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EMILIO ACERBI

IRENE FONSECA

and

NICOLA FUSCO

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REGULARITY OF MINIMIZERS FOR A CLASS OF MEMBRANE ENERGIES

Emilio Acerbi [1] Irene Fonseca [2] Nicola Fusco [3]

Abstract Regularity properties for (local) minimizers of elastic energies have been challenging mathematical techniques for many years. Recently the interest has resurfaced due in part to the fact that existing partial regularity results do not suffice to ensure existence of (classical) solutions to problems involving free discontinuity sets. The analysis of such questions was started with the fundamental work of De Giorgi in the early 80's in connection with the Mumford-Shah model for image segmentation in computer vision, and later applied to some models for fracture mechanics, thin films, and membranes ([1], [18], [20]). In this paper it is shown that local minimizers in $W^{1,2}(\Omega;\mathbb{R}^d)$ of the functional

$$\mathcal{F}_0(u,\Omega):=\int_\Omega\left[\tfrac{1}{2}|Du|^2+f(|\nu(u)|)\right]dx$$

are Hölder continuous of any exponent $\gamma\in(0,1)$, where $\Omega\subset\mathbb{R}^2$ is an open, bounded set, f is a (not necessarily convex) function growing linearly at infinity, and $\nu(u)$ stands for the vector of all 2×2 minors of Du. As a consequence, it is possible to obtain existence of "classical" minimizers in $SBV(\Omega;\mathbb{R}^d)$ of

$$\mathcal{F}(u,\Omega):=\int_\Omega\left[\tfrac{1}{2}|\nabla u|^2+f(|\nu(u)|)\right]dx+\beta\int_\Omega|u-g|^q\,dx+\gamma H^{N-1}(S_u\cap\Omega)$$

where $g\in L^\infty(\Omega;\mathbb{R}^d)$, $q>1$, $\beta,\gamma>0$. These minimizers are "classical minimizers" in the sense that $H^1\left((\overline{S_u}\setminus S_u)\cap\Omega\right)=0$ and $u\in W^{1,2}(\Omega\setminus\overline{S_u})$.

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[1] Dipartimento di Matematica, Via Massimo D'Azeglio 85/A, 43100 Parma (Italy). Research supported by MURST, Gruppo Nazionale 40%.
[2] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213. Research partially supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis, and by the National Science Foundation under Grant No. DMS-9500531.
[3] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50131 Firenze (Italy). Research supported by MURST, Gruppo Nazionale 40%.

It is not restrictive to assume that

$$0 < \alpha < 1$$

and in what follows we will work under this assumption. Also, in order to simplify the notation the value of the constant C may change from one line to the next, and B_R, $R > 0$, will denote a generic open ball of radius R, centered at $x \in \Omega$, and such that $B_R \subset \Omega$.

Given $u \in SBV(\Omega;\mathbb{R}^d)$ we define

$$\nu(u) := \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2},$$

the 2-covector whose components are the 2×2 subdeterminants of ∇u.

Consider the energies

$$\mathcal{G}(K,u) := \int_{\Omega \setminus K} \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta \int_{\Omega \setminus K} |u-g|^q \, dx + \gamma H^1(\Omega \cap K),$$

$$\mathcal{F}(u,\Omega) := \int_{\Omega} \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta \int_{\Omega} |u-g|^q \, dx + \gamma H^1(S_u \cap \Omega),$$

and

$$\mathcal{F}_0(u,\Omega) := \int_{\Omega} \left[\frac{1}{2}|Du|^2 + f(|\nu(u)|)\right] dx.$$

Definition 2.1. *We say that $u \in W^{1,2}(\Omega;\mathbb{R}^d)$ is a $W^{1,2}$-*local minimizer *of*

$$I(v,\Omega) := \int_{\Omega} F(\nabla v)\, dx\,, \qquad v \in W^{1,2}(\Omega;\mathbb{R}^d)$$

if

$$I(u,B_R(x_0)) = \min\left\{I(v,B_R(x_0)) \ : \ v \in u + W_0^{1,2}(B_R(x_0);\mathbb{R}^d)\right\}$$

for all balls $B_R(x_0) \subset \Omega$.

The main result of this paper is the following theorem.

Theorem 2.2. *If $u \in W^{1,2}(\Omega,\mathbb{R}^d)$ is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then $u \in C^{0,\gamma}_{\mathrm{loc}}$ for all $\gamma \in (0,1)$.*

In the proof of Theorem 2.2 we will use classical arguments of regularity theory within the framework of the *Morrey spaces* $L^{p,\lambda}$; for a detailed study of these methods we refer the reader to [21], [24].

Definition 2.3. *Given $\lambda \geq 0$ we say that $f \in L^{p,\lambda}(\Omega;\mathbb{R})$ if there exists a constant $C > 0$ such that*

$$\int_{B_\rho(x)\cap\Omega} |f|^p \, dx \leq C\rho^\lambda$$

for all $x \in \Omega$ *and* $0 < \rho < \operatorname{diam} \Omega$. *The function* f *is said to be in* $L^{p,\lambda}_{\rm loc}(\Omega)$ *if* $f \in L^{p,\lambda}(\Omega')$ *for all* $\Omega' \subset\subset \Omega$.

It can be shown that, with $\Omega \subset \mathbb{R}^2$,

$$L^{p,0}(\Omega) = L^p(\Omega),\ L^{p,2}(\Omega) = L^\infty(\Omega),\ L^{p,\lambda}(\Omega) = \{0\} \quad \text{if } \lambda > 2.$$

and that $L^{p,\lambda}(\Omega)$ is a Banach space endowed with the norm

$$||f||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \int_{B_\rho(x)\cap\Omega} |f|^p \, dx \right\}^{\frac{1}{p}}.$$

Morrey proved that (see Theorem 3.5.2, [24])

Lemma 2.4. *If* $u \in W^{1,2}_{\rm loc}(\Omega)$ *and* $Du \in L^{2,\lambda}_{loc}(\Omega)$ *for some* $0 < \lambda < 2$ *then* $u \in C^{0,\lambda/2}_{\rm loc}(\Omega)$.

In light of Lemma 2.4, we will prove Theorem 2.2 by showing that if u is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then for all $0 \le \lambda < 2$

$$\int_{B_\rho} |Du|^2 \, dx \le C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Du|^2 \, dx + C\rho^\lambda \tag{2.1}$$

for all $0 < \rho < R$ with $B_R \subset\subset \Omega$.

As a corollary we obtain,

Corollary 2.5. *Let* $u \in SBV(\Omega; \mathbb{R}^d)$ *be a minimizer for* $\mathcal{F}$. *Then* $(\overline{S_u}, u)$ *is a minimizer for* $\mathcal{G}$ *among all pairs* (K, v) *with* $K \subset \Omega$ *closed and* $v \in W^{1,2}(\Omega \setminus K; \mathbb{R}^d)$. *Moreover,*

$$H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

Following the argument introduced by De Giorgi, Carriero and Leaci [16], and outlined in [1], the corollary holds provided we can show that $W^{1,2}$–local minimizers of

$$v \in W^{1,2}(B_1; \mathbb{R}^d) \mapsto \int_{B_1} \left[\frac{1}{2}|Dv|^2 + M|\nu(v)|\right] dx$$

satisfy an estimate of the type

$$\int_{B_\rho} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx \le C\rho^\lambda \int_{B_1} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx + C\rho^\lambda,$$

for some $0 < \lambda < 2$ and $0 < \rho \le 1$ or, equivalently,

$$\int_{B_\rho} |Du|^2 \, dx \le C\rho^\lambda \int_{B_1} |Du|^2 \, dx + C\rho^\lambda.$$

We conclude that the assertion of the corollary holds true provided we prove (2.1).

The following two lemmas may be found in [21] (see Chapter 3, Theorem 3.1, page 87, and Lemma 2.1, respectively).

Lemma 2.6. *Let $\lambda < 2$, let $f \in L^{2,\lambda}(B_R;\mathbb{R}^2)$, and let $v \in W^{1,2}(B_R;\mathbb{R})$ satisfy*

$$\Delta v = \operatorname{div} f \quad in\ B_R.$$

Then $Dv \in L^{2,\lambda}_{\mathrm{loc}}(B_R;\mathbb{R}^2)$, and for every $\rho \le R$

$$\int_{B_\rho} |Dv|^2\,dx \le C\left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Dv|^2\,dx + C\rho^\lambda\,\|f\|^2_{L^{2,\lambda}(B_R)}.$$

Lemma 2.7. *Let $\phi : [0,+\infty) \to [0,+\infty)$ be a nonnegative, nondecreasing function, such that*

$$\phi(\rho) \le H\left[\left(\frac{\rho}{R}\right)^\gamma + \varepsilon\right]\phi(R) + KR^\beta$$

for all $0 < \rho < R \le R_0$ and for some constants $H, K \ge 0$ and $0 < \beta < \gamma$. Then there exist constants $\varepsilon_0 = \varepsilon_0(H,\gamma,\beta), C = C(H,\gamma,\beta)$ such that

$$\phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^\beta \phi(R) + K\rho^\beta\right]$$

for all $0 < \rho < R \le R_0$.

Lemma 2.8. *Let $p > 1$ and $0 \le \lambda < 2$. If $f_{ij} \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ for $i,j \in \{1,2\}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is a distributional solution of*

$$\Delta u = \sum D_{ij} f_{ij}$$

then $u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$.

Proof. Let $B_R \subset\subset \Omega$ and for every i,j let v_{ij} be the solution of (see Theorem 9.15 and Lemma 9.17, [22])

$$\begin{cases} \Delta v_{ij} = f_{ij} \\ v_{ij} \in W^{1,p}_0(B_R) \cap W^{2,p}(B_R), \end{cases}$$

and we set

$$w := \sum D_{ij} v_{ij}.$$

Then $w \in L^p(B_R)$ and $\|w\|_{L^p(B_R)} \le C \sum \|f_{ij}\|_{L^p(B_R)}$. In addition, $\Delta w = \sum D_{ij} f_{ij}$ in $\mathcal{D}'$, so that the function

$$v := u - w$$

is harmonic, i.e. $\Delta v = 0$. Hence

$$\sup_{B_{R/2}} |v| \le C(p) \left(\frac{1}{|B_R|} \int_{B_R} |v|^p \, dx \right)^{1/p},$$

from which we deduce that for every $\rho \le R/2$ (thus, for all $0 < \rho \le R$)

$$\int_{B_\rho} |v|^p \, dx \le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |v|^p \, dx.$$

We have

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \int_{B_\rho} (|v|^p + |w|^p) \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |v|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |u|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |u|^p \, dx + C R^\lambda. \end{aligned}$$

By Lemma 2.7 we deduce that for all $0 < \rho \le R$

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \left(\frac{\rho}{R} \right)^\lambda \int_{B_R} |u|^p \, dx + C \rho^\lambda \\ &\le \rho^\lambda \left[\frac{C}{R^\lambda} \int_{B_R} |u|^p \, dx + C \right], \end{aligned}$$

and so $u \in L^{p,\lambda}_{\text{loc}}(\Omega)$. □

We end this section with a list of algebraic inequalities, following an argument introduced in [8] (see also [17]).

Let $P, Q \in \mathbb{R}^d$ and set

$$A := \frac{|P|^2 - |Q|^2}{2}, \quad B := P \cdot Q, \quad \nu := P \wedge Q.$$

Lemma 2.9. *We have*

i) $2\sqrt{A^2 + B^2} \le |P|^2 + |Q|^2$;

ii) $0 \le |P|^2 + |Q|^2 - 2|\nu| \le 2\sqrt{A^2 + B^2}$;

iii) if $\nu = 0$ *then* $|P|^2 + |Q|^2 = 2\sqrt{A^2 + B^2}$;

iv) if $\alpha, \beta \in \mathbb{R}^d$ *and* $\nu \ne 0$ *then*

$$\left| \frac{1}{|\nu|} \nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \right|^2 \le 4\sqrt{A^2 + B^2} \, (|\alpha|^2 + |\beta|^2).$$

Proof. Since

$$|\nu|^2 = \sum_{i<j} |P_i Q_j - P_j Q_i|^2 = \frac{1}{2} \sum_{i,j} |P_i Q_j - P_j Q_i|^2 = |P|^2 |Q|^2 - (P \cdot Q)^2,$$

we have

$$|P|^2|Q|^2 = B^2 + |\nu|^2,$$

and so

$$4A^2 = (|P|^2 + |Q|^2)^2 - 4|P|^2|Q|^2 = (|P|^2 + |Q|^2)^2 - 4(B^2 + |\nu|^2),$$

and

$$4(A^2 + B^2) = (|P|^2 + |Q|^2)^2 - 4|\nu|^2.$$

Clearly i) and iii) follow. In addition, we have that

$$(|P|^2 + |Q|^2)^2 - 4|\nu|^2 \geq 0$$

hence

$$0 \leq |P|^2 + |Q|^2 - 2|\nu| \leq \sqrt{(|P|^2 + |Q|^2)^2 - 4|\nu|^2} = 2\sqrt{A^2 + B^2},$$

which yields assertion ii).

Now remark that if $\nu \neq 0$ then $P \neq 0$ and, setting

$$Q' := Q - \frac{P \cdot Q}{|P|^2} P,$$

then also $Q' \neq 0$. Define the orthonormal vectors

$$P_1 := \frac{P}{|P|}, \quad Q_1 := \frac{Q'}{|Q'|}.$$

We write

$$P = p\, P_1, \quad Q = s\, P_1 + q\, Q_1$$

with

$$p := |P|, \quad q := |Q'|, \quad s := \frac{P.Q}{|P|}.$$

Note that

$$\nu = pq\, P_1 \wedge Q_1, \quad |\nu| = pq,$$

and that if $v \in \mathbb{R}^d$ then

$$(P_1 \wedge Q_1) \cdot (P_1 \wedge v) = v \cdot Q_1, \quad (P_1 \wedge Q_1) \cdot (v \wedge Q_1) = v \cdot P_1.$$

We have

$$\begin{aligned}
&\frac{1}{|\nu|} \nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \\
&= (P_1 \wedge Q_1) \cdot (p\, P_1 \wedge \beta - s\, P_1 \wedge \alpha + q\, \alpha \wedge Q_1) - (p\, P_1 \cdot \alpha + s\, P_1 \cdot \beta + q\, Q_1 \cdot \beta) \\
&= [(q - p)\, P_1 - s\, Q_1] \cdot \alpha + [-s\, P_1 + (p - q)\, Q_1] \cdot \beta \\
&= v_1 \cdot \alpha + v_2 \cdot \beta,
\end{aligned}$$

with

$$v_1 := (q - p)\, P_1 - s\, Q_1 \qquad \text{and} \quad v_2 := -s\, P_1 + (p - q)\, Q_1.$$

We have

$$|v_1 \cdot \alpha + v_2 \cdot \beta|^2 \leq (|v_1|^2 + |v_2|^2)(|\alpha|^2 + |\beta|^2) = 2(|P|^2 + |Q|^2 - 2|\nu|)(|\alpha|^2 + |\beta|^2),$$

which, together with ii), concludes the proof of iv). □

3. Proof of the Regularity Theorem

In this section we assume that $u \in W^{1,2}(\Omega;\mathbb{R}^d)$ is a local minimizer of $\mathcal{F}_0$.

Proposition 3.1. *If $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbb{R}^d)$ for some $0 \le \lambda < 2$ then $Du \in L^{2,q_0(\lambda)}_{\rm loc}(\Omega;\mathbb{R}^d)$, where $q_0(\lambda) := \alpha + \lambda(1-\alpha/2)$.*

Before proceeding with the proof of this result, we remark that using an iterative scheme where

$$\lambda_0 := 0, \quad \lambda_{k+1} := q_0(\lambda_k),$$

then

$$\lim_{k\to+\infty} \lambda_k = \lim_{k\to+\infty} \alpha \sum_{i=0}^{k} \left(1-\frac{\alpha}{2}\right)^i = 2,$$

hence (2.1) will follow for all $0 \le \lambda < 2$ and, as justified in Section 2, this suffices to assert Theorem 2.2.

The proof of Proposition 3.1 uses higher integrability properties of the functions

$$A := \frac{|D_1u|^2 - |D_2u|^2}{2}, \quad B := (D_1u)\cdot(D_2u).$$

where D_1u and D_2u stand for the column vectors in $\mathbb{R}^d$ of the derivatives of u with respect to x_1 and to x_2, respectively.

Proposition 3.2. *The functions A and B solve the system*

$$\begin{cases} \Delta A = D^2_{11}g - D^2_{22}g \\ \Delta B = 2D^2_{12}g. \end{cases}$$

where

$$g := f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|).$$

In addition, if $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbb{R}^{2d})$ for some $0 \le \lambda < 2$ then $\sqrt{|A|+|B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\rm loc}(\Omega;\mathbb{R})$.

Proof. Consider $\Phi := (\varphi,\psi) \in C^1_0(\Omega;\mathbb{R}^2)$, and let $\varepsilon > 0$ be small enough so that with $\Phi_\varepsilon(x) := x + \varepsilon\Phi(x)$, then $\Phi_\varepsilon : \Omega \to \Omega$ is a smooth diffeomorphism satisfying

$$\det D\Phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div}\Phi(x) + \omega_1(x,\varepsilon),$$
$$\det D\Phi_\varepsilon^{-1}(y) = 1 - \varepsilon \operatorname{div}\Phi(\Phi_\varepsilon^{-1}(y)) + \omega_2(y,\varepsilon),$$

where $\omega_i(\cdot,\varepsilon)/\varepsilon \to 0$, as $\varepsilon \to 0$, uniformly in Ω. Set

$$u_\varepsilon(y) := u\left(\Phi_\varepsilon^{-1}(y)\right), \quad y \in \Omega.$$

We have

$$\begin{aligned}\int_{\Omega} |Du_{\varepsilon}(y)|^2 dy &= \int_{\Omega} |Du(\mathbb{I} - \varepsilon D\Phi)|^2 (\Phi_{\varepsilon}^{-1}(y))\, dy + o(\varepsilon)\\ &= \int_{\Omega} |Du(\mathbb{I} - \varepsilon D\Phi)|^2 (1 + \varepsilon \operatorname{div} \Phi)\, dx + o(\varepsilon)\\ &= \int_{\Omega} |Du|^2\, dx + \varepsilon \int_{\Omega} \left[|Du|^2 \operatorname{div} \Phi - 2DuD\Phi \cdot Du\right]\, dx + o(\varepsilon),\end{aligned}$$

where the inner product of two $d \times 2$ matrices ξ and η is defined as $\xi \cdot \eta := \operatorname{trace}(\xi^T \eta)$.

On the other hand, since

$$\nu(u_{\varepsilon}(y)) = [\det D\Phi_{\varepsilon}^{-1}(y)]\, \nu(u)(\Phi_{\varepsilon}^{-1}(y)),$$

we also have that, setting $\Omega_{\varepsilon} := \{x \in \Omega : |\varepsilon \operatorname{div} \Phi - \omega_2||\nu(u)| \neq 0\}$,

$$\begin{aligned}&\int_{\Omega} f(|\nu(u_{\varepsilon}(y))|)\, dy = \int_{\Omega} f\left((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\right)\, \det D\Phi_{\varepsilon}\, dx\\ &= \int_{\Omega_{\varepsilon}} [f(|\nu(u)|) + (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|)] \det D\Phi_{\varepsilon}\, dx\\ &\quad + \int_{\Omega_{\varepsilon}} \left[\frac{f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|) - f(|\nu(u)|)}{(-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|} - f'(|\nu(u)|)\right]\\ &\qquad\qquad (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|) \det D\Phi_{\varepsilon}\, dx\\ &\quad + \int_{\Omega \setminus \Omega_{\varepsilon}} f(|\nu(u)|) \det D\Phi_{\varepsilon}\, dx\\ &= \int_{\Omega} f(|\nu(u)|) \det D\Phi_{\varepsilon}\, dx\\ &\quad + \int_{\Omega} (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|) \det D\Phi_{\varepsilon}\, dx + o(\varepsilon),\\ &= \int_{\Omega} f(|\nu(u)|)\, dx + \varepsilon \int_{\Omega} [f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|)] \operatorname{div} \Phi\, dx + o(\varepsilon),\end{aligned}$$

because by Lebesgue's dominated convergence, by (H1), and due to the boundedness of f',

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} &\left|\frac{f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|) - f(|\nu(u)|)}{(-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|} - f'(|\nu(u)|)\right|\\ &|\nu(u)| \left|\operatorname{div} \Phi - \frac{\omega_2}{\varepsilon}\right| |1 + \varepsilon \operatorname{div} \Phi + \omega_1|\, dx = 0.\end{aligned}$$

By the local minimality of u we have $\mathcal{F}_0(u_{\varepsilon}) - \mathcal{F}_0(u) \geq 0$, from which the Euler-Lagrange equation can be easily obtained,

$$\int_{\Omega} \left[\frac{1}{2}|Du|^2 \operatorname{div} \Phi - DuD\Phi \cdot Du\right] dx = \int_{\Omega} [\, |\nu(u)| f'(|\nu(u)|) - f(|\nu(u)|)\,] \operatorname{div} \Phi\, dx$$

for every $\Phi = (\varphi, \psi) \in C_0^1(\Omega; \mathbb{R}^2)$. This equation may be rewritten as

$$\int_{\Omega}[A(D_2\psi - D_1\varphi) - B(D_1\psi + D_2\varphi)]\,dx = \int_{\Omega} -g(D_1\varphi + D_2\psi)\,dx,$$

that is,

$$\begin{cases} D_1A + D_2B = D_1g \\ D_2A - D_1B = -D_2g, \end{cases}$$

and the first assertion follows. By (H3)

$$|g| \le C(1 + |\nu(u)|^{1-\alpha})$$

and so, assuming that $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbb{R}^{2d})$ we have that $|\nu(u)| \in L^{1,\lambda}_{\rm loc}(\Omega;\mathbb{R})$ and

$$g \in L^{\frac{1}{1-\alpha},\lambda}_{\rm loc}(\Omega).$$

We may now use Lemma 2.8 to obtain that

$$A, B \in L^{\frac{1}{1-\alpha},\lambda}_{\rm loc}(\Omega),$$

and by Hölder inequality we conclude that

$$\sqrt{|A| + |B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\rm loc}(\Omega).$$

□

Finally, in order to prove Proposition 3.1 we introduce the following notation:

$$q(\lambda) := 2\alpha + \lambda(1-\alpha),$$

$$\Omega_0 := \{x \in \Omega : |\nu(u)| = 0\},$$

$$\Omega_0' := \{x \in \Omega : |\nu(u)| > 0\},$$

$$\Omega_K := \{x \in \Omega : 0 < |\nu(u)| \le K\},$$

$$\Omega_K' := \{x \in \Omega : |\nu(u)| > K\}.$$

Proof of Proposition 3.1. Fix $\phi \in W^{1,2}_0(\Omega;\mathbb{R}^d)$ and assume that $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbb{R}^{2d})$ for some $0 \le \lambda < 2$. For $\varepsilon \in \mathbb{R}$ set $u_\varepsilon(x) := u(x) + \varepsilon\phi(x)$. Define

$$P := D_1u, \quad Q := D_2u, \quad \alpha := D_1\phi, \quad \beta = D_2\phi, \quad \nu := \nu(u).$$

Since

$$\nu(u_\varepsilon) = \nu(u) + \varepsilon P \wedge \beta + \varepsilon\alpha \wedge Q + \varepsilon^2\alpha \wedge \beta,$$

we have

$$\int_\Omega f(|\nu(u_\varepsilon)|)\,dx - \int_\Omega f(|\nu|)\,dx = \varepsilon \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx$$
$$+ |\varepsilon| \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx + o(\varepsilon).$$

Local minimality of u entails

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{F}_0(u_\varepsilon,\Omega)-\mathcal{F}_0(u,\Omega)}{\varepsilon} \le 0,$$

and so

$$\int_\Omega Du\cdot D\phi\,dx + \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx \le \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx.$$

We have

$$(M+1)\int_\Omega Du\cdot D\phi\,dx + M\int_{\Omega_0'}\left[\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q) - (P\cdot\alpha+Q\cdot\beta)\right]dx$$
$$+ \int_{\Omega_0'\cap\Omega_K} (f'(|\nu|)-M)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx$$
$$\le C\int_{\Omega_0}|Du||D\phi|\,dx + \omega_K\int_{\Omega_K'}|Du||D\phi|\,dx,$$

where

$$\omega_K := \sup_{t\ge K}|M-f'(t)|.$$

We recall that by (H2)

$$\omega_K \to 0 \qquad \text{as } K\to+\infty.$$

By Lemma 2.9 iii), iv), we deduce that

$$(M+1)\int_\Omega Du\cdot D\phi\,dx + \int_\Omega G\cdot D\phi\,dx$$
$$\le C\int_\Omega \sqrt{|A|+|B|}\,|D\phi|\,dx + \omega_K\int_\Omega |Du||D\phi|\,dx \tag{3.1}$$

with $G=(G_1,G_2)$ and

$$G_1 := \chi_{\Omega_0'\cap\Omega_K}(M-f'(|\nu|))\frac{\nu}{|\nu|}\wedge Q$$
$$G_2 := \chi_{\Omega_0'\cap\Omega_K}(f'(|\nu|)-M)\frac{\nu}{|\nu|}\wedge P,$$

and where χ_A stands for the characteristic function of the set A. By Lemma 2.9 ii), iii), and recalling that on Ω_K we have $|\nu|\le K$, we have

$$|G| \le C(K)(1+\sqrt{|A|+|B|}), \qquad \text{a.e. in } \Omega,$$

and by Proposition 3.2 we deduce that $G \in L^{2,q(\lambda)}(\Omega;\mathbb{R}^d)$. Next, for a fixed ball $B_R \subset\subset \Omega$ we compare u with the solution of the Dirichlet problem

$$\begin{cases} (M+1)\Delta v = \operatorname{div} G & \text{in } B_R \\ v - u \in W_0^{1,2}(B_R;\mathbb{R}). \end{cases} \tag{3.2}$$

By Lemma 2.6 $Dv \in L^{2,q(\lambda)}_{\mathrm{loc}}(B_R;\mathbb{R}^2)$ and for all $0 < \rho \le R$

$$\int_{B_\rho} |Dv|^2\,dx \le C\left(\frac{\rho}{R}\right)^{q(\lambda)} \int_{B_R} |Dv|^2\,dx + C(K)\rho^{q(\lambda)}. \tag{3.3}$$

From (3.1) and (3.2) we have for all $\phi \in W_0^{1,2}(B_R;\mathbb{R}^d)$

$$(M+1)\int_{B_R}(Du-Dv)\cdot D\phi dx \le C\int_{\Omega\cap B_R}\sqrt{|A|+|B|}|D\phi|dx + \omega_K\int_{B_R}|Du||D\phi|dx.$$

Therefore, taking $\phi := u - v$, and using the fact that by the definition of G and by (3.2)

$$|G| \le C|Du|, \qquad \int_{B_R}|Dv|^2 \le C\int_{B_R}|Du|^2,$$

we have

$$\int_{B_R}|Du-Dv|^2\,dx \le C\int_{B_R}(|A|+|B|)\,dx + C\omega_K\int_{B_R}|Du|^2\,dx.$$

Using (3.3) we now obtain

$$\int_{B_\rho}|Du|^2\,dx \le C\left[\left(\frac{\rho}{R}\right)^{q(\lambda)} + \omega_K\right]\int_{B_R}|Du|^2\,dx + C(K)R^{q(\lambda)},$$

and if K is large enough, so that ω_K is small, from Lemma 2.7 we conclude that for all $0 < \lambda' < q(\lambda)$

$$\int_{B_\rho}|Du|^2\,dx \le C\left(\frac{\rho}{R}\right)^{\lambda'}\int_{B_R}|Du|^2\,dx + C\rho^{\lambda'}, \tag{3.4}$$

and thus (3.4) holds true for $\lambda' = q_0(\lambda)$. □

References

1 ACERBI, E., I. FONSECA, N. FUSCO, Regularity results for equilibria in a variational model for fracture. To appear in *Proc. R. Soc. Edin.*

2 AMBROSIO, L., A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857–881.

3 AMBROSIO, L., A new proof of the SBV compactness theorem, *Calc. Var.* **3** (1995), 127–137.

4 AMBROSIO, L., On the lower semicontinuity of quasiconvex integrals in $SBV(\Omega,\mathbb{R}^k)$, *Nonlinear Anal.* To appear.

5 AMBROSIO, L., N. FUSCO and D. PALLARA, Partial regularity of free discontinuity sets II. To appear in *Ann. Scuola Norm. Sup. di Pisa*
6 AMBROSIO, L. and D. PALLARA, Partial regularity of free discontinuity sets I. To appear in *Ann. Scuola Norm. Sup. di Pisa*
7 BHATTACHARYA, K., R. JAMES, in preparation.
8 BAUMAN, P., N. C. OWEN and D. PHILLIPS, Maximum principles and apriori estimates for a class of problems from nonlinear elasticity, *Ann. Inst. H. Poincaré* **8** (1991), 119–157.
9 BLAKE, A., and A. ZISSERMAN, Visual Reconstruction, *The MIT Press, Cambridge, Massachussets, 1985.*
10 BONNET, A., On the regularity of edges in the Mumford-Shah model for image segmentation. To appear.
11 CARRIERO, M. and A. LEACI, S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities, *Ann. Scuola Norm. Sup. di Pisa* **18** (1991), 321–352.
12 CIARLET, P. G., P. DESTUYNDER, A justification of a nonlinear model in plate theory, *Comput. Methods Appl. Mech. Engrg.* **17/18** (1979), 227-258.
13 DAVID, G. and S. SEMMES, On the singular set of minimizers of the Mumford-Shah functional. To appear in *J. Math. Pures et Appl.*
14 DE GIORGI, E., *Free Discontinuity Problems in the Calculus of Variations*, a collection of papers dedicated to J. L. Lions on the occasion of his 60^{th} birthday, North Holland (R. Dautray ed.), 1991.
15 DE GIORGI, E. and L. AMBROSIO, Un nuovo tipo di funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.
16 DE GIORGI, E., M. CARRIERO and A. LEACI, Existence theorem for a minimum problem with free discontinuity set, *Arch. Rat. Mech. Anal.* **108** (1989), 195–218.
17 DOUGHERTY, M., Higher integrability of the gradient for minimizers of certain polyconvex functionals in the calculus of variations. Preprint.
18 FONSECA, I. and G. FRANCFORT, Relaxation in BV versus quasiconvexification in $W^{1,p}$; a model for the interaction between fracture and damage, *Calc. Var.* **3** (1995), 407-446.
19 FONSECA, I. and G. FRANCFORT, Optimal design problems in elastic membranes. To appear.
20 FONSECA, I. and N. FUSCO, Regularity results for anisotropic image segmentation models, To appear in *Ann. Scuola Norm. Sup. di Pisa*
21 GIAQUINTA, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, Princeton University Press, 1983.
22 GILBARG, D., N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.
23 LE DRET, H., A. RAOULT, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures et Appl.* **74** (1995), 549-578.
24 MORREY, C. B., *Multiple integrals in the Calculus of Variations*, Springer, Berlin 1966.
25 MUMFORD, D. and J. SHAH, Boundary detection by minimizing functionals, *Proc. IEEE Conf. on Computer Vision and Pattern Recognition (San Francisco, 1985).*

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