RATIONAL COMPARISON OF PROBABILITIES
VIA A BLOW-UP CONJUGACY

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Rational Comparison of Probabilities via a Blow-Up Conjugacy

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Abstract

We consider the problem of producing a subjectively acceptable quantitative measure of the difference between two numerical measurements taking values between -1 and $+1$. We begin by listing the required properties of any such *subjective difference measure*. Then we define a family of measures via conjugacy in terms of a *blow-up transformation* β which we initially leave unspecified, and describe the *blown-up metric* corresponding to a given β. We establish necessary and sufficient conditions on β in order for the associated subjective difference measure to satisfy the properties of an admissible measure described previously, and we give several concrete choices for β leading to admissible measures. We show that one of these measures has interesting connections with probability, Dempster-Shafer evidence theory, and special relativity. We show how to find a blow-up transformation β leading to a given measure if one exists, thus in particular establishing the equivalence of the basic objects of our theory, namely the difference measure, the blow-up transformation, and the metric. We close by illustrating the application of one of our measures to lateralization assessment in a computational simulation of sensory map formation in a bihemispheric brain.

Introduction

The problem of assigning a meaningful and useful value to the difference between two numerical measurements is an important one that arises in a variety of contexts. Examples include the determination of the degree of directional asymmetry in zoology [10] and in high-energy physics [5], the measurement of lateralization in neurological and psychological studies of handedness [7], language [3], and vision [8], and the combination of numerical valuations representing degrees of belief and disbelief in uncertain reasoning [12] as used for instance in knowledge-based systems [2]. In contexts such as these it is desirable to have a quantitative measure of the difference between measurement values and not merely the binary decision regarding which value is greater. Of course, one could simply subtract the two measurement values in an attempt to compare them. However, this arithmetic difference does not in general produce subjectively acceptable results, as the following simple example shows.

Example 0.1. Suppose that during the course of a game one must choose between two doors. Consider the (possibly overlapping) statements:

$$A: \quad \text{there is a prize behind the first door} \tag{1}$$

$$B: \quad \text{there is a prize behind the second door} \tag{2}$$

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No certain knowledge of the outcomes is assumed to be available *a priori*, and so we model A and B as stochastic events. Based on the information at hand one may be able to assign subjective values to the probabilities $P(A)$ and $P(B)$. Two possible scenarios for these probabilities are:

$$P(A) = .5, \quad P(B) = .6 \tag{3}$$

$$P(A) = 0., \quad P(B) = .1 \tag{4}$$

Certainly the first scenario is more encouraging, in the sense that the net probability of winning a prize is greater in the first scenario than in the second, regardless of which door is chosen. However, consider now the question of measuring, for a given scenario, the difference in desirability between the two doors. In the second scenario it seems clear that B is far better than A, while in the first scenario one feels that although B is better, the difference between B and A is not all that great. One way of quantifying these informal impressions is to consider the expected number of times that each door would have to be examined in a sequence of independent trials until a prize is found; this expected value is the reciprocal $1/P(X)$ of the probability that there is a prize behind the chosen door, and in this sense the "distance" between $P(A)$ and $P(B)$ is infinite in the first scenario while in the second it is only $\frac{1}{3}$. Yet despite this contrast between the two scenarios, the arithmetic difference $P(A) - P(B)$ between $P(A)$ and $P(B)$ is the same for both. Thus, the arithmetic difference is not suitable as a means to quantitatively compare probabilities, and a more sophisticated approach is required.

Scope of the paper

In the present paper we construct a mathematical framework within which subjectively acceptable difference measures may be defined and studied. This framework not only unifies a number of measures previously developed in a more ad-hoc manner but also yields various new measures. We are motivated by the comparison of probabilities, understood loosely as numbers between 0 and 1 representing an asymptotic relative frequency of occurrence, or a degree of belief or other subjective valuation. However, our results apply also to other contexts in which the measurement values to be compared lie in a predetermined bounded interval $[A, B]$ of non-negative real numbers. Although in certain contexts it is perhaps best to think of difference measures as operating explicitly on *events* rather than on the numbers which represent the probabilities of such events, for simplicity in most of the present paper we will adopt the point of view that the measures have numbers as their arguments. In order to use a standard scale, we transform the measurement interval into the interval $[-1, +1]$ and we produce a difference measure taking values in $[-1, +1]$ also. Heuristically, numbers in this range may be interpreted as representing either belief (positive) or disbelief (negative).

We will present an approach to constructing subjective difference measures based on the mathematical notion of *conjugacy*. This concept is a generalization of the notion of *equivalence* which relates representations of the same object in different coordinate systems. In linear algebra for example, one motivation for studying equivalence is the desire to find particularly simple and fundamental representations of a given matrix, such as that provided by the singular value decomposition [6]. As will become clear, the essence of our approach to difference measurement is the idea that in the appropriate coordinates a suitable measure of the difference between numerical values is obtained simply by using the standard arithmetic difference. This approach provides a mathematically elegant route to subjective difference measurement. We will also describe a metric associated with each such choice of coordinates, as well as the analog of scalar multiplication in the chosen coordinate system. The latter provides a nonlinear scaling mechanism that is useful in tuning the measures using experimental data of interest.

In brief, the idea underlying our approach is the following. As the above Example suggests, there is a sense in which the minimum probability value 0 is "infinitely far" from any nonzero value. In the probabilistic interpretation, it is reasonable to assume that the "distance" between events should change only in sign under complementation, and thus the maximum probability value 1 should also be considered to be infinitely far from all other values. One can make these ideas concrete by identifying the standard measurement interval $[-1,+1]$ with the extended real line $[-\infty,+\infty]$ by means of a *blow-up transformation* β which maps -1 and $+1$ to $-\infty$, $+\infty$ respectively. Distances between measurements p and q can then be judged by computing the arithmetic difference $\beta(p)-\beta(q)$ between their images in the blown-up interval $[-\infty,+\infty]$, and then mapping this difference back to the range $[-1,+1]$ through the inverse β^{-1} of the blow-up transformation. The final subjective difference measure is given by:

$$f(p,q)=\beta^{-1}\left(\beta(p)-\beta(q)\right) \tag{5}$$

We begin the paper by describing the properties that an admissible subjective difference measure is required to satisfy. Next we define a general measure of subjective difference using the notion of conjugacy relative to a blow-up transformation. We proceed to uncover the requirements for an admissible blow-up transformation, and describe some of the resulting subjective difference measures. We show that one of our measures has remarkable interpretations in terms of probability, Dempster-Shafer evidence theory, and special relativity. We show how to find a blow-up transformation producing a given difference measure by conjugacy if one exists. Finally, we give an example of the application of one of the measures to the problem of lateralization assessment in a simple computational model of a bihemispheric brain, using the data provided by human subjects in [1].

1 Admissible measures of subjective difference

In order to simplify the subsequent analysis we first transform the measurement interval $[A,B]$ into the standard interval $[-1,1]$. We do this by using the following affine normalizing transformation:

$$\theta(x)=\frac{2}{B-A}\left(x-\frac{A+B}{2}\right) \tag{6}$$

Thus, $\theta(x)$ is negative if x is less than the midpoint $\frac{A+B}{2}$ of the measurement interval and positive otherwise. Any function $f:[-1,+1]\to[-1,+1]$ defined on the standard measurement interval $[-1,+1]$ has a corresponding version $\tilde{f}:[0,1]\to[-1,+1]$ defined on the probability interval $[0,1]$. This probability version is obtained by applying the normalizing transformation of Eq. 6 to the arguments of the given measure f:

$$\tilde{f}(p,q)=f\left(\theta(p),\theta(q)\right) \tag{7}$$

We now describe the properties to be satisfied by the measures that we seek, in terms of the normalized measurement interval $[-1,+1]$.

Definition 1.1. A function $f : [-1,+1] \times [-1,+1] \to [-1,+1]$ is an *admissible difference measure* if and only if it satisfies the following properties:

$$\begin{aligned}
f \quad & \text{is continuous away from the four "corners" } (\pm 1, \pm 1) \qquad (8)\\
1 = f(1,q) > f(p_1,q) \quad & > f(p_2,q) > f(-1,q) = -1 \text{ if } 1 > p_1 > p_2 > -1, \;\; q \neq \pm 1\\
f(q,p) \quad & = -f(p,q)\\
f(-p,-q) \quad & = -f(p,q)\\
f(p_1+\delta, p_1) \quad & > f(p_2+\delta, p_2) \text{ whenever } p_1 > p_2, \text{ for every fixed } \delta > 0
\end{aligned}$$

The first three properties, continuity, monotonicity, and antisymmetry, are modelled directly on the corresponding properties for the usual arithmetic difference and certainly seem to be reasonable requirements for a general difference measure. The two remaining properties are motivated by the Example of the Introduction. By examining the probability version $\tilde{f}$ of f as in Eq. 7, we see that the first of these two properties corresponds to the symmetry requirement

$$\tilde{f}(1-p, 1-q) = -\tilde{f}(p,q) \tag{9}$$

which is a very reasonable statement about the relation between the difference of two probabilities and the difference of the complementary probabilities. The last property encodes the intuitively desirable requirement that differences should be larger when the values involved are near the ends of the measurement interval $[-1,+1]$ ($[0,1]$ in terms of the probability version) than when they are closer to the middle of the interval.

Example 1.1. An example of an admissible difference measure is the following one, which has been used in a study of auditory lateralization [3]:

$$\tilde{f}(p,q) = \frac{p-q}{((p+q)(2-p-q))^{\frac{1}{2}}} \tag{10}$$

The version of this measure on the standard interval $[-1,+1]$ is:

$$f(p,q) = \frac{p-q}{(4-(p+q)^2)^{\frac{1}{2}}} \tag{11}$$

It is a simple matter to verify that all the properties of Definition 1.1 hold for this measure.

Example 1.2. One rather widely used measure is the following one, variously referred to as an "asymmetry index", a "laterality index", a "handedness index", etc. (e.g. [8], [7], [5]):

$$f(x,y) = \frac{x-y}{x+y} \tag{12}$$

The numbers x and y have various interpretations and ranges depending on the context. They are usually taken to be non-negative and so by letting $p = x/(x+y)$ and $q = y/(x+y)$ we may assume that their values lie between 0 and 1. In terms of the normalized values p and q, the measure of Eq. 12 equals the arithmetic difference. The division by $x+y$ in Eq. 12 scales the difference so that the resulting values lie between -1 and $+1$. The corresponding measure on the interval $[-1,+1]$ is found by using Eq. 7 "in reverse":

$$f(p,q) = \tilde{f}(\theta^{-1}(p), \theta^{-1}(q)) = \frac{p-q}{p+q+2} \tag{13}$$

It follows from this expression that the properties of continuity, monotonicity, and skew-symmetry hold, while the last two properties of Definition 1.1 fail for the asymmetry index.

2 Conjugacy and subjective differences

As described in the Introduction, our approach to defining suitable difference measures is based on the concept of conjugacy via a blow-up transformation which esentially constitutes a special choice of coordinates on the interval $[-1,+1]$. We give this construction below, and describe the metric associated with each choice of coordinates, as well as the analog of scalar multiplication in the chosen coordinate system.

2.1 Conjugacy

Definition 2.1. Let $f : X' \to Y'$ be a mapping, and let $\beta_X : X \to X'$ and $\beta_Y : Y \to Y'$ be mappings onto the domain and codomain of f respectively. We say that a mapping $g : X \to Y$ is *conjugate to f via the pair* $\beta = (\beta_X, \beta_Y)$ if and only if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\beta_X} & X' \\ \downarrow g & & \downarrow f \\ Y & \xrightarrow{\beta_Y} & Y' \end{array}$$

We will be interested mainly in the case in which β_X and β_Y are invertible mappings. There is then a unique operation on X that is conjugate to f via β, namely $\beta^{\leftarrow}f := \beta_Y^{-1} \circ f \circ \beta_X$. The conjugate $\beta^{\leftarrow}f$ is also called the *pullback* of f to X via β, and is sometimes denoted $\beta^{*}f$. Conceptually, the conjugate of f via β represents the operation f as viewed in the coordinate frames on X' and Y' defined by β_X and β_Y.

Example 2.1. Suppose that the mapping f is an n-ary operation $f : Z^n \to Z$ on a space Z, and let $\beta : W \to Z$ be a mapping of some space W onto the space Z. Then the mapping conjugate to f via (β^n, β), which for simplicity we simply denote $\beta^{\leftarrow}f : W^n \to W$, is an n-ary operation on W. The associated commutative diagram is:

$$\begin{array}{ccc} W^n & \xrightarrow{\beta^n} & Z^n \\ \downarrow \beta^{\leftarrow}f & & \downarrow f \\ W & \xrightarrow{\beta} & Z \end{array}$$

Here, β^n denotes the mapping taking an n-tuple $(w_1, w_2, ..., w_n)$ of elements of W to the n-tuple $(\beta(w_1), \beta(w_2), ..., \beta(w_n))$ consisting of the images of the w_i in Z via the mapping β.

Example 2.2. If in Definition 2.1 one chooses f to be a metric on the space X' with target space Y' equal to the real line R, if Y is chosen to be R and if β_Y is the identity map of R, then assuming that β is one-to-one, the pullback $\beta^{\leftarrow}f$ is a metric on X and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\beta} & X' \\ \downarrow \beta^{\leftarrow}f & & \downarrow f \\ R & \xrightarrow{I} & R \end{array}$$

2.2 Subjective differences as conjugated differences

We now wish to use an appropriate conjugacy to produce a reasonable measure of the difference between two probabilities. We propose to consider as a measure of subjective difference on the normalized measurement interval $[-1,+1]$, the binary operation on $[-1,+1]$ that is conjugate to the standard difference operation $f(y_1, y_2) = y_2 - y_1$ on the interval $[-L,+L]$ (typically $[-\infty,+\infty]$) via an appropriate *blow-up transformation* β from $[-1,+1]$ onto $[-L,+L]$. The associated commutative diagram is shown below, where $-$ denotes the usual arithmetic difference operator on $(-L,+L)$:

$$\begin{array}{ccc} (-1,+1)\times(-1,+1) & \xrightarrow{\beta} & (-L,+L)\times(-L,+L) \\ \downarrow{-_\beta} & & \downarrow{-} \\ (-1,+1) & \xrightarrow{\beta} & (-L,+L) \end{array}$$

In other words, the difference measure $-_\beta := \beta^{\leftarrow}-$ on $[-1,+1]$ is defined by

$$a\ -_\beta\ b = \beta^{-1}(\beta(a) - \beta(b)) \tag{14}$$

In the general case in which the measurement interval is $[A, B]$ rather than $[-1,+1]$, a normalization is performed first as in Eq. 6, yielding the difference measure

$$a\ -_\beta\ b = \beta^{-1}(\beta(\theta(a)) - \beta(\theta(b))) \tag{15}$$

2.3 The pulled-back metric

As explained in the Introduction, the definition of the difference measure in Eq. 14 is based on the idea that the endpoints of the measurement interval should be considered to be infinitely far from the rest of the points of the interval. Here we briefly comment on the distance function underlying the difference measure constructed above.

One may view Eq. 14 as involving two distinct steps. In the first, the pair (a, b) is mapped to the difference $\beta(a) - \beta(b)$, which is simply the *signed* version of the pullback to $[-1,1]$ via (β, I_R) of the Euclidean metric on the real line R (see Example 2.2). In the second step, this signed distance function is pulled back to a map on $[-1,+1]$ via β. Explicitly, the pulled-back metric referred to here is given by:

$$d(a,b) = |\beta(a) - \beta(b)| \tag{16}$$

Assuming that β is differentiable, the pulled-back metric is a Riemann metric on $[-1,+1]$ with length element ds given by:

$$ds = \beta'(x)dx \tag{17}$$

We will require the length element $\beta'(x)dx$ to "blow up" at the endpoints ± 1 of the standard measurement interval; this provides a rigorous meaning to our informal comments in the Introduction regarding distances.

2.4 New measures via stretching

Given a measure $g : [-1,+1] \to [-1,+1]$ obtained via a blow-up transformation $\beta : [-1,+1] \to [-\infty,+\infty]$, a new measure is obtained by letting the group of scalings $x \mapsto tx$ for $t \in R^+$ act on $[-1,+1]$ via conjugation by the blow-up transformation β. Thus, as in Example 2.1, we have the commutative diagram shown below:

$$\begin{array}{ccc} [-1,+1] & \xrightarrow{\beta} & [-\infty,+\infty] \\ \downarrow{\scriptstyle \beta^{\leftarrow} t} & & \downarrow{\scriptstyle x \mapsto tx} \\ [-1,+1] & \xrightarrow{\beta} & [-\infty,+\infty] \end{array}$$

The collection of pullbacks $\beta^{\leftarrow} t$ forms a group of nonlinear scalings of the measurement interval $[-1,+1]$. The pulled-back scaling by t is given by:

$$(\beta^{\leftarrow} t)\, x = \beta^{-1}\left(t\beta(x)\right) \tag{18}$$

If we let the pulled-back scaling act on the measure f on $[-1,+1]$ conjugate to the difference operator on $[-L,+L]$ via β, we obtain the following new measure:

$$f_t(p,q) = \beta^{-1}\left(t\beta\left(\beta^{-1}(\beta(p)-\beta(q))\right)\right) = \beta^{-1}\left(t(\beta(p)-\beta(q))\right) \tag{19}$$

The associated blown-up metric on $[-1,+1]$ is:

$$ds = t\beta'(x)dx \tag{20}$$

2.5 Weighted combinations

Although our main interest in the present paper is to consider differences in various forms, it is clear that given any pair of scalars (α,β) one may consider the conjugate operation on $(-1,+1)$ of the linear combination operator $(x,y) \mapsto \alpha x + \beta y$ on $(-\infty,+\infty)$. In applications, this would correspond to combining two measures using different *weights* and may prove to be useful.

3 Admissible blow-up transformations

We now consider the question of determining the properties that must be satisfied by a blow-up transformation $\beta : [-1,+1] \to [-L,+L]$ so that the difference measure associated to β via conjugation as in Eq. 15 is admissible in the sense of Definition 1.1. For simplicity we assume that β is an invertible and increasing map of $[-1,+1]$ onto $[-L,+L]$; as we will see below, the symmetry about 0 of the target interval follows from the other assumptions.

Proposition 3.1. *An invertible, increasing mapping $\beta : [-1,+1] \to [-L,+L]$ is admissible if and only if the following properties hold:*

1. *β and its inverse β^{-1} are continuous*
2. *β has odd symmetry about 0, i.e. $\beta(-x) = -\beta(x)$*
3. *$\beta(x)$ is strictly convex for $x > 0$*

Proof. Since β is monotone, at each point $x \in [-1,+1]$ both the left and right limits $\beta(x\pm) = \lim_{y\to x\pm} \beta(y)$ exist. But since the range of β is an interval, these lateral limits must both be equal to the value $\beta(x)$ at x, for the intervals $(\beta(x-), \beta(x))$ and $(\beta(x), \beta(x+))$ lie outside the range of β since β is increasing. This proves that β is continuous, and the analogous argument shows that β^{-1} is continuous. Conversely, continuity of β and β^{-1} obviously imply continuity of the associated difference measure f.

The odd symmetry of β is a simple consequence of the assumption in Eq. 8 that the corresponding difference measure should be antisymmetric:

$$f(q,p) = -f(p,q)$$

Recall the definition of f in terms of β from Eq. 14:

$$f(p,q) = \beta^{-1}(\beta(p) - \beta(q))$$

Let $q = p$. Thus, since $f(p,p) = 0$, we have:

$$\beta(0) = 0 \tag{21}$$

Antisymmetry of f is equivalent to the identity:

$$\beta^{-1}(\beta(q) - \beta(p)) = -\beta^{-1}(\beta(p) - \beta(q))$$

Let $q = 0$ in this equation. Since $\beta(0) = 0$ by Eq. 21, we obtain:

$$\beta^{-1}(-\beta(p)) = -\beta^{-1}(\beta(p)) \tag{22}$$

Applying β to both sides of Eq. 22, we obtain:

$$-\beta(p) = \beta(-p) \tag{23}$$

This proves that β has odd symmetry about 0. Conversely, if we know that β has odd symmetry about 0, then we see by Eq. 14 that the corresponding difference measure f is antisymmetric.

Finally, we prove that convexity of β for positive arguments is equivalent to the final condition in Eq. 8. Assume first that the latter condition holds. We proved above that β is continuous. It therefore suffices (e.g. [11], chapter 3, exercise 3) to prove that for every pair x_1, x_2 of numbers in the interval $[0,1]$ one has:

$$\beta\left(\frac{x_1 + x_2}{2}\right) < \frac{\beta(x_1) + \beta(x_2)}{2} \tag{24}$$

Since β is a strictly increasing function by assumption, the property of Eq. 8 and the definition of the difference measure f of Eq. 15 imply:

$$\beta(p, p+\delta) \quad \text{is a strictly increasing function of } p \text{ for every fixed } \delta \tag{25}$$

Suppose for definiteness that $x_1 < x_2$. Let $\bar{x} := \frac{x_1+x_2}{2}$ and $\delta := \frac{x_2-x_1}{2}$. Then $\bar{x} = x_1 + \delta$, $x_2 = \bar{x} + \delta$, and by Eq. 25 applied to the points $p_1 = x_1$ and $p_2 = \bar{x}$, which satisfy $p_1 < p_2$, we obtain:

$$\beta(\bar{x}) - \beta(x_1) < \beta(x_2) - \beta(\bar{x}) \tag{26}$$

Eq. 24 follows, so β is convex. Conversely, convexity of β implies the condition of Eq. 8. For example, if β is differentiable then convexity implies that $\beta'(x)$ is an increasing function of x and the condition follows. This completes the proof of the Proposition.

3.1 Inverse hyperbolic tangent blow-up, $\beta(x) = \tanh^{-1}(x)$.

This choice for β in Eq. 14 leads to the following very simple expression for the associated subjective difference measure:

$$f(p,q) = \tanh\left(\tanh^{-1} p - \tanh^{-1} q\right) = \frac{p-q}{1-pq} \tag{27}$$

The following similar measure has been successfully used to combine positive and negative degrees of certainty in the medical expert system MYCIN (see [2], chapter 10):

$$f(p,q) = \tanh\left(\tanh^{-1} p - \tanh^{-1} q\right) = \frac{p-q}{1-\min(p,q)} \tag{28}$$

Eq. 28 may be viewed as an instance of Eq. 27 by interpreting the minimum operator as a non-standard multiplication operator.

Nonlinear scaling by t as in Eq. 18 embeds the measure of Eq. 27 as the case $t = 1$ of the family:

$$f_t(p,q) = \tanh\left(t\left(\tanh^{-1}(p) - \tanh^{-1}(q)\right)\right) = \frac{\left(\frac{1+p}{1-p}\right)^t - \left(\frac{1+q}{1-q}\right)^t}{\left(\frac{1+p}{1-p}\right)^t + \left(\frac{1+q}{1-q}\right)^t} \tag{29}$$

The pulled back metric defined by the argument of tanh in Eq. 29 is given as in Eq. 17 by:

$$ds = t\left(\tanh^{-1}\right)'(x)dx = \frac{tdx}{1-x^2} \tag{30}$$

We may also express the difference measure and the corresponding metric in terms of probability values in $[0,1]$ rather than the standardized values in $[-1,+1]$, by performing a preliminary normalization of the arguments as in Eq. 6. For the difference measure we obtain:

$$f_t(p,q) = \tanh\left(t\left(\tanh^{-1}(2p-1) - \tanh^{-1}(2q-1)\right)\right) = \frac{\left(\frac{p}{1-p}\right)^t - \left(\frac{q}{1-q}\right)^t}{\left(\frac{p}{1-p}\right)^t + \left(\frac{q}{1-q}\right)^t} \tag{31}$$

This reduces when $t = 1$ to:

$$f(p,q) = \frac{p-q}{p+q-2pq} \tag{32}$$

The pulled back metric on $[0,+1]$ is:

$$ds = t\left(\tanh^{-1}\right)'(x)dx = \frac{2tdx}{1-(2x-1)^2} = \frac{tdx}{2x(1-x)} \tag{33}$$

We discuss the measure of Eq. 27 further in the next section, and provide examples of the measurement values produced by it in the last section of the paper.

3.2 Tangent blow-up, $\beta(x) = \frac{2}{\pi}\tan(\frac{\pi}{2}x)$.

This choice for β yields the following alternative family of subjective difference measures:

$$f_t(p,q) = \frac{2}{\pi}\tan^{-1}\left(\frac{t\sin(\frac{\pi}{2}(p-q))}{\cos(\frac{\pi}{2}p)\cos(\frac{\pi}{2}q)}\right) \tag{34}$$

The pulled back metric on the standard interval $[-1,+1]$ is:

$$ds = \frac{tdx}{\cos^2(\frac{\pi}{2}x)} \tag{35}$$

The version of the difference measure for probability values in $[0,1]$ is:

$$f(p,q) = \frac{2}{\pi}\tan^{-1}\left(\frac{t\sin(\pi(p-q))}{\sin(\pi p)\sin(\pi q)}\right) \tag{36}$$

This corresponds to the following pulled back metric on $[0,+1]$:

$$ds = \frac{2tdx}{\sin^2(\pi x)} \tag{37}$$

3.3 Discussion

How do the above measures perform on the two scenarios described in the Example of the Introduction? As it turns out, *the two formulas give identical results.* Indeed, a computation shows that, with either one of the two formulas for f, we have for the first scenario:

$$f(.5,.6) = -1/5, \tag{38}$$

while for the second scenario,

$$f(0,.1) = -1 \tag{39}$$

In fact, we obtain the same results independently of the choice of blow-up function β in the definition of the subjective difference measure f! To see this, start with a general β and consider the associated subjective difference measure as in Eq. 15:

$$f(p,q) = \beta^{-1}(\ \beta(\theta(p)) - \beta(\theta(q))\ ) \tag{40}$$

Letting p=0, we obtain

$$f(0,q) = \beta^{-1}(\ \beta(-1) - \beta(\theta(q))\ ) = \beta^{-1}(\ -L\ ) = -1, \tag{41}$$

which explains the result on the first scenario, while if we let $p = \frac{1}{2}$, then we have

$$f(p,q) = \beta^{-1}(\ \beta(0) - \beta(\theta(q))\ ) = \beta^{-1}(\ -\beta(\theta(q))\ ) = -\theta(q) = -(2q-1), \tag{42}$$

yielding in particular the value $f(.5,.6) = -.2$, corresponding to the second scenario.

4 Probabilistic, evidential, and relativistic interpretations in the case $\beta(x) = \tanh x$

4.1 Subjective difference as relative likelihood

If we choose $\beta(x) = \tanh^{-1}(x)$, then Eq. 32 for the corresponding subjective difference measure may be written as:

$$f(p,q) = \frac{pq' - qp'}{pq' + qp'}, \tag{43}$$

where the primes denote the probabilities of the complementary events, that is, $p' = 1-p$, $q' = 1-q$. Assuming that the events

$$\begin{aligned} &A: \text{ there is a prize behind the first door} \\ &B: \text{ there is a prize behind the second door} \end{aligned} \tag{44}$$

are independent, the term pq' represents the probability that there is a prize behind the first door but not behind the second, and, similarly, qp' is the probability that only the second door hides a prize. Thus, the denominator $pq' + qp'$ is the probability that there is a prize behind exactly one of the two doors, and the subjective difference may be written as the difference of two conditional probabilities:

$$f(p,q) = P(A|A\Delta B) - P(B|A\Delta B), \tag{45}$$

where Δ is the symmetric difference operator. This result is intuitively satisfying. Furthermore, observe that the right hand side of Eq. 45 is defined even if the events A and B are not independent.

We must stop short of attempting to interpret the right-hand side of Eq. 45 as an operation defined on the numbers p and q, since these numbers alone do not determine the value of the right hand side. Rather, in the general case we must think of the subjective difference as a measure of the *relative likelihood of two events.* On the other hand, once an expression is specified for the probability $P(A \cap B)$ of the intersection of the events A and B in terms of the numbers $p = P(A)$ and $q = P(B)$, then the right hand side of Eq. 45 above becomes a well–defined function of p and q. The case of independent events, in which $P(A \cap B) = P(A)P(B)$, was already discussed above. As another example, consider the case of mutually exclusive events. In that case we have $P(A \cap B) = 0$, and substituting into the formula for the relative likelihood, Eq. 45, we obtain the asymmetry index described in Example 1.2:

$$f(p,q) = \frac{p-q}{p+q} \tag{46}$$

We stress the fact that the simple relative likelihood expression of Eq. 45 arises from the subjective difference measure only if the blow-up function $\beta(x) = \tanh^{-1}(x)$ is chosen. The author is not aware of any simple probabilistic interpretation for the subjective difference measures associated with other choices for the blow-up function.

4.2 Relation to the Dempster-Shafer theory

We briefly comment on the connection between the difference measure associated to the inverse hyperbolic tangent blow-up transformation and the theory of evidence developed in [12], usually referred to as the Dempster-Shafer theory. Dempster-Shafer theory provides a way of combining two numbers, each representing a degree of belief relative to some point of view, in order to obtain a single number representing a degree of belief that takes into account both points of view. Consider the following table, which we will call the Dempster-Shafer matrix corresponding to the events A and B.

	A	A'
B	pq	$(1-p)q$
B'	$p(1-q)$	$(1-p)(1-q)$

(47)

The symbols A' and B' denote the events complementary to A and B respectively, and each entry of the matrix has been filled in with the product of the probabilities of the events corresponding to the row and column in which the entry appears.

In Dempster-Shafer theory, a "basic probability" is assigned to events (subsets of the set $\{A, B\}$) based on the above matrix. The resulting basic probability value of an event may be interpreted as a combined degree of belief in the event that takes into account the belief information about A and B separately that is encoded in the values $p,\ 1-p$ and $q,\ 1-q$. In the case in which the events A and B are disjoint, the basic probabilities $m(A)$, $m(B)$ are given by ([12], chapter 3):

$$\begin{aligned} m(A) &= \frac{p(1-q)}{1-pq} \\ m(B) &= \frac{q(1-p)}{1-pq} \end{aligned} \tag{48}$$

The numerator of $m(E)$ in each of the two above cases is the sum of all entries of the Dempster-Shafer matrix corresponding to row/column event pairs whose intersection equals E. The denominator is the sum of all matrix entries corresponding to row/column event pairs whose intersection is nonempty. We see that for disjoint events A and B the difference measure $f(p, q)$ of Eq. 27 is related to the Dempster-Shafer probabilities very simply:

$$f(p, q) = m(A) - m(B) \tag{49}$$

Observe however that we use the standard version of $f(p, q)$ here in which p and q are assumed to range between -1 and $+1$, while in the computation of the Dempster-Shafer basic probabilities the arguments p, q are assumed to range between 0 and 1.

4.3 Subjective difference as relativistic difference of velocities

In the preceding subsection we showed that the subjective difference measure based on the blow-up function $\beta(x) = \tanh^{-1}(x)$ fits naturally into a probabilistic framework. The same difference measure arises in a problem of physics which on the surface appears to be totally unrelated to the measurement of subjective differences. This is the kinematical problem of the difference of two velocities, or, more correctly, of the relative velocity of two objects.

According to the theory of special relativity [4], the speed of an object relative to any given reference frame cannot be greater than the speed of light. If this is true, then the arithmetic difference of the velocities cannot possibly yield in general the speed of two objects relative to each other, since otherwise if the objects were moving in opposite directions it would be possible for their relative speed to be greater than the speed of light.

The correct relativistic formula for the relative speed of two objects moving at speeds v and w with respect to a given reference frame is the following [4]:

$$\mathrm{RDiff}(v, w) = \frac{v - w}{1 + \frac{vw}{c^2}}, \tag{50}$$

where c denotes the speed of light in the chosen system of units. If we take the units to be such that $c = 1$, then by an identity for the hyperbolic tangent function one has

$$\mathrm{RDiff}(v, w) = \tanh\left(\tanh^{-1} v - \tanh^{-1} w\right) \tag{51}$$

See [9], chapter 6, for a discussion in terms of Minkowski spacetime geometry.

Notice that Eq. 51 is identical to our definition of the subjective difference measure f in Eq. 14 corresponding to the blow-up transformation $\beta(x) = \tanh^{-1}(x)$! So, we see that this idea of comparing numbers by blowing up the interval, subtracting, and mapping back to the interval as in section 2 is, quite literally, very natural. In relativity, the speed of light, which we assumed to be numerically equal to 1 in our system of units, is in a sense infinitely far from any other speed (for example, it takes an infinite amount of energy to accelerate an object having nonzero mass to the speed of light). This is reminiscent of the probability Example discussed in the Introduction.

5 Recovering the blow-up function β from the difference measure

Mathematically, the following are natural questions. Which difference measures $f : [-1, +1] \to [-1, +1]$ arise via conjugacy as in our construction above? How does one determine the blow-up function corresponding to a given measure if it exists? In this section we answer these questions.

Observe first that the blow-up function β is very easy to recover from the blown-up metric $ds = \beta'(x)dx$. Since $\beta(0) = 0$ by antisymmetry (Proposition 3.1), we have simply:

$$\beta(x) = \int_0^x \beta'(u)du \tag{52}$$

We will now show that the metric $\beta'(x)dx$ may be recovered from the difference measure $f(p, q)$, thereby establishing that the basic objects of our theory are equivalent:

$$\begin{array}{ccc} f & \xrightarrow{\text{Prop. 5.1}} & \beta'(x)\,dx \\ \downarrow{=} & & \downarrow \\ f & \xleftarrow{\text{conjugacy}} & \beta \end{array}$$

Proposition 5.1. *Let $f(p,q) = \beta^{-1}(\beta(p) - \beta(q))$ be an admissible difference measure as in Definition 1.1. Suppose that f and β are continuously differentiable. Then there exists a constant $C > 0$ such that the blown-up metric is given by:*

$$\beta'(x) = C\partial_1 f(q,q) \tag{53}$$

and therefore the blow-up transformation β is given by:

$$\beta(x) = C\int_0^x \partial_1 f(q,q)\, dq \tag{54}$$

Any constant $C > 0$ in Eq. 54 yields the given measure f.

Proof. We have by assumption:

$$\beta(f(p,q)) = \beta(p) - \beta(q) \tag{55}$$

Since β and f are differentiable, we have:

$$\beta'(f(p,q))\frac{\partial f}{\partial p}(p,q) = \beta'(p) \tag{56}$$

Letting $p = q$, and recalling that $f(p,p) = 0$,

$$\beta'(0)\partial_1 f(q,q) = \beta'(q) \tag{57}$$

Thus:

$$\beta'(q) = \beta'(0)\partial_1 f(q,q) \tag{58}$$

This expresses the blown-up metric β' in terms of the given difference measure f. We now recover the blow-up transformation β as in the statement of the Proposition:

$$\beta(x) = \beta'(0)\int_0^x \partial_1 f(q,q)\, dq \tag{59}$$

Finally, we observe that since

$$(k\beta)^{-1}(k\beta(p) - k\beta(q)) = \beta^{-1}(\beta(p) - \beta(q)),$$

the value of the constant in front of the integral in Eq. 54 has no effect on the resulting difference measure. This concludes the proof.

The preceding Proposition makes it easy to determine if a given measure corresponds to some blow-up transformation β.

Corollary 5.1. *In order for an admissible measure $f(p,q)$ to be a conjugate measure of the form $f(p,q) = \beta^{-1}(\beta(p) - \beta(q))$, it is necessary and sufficient that the transformation β defined by Eq. 54 with $C = 1$ satisfy the properties of an admissible blow-up transformation as in Proposition. 3.1. In particular, β should have odd symmetry about 0 and should be convex for positive arguments.*

Example 5.1. Consider the following difference measure, which has been used in studies of functional directional asymmetry [3]:

$$\tilde{f}(R,L) = \frac{R-L}{((R+L)(2T-R-L))^{\frac{1}{2}}} \tag{60}$$

Here, R, L, T denote the number of correct right ear responses, the number of correct left ear responses, and the total number of trials, respectively (the subjects were presented with a sequence of pairs of auditory signals, one to each of the ears, and then asked to identify the signals; see [3]).

Dividing the numerator and denominator of the right-hand side of Eq. 60 by T, we obtain the version of the measure for the probabilities $p = R/T$, $q = L/T$ given in Eq. 10. Recall that the corresponding expression on the standard interval $[-1, +1]$ is:

$$f(p,q) = \frac{p-q}{(4-(p+q)^2)^{\frac{1}{2}}} \tag{61}$$

We wish to determine if this measure corresponds to a blow-up conjugacy relative to the standard difference operator. As in the above Proposition and its Corollary, we attempt to compute the blown-up metric:

$$\beta'(x)dx = \partial_1 f(x,x)dx = \frac{dx}{2(1-x^2)^{\frac{1}{2}}} \tag{62}$$

The blow-up transformation is given by:

$$\beta(x) = \int_0^x \frac{du}{2(1-u^2)^{\frac{1}{2}}} = \frac{1}{2}\sin^{-1} u \tag{63}$$

This β satisfies all the requirements for an admissible blow-up transformation, and we conclude that the given measure indeed fits into our conjugacy framework.

6 Application to cerebral lateralization

We consider the problem of measuring *cerebral lateralization.* The term cerebral lateralization refers to anatomical and functional directional asymmetry between the two hemispheres of the brain. This is usually considered in the context of a specific task or ability. For instance, it is known that on average the left hemisphere in humans is more competent than the right hemisphere in interpreting verbal descriptions. It is thus said that there is a net functional lateralization toward the left for this task. In this section we briefly discuss the measurement of lateralization in the context of sensory differentiation of spatial location. We begin by describing a visual representation of the topographic maps which encode hemispheric organization in this context. We then proceed to apply the measure of Eq. 32 to human assessments of the degrees of topographic map organization based on examples of bihemispheric topographic maps from the study [1].

6.1 Topographic maps

In order to use the subjective difference measures defined above as a basis to define measures of cerebral lateralization, one first associates with each cerebral hemisphere a number between 0 and 1 that measures the net *performance* of the hemisphere in the context of a task of interest. The subjective difference between the values of the performance measure for the two hemispheres may then be interpreted as a measure of lateralization. The hemispheric performances are obtained by analyzing graphical *topographic maps* that encode the hemispheres' abilities to determine the spatial location of sensory stimuli. See the Figure.

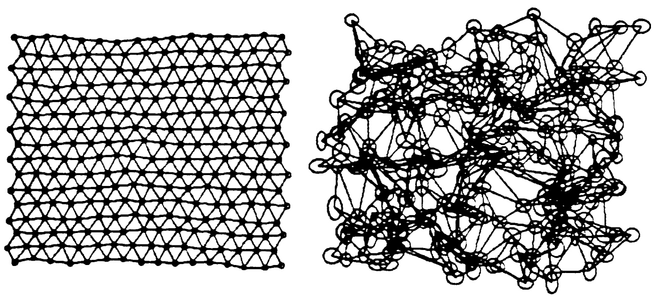
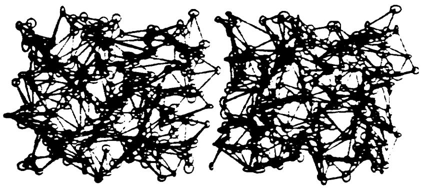


Figure 1: A bihemispheric topographic map

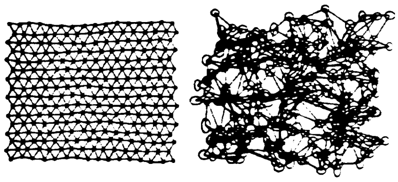
Each node of the map corresponds to a cortical column, a collection of brain cells (neurons) that are assumed to operate as a unit. The *location* of a node within the map points roughly to the location on the sensory organ which the corresponding cortical column responds to with the greatest sensitivity. The *size* of the node indicates the size of the region on the sensory organ to which the cortical column responds significantly. Collectively, the *receptive fields* so defined, for all nodes in each hemisphere, encode the ability of the hemisphere to pinpoint locations on the sensory organ. This ability is greatest when receptive fields are small and their centers form a regular lattice covering the sensory organ. The performance measures are designed to measure the degree to which such an ideal situation is attained for a given hemisphere. See [1] for details.

6.2 Examples of lateralization values obtained

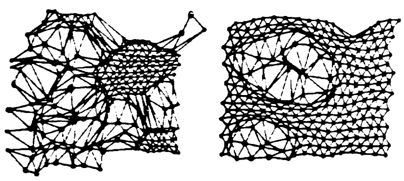
Below we give examples of bihemispheric maps and the associated mean values of map organization (denoted p_L, p_R) for each hemisphere and lateralization (h_lat) of the bihemispheric system as judged by human subjects in the study [1], together with the lateralization values (lat) obtained by applying the subjective difference measure of Eq. 32 with $t = 0.7$ to the mean organization values p_L, p_R for the two hemispheres.



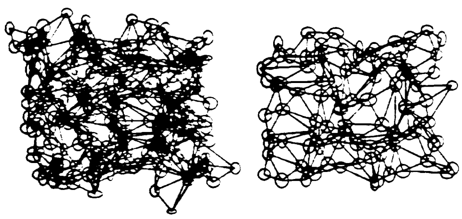
$p_L = .01$, $p_R = .01$, h_lat = 0, lat = 0



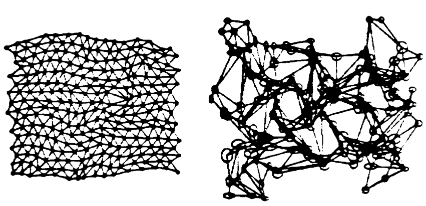
$p_L = .98$, $p_R = .03$, h_lat = −1, lat = −.99



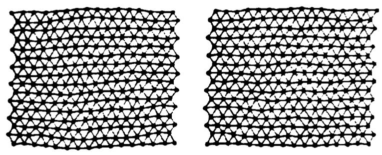
$p_L = .39$, $p_R = .71$, h_lat = .44, lat = .44



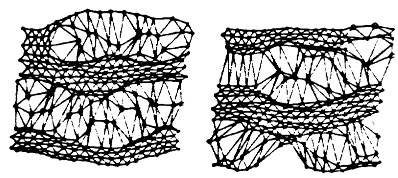
$p_L = .13$, $p_R = .19$, h_lat = .14, lat = .16



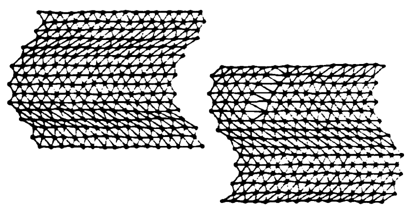
$p_L = .82$, $p_R = .11$, h_lat = −.82, lat = −.85



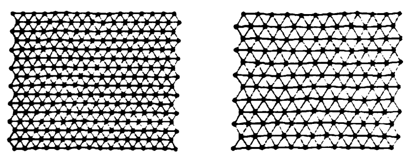
$p_L = .99$, $p_R = .99$, h_lat = 0, lat = 0



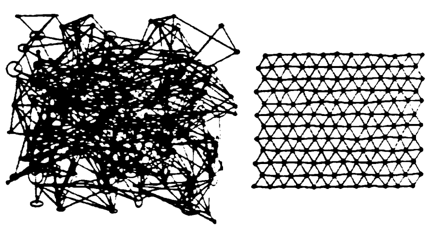
$p_L = .60$, $p_R = .60$, h_lat = −.03, lat = 0



$p_L = .88$, $p_R = .83$, h_lat = −.04, lat = −.14



$p_L = 1.0$, $p_R = .99$, h_lat = 0, lat = −1.0



$p_L = .03$, $p_R = .98$, h_lat = .98, lat = .99

Although a detailed discussion of the validity of our approach in the context of cerebral lateralization is beyond the scope of the present paper, we should say that the above results show generally excellent agreement with the mean lateralization values assigned to the examples by humans familiar with the visual representation of bihemispheric topographic maps used above. The sole exception is the map on the right in the next-to-last row, for which the left hemisphere's performance value of exactly -1.0 has forced a lateralization value of -1; this points to the need to employ higher precision in the measurement values when these values are close to ± 1. Referring to the above map examples we see that if we omit the map on the right in the next-to-last row above, the average over all the maps of the absolute value of the difference between the mean human lateralization value h_lat and the value lat computed by the measure of Eq. 32 is approximately .02, which is considerably less than the .08 average standard deviation of the human values in [1].

Conclusions

We have presented an approach to subjective difference measurement based on the notion of conjugacy via a blow-up transformation. The blow-up transformation defines a non-Euclidean metric on the interval $[-1, +1]$ of measurement values and induces a pullback to $[-1, +1]$ of the standard arithmetic difference operator on the blown-up interval, which is typically $[-\infty, +\infty]$. The pulled-back difference operator provides an admissible subjective difference measure under mild conditions on the blow-up transformation. Our approach unifies a number of previously defined measures within a single theoretical framework, and yields various families of new measures, one of which we have shown to have natural interpretations in terms of probability, Dempster-Shafer evidence theory, and special relativity.

In hindsight, a hidden blow-up transformation may be seen lurking underneath the surface in some previous approaches. Shafer [12] uses the concept of *weight of the evidence* which is a measure taking values between 0 and ∞, and in [1] some unary measures of organization are defined in terms of *disorganization measures* that may be unbounded. In the present paper we have not only made the blow-up transformation explicit but also, more importantly, we have uncovered its fundamental role in permitting a single normal form to be used to produce a variety of subjective difference measures. We have proved the basic objects of our theory, the subjective difference measure, the blow-up transformation, and the blown-up metric, to be equivalent and a process has been given to move back and forth between them. The resulting theory has a degree of elegance and simplicity not shared by the previous approaches of which we are aware.

We have presented preliminary results indicating that our framework produces measures that perform well in practice. In future work we plan to address the experimental performance of our measures in greater detail.

Acknowledgement

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