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MOTION OF A CLOSED CURVE BY MINUS THE SURFACE LAPLACIAN OF CURVATURE

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Motion of a closed curve by minus the surface Laplacian of curvature [1]

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Abstract

The phenomenon of surface diffusion is of interest in a variety of physical situations [8]. Surface diffusion is modelled by a fourth-order quasilinear parabolic partial differential equation associated with the negative of the surface Laplacian of curvature operator. We address the well-posedness of the corresponding initial value problem in the case in which the interface is a smooth closed curve Γ contained in a tubular neighborhood of a fixed simple closed curve Γ_0 in the plane. We prove existence and uniqueness, as well as analytic dependence on the initial data of classical solutions of this problem locally in time, in the spaces E^h of functions f whose Fourier transform $(\hat{f}_k)_{k\in Z}$ decays faster than $|k|^{-h}$, for $h > 5$. Our results are based on the machinery developed in [1], [2], [3], which allows the application of the method of maximal regularity [11], [14], [4] in the spaces E^h.

1 Introduction

The free-boundary problem of evolution of a closed surface by minus the surface Laplacian of the mean curvature arises in various contexts of great physical interest [8]. Some examples include sharp interface limits of the Cahn-Hilliard equation with concentration-dependent mobility [7], and the study of smectic A liquid crystal configurations [13]. The negative surface Laplacian operator models *surface diffusion*, that is, diffusion taking place within the evolving surface itself. The evolution may be thought of as being due to surface diffusion of the mass of a body enclosed by the surface [9].

The problem we are concerned with is defined as follows. A simple closed curve Γ_0 in the plane is given, and one seeks a family $(\Gamma(t))_{t\in[0,T]}$ of closed curves evolving according to the following prescription:

$$\begin{aligned} V_n &= -\Delta_{\Gamma(t)}\kappa(\Gamma(t)) \qquad \text{for } t \in (0,T) \\ \Gamma(0) &= \Gamma_0 \end{aligned} \tag{1}$$

Here, V_n is the normal velocity of the family $\Gamma(t)$ at time t, $\Delta_{\Gamma(t)}$ is the Laplace operator relative to the arclength metric on $\Gamma(t)$, and $\kappa(\Gamma(t))$ is the curvature of $\Gamma(t)$.

In recent work [9], [10], Coleman and his collaborators have numerically uncovered some interesting behavior of the solutions to this problem, including loss of convexity. The latter phenomenon points to a significant analytical obstacle to the study of well-posedness for motion by surface diffusion, namely, the lack of a maximum principle. The issues of existence, uniqueness, and dependence of solutions on the initial data have to our knowledge not previously been addressed for this problem. Assuming small initial data, Baras, Duchon, and Robert obtained in [6] existence and uniqueness

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of solutions globally in time for the analogous problem in the case of an infinite curve. However, their hypotheses require that the initial curvature either be a measure with finite total variation ([6], Theorem 2.1) or else belong to some L^p space with $1 < p \leq 2$ ([6], Theorem 2.2) and so their results do not apply in the periodic case with which we are concerned here.

In the present paper we exploit the framework developed in earlier work of Alvarez and Pego [1] to provide not only local existence and uniqueness of classical solutions for small initial data, but also analytic dependence of solutions on the initial data in spaces of periodic functions with algebraic Fourier decay. We will assume that the interface $\Gamma(t)$ is parametrized by a function $d(t) \in E^h$, where the spaces E^h are defined below in 1.1. This definition of the spaces E^h originally appeared in [1].

Definition 1.1. For $h > 0$, let E^h denote the space of distributions $f : S^1 \to \mathrm{R}$ whose Fourier transform $\hat{f} : \mathrm{Z} \to \mathrm{R}$ satisfies $\hat{f}_k = o(|k|^{-h})$ as $|k| \to \infty$, equipped with the Banach norm

$$||f||_{E^h} = \sup_{k \in \mathrm{Z}} (1 + |k|^h)|\hat{f}_k|$$

The index h gives roughly the degree of smoothness of the functions that belong to the space E^h. For example ([1], [2]), E^h embeds compactly into the Sobolev space $W^{h-\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$, as well as into the space $C^{h-1-\epsilon}$ of Hölder-continuous functions of exponent $h - 1 - \epsilon$.

The main result of the present paper is the following.

Theorem 1.1. *Let $h > 5$. Assume that the curve parametrized by $d = 0$ is of smoothness class C^∞. Then there exist $R > 0$ and $T > 0$ such that if d_0 is an element of E^h having E^h norm less than R, then there is a map $d \in C([0,T], E^h)$ solving the initial-value problem given above in Eq. 1, with $\Gamma(t) = \Gamma(d(t))$ and $\Gamma_0 = \Gamma(d_0)$. Furthermore, the mapping $d_0 \mapsto d$ is analytic from the space E^h to the space $C([0,T], E^h)$.*

Our approach is based on showing that the method of maximal regularity [11], [4], [14] applies in the phase space pair (E^{h-4}, E^h). This involves proving that the nonlinear minus surface Laplacian of curvature operator maps E^h into E^{h-4} continuously with analytic dependence on the interface parametrization d, that the linearization L of this operator around the reference curve $d = 0$ generates an analytic forward-time flow in E^{h-4}, and that L satisfies the following condition for the variation of constants integral associated with the forced dynamics for L relative to the pair of spaces (E^{h-4}, E^h).

Definition 1.2. Let X and Y be Banach spaces with Y densely embedded in X. A linear operator $L : Y \to X$ is said to satisfy the *maximal regularity property relative to the phase pair* (X, Y) iff the linear operator T defined by

$$(Tf)(t) = \int_0^t e^{(t-s)L} f(s)\, ds \tag{2}$$

maps the space $C([0,T], X)$ continuously into the space $C([0,T], Y)$ for some $T > 0$.

Roughly, the idea behind the maximal regularity property is that the solution of the nonlinear initial value problem

$$\begin{aligned} \dot{x}(t) &= G(x(t)) \\ x(0) &= 0 \end{aligned} \tag{3}$$

where $G : Y \to X$ is a vector field of class C^k whose differential at 0 equals L, may be found in terms of the solutions of the corresponding forced linearized problem by writing the nonlinear dynamics as the linear dynamics plus a forcing term containing the higher-order terms of the nonlinear vector field:

$$\dot{x}(t) = Lx(t) + N(x(t)) \tag{4}$$

The solution of Eq. 4 with $x(0) = 0$ satisfies the variation-of-constants formula:

$$x(t) = \int_0^t e^{(t-s)L} N(x(s))\, ds \tag{5}$$

If the maximal regularity property holds for L, then a fixed-point argument using the inverse function theorem yields existence and uniqueness locally in time for the initial value problem associated with the nonlinear evolution equation of Eq. 4, as well as C^k dependence on the initial data. The index k can be any natural number $k \geq 1$, or $k = \infty$, or even $k = \omega$ (analytic).

Theorem 1.2. ([4], Theorem 2.7) *If the Fréchet derivative $dG(x)$ satisfies the maximal regularity property relative to (X, Y) for every x in some X-neighborhood O of 0, then the nonlinear initial value problem of Eq. 3 has a unique solution on some small enough interval $[0, T]$.*

Theorem 1.3. ([4], Corollary 2.9) *If $G \in C^k(O, X)$ and $dG(x)$ satisfies the maximal regularity property for each $x \in O$, then the local semiflow on O generated by the initial value problem of Eq. 3 is of class C^k.*

We will show that the minus surface Laplacian of curvature vector field depends analytically on the interface parametrization d in the context of the E^h spaces, and thus by Theorems 1.2 and 1.3 we will obtain local existence, uniqueness, and analytic data dependence for this evolution problem. Since the set of operators having the maximal regularity property relative to a phase pair (X, Y) is open in the space $L(Y, X)$ of bounded linear operators from Y to X ([4], Lemma 2.1), it will be sufficient to show that the maximal regularity property holds for the single operator $L = dG(0)$, where G is the negative Laplacian of curvature field.

The reader may wonder why we work in the spaces E^h rather than in, say, L^p Sobolev spaces. The reason is that the maximal regularity property of Definition 1.2 fails in the latter spaces. In fact, a theorem by Baillon [5] (also see [12]) states that any space X on which there exists some densely defined linear operator L satisfying the maximal regularity property, must contain an embedded copy of the space c_0 of sequences vanishing at infinity. In particular, no suitable phase space can be reflexive.

We now give two key properties of the spaces E^h which are used in our analysis of motion by minus the surface Laplacian of curvature. These results were obtained in [1], [2], [3].

Theorem 1.4. *Multiplication, $(f, g) \mapsto fg$, is an analytic mapping $E^a \times E^b \to E^{\min(a,b)}$ for all pairs (a, b) satisfying $a, b \geq 0$, $\max(a, b) > 1$.*

Theorem 1.5. *Let $\Phi : S \to \mathbb{R}$ be analytic on some open set S in $\mathbb{R}$. Then composition with Φ, $f \mapsto \Phi \circ f$, is an analytic mapping from the open subset of E^h consisting of functions f whose image is contained in S, to E^h itself, for any $h > 2$.*

The proof of Theorem 1.4 is based on directly estimating the decay rate as $|k| \to \infty$ of the convolution of two sequences which decay faster than $|k|^{-a}$, respectively $|k|^{-b}$. Theorem 1.5 is proved by making use of embedding results relating the E^h spaces with the Hölder spaces C^μ, plus an analyticity result for multiplication in Hölder spaces which relies on Cauchy estimates for the analytic function Φ. Complete proofs may be found in [1], [2], [3].

The above results allow us to control nonlinear terms arising in the evolution equation, and in particular to show, in the next section of the paper, that the nonlinear vector field is well-defined and analytic as a map from E^h to E^{h-4}. After proving analyticity of the nonlinear vector field, we will focus our attention on showing that the linearized field generates an analytic semiflow on E^{h-4} and satisfies the maximal regularity property in Definition 1.2 relative to the phase pair (E^{h-4}, E^h). As the reader will see, the latter property is particularly easy to prove in the spaces E^h.

2 Analyticity of the nonlinear vector field

In this section we compute the evolution equation in tubular coordinates around a reference curve Γ_0, and show that the associated vector field is analytic on the phase pair (E^{h-4}, E^h) if $h > 5$. We describe the reference curve by its arclength parametrization $x_0(\theta)$, which after scaling Γ_0 may be assumed to be defined for θ on the unit circle S^1, and which satisfies

$$|x_0{}'(\theta)| = 1 \tag{6}$$

Let $n_0(\theta)$ denote the outward pointing unit normal vector to Γ_0 at the point $x_0(\theta)$. Each smooth function $d : S^1 \to \mathrm{R}$ then corresponds to the curve $\Gamma(d)$ parametrized as

$$x(\theta) = x_0(\theta) + d(\theta)n_0(\theta) \tag{7}$$

We will find an evolution equation for d representing the problem of motion by minus the surface Laplacian of curvature for curves $\Gamma(d)$ lying inside a tubular neighborhood of the reference curve Γ_0. We will prove the following result.

Theorem 2.1. *Let $h > 5$. There exists $R > 0$ such that the surface Laplacian of curvature operator $d \mapsto \Delta_{\Gamma(d)}\kappa(\Gamma(d))$ maps the ball of radius R in E^h analytically to E^{h-4}.*

Proof. We restrict the values $d(\theta)$ to be sufficiently small so that the map x associated with d as in Eq. 7 is a diffeomorphism from the unit circle S^1 onto $\Gamma(d)$. We now proceed to compute the surface Laplacian of curvature operator which defines the evolution of Γ. At each point $x(\theta)$ of Γ we consider the positively oriented orthonormal basis $(n_0(\theta), \tau_0(\theta))$, where $\tau_0(\theta) = x_0'(\theta)$ is the unit tangent vector to Γ_0 in the clockwise direction, and $n_0(\theta)$ is the unit normal vector as before. In terms of the curvature $\kappa_0(\theta)$ of Γ_0 we have the basic relations

$$\begin{aligned} \tau_0{}'(\theta) &= -\kappa_0(\theta)n_0(\theta) \\ n_0{}'(\theta) &= \kappa_0(\theta)\tau_0(\theta) \end{aligned} \tag{8}$$

Using these relations one finds the s-derivatives of $x(\theta)$:

$$x'(\theta) = (1 + d(\theta)\kappa_0(\theta))\,\tau_0(\theta) \;+\; d'(\theta)n_0(\theta) \tag{9}$$

and

$$x''(\theta) = \left(2d'(\theta)\kappa_0(\theta) + d(\theta)\kappa_0'(\theta)\right)\tau_0(\theta) \; + \; \left(d''(\theta) - (1 + d(\theta)\kappa_0(\theta))\,\kappa_0(\theta)\right) n_0(\theta) \tag{10}$$

From Eq. 9 one obtains the orthonormal basis $(n(\theta), \tau(\theta))$ consisting of the unit normal and tangent vectors to the curve $\Gamma(d)$ parametrized by $x(\theta)$ in Eq. 7:

$$\begin{aligned} \tau(\theta) &= \frac{(1 + d(\theta)\kappa_0(\theta))\,\tau_0(\theta) \; + \; d'(\theta)n_0(\theta)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{1/2}} \\ n(\theta) &= \frac{-d'(\theta)\tau_0(\theta) \; + \; (1 + d(\theta)\kappa_0(\theta))\,n_0(\theta)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{1/2}} \end{aligned} \tag{11}$$

This yields the normal velocity of the interface:

$$V_n = \frac{1 + d(\theta)\kappa_0(\theta)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{1/2}} \frac{\partial d}{\partial t} \tag{12}$$

Using Eqs. 9 and 10 together, one obtains the curvature $\kappa(\theta)$ of the interface $\Gamma(d)$:

$$\kappa(\theta) = \frac{\kappa_0(\theta) - d''(\theta) + 2\kappa_0^2(\theta)d(\theta) + (2d'(\theta)^2 - d(\theta)d''(\theta)\kappa_0(\theta))\kappa_0(\theta) + d(\theta)d'(\theta)\kappa_0'(\theta) + d^2(\theta)\kappa_0^3(\theta)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{3/2}} \tag{13}$$

In order to compute the Laplacian with respect to arclength s within the interface, we use the formula

$$\frac{\partial}{\partial s} = \frac{\frac{\partial}{\partial \theta}}{\frac{ds}{d\theta}} = \frac{\frac{\partial}{\partial \theta}}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{1/2}} \tag{14}$$

By applying the differential operator in Eq. 14 twice to Eq. 13, we obtain the surface Laplacian of the curvature function of the interface Γ. The result may be written in the form

$$\begin{aligned} \frac{\partial^2 \kappa(\theta)}{\partial s^2} &= \frac{P_4(d)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{5/2}} + \frac{P_3(d)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{7/2}} \\ &+ \frac{P_2(d)}{\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{9/2}}, \end{aligned} \tag{15}$$

where each P_i, $i = 2, 3, 4$, is a differential polynomial of i-th order (in the θ-derivatives of d):

$$\begin{aligned} P_4(d) &= dd'\kappa_0''' + 2d'^2\kappa_0^3 + 8\kappa_0'\kappa_0 d' + 2dd''\kappa_0'' + 4\kappa_0'^2 d + 4\kappa_0''\kappa_0 d \\ &+ 3d'd''\kappa_0' + 2d''\kappa_0^2 + 6d^2\kappa_0'^2\kappa_0 + \kappa_0'' - d'''' + (4d'''d' + 4d''^2 \\ &- d''^2\kappa_0 - 2d'd'''\kappa_0 - 2d'd''\kappa_0' - dd''''\kappa_0 - 2dd'''\kappa_0' - dd''\kappa_0'')\kappa_0 \\ &+ 2\alpha_6\kappa_0' + dd'''\kappa_0' + 2d''d\kappa_0^3 + 3d^2\kappa_0''\kappa_0^2 + \alpha_3\kappa_0'' + 2d'^2\kappa_0'' + 12d'd\kappa_0'\kappa_0^2 \\ P_3(d) &= -3(\kappa_0' - d''' + 4\kappa_0'\kappa_0 d + 2\kappa_0^2 d' + \alpha_6\kappa_0 + \alpha_3\kappa_0' + d'^2\kappa_0' + \alpha_5 + dd'\kappa_0'' \\ &+ 2d'd\kappa_0^3 + 3d^2\kappa_0'\kappa_0^2)\alpha_2 - \frac{3}{2}\alpha_4(2(d''\kappa_0 + 2d'\kappa_0' + d\kappa_0'')(1 + d\kappa_0). \\ &+ 2\alpha_1^2 + 2d'''d' + 2d''^2) - \frac{1}{2}\alpha_2(\kappa_0' - d''' + 4\kappa_0'\kappa_0 d + 2\kappa_0^2 d' + \alpha_6\kappa_0 \\ &+ \alpha_3\kappa_0' + d'^2\kappa_0' + \alpha_5 + dd'\kappa_0'' + 2d'd\kappa_0^3 + 3d^2\kappa_0'\kappa_0^2) \\ P_2(d) &= \frac{9}{2}\alpha_4\alpha_2^2 \end{aligned} \tag{16}$$

with the α_i defined as follows:

$$\begin{aligned}
\alpha_1 &:= d'\kappa_0 + d\kappa_0' \\
\alpha_2 &:= 2\alpha_1(1 + d\kappa_0) + 2d''d' \\
\alpha_3 &:= 2d'^2 - dd''\kappa_0 \\
\alpha_4 &:= \kappa_0 - d'' + 2\kappa_0^2 d + \alpha_3\kappa_0 + dd'\kappa_0' + d^2\kappa_0^3 \\
\alpha_5 &:= dd''\kappa_0' \\
\alpha_6 &:= 4d''d' - d'd''\kappa_0 - dd'''\kappa_0 - \alpha_5
\end{aligned} \tag{17}$$

We obtain the desired equation of evolution by equating the expressions in Eqs. 12 and 15:

$$\begin{aligned}
\frac{\partial d}{\partial t} = &- \frac{((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2)^{1/2}}{1 + d(\theta)\kappa_0(\theta)}\{\frac{P_4(d)}{((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2)^{5/2}} \\
&+ \frac{P_3(d)}{((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2)^{7/2}} + \frac{P_2(d)}{((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2)^{9/2}}\}
\end{aligned}$$

It follows immediately from Theorems 1.4 and 1.5 that the nonlinear vector field in d on the right-hand side of this equation maps E^h to E^{h-4} analytically if $h > 5$. In other words, the nonlinear vector field defining the problem of motion by minus the surface Laplacian of curvature is analytic with respect to the phase pair (E^{h-4}, E^h). This completes the proof of Theorem 2.1.

3 Linearization and maximal regularity

As explained in the introduction, we must now show that the linearized vector field generates an analytic forward-time flow and that it satisfies the maximal regularity property given above in Definition 1.2. This is the content of the following result.

Theorem 3.1. *Let $h > 5$. The linearization of minus the surface Laplacian of curvature operator generates an analytic semiflow on E^{h-4} and satisfies the maximal regularity property relative to the phase pair (E^{h-4}, E^h).*

Proof. The statement that a continuous linear operator L generates an analytic semiflow is equivalent ([15], section 2.5, Theorem 5.2) to the *sectoriality condition* that the spectrum of L is contained in some convex wedge in the complex plane with vertex on the real line and opening angle strictly less than π toward the left half-plane, and that there is a constant $M \in \mathrm{R}^+$ such that for all complex numbers λ lying inside the wedge C the resolvent $R(\lambda, L)$ satisfies:

$$||R(\lambda, L)|| \leq \frac{M}{|\Im\lambda|} \tag{18}$$

We will verify that this condition holds when L is the linearized minus surface Laplacian of curvature operator.

From Eqs. 16 and 17 we find the linearization of minus the surface Laplacian of curvature operator around the reference curve $d = 0$. Indeed, we observe that for the purpose of linearizing around $d = 0$, the expressions α_i of Eq. 17 are equivalent to:

$$\begin{aligned} \alpha_1 &= d'\kappa_0 + d\kappa_0' \\ \alpha_2 &= 2\alpha_1 \\ \alpha_3 &= 0 \\ \alpha_4 &= \kappa_0 - d'' + 2\kappa_0^2 d \\ \alpha_5 &= 0 \\ \alpha_6 &= 0 \end{aligned} \tag{19}$$

and the P_i of Eq. 16 are equivalent to:

$$\begin{aligned} P_4(d) &= 8\kappa_0'\kappa_0 d' + 4{\kappa_0'}^2 d + 4\kappa_0''\kappa_0 d + 2d''\kappa_0^2 + \kappa_0'' - d'''' \\ P_3(d) &= -7\kappa_0'(d'\kappa_0 + d\kappa_0') - 3\kappa_0(d''\kappa_0 + 2d'\kappa_0' + d\kappa_0'') \\ P_2(d) &= 0 \end{aligned} \tag{20}$$

Finally, we use the fact that for all $z \in \mathrm{R}$ we have:

$$\left((1 + d(\theta)\kappa_0(\theta))^2 + (d'(\theta))^2\right)^{z/2} = 1 + zd(\theta)\kappa(\theta) + o(d) \quad \text{as } d \to 0 \tag{21}$$

We obtain for the linearized field which appears on the right-hand side of Eq. 18:

$$((dG)(0))d = -d'''' - \kappa_0^2 d'' - 5\kappa_0\kappa_0' d' - \left(4\kappa_0\kappa_0'' + 3{\kappa_0'}^2\right) d \tag{22}$$

(The term of order 0, which equals $-\kappa_0''$, has been omitted here.)

Consider first the term of highest order, $-d''''$, in Eq. 22. In order to compute its spectrum, we apply the Fourier transform to the equation

$$- f'''' - \lambda f = g, \tag{23}$$

obtaining the following family of equations indexed by the dual variable $k \in \mathrm{Z}$:

$$- k^4 \hat{f}_k - \lambda \hat{f}_k = \hat{g}_k, \tag{24}$$

We see that if $k^4 + \lambda$ remains bounded away from 0 as k ranges over all integers, then we may solve for f in terms of g in Eq. 24 and thus also in Eq. 23. Indeed, we will have:

$$\hat{f}_k = -\frac{\hat{g}_k}{k^4 + \lambda}, \tag{25}$$

and the resulting function g will belong to the space E^h if g belongs to E^{h-4}. It follows that the spectrum of the operator $-D^4$ is contained in the closure of the set of values $-k^4$ for $k \in \mathrm{Z}$, i.e. in the nonpositive real line. Furthermore, the operator $-D^4$ satisfies:

$$\begin{aligned} \|R(\lambda, D^4)\| &= \sup_{f \in E^h, \|f\|_{E^h} \leq 1} \sup_{k \in \mathrm{Z}} \frac{|k|^h \hat{f}_k}{|\lambda + k^4|} \\ &\leq \sup_{z \in c_0, \|z\|_{c_0} \leq 1} \sup_{k \in \mathrm{Z}} \frac{|z_k|}{|\lambda + k^4|} \\ &\leq \sup_{k \in \mathrm{Z}} \frac{1}{|\lambda + k^4|} \\ &\leq \frac{1}{|\Im \lambda|} \end{aligned} \tag{26}$$

Thus, the resolvent bound of Eq. 18 holds for $-D^4$, so this operator is sectorial, and therefore generates an analytic semiflow on the space E^{h-4}.

We now show that the remaining terms in Eq. 22 do not affect the sectoriality of the spectrum. In order to do this, we make use of the following result.

Lemma 3.2. *([15], section 3.2, Theorem 2.1). Let A be the infinitesimal generator of an analytic semigroup. Let B be a closed linear operator satisfying $D(B) \supseteq D(A)$ and*

$$||Bx|| \le a||Ax|| + b||x|| \qquad \forall x \in D(A) \tag{27}$$

There exists a positive number δ such that if $0 \le a \le \delta$ then $A + B$ is the infinitesimal generator of an analytic semigroup.

We can easily show that if j is any non-negative integer less than 4, then Eq. 27 holds for $A = D^4$, $B = D^j$, with $a > 0$ arbitrarily small. To see this, let $\epsilon > 0$ and choose $k_0 \in Z^+$ such that ${k_0}^{j-4} < \epsilon$. We then have the following inequalities in the E^{h-4} norm for any $x \in E^h$:

$$\begin{aligned}
||D^j x|| &= \sup_{k\in Z} \left(1 + |k|^{h-4}\right) |k|^j |\hat{x}_k| \\
&\le \sup_{|k|>k_0} \left(1 + |k|^{h-4}\right) |k|^j |\hat{x}_k| \;+\; \sup_{|k|\le k_0} \left(1 + |k|^{h-4}\right) |k|^j |\hat{x}_k| \\
&\le \epsilon \sup_{|k|>k_0} \left(1 + |k|^{h-4}\right) |k|^4 |\hat{x}_k| \;+\; k_0^4 \sup_{|k|\le k_0} \left(1 + |k|^{h-4}\right) |\hat{x}_k| \\
&\le \epsilon||D^4 x|| + k_0^4||x||
\end{aligned} \tag{28}$$

By Lemma 3.2, it follows that the lower order terms in Eq. 22 do not affect the analyticity of the generated semigroup in E^{h-4}, and so the linearized minus surface Laplacian of curvature operator generates an analytic semiflow on E^{h-4}.

The verification of the maximal regularity property uses a perturbation argument also. First we show that the negative of the fourth-derivative operator has the maximal regularity property (Definition 1.2). Given $f \in C([0,T], E^{h-4})$, the Fourier transform of the value at f of the associated variation of constants integral operator R satisfies:

$$\begin{aligned}
|(Rf)^{\wedge}_k(t)| &= \left| \left(\int_0^t e^{-(t-s)\frac{\partial^4}{\partial\theta^4}} f(s)\, ds \right)^{\wedge}_k \right| \\
&= \left| \int_0^t e^{-(t-s)k^4} \hat{f}_k(s)\, ds \right| \\
&\le ||f||_{C([0,T],E^{h-4})} |k|^{-(h-4)} \int_0^t e^{-(t-s)k^4}\, ds \\
&\le ||f||_{C([0,T],E^{h-4})} |k|^{-h}
\end{aligned} \tag{29}$$

Notice that the right-hand side of Eq. 29 is independent of t, and that the simplicity of the above derivation of this uniform estimate relies heavily on the definition of the spaces E^h. We now invoke the following fact:

Lemma 3.3. (see the proof of Theorem 3.2.1 in [1].) *The space $C([0,T], E^\infty)$ is dense in the space $C([0,T], E^h)$.*

The proof of Lemma 3.3 involves truncating the Fourier transform of a given $g \in C([0,T],E^h)$ to a bounded interval $|k| \leq k_0(t)$ to obtain an E^∞ approximation of $g(t)$ for each t and then using compactness of $[0,T]$ and continuity in t of $g(t)$ in the E^h norm to obtain a uniform bound on the numbers $k_0(t)$ for t in $[0,T]$.

In view of Lemma 3.3, Eq. 29 implies that $Rf(t)$ belongs to E^h for each $t \in [0,T]$, i.e. that the Fourier transform of $Rf(t)$ decays strictly faster than $|k|^{-h}$. The continuous dependence of $Rf(t)$ on t in the E^h norm also follows from the density property of Lemma 3.3 (for details see the proof of Theorem 3.2.1 in [1]). Moreover, Eq. 29 proves continuity of R as a mapping from $C([0,T],E^{h-4})$ to $C([0,T],E^h)$. We see, therefore, that the variation-of-constants operator R maps the space $C([0,T],E^{h-4})$ continuously into the space $C([0,T],E^h)$ as required by the maximal regularity property relative to the phase pair (E^{h-4},E^h).

The lower-order terms of the linearized surface Laplacian of curvature can now be included by a perturbation argument as was done above in the proof that the generated semiflow is analytic. The relevant perturbation result is the following version of Lemma 2.5 in [4] (the statement in [4] is slightly different, but the present version follows from the proof given there).

Lemma 3.4. *Let A have the maximal regularity property relative to the phase pair (X,Y), and let $B : Y \to X$ be such that*

$$\|Bx\|_X \leq a\|x\|_Y + b\|x\|_X \qquad \forall x \in D(A) \tag{30}$$

There exists a positive number δ such that if $0 \leq a \leq \delta$ then $A+B$ also has the maximal regularity property relative to (X,Y).

The fact that Eq. 30 holds for $A = -D^4$, $B = D^j$ for any $j < 4$ relative to the phase pair (E^{h-4},E^h), is proved exactly as was done above for Eq. 28. We conclude by Lemma 3.4 that the lower order terms of the linearized vector field in Eq. 22 do not affect the maximal regularity property. This completes the proof of Theorem 3.1, and therefore also of Theorem 1.1.

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