant solutions, these can be expected to be saddles rather than local maxima or minima. These critical values can be characterized as follows. Let $\sum_n, n = 1, 2, \ldots, N$ denote the collection of compact, balanced (i.e. invariant under the map $\mathbf{u} \to -\mathbf{u}$) subsets $S \subseteq \mathcal{M}_N \backslash \{0\}$ of genus $\leq n$. The genus of a compact, balanced subset of $\mathcal{M}_N \backslash \{0\}$ is the least integer n such that there exists an odd map $f : S \to \mathbf{S}^{n-1}$ (the $(n-1)$-sphere); clearly for S as above the genus is $\leq N$ and by the Borsuk-Ulam theorem an n-sphere has genus $n+1$; for more details see [4], [5].

Put

$$
c_n = \min_{S \in \sum_n} \max_{\mathbf{u} \in S} \mathcal{J}(\mathbf{u}), \quad n = 1, \ldots N, \tag{3.43}
$$

then

$$
\frac{1}{2} \Lambda A(\Omega)\left(\Psi(1) - 1\right) = c_1 \leq c_2 \leq \ldots \leq c_N < 0. \tag{3.44}
$$

The left-most identity follows from (3.30), the monotonicity of the c_n's from the definition (3.43) and the last inequality from the fact that $\mathbf{u} = 0$ is an isolated local maximum. It can also be shown that $c_1 < c_2$.

By a standard application of the Lyusternik-Schirelman theory it follows that the c_n's defined by (3.43) are critical values of $\mathcal{J}(\mathbf{u})$. If these numbers are distinct, this implies the existence of at least N pairs of solutions to (3.42). If there is repetition, i.e. if

$$c_j = c_{j+1} = \ldots = c_{j+k-1},$$

for some $j : 1 \leq j \leq N - k + 1$ then the set of solutions to (3.42) on the level c_j is a set of genus k. In particular a set of solutions of genus k will contain k pairs $(\mathbf{u}_i, -\mathbf{u}_i)$, $i = 1, \ldots, k$ with inner product $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$.

This concludes section 3.

4. Concluding Remarks

We conclude with some remarks about the ferromagnetic case. We assume that

$$\mathbf{B} = \mu\left(\mathbf{H} + \mathbf{M}\right) \tag{4.1}$$

where

$$\mathbf{H} = ah\mathbf{e}_3 \text{ and } \mathbf{M} = am\mathbf{e}_3\ , \tag{4.2}$$

that

$$\mathbf{E} = b\left(e_1\mathbf{e}_1 + e_2\mathbf{e}_2\right), \tag{4.3}$$

and that

$$\mathbf{D} = \epsilon\mathbf{E}. \tag{4.4}$$

Again, $a > 0,\ b > 0,\ \frac{b}{a} = \sqrt{\frac{\epsilon}{\mu}},\ c = \frac{1}{\sqrt{\epsilon\mu}},$ and $\sigma_1 = \frac{\sigma}{\epsilon}$ and all fields are functions of $x, y,$ and t. In this case Maxwell's equations reduce to

$$\left.\begin{aligned} h_t + m_t &= c\left(e_{1y} - e_{2x}\right), \\ e_{1t} + \sigma_1 e_1 &= ch_y, \\ \text{and} \qquad & \\ e_{2t} + \sigma_1 e_2 &= -ch_x, \end{aligned}\right\} \tag{4.5}$$

and we supplement this system with the following evolution equation for m:

$$\delta^2 m_{tt} + \alpha\delta m_t - \beta\left(h - g(m)\right) = \delta^2\lambda^2\Delta m. \tag{4.6}$$

We assume these equations hold in a simply connected domain Ω. On m we impose the natural boundary condition

$$\frac{\partial m}{\partial n}\left(x^B, y^B\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{4.7}$$

We further assume that

$$\mathbf{E}\cdot\mathbf{n}^{\perp} = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega \tag{4.8}$$

where once again $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ is the unit exterior normal to $\partial\Omega$ and $\mathbf{n}^{\perp} = -n_2\mathbf{e}_1 + n_1\mathbf{e}_2$ in the unit tangent to $\partial\Omega$. The latter boundary condition implies that h also satisfies the zero flux condition

$$\frac{\partial h}{\partial n}\left(x^B, y^B\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{4.9}$$

The analysis of the above problem is essentially identical to the analysis ferroelectric equations when the boundary conditions (1.10) and (2.42) obtain. We assume that the initial data for $\mathbf{E}$ is divergence free and thus we are guaranteed that for all $t > 0$

$$e_{1x} + e_{2y} = 0 \quad \text{and} \quad \int_{\partial\Omega} \left(e_1 n_1 + e_2 n_2\right) ds = 0. \tag{4.10}$$

Our results for the ferromagnetic problem are summarized below:

(*i*) Independently of whether $\sigma_1 > 0$ or $\sigma_1 = 0$ $\mathbf{E} = e_1\mathbf{e}_1 + e_2\mathbf{e}_2$ converges strongly to zero in $L_2(\Omega)$ as $t \to \infty$.

(*ii*) h converges to a constant, h^∞, and m converges to a stationary distribution m^∞ and both are related by

$$\Delta m^\infty + \frac{\beta}{\lambda^2\delta^2}\left(h^\infty - g\left(m^\infty\right)\right) = 0 \quad , \quad (x,y) \in \Omega, \tag{4.11}$$

$$h^\infty = \frac{1}{A(\Omega)} \iint_\Omega \left(h(x,y,0) + m(x,y,0) - m^\infty(x,y)\right) dxdy, \tag{4.12}$$

and

$$\frac{\partial m^\infty}{\partial n}\left(x^B, y^B\right) = 0 \quad , \quad \left(x^B, y^B\right) \in \partial\Omega. \tag{4.13}$$

(iii) The energy

$$E(t) \stackrel{def}{=} \iint_\Omega \left(\frac{\beta}{2}\left(h^2 + e_1^2 + e_2^2\right) + \beta\Phi\left(m\right) + \frac{\delta^2}{2}\left(m_t^2 + \lambda^2 |\nabla m|^2\right)\right)(x,y,t)dxdy \tag{4.14}$$

is monotone decreasing in t and for any $T > 0$ the averaged energy

$$E(t,T) \stackrel{def}{=} \int_0^T E(t+s)ds \tag{4.15}$$

satisfies $\lim\limits_{t\to\infty} E(t,T) = T\mathcal{E}_\infty$ where

$$\mathcal{E}_\infty = \iint_\Omega \left(\beta\Phi\left(m^\infty\right) + \frac{\delta^2\lambda^2}{2}|\nabla m^\infty|^2\right) dxdy + \frac{\beta A(\Omega)\left(h^\infty\right)^2}{2} \tag{4.16}$$

and h^∞ and m^∞ are defined in (4.11) - (4.13) above. We note that the equilibrium energy defined in (4.16) is the one used in [3] in the study of steady state ferromagnetism.

This concludes our paper.

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