A GLOBAL METHOD FOR RELAXATION

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Abstract. A new method for the identification of the integral representation of some class of functionals defined on $BV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$ (where $\mathcal{A}(\Omega)$ represents a family of open subsets of Ω) is presented. Applications are derived, such as the integral representation of the relaxed energy in $BV(\Omega;\mathbb{R}^d)$ corresponding to a functional defined in $W^{1,1}(\Omega;\mathbb{R}^d)$ with a discontinuous integrand with linear growth ; relaxation and homogenization results in $SBV(\Omega;\mathbb{R}^d)$ are recovered in the case where bulk and surface energies are present.

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1. Introduction.

Several problems in phase transitions, fracture mechanics, plasticity and image segmentation, may be studied within a framework where the underlying energy is given by a functional of the type

$$\mathcal{F} : BV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\longrightarrow[0,+\infty],$$

where $\mathcal{A}(\Omega)$ stands for the family of open subsets A of a fixed bounded domain Ω of $\mathbb{R}^N$, with Lipschitz boundary ∂A, and $\mathcal{F}$ satisfies the following properties :

i) $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure ;

ii) $\mathcal{F}(\cdot;A)$ is $L^1(A;\mathbb{R}^d)$-lower semicontinuous ;

iii) there exists $C>0$ such that, for some $p\geq 1$,

$$0\ \leq\mathcal{F}(u;A)\ \leq C\left\{\int_A(1+|\nabla u|^p)\,dx+|D_s u|(A)\right\}.$$

The case where $p>1$ will be studied in a forthcoming paper. Here we treat the case where $p=1$.

An important example of such functionals is given by the relaxed energy corresponding to a discontinuous bulk energy density, precisely

$$\mathcal{F}(u;A) := \inf\Big\{ \liminf_{n\to+\infty} \int_A f_0(x,u_n,\nabla u_n)\, dx \mid u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d), \\ u_n \in W^{1,1}(\Omega;\mathbb{R}^d)\Big\}.$$

We may also consider the case where both bulk and surface energies are present in the underlying functional, namely

$$\mathcal{F}(u;A) = \inf\Big\{ \liminf_{n\to+\infty} \int_A f_0(x,u_n,\nabla u_n)\, dx + \int_{A\cap S(u_n)} g_0(x,u_n^+,u_n^-,\nu_{u_n}) dH^{N-1} \mid \\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d) \quad , \quad u_n \in SBV(\Omega;\mathbb{R}^d)\Big\},$$

where $S(u_n)$ denotes the jump set of u_n. Another example of a functional $\mathcal{F}$ to which our theory may be applied is provided by a sequence of functionals F_ε (for instance, in the context of homogenization theory), where the energy $\mathcal{F}(u,A)$ that we want to identify reduces to the limit of (F_ε) in the sense of Γ – *convergence.*

A natural question at the core of the Calculus of Variations concerns the search for an integral representation of $\mathcal{F}(u;A)$. In this paper we propose a new method suitable to the study of all situations mentioned before ; the main idea of this method consists in showing that $\mathcal{F}(u;A)$ can be reconstructed in terms of the set function $m(u,\cdot)$ defined on $\mathcal{A}(\Omega)$ by

$$m(u;A) := \inf\Big\{\mathcal{F}(v;A) \mid v|_{\partial A} = u|_{\partial A}, v \in BV(\Omega;\mathbb{R}^d)\Big\}.$$

The reduction of the relaxed problem to a local Dirichlet type of question has already been used in the context of homogenization or quasiconvexification theories. The main point proved in Section 3 (see Lemma 3.3) is that $m(u;A)$ behaves as $\mathcal{F}(u;A)$ when A is a cube of small size. Then the bulk and the jump local densities of the energy can be recovered from $m(u,.)$ by using Besicovitch Differentiation Theorem (see Theorem 3.4). An explicit identification of these densities comes easily by means of a blow-up argument and using the Lipschitz behaviour of $m(u,A)$ with respect to the norm in $L^1(\partial A)$ of the trace of u (see Lemma 3.1).

In Theorem 3.4 we obtain a representation formula of the form

$$\mathcal{F}(u;A) = \int_A f(x,u,\nabla u)\, dx + \int_{S(u)\cap A} g(x,u^+,u^-,\nu_u)\, d\mathcal{H}^{N-1} \tag{1.1}$$

for every u in the space $SBV(\Omega;\mathbb{R}^d)$ of all functions with bounded variation whose distributional derivative may be written as
$Du = \nabla u\ \mathcal{L}^N + [(u^+ - u^-)\otimes \nu_u]\ \mathcal{H}^{N-1}\lfloor S(u)$ (see [Am2]). Here, and in what

follows, $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure, and $\mathcal{H}^{N-1}$ stands for the $N-1$-dimensional Hausdorff measure (see Section 2). For general BV functions we have also to take into account an extra term in the decomposition of Du, $Du = \nabla u \; \mathcal{L}^N + [(u^+ - u^-) \otimes \nu_u] \; \mathcal{H}^{N-1} \lfloor S(u) + C(u)$, where $C(u)$ denotes the Cantor part of Du. The characterization of the density of $\mathcal{F}$ with respect to $C(u)$ seems to be very difficult to obtain in general (see [BDM] in the scalar case). Under an additional assumption of continuity of $\mathcal{F}$ with respect to vertical and horizontal translations (see condition (2.4)), we obtain in Theorem 3.10 the full integral representation for $u \in BV(\Omega; \mathbb{R}^d)$,

$$\begin{aligned}\mathcal{F}(u; A) = \int_A f(x, u, \nabla u) \; dx + \int_{S(u) \cap A} g(x, u^+, u^-, \nu_u) \; d\mathcal{H}^{N-1} \\ + \int_A f^\infty \Big(x, u, \frac{dC(u)}{d|C(u)|}\Big) \; d|C(u)| \; .\end{aligned} \tag{1.2}$$

In Section 4 we apply the latter characterizations to some specific situations. In Subsection 4.1 we provide a new integral representation of the relaxed energy for a discontinuous integrand with linear growth conditions and in the vectorial case, recovering the results of [FM1] and [FM2] in the case of non degenerate coercivity assumptions. The corresponding scalar case, previously treated by Bouchitté and Dal Maso [BDM], and by Braides and Coscia [BC], follows as a corollary. In Subsection 4.2 we extend the results of Barroso, Bouchitté, Buttazzo and Fonseca [BBBF] concerning the relaxation in SBV of an energy involving bulk and interfacial contributions. In Subsection 4.3 we obtain the characterization of the homogenized energy associated with a sequence of free discontinuity problems with a linear growth condition. This problem was treated by Braides, Defranceschi and Vitali [BDV] in the case $p > 1$.

2. Preliminaries.

Let Ω represent an open bounded subset of $\mathbb{R}^N$. In the sequel we use the standard notations for bounded variation, Sobolev and Lebesgue spaces, denoted, respectively, by $BV(\Omega; \mathbb{R}^d), W^{1,p}(\Omega; \mathbb{R}^d)$ and $L^p(\Omega; \mathbb{R}^d)$. $\mathcal{A}(\Omega)$ stands for the family of all open subsets A of Ω with Lipschitz boundary ∂A, and $\mathcal{B}(\Omega)$ is the collection of all Borel subsets of Ω. The Lebesgue measure and the Hausdorff (N-1)-dimensional measure in $\mathbb{R}^N$ are designated by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively. C will denote a generic constant which may vary from line to line.

To each $\nu \in S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ we associate a rotation R_ν such that $R_\nu(e_N) = \nu$, where $(e_i)_{i=1,\cdots,N}$ stands for the canonical basis in $\mathbb{R}^N$. We may choose $\nu \mapsto R_\nu$ so that R_{e_N} is the identity and $\nu \mapsto R_\nu(e_i)$ is continuous in $S^{N-1} \setminus \{e_N\}$, for all $i = 1, \cdots, N-1$. We define $Q_\nu := R_\nu(Q)$, where $Q := \{x \in \mathbb{R}^N \mid |x \cdot e_i| < 1/2, \; i = 1, \cdots, N\}$ and we set $Q_\nu(x, \varepsilon) := x + \varepsilon Q_\nu$, for $\varepsilon > 0$. We will omit the subscript ν whenever ν coincides with e_N.

In what concerns general BV space theory we follow Evans and Gariepy [EG], Federer [F], Giusti [G], and Ziemer [Z]. We represent by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure (or Radon Nikodym derivative), and $S(u)$, *the jump set*, is the complement of the set of Lebesgue points, i.e. the set of points x where the *approximate upper limit* $u_i^+(x)$ is different from the *approximate lower limit* $u_i^-(x)$, for some $i \in \{1, \dots, d\}$, namely

$$S(u) = \bigcup_{i=1}^{d} \left\{ x \in \Omega \mid u_i^-(x) < u_i^+(x) \right\}.$$

Choosing a normal $\nu_u(x)$ to $S(u)$ at x (defined uniquely, up to sign, for $\mathcal{H}^{N-1}$ a.e. x), we set $[u](x) := u^+(x) - u^-(x)$ the difference between the traces of u at $x \in S(u)$, oriented by $\nu_u(x)$. Representing by $C(u)$ the Cantor part of the measure Du, the following decomposition holds :

$$Du = \nabla u \, \mathcal{L}^N + ([u] \otimes \nu_u) \, \mathcal{H}^{N-1} \lfloor S(u) + C(u).$$

We represent by $SBV(\Omega; \mathbb{R}^d)$ the space of *special functions of bounded variation* introduced by De Giorgi and Ambrosio (see [ADG]), i.e. the space of all functions in $BV(\Omega; \mathbb{R}^d)$ such that $C(u) = 0$.

In what follows we consider a functional

$$\mathcal{F} : BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \longrightarrow [0, +\infty]$$

satisfying

$$\mathcal{F}(u; \cdot) \text{ is the restriction to } \mathcal{A}(\Omega) \text{ of a Radon measure,} \tag{2.1}$$

$$\mathcal{F}(\cdot; A) \text{ is } L^1(A; \mathbb{R}^d) - \text{lower semicontinuous,} \tag{2.2}$$

there exist $C > 0$ such that

$$0 \leq \mathcal{F}(u; A) \leq C\Big(\mathcal{L}^N(A) + |Du|(A)\Big) \,. \tag{2.3}$$

In order to characterize the density energy corresponding to the Cantor part of the measure $\mathcal{F}(u; \cdot)$, we will need to assume further that the functional $\mathcal{F}$ depends continuously both on horizontal and vertical translations in the following sense :

There exists a modulus of continuity $\Phi(t)$ satisfying

$$|\mathcal{F}(u(\cdot - z) + b; z + A) - \mathcal{F}(u; A)| \leq \Phi(|b| + |z|) \left(\mathcal{L}^N(A) + |Du|(A)\right), \tag{2.4}$$

for all $(u, A, b, z) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \times \mathbb{R}^d \times \mathbb{R}^N$, such that $z + A \subset \Omega$.

Remark 2.1. Condition (2.2) implies that $\mathcal{F}$ is *local*, i.e.,

$$\text{if } \; u = v \;\; \mathcal{L}^N \text{a.e. } x \in A, \text{ then } \mathcal{F}(u; A) = \mathcal{F}(v; A) \quad \text{for all } A \in \mathcal{A}(\Omega) \,. \tag{2.2'}$$

Remark 2.2. Without loss of generality we may assume that coercivity holds, and so we replace (2.3) by the condition

$$\frac{1}{C}\,|Du|(A) \leq \mathcal{F}(u;A) \leq C\left(\mathcal{L}^N(A) + |Du|(A)\right) \qquad \text{for some } C>0. \qquad (2.3')$$

Indeed, if we are able to identify the integral representation under (2.1), (2.2) and (2.3′), given $\mathcal{F}$ satisfying (2.1), (2.2) and (2.3), it suffices to define

$$\mathcal{F}_1(u;A) := \mathcal{F}(u;A) + |Du|(A)\;.$$

By virtue of the lower semicontinuity property of the total variation, it is clear that $\mathcal{F}_1$ is under conditions (2.1), (2.2) and (2.3′), and so we are able to find densities f_1, g_1, h_1 such that

$$\begin{aligned}\mathcal{F}_1(u;A) = \int_A f_1(x,u,\nabla u)\,dx + \int_{S(u)\cap A} g_1(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}\\ + \int_A h_1\left(x,u,\frac{dC(u)}{d|C(u)|}\right)\,d|C(u)|\;.\end{aligned}$$

We deduce that for every $u \in BV(\Omega;\mathbb{R}^d)$ we have

$$\begin{aligned}\mathcal{F}(u;A) = &\int_A \left[f_1(x,u,\nabla u) - |\nabla u|\right]\,dx\\ &+ \int_{S(u)\cap A}\left[g_1(x,u^+,u^-,\nu_u) - |u^+-u^-|\right]\,d\mathcal{H}^{N-1}\\ &+ \int_A\left[h_1\left(x,u,\frac{dC(u)}{d|C(u)|}\right) - 1\right]\,d|C(u)|,\end{aligned}$$

which provides the representation formula (1.2).

We now state some technical results that will be used in the sequel. Given $(u;A) \in BV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$, we represent by tr u or $u|_{\partial A}$ the *trace* of u restricted to A. The proof of the first two lemmas may be found in [G].

Lemma 2.3. *Let $A \in \mathcal{A}(\Omega)$ and let $u_n, u \in BV(A;\mathbb{R}^d)$ be such that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$ and $|Du_n|(A) \to |Du|(A)$. Then*

$$\int_{\partial A} |\mathrm{tr}\ u_n - \mathrm{tr}\ u|\,d\mathcal{H}^{N-1} \to 0.$$

Lemma 2.4. *Let $A \in \mathcal{A}(\Omega)$ and let $\theta \in L^1(\partial A)$. For every $\varepsilon > 0$ there exists $w_\varepsilon \in W^{1,1}(A)$ and a constant C, depending only on ∂A, such that*

$$w_\varepsilon|_{\partial A} = \theta,\ \int_A |w_\varepsilon|\,dx \leq \varepsilon\int_{\partial A}|\theta|\,d\mathcal{H}^{N-1},\ \int_A |\nabla w_\varepsilon|\,dx \leq C\int_{\partial A}|\theta|\,d\mathcal{H}^{N-1}.$$

Next we prove a density result in BV under Dirichlet boundary conditions.

Lemma 2.5. *Let $A \in \mathcal{A}(\Omega)$. Given $u \in BV(A;\mathbb{R}^d)$ we may find $v_n \in W^{1,1}(A;\mathbb{R}^d)$ such that*

$$v_n|_{\partial A} = u|_{\partial A}, \quad \|v_n - u\|_{L^1(A;\mathbb{R}^d)} \to 0, \quad |Dv_n|(A) \to |Du|(A).$$

Proof. Let $\theta_n \in C^\infty(A;\mathbb{R}^d)$ satisfy $\theta_n \to u$ in $L^1(A;\mathbb{R}^d)$ and $\int_A |\nabla\theta_n| dx \to |Du|(A)$. By Lemma 2.3 we have

$$\int_{\partial A} |\text{tr } \theta_n - \text{tr } u| \, d\mathcal{H}^{N-1} \to 0. \tag{2.5}$$

Using Lemma 2.4, for each n consider $w_n \in W^{1,1}(A;\mathbb{R}^d)$ such that

$$\begin{gathered} w_n|_{\partial A} = (\theta_n - u)|_{\partial A}, \quad \int_A |w_n| \, dx \le \int_{\partial A} |\text{tr } \theta_n - \text{tr } u| \, d\mathcal{H}^{N-1}, \\ \int_A |\nabla w_n| \, dx \le C \int_{\partial A} |\text{tr } \theta_n - \text{tr } u| \, d\mathcal{H}^{N-1}. \end{gathered} \tag{2.6}$$

Let $v_n := \theta_n - w_n$. Then $v_n|_{\partial A} = u|_{\partial A}$ and

$$\|v_n - u\|_{L^1(A;\mathbb{R}^d)} \le \|\theta_n - u\|_{L^1(A;\mathbb{R}^d)} + \|w_n\|_{L^1(A;\mathbb{R}^d)}. \tag{2.7}$$

From (2.5) and (2.6) we conclude that

$$w_n \to 0 \quad \text{in } W^{1,1}(A;\mathbb{R}^d),$$

and so, by (2.7) we have

$$v_n \to u \quad \text{in } L^1(A;\mathbb{R}^d), \ \lim_{n\to+\infty} \int_A |\nabla v_n| \, dx = \lim_{n\to+\infty} \int_A |\nabla\theta_n| \, dx = |Du|(A).$$

□

The following result is a version of the *Slicing Lemma* of E. De Giorgi.

Lemma 2.6. *Let $F : BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be a functional satisfying conditions (2.1), (2.2′) and (2.3). Let $u \in BV(\Omega;\mathbb{R}^d)$ and let (v_n) be a sequence in $BV(\Omega;\mathbb{R}^d)$ such that $v_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. Then, for every $A \in \mathcal{A}(\Omega)$ we can find a sequence $w_n \in BV(\Omega;\mathbb{R}^d)$ such that*

$$\|w_n - u\|_{L^1(\Omega;\mathbb{R}^d)} \to 0 \ , \ w_n = u \ \text{ on } \ \partial A \ , \ \limsup_{n\to+\infty} F(w_n;A) \le \liminf_{n\to+\infty} F(v_n;A) \ .$$

Proof. Let $v_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. Up to a subsequence, we may assume that $\liminf_{n\to+\infty} F(v_n;A) = \lim_{n\to+\infty} F(v_n;A)$. For each $k \in \mathbb{N}$ define

$$A_k := \{x \in A | \ \text{dist}(x, \partial A) > 1/k\}$$

and consider the layer $L_k := A_{k-1} \setminus A_k$. For each $n \in \mathbb{N}$ set $M_n := n + |Dv_n|(A)$.

Representing by $[\![a]\!]$ the integer part of $a \in \mathbb{R}$, we split each L_k into $[\![M_n]\!]^2$ layers $L_{k,i}$, $i = 1, \cdots, [\![M_n]\!]^2$, of thickness $\left[k(k-1)[\![M_n]\!]^2\right]^{-1}$, $L_k = \bigcup_i L_{k,i}$, and where the layers $L_{k,i}$ are labeled so that $L_{k,i}$ is closer to the boundary of A than $L_{k,j}$ if $i > j$, $i, j \in \{1, \cdots, [\![M_n]\!]^2\}$.

To each layer $L_{k,i}$ we assign a *cut-off* function $\varphi_{k,i}$ with $0 \leq \varphi_{k,i} \leq 1$, $\varphi_{k,i} = 0$ in $\Omega \setminus \left[\bigcup_{j \leq i}(L_{k,j} \cup A_k)\right]$ and $\varphi_{k,i} = 1$ in $\bigcup_{j<i}(L_{k,j} \cup A_k)$. We have $\|\nabla \varphi_{k,i}\|_\infty = O(k^2[\![M_n]\!]^2)$.

Defining

$$w_{n,k,i} := \varphi_{k,i} v_n + (1 - \varphi_{k,i})u, \tag{2.8}$$

and using (2.1) we will have

$$\begin{aligned} F(w_{n,k,i}; A) &\leq F(v_n; A) + F(u; A \setminus A_k) + F(w_{n,k,i}; L_{k,i}) \\ &\leq F(v_n; A) + C\left(\mathcal{L}^N + |Du|\right)(A \setminus A_k) + C\left(\mathcal{L}^N + |Dw_{n,k,i}|\right)(L_{k,i}) \end{aligned}$$

where we have used (2.3). On the other hand, for fixed k,

$$\begin{aligned} \frac{1}{[\![M_n]\!]^2} \sum_i |Dw_{n,k,i}|(L_{k,i}) \leq & \frac{C}{[\![M_n]\!]^2}\Big(|Dv_n|(A) + |Du|(A)\Big) \\ & + \frac{1}{[\![M_n]\!]^2} O(k^2[\![M_n]\!]^2)\|v_n - u\|_{L^1(A;\mathbb{R}^d)}. \end{aligned} \tag{2.9}$$

Since the right hand side of (2.9) goes to zero as $n \to +\infty$, we can construct a sequence $n_k \to +\infty$ such that

$$\begin{aligned} &\|w_{n_k,k,i} - u\|_{L^1(A;\mathbb{R}^d)} \leq \|v_{n_k} - u\|_{L^1(A;\mathbb{R}^d)} \leq \frac{1}{k} \quad \text{for all } \; i \in \{1, \cdots, [\![M_{n_k}]\!]^2\}, \\ &\frac{1}{[\![M_{n_k}]\!]^2} \sum_i |Dw_{n_k,k,i}|(L_{k,i}) < \frac{1}{k}, \end{aligned}$$

and then choose i_k such that

$$|Dw_{n_k,k,i_k}|(L_{k,i_k}) < \frac{1}{k}.$$

Defining $w_k := w_{n_k,k,i_k}$ we obtain $w_k \to u$ in $L^1(\Omega; \mathbb{R}^d)$,

$$F(w_k; A) \leq F(v_{n_k}; A) + O(1/k)$$

and, consequently,

$$\limsup_{k \to +\infty} F(w_k; A) \leq \limsup_{k \to +\infty} F(v_{n_k}; A) = \liminf_{n \to +\infty} F(v_n; A),$$

which completes the proof. □

Remark 2.7. Having in mind the applications treated in Section 4, we mention some extensions of Lemma 2.6.

1) If the sequence (v_n) is in $W^{1,1}(\Omega;\mathbb{R}^d)$ (respectively in $SBV(\Omega;\mathbb{R}^d)$), then the sequence (w_n) can be constructed in $W^{1,1}(\Omega;\mathbb{R}^d)$ (respectively in $SBV(\Omega;\mathbb{R}^d)$).

Indeed, using Lemma 2.5 we can replace u in (2.8) by a sequence (u_n) in $W^{1,1}(\Omega;\mathbb{R}^d)$ satisfying

$$u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d), \quad u_n = u \text{ on } \partial A\,, \text{ and } |Du_n|(A) \to |Du|(A).$$

Then, by the lower semicontinuity of the total variation in open sets we obtain

$$\begin{aligned}\limsup_{n\to+\infty} |Du_n|(A\setminus A_k) &= \lim_{n\to+\infty} |Du_n|(A) - \liminf_{n\to+\infty} |Du_n|(A_k)\\ &\le |Du|(A) - |Du|(A_k) = |Du|(A\setminus A_k) = O(1/k).\end{aligned}$$

2) We can also extend the results stated in Lemma 2.6 and in the previous remark to a sequence of functionals (F_n) satisfying conditions (2.1), (2.2′) and (2.3) uniformly in n. In this case, if $u \in BV(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$, for each sequence $v_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ we can find a sequence of indexes (n_k) and a sequence $w_k \in BV(\Omega;\mathbb{R}^d)$ such that

$$\|w_k - u\|_{L^1(\Omega;\mathbb{R}^d)} \to 0\,,\ w_k = u \text{ on } \partial A,\ \limsup_{k\to+\infty} F_{n_k}(w_k;A) \le \liminf_{n\to+\infty} F_n(v_n;A)\,.$$

Finally we state the following truncation lemma (see Lemma 3.7 in [BBBF]) :

Lemma 2.8. *Let $F : BV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega) \to [0,+\infty]$ be a functional satisfying conditions (2.1), (2.2′) and (2.3). If $u_0 \in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ and if $\varepsilon > 0$ then, for every $R > 0$ there exists $M = M\left(\varepsilon, R, C, \|u_0\|_{L^\infty(\Omega;\mathbb{R}^d)}\right)$ such that for every $u \in BV(\Omega;\mathbb{R}^d)$ (resp. $u \in SBV(\Omega;\mathbb{R}^d)$ or $u \in W^{1,1}(\Omega;\mathbb{R}^d)$) with $\|u\|_{BV(\Omega;\mathbb{R}^d)} \le R$ and $u = u_0$ on $\partial\Omega$, there exists $\bar u \in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ (resp. $\bar u \in SBV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ or $\bar u \in W^{1,1}(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$) such that*

i) $\|\bar u\|_{L^\infty(\Omega;\mathbb{R}^d)} \le M$; *ii)* $\bar u = u_0$ *on* $\partial\Omega$;

iii) $|D\bar u|(\Omega) \le |Du|(\Omega)$; *iv)* $F(\bar u;\Omega) \le F(u;\Omega) + \varepsilon$.

3. The General Method.

In this section we identify the bulk and jump densities of a functional $\mathcal{F}$ satisfying conditions (2.1), (2.2) and (2.3′) (Theorem 3.4). In case condition (2.4) holds, we can also characterize the Cantor part and conclude with the full representation of $\mathcal{F}$ (see Theorem 3.10).

Given $(u; A) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$ we introduce

$$m(u; A) := \inf \Big\{ \mathcal{F}(v; A) |\ v|_{\partial A} = u|_{\partial A}, v \in BV(\Omega; \mathbb{R}^d) \Big\}. \tag{3.1}$$

The basic idea of our method consists in comparing, for fixed $x_0 \in \Omega$, the asymptotic behaviors of $m(u; Q(x_0, \varepsilon))$ and $\mathcal{F}(u; Q(x_0, \varepsilon))$ when ε goes to 0. This is made clear in Lemma 3.3 below where, via a blow-up argument, it is shown that as ε gets small we may conclude that relaxation reduces to solving a Dirichlet problem. An important tool of this method is Lemma 3.1, which allows us to replace u by its limit obtained by a blow-up at x_0.

Lemma 3.1. *There exists a constant C such that*

$$|m(u_1; A) - m(u_2; A)| \leq C \int_{\partial A} |\text{tr } (u_1 - u_2)| \, d\mathcal{H}^{N-1} \tag{3.2}$$

for all $u_1, u_2 \in BV(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.

Proof. Let $u_1, u_2 \in BV(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$. For $\delta > 0$ small enough, set

$$A_\delta := \{x \in A |\ \text{dist } (x, \partial A) > \delta\}.$$

Given $v \in BV(\Omega; \mathbb{R}^d)$ with $v|_{\partial A} = u_2|_{\partial A}$, define v_δ such that $v_\delta = v$ in A_δ, and $v_\delta = u_1$ in $\Omega \setminus A_\delta$. In view of (3.1) and (2.1) one has

$$\begin{aligned} m(u_1; A) &\leq \mathcal{F}(v_\delta; A) \\ &= \mathcal{F}(v_\delta; A_\delta) + \mathcal{F}(v_\delta; A \setminus A_\delta) \\ &\leq \mathcal{F}(v; A) + \mathcal{F}(v_\delta; A \setminus A_\delta). \end{aligned} \tag{3.3}$$

From (2.3′), which still holds for Borel sets, we obtain

$$\mathcal{F}(v_\delta; A \setminus A_\delta) \leq C \int_{A \setminus A_\delta} (1 + |\nabla u_1|) \, dx + C|D_s u_1|(A \setminus \bar{A}_\delta) + C|D_s v_\delta|(\partial A_\delta). \tag{3.4}$$

As δ goes to zero, one has immediately

$$\int_{A \setminus A_\delta} (1 + |\nabla u_1|) \, dx \to 0, \quad |D_s u_1|(A \setminus \bar{A}_\delta) \to 0, \tag{3.5}$$

and, using the definition of trace and Green's formula (see [EG], 5.4),

$$|D_s v_\delta|(\partial A_\delta) = \int_{\partial A_\delta} |\text{tr } (u_1|_{A \setminus \bar{A}_\delta} - v|_{A_\delta})| d\mathcal{H}^{N-1} \to \int_{\partial A} |\text{tr } (u_1 - u_2)| d\mathcal{H}^{N-1}. \tag{3.6}$$

From (3.3) - (3.6) we conclude that

$$m(u_1; A) \leq \mathcal{F}(v; A) + C \int_{\partial A} |\text{tr } (u_1 - u_2)| \, d\mathcal{H}^{N-1}.$$

Taking the infimum over v and interchanging the roles of u_1 and u_2, inequality (3.2) follows. $\square$

Fix $u \in BV(\Omega; \mathbb{R}^d)$, $\nu \in S^{N-1}$, and define $\mu := \mathcal{L}^N + |D_s u|$. Let

$$\mathcal{A}_\nu := \{Q_\nu(x, \varepsilon) | \; x \in \Omega, \; \varepsilon > 0\} \tag{3.7}$$

and for $\delta > 0$ set

$$m^\delta(u; A) := \inf \Big\{ \sum_{i=1}^{\infty} m(u; Q_i) | \; Q_i \in \mathcal{A}_\nu, \; Q_i \cap Q_j = \emptyset, \; Q_i \subset A,$$
$$\text{diam}(Q_i) < \delta, \; \mu(A \setminus \cup_{i=1}^{\infty} Q_i) = 0 \Big\}.$$

Besicovitch's Covering Theorem guarantees the existence of such coverings of A. Given that $\delta \mapsto m^\delta(u; A)$ is a decreasing function, we define

$$\begin{aligned} m^*(u; A) : &= \sup \; \{m^\delta(u; A) | \; \delta > 0\} \\ &= \lim_{\delta \to 0} \; m^\delta(u; A). \end{aligned}$$

Lemma 3.2. *Under hypotheses (2.1), (2.2) and (2.3′),*

$$\mathcal{F}(u; A) = m^*(u; A).$$

Proof. Since $\mathcal{F}(u; \cdot)$ is a Radon measure (see (2.1)), and because $m(u; A) \leq \mathcal{F}(u; A)$, the inequality $m^*(u; A) \leq \mathcal{F}(u; A)$ is obvious. We prove that

$$\mathcal{F}(u; A) \leq m^*(u; A).$$

Fix $\delta > 0$ and let (Q_i^δ) be an admissible sequence in the sense of the definition of $m^\delta(u; A)$, such that

$$\sum_{i=1}^{\infty} m(u; Q_i^\delta) < m^\delta(u; A) + \delta. \tag{3.8}$$

Using the definition of m, choose $v_i^\delta \in BV(\Omega; \mathbb{R}^d)$ such that

$$v_i^\delta|_{\partial Q_i^\delta} = u|_{\partial Q_i^\delta}, \quad \mathcal{F}(v_i^\delta; Q_i^\delta) \leq m(u; Q_i^\delta) + \delta \; \mathcal{L}^N(Q_i^\delta). \tag{3.9}$$

Set

$$v^\delta := \sum_{i=1}^{\infty} v_i^\delta \chi_{Q_i^\delta} + u\chi_{N_0^\delta},$$

where $N_0^\delta := \Omega \setminus \cup_{i=1}^{\infty} Q_i^\delta$. From (3.8), (3.9) and the coercivity hypothesis (2.3′), it follows that $v^\delta \in BV(\Omega; \mathbb{R}^d)$ (*),

$$\begin{aligned} Dv^\delta &= \sum_{i=1}^{\infty} Dv_i^\delta \lfloor Q_i^\delta + Du \lfloor N_0^\delta, \\ |Dv^\delta| \lfloor N^\delta &= 0, \quad \mu(N^\delta) = 0, \end{aligned} \tag{3.10}$$

where $N^\delta := A \cap N_0^\delta$, and

$$\mathcal{F}(v^\delta; N^\delta) \le C\left(\mathcal{L}^N(N^\delta) + |Dv^\delta|(N^\delta)\right) = 0 \, .$$

Using (2.1), (3.8) and (3.9), we deduce that

$$\begin{aligned} \mathcal{F}(v^\delta; A) &= \sum_{i=1}^{\infty} \mathcal{F}(v_i^\delta; Q_i^\delta) + \mathcal{F}(v^\delta; N^\delta) \\ &\le m^\delta(u; A) + \delta + \delta\, \mathcal{L}^N(A). \end{aligned} \tag{3.11}$$

We claim that $v^\delta \to u$ in $L^1(A)$. If so, using hypothesis (2.2) we have

$$\mathcal{F}(u; A) \le \liminf_{\delta \to 0} \mathcal{F}(v^\delta; A),$$

which, together with (3.11), yields

$$\mathcal{F}(u; A) \le \liminf_{\delta \to 0} m^\delta(u; A) = m^*(u; A).$$

It remains to prove the claim : $v^\delta \to u$ in $L^1(A)$. By Poincaré's inequality there exists a constant C such that

$$\|v^\delta - u\|_{L^1(Q_i^\delta)} \le C\delta\, |Dv^\delta - Du|(Q_i^\delta),$$

(*) For every $\varphi \in C_0(\Omega)$, integrating by parts on every Q_i^δ and recalling that $v_i^\delta = u$ on ∂Q_i^δ, we can write

$$\langle D(v^\delta - u), \varphi \rangle = -\sum_{i=1}^{\infty} \int_{Q_i^\delta} (v_i^\delta - u) \otimes \nabla\varphi \, dx = \sum_{i=1}^{\infty} \int_{Q_i^\delta} \varphi . (Dv_i^\delta - Du) \, .$$

thus

$$\begin{aligned}\|v^\delta - u\|_{L^1(A)} &= \sum_{i=1}^{\infty} \|v_i^\delta - u\|_{L^1(Q_i^\delta)} \\ &\leq C\delta\ |Dv^\delta - Du|(\cup_{i=1}^{\infty} Q_i^\delta) \\ &\leq C\delta\ \Big(|Dv^\delta|(A) + |Du|(A)\Big).\end{aligned}$$

In view of the coercivity condition (2.3′) and by (3.11), $|Dv^\delta|(A)$ is bounded and we conclude that $\|v^\delta - u\|_{L^1(A)} \to 0$. □

Lemma 3.3. *Under (2.1),(2.2) and* (2.3′), *the following equality holds*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{F}(u; Q_\nu(x_0,\varepsilon))}{m(u; Q_\nu(x_0,\varepsilon))} = 1, \quad \mu \text{ a.e. } x_0 \in \Omega \quad \textit{and for all } \nu \in S^{N-1}. \tag{3.12}$$

Proof. Since $m(u; Q_\nu(x_0,\varepsilon)) \leq \mathcal{F}(u; Q_\nu(x_0,\varepsilon))$, we have

$$1 \leq \liminf_{\varepsilon\to 0} \frac{\mathcal{F}(u; Q_\nu(x_0,\varepsilon))}{m(u; Q_\nu(x_0,\varepsilon))}.$$

We only need to prove that, μ a.e. $x_0 \in \Omega$ and for all $\nu \in S^{N-1}$,

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{F}(u; Q_\nu(x_0,\varepsilon))}{m(u; Q_\nu(x_0,\varepsilon))} \leq 1.$$

For each $t > 1$ let E_t be defined by

$$E_t := \Big\{x \in \Omega \mid \text{there exist } \nu \in S^{N-1} \text{ and } \varepsilon_h \to 0 \text{ such that } \mathcal{F}(u; Q_\nu(x,\varepsilon_h)) > t\ m(u; Q_\nu(x,\varepsilon_h)) \text{ for all } h\Big\}.$$

Our aim is to show that $\mu(E_t) = 0$. Consider an open set ω and a compact set K such that $K \subset E_t \subset \omega$. Fix $\delta > 0$ and define

$$X^\delta := \Big\{Q_\nu(x,\varepsilon) | \varepsilon < \delta, x \in K, Q_\nu(x,\varepsilon) \subset \omega,\ \mathcal{F}(u; Q_\nu(x,\varepsilon)) > t\ m(u; Q_\nu(x,\varepsilon))\ \Big\}$$

$$Y^\delta := \Big\{Q_\nu(x,\varepsilon) |\ \varepsilon < \delta,\ Q_\nu(x,\varepsilon) \subset \omega \setminus K\Big\}.$$

By virtue of the definition of E_t, if $x \in K$ there exists $\varepsilon < \delta$ such that $Q_\nu(x,\varepsilon) \in X^\delta$ and so

$$\omega = \Big(\bigcup_{Q_\nu(x,\varepsilon)\in X^\delta} Q_\nu(x,\varepsilon)\Big) \cup \Big(\bigcup_{Q_\nu(x,\varepsilon)\in Y^\delta} Q_\nu(x,\varepsilon)\Big).$$

Using Besicovich's Covering Theorem, we may find a subcovering of ω such that

$$\omega = \Big(\bigcup_{i\in I} Q_i^{X^\delta}\Big) \cup \Big(\bigcup_{j\in J} Q_j^{Y^\delta}\Big) \cup N,$$

where I and J are countable, $Q_i^{X^\delta} \in X^\delta$, $Q_j^{Y^\delta} \in Y^\delta$, the sets $Q_i^{X^\delta}$ and $Q_j^{Y^\delta}$ are mutually disjoint, and $\mu(N) = 0$. Since $m(u;\cdot) \le \mathcal{F}(u;\cdot)$ and $\mathcal{F}(u;\cdot)$ is absolutely continuous with respect to μ, we have

$$\begin{aligned}\mathcal{F}(u;\omega) &= \sum_{i\in I} \mathcal{F}(u; Q_i^{X^\delta}) + \sum_{j\in J} \mathcal{F}(u; Q_j^{Y^\delta}) \\ &\ge \sum_{i\in I} t\, m(u; Q_i^{X^\delta}) + \sum_{i\in J} m(u; Q_j^{Y^\delta}) \\ &= t\,\Big(\sum_{i\in I} m(u; Q_i^{X^\delta}) + \sum_{j\in J} m(u; Q_j^{Y^\delta})\Big) + (1-t) \sum_{j\in J} m(u; Q_j^{Y^\delta}) \\ &\ge t\, m^\delta(u;\omega) + (1-t)\, \mathcal{F}(u; \omega\setminus K),\end{aligned}$$

and letting $\delta \to 0$ we deduce that

$$\begin{aligned}\mathcal{F}(u;\omega) &\ge t\, m^*(u;\omega) + (1-t)\, \mathcal{F}(u;\omega\setminus K) \\ &= t\, \mathcal{F}(u;\omega) + (1-t)\, \mathcal{F}(u;\omega\setminus K),\end{aligned}$$

where we have used Lemma 3.2. Letting $\omega \searrow E_t$, $K \nearrow E_t$, and using the regularity of $\mathcal{F}(u;\cdot)$, we get $\mathcal{F}(u; E_t) = 0$, hence $\mu(E_t) = 0$ due to the coercivity assumption. □

We now prove the following representation theorem.

Theorem 3.4. *Under hypotheses (2.1), (2.2) and (2.3′), for every $u \in SBV(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u;A) = \int_A f(x,u,\nabla u)\, dx + \int_{S(u)\cap A} g(x,u^+,u^-,\nu_u)\, d\mathcal{H}^{N-1}$$

where

$$f(x_0,a,\xi) := \limsup_{\varepsilon\to 0} \frac{m(a+\xi\,(\cdot - x_0); Q(x_0,\varepsilon))}{\varepsilon^N}, \tag{3.13}$$

$$g(x_0,\lambda,\theta,\nu) := \limsup_{\varepsilon\to 0} \frac{m(u_{\lambda,\theta,\nu}\,(\cdot - x_0); Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}, \tag{3.14}$$

for all $x_0 \in \Omega$, a, $\theta, \lambda \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, and where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \theta & \text{otherwise.}\end{cases}$$

Remark 3.5. for $\mathcal{L}^N$ almost

1) If for all $(u, A, b) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \times \mathbb{R}^d$ we have $\mathcal{F}(u+b; A) = \mathcal{F}(u; A)$, then $f = f(x, \xi)$ and $g = g(x, \lambda - \theta, \nu)$.

2) If for all $(u, A, z) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \times \mathbb{R}^N$ we have $\mathcal{F}(u(\cdot - z); z + A) = \mathcal{F}(u; A)$, then $f = f(a, \xi)$ and $g = g(\lambda, \theta, \nu)$.

3) In case both conditions 1) and 2) are satisfied we find that the upper-limits in (3.13) and (3.14) are indeed limits.

Remark 3.6. It is easy to check that the conclusions of Lemma 3.3 still hold if we replace the hypercube $Q_\nu(x_0, \varepsilon)$ by $K(x_0, \varepsilon) := x_0 + \varepsilon K$, where K is any bounded, open, convex subset of $\mathbb{R}^N$ containing the origin. This remark will be useful to obtain the characterization of the Cantor part of $\mathcal{F}(u; A)$ when $u \in BV(\Omega; \mathbb{R}^d)$.

Proof. We first prove (3.13). For $u \in BV(\Omega; \mathbb{R}^d)$ and $\nu \in S^{N-1}$ (in particular for $\nu = e_N$) it is known that for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} |Du|(Q_\nu(x_0, \varepsilon)) = |\nabla u(x_0)|, \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} |D_s u|(Q_\nu(x_0, \varepsilon)) = 0, \tag{3.15}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N+1}} \int_{Q_\nu(x_0, \varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| \, dx = 0, \tag{3.16}$$

$$\frac{d\mathcal{F}(u; \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(u; Q_\nu(x_0, \varepsilon))}{\varepsilon^N},$$

and, in view of Lemma 3.3,

$$\frac{d\mathcal{F}(u; \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0} \frac{m(u; Q_\nu(x_0, \varepsilon))}{\varepsilon^N}. \tag{3.17}$$

Let

$$u_\varepsilon(y) := \frac{u(x_0 + \varepsilon y) - u(x_0)}{\varepsilon}.$$

By (3.16) we have $u_\varepsilon \to \nabla u(x_0) y$ in $L^1(Q_\nu; \mathbb{R}^d)$. We claim that

$$|Du_\varepsilon|(Q_\nu) \to |\nabla u(x_0)|. \tag{3.18}$$

If so, by Lemma 2.3 we obtain

$$\int_{\partial Q_\nu} |\text{tr } (u_\varepsilon(y) - \nabla u(x_0) y)| \, d\mathcal{H}^{N-1}(y)$$
$$= \frac{1}{\varepsilon^N} \int_{\partial Q_\nu(x_0, \varepsilon)} |\text{tr } [u(x) - u(x_0) - \nabla u(x_0)(x - x_0)]| \, d\mathcal{H}^{N-1}(x) \to 0,$$

and, consequently, using Lemma 3.1 we obtain from (3.17)

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0} \frac{m(u(x_0)+\nabla u(x_0)(x-x_0); Q_\nu(x_0,\varepsilon))}{\varepsilon^N}$$
$$= f(x_0, u(x_0), \nabla u(x_0)).$$

We now prove claim (3.18). By definition of $|Du_\varepsilon|(Q_\nu)$,

$$\begin{aligned}|Du_\varepsilon|(Q_\nu) &= \sup_{\substack{\phi\in C_0^1(Q_\nu)\\ \|\phi\|_\infty\le 1}} \int_{Q_\nu} \frac{u(x_0+\varepsilon y)-u(x_0)}{\varepsilon} \operatorname{div}\phi(y)\,dy,\\ &= \sup_{\substack{\varphi\in C_0^1(Q_\nu(x_0,\varepsilon))\\ \|\varphi\|_\infty\le 1}} \frac{1}{\varepsilon^N}\int_{Q_\nu(x_0,\varepsilon)} [u(x)-u(x_0)] \operatorname{div}\varphi(x)\,dx,\\ &= \frac{1}{\varepsilon^N}|Du|(Q_\nu(x_0,\varepsilon)),\end{aligned}$$

where we took $\varphi(x) := \phi\left(\frac{x-x_0}{\varepsilon}\right)$. Therefore, by (3.15) $|Du_\varepsilon|(Q_\nu)$ converges to $|\nabla u(x_0)|$, and the proof of the claim is complete.

Finally, we prove (3.14). For $u \in BV(\Omega;\mathbb{R}^d)$ it is known that for $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$

$$|[u](x_0)| = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}}|Du|(Q_\nu(x_0,\varepsilon)), \tag{3.19}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{Q_\nu^+(x_0,\varepsilon)} |u(x)-u^+(x_0)|\,dx = 0, \tag{3.20}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{Q_\nu^-(x_0,\varepsilon)} |u(x)-u^-(x_0)|\,dx = 0, \tag{3.21}$$

where $\nu = \nu_u(x_0)$ is the normal to $S(u), Q_\nu^+(x_0,\varepsilon) := \{x \in Q_\nu(x_0,\varepsilon) | (x-x_0)\cdot \nu(x_0) > 0\}$ and $Q_\nu^-(x_0,\varepsilon) := \{x \in Q_\nu(x_0,\varepsilon) | (x-x_0)\cdot\nu(x_0) < 0\}$,

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{\varepsilon\to 0}\frac{\mathcal{F}(u;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}$$

and, in view of Lemma 3.3,

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{\varepsilon\to 0}\frac{m(u;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}. \tag{3.22}$$

Defining, for each $y \in Q_\nu$,

$$u_\varepsilon(y) := u(x_0+\varepsilon y) \text{ and } \bar{u}_{x_0,\nu}(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0\\ u^-(x_0) & \text{if } y\cdot\nu \le 0,\end{cases}$$

from (3.20) and (3.21) we have that $u_\varepsilon \to \bar{u}_{x_0,\nu}$ in $L^1(Q_\nu)$ and, by the same argument used to prove (3.18) and by (3.19), we obtain that

$$|Du_\varepsilon|(Q_\nu) = \frac{1}{\varepsilon^{N-1}}|Du|(Q_\nu(x_0,\varepsilon)) \to |[u](x_0)| = |D\bar{u}_{x_0,\nu}|(Q_\nu).$$

In light of Lemma 2.3, we have

$$\int_{\partial Q_\nu} |\text{tr } (u_\varepsilon - \bar{u}_{x_0,\nu})| \, d\mathcal{H}^{N-1} = \frac{1}{\varepsilon^{N-1}} \int_{\partial Q_\nu(x_0,\varepsilon)} |\text{tr } (u - \bar{u}_{x_0,\nu}(\cdot - x_0))| \, d\mathcal{H}^{N-1} \to 0$$

and, by (3.22) and Lemma 3.1, we conclude that

$$\begin{aligned} \frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &= \lim_{\varepsilon\to 0} \frac{m(\bar{u}_{x_0,\nu}(\cdot - x_0); Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \\ &= g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)). \end{aligned}$$

□

In order to complete the integral representation on all $BV(\Omega;\mathbb{R}^d)$, it remains to obtain the characterization of the energy density with respect to the Cantor part of Du, $C(u)$. By Lemma 3.3, this problem reduces to the computation of

$$\frac{d\mathcal{F}(u;\cdot)}{d|C(u)|}(x_0) = \lim_{\varepsilon\to 0} \frac{m(u; x_0 + \varepsilon K)}{|Du|(x_0 + \varepsilon K)} \tag{3.23}$$

at $C(u)$-almost all $x_0 \in \Omega$, where (see Remark 3.6) K is any convex bounded open subset containing the origin in its interior. Recall that, by Alberti's result [A], the Cantor measure $C(u)$ is rank one, precisely,

$$\frac{dC(u)}{d|C(u)|}(x_0) = a_u(x_0) \otimes \nu_u(x_0) \tag{3.24}$$

for $|C(u)|$ a.e. x_0 and for suitable $(a_u(x_0), \nu_u(x_0)) \in \mathbb{R}^d \times S^{N-1}$. In Lemma 3.7 below we will use (3.23) taking for K the hypercube $Q_\nu^{(k)}$, with $\nu = \nu_u(x_0)$, obtained from Q_ν by a dilatation of amplitude k ($k \in \mathbb{N}$ will tend to $+\infty$) in the directions orthogonal to ν, precisely,

$$Q_\nu^{(k)} = R_\nu\left(\left(-\frac{k}{2},\frac{k}{2}\right)^{N-1} \times \left(-\frac{1}{2},\frac{1}{2}\right)\right),$$

where R_ν denotes a rotation such that $R_\nu(e_N) = \nu$ (see Section 2).

Lemma 3.7. *Given $u \in BV(\Omega;\mathbb{R}^d)$, for $|C(u)|$ almost all $x_0 \in \Omega$ there exists a double indexed sequence $(t_\varepsilon^{(k)}, b_\varepsilon^{(k)}) \in (0,+\infty) \times \mathbb{R}^d$ such that, for every k,*

$$t_\varepsilon^{(k)} \to +\infty, \qquad \varepsilon\, t_\varepsilon^{(k)} \to 0, \qquad b_\varepsilon^{(k)} \to u(x_0), \quad \text{as } \varepsilon \to 0,$$

and

$$\frac{d\mathcal{F}(u;\cdot)}{d|C(u)|}(x_0) = \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{m(b_\varepsilon^{(k)} + t_\varepsilon^{(k)}\, a \otimes \nu(\cdot - x_0); x_0 + \varepsilon Q_\nu^{(k)})}{k^{N-1}\, \varepsilon^N\, t_\varepsilon^{(k)}}, \tag{3.25}$$

where $a = a_u(x_0)$ and $\nu = \nu_u(x_0)$ satisfy (3.24).

Proof. Let us apply (3.23) with $K = Q_\nu^{(k)}$ and set $Q_\nu^{(k)}(x_0,\varepsilon) := x_0 + \varepsilon Q_\nu^{(k)}$. There exists a $|C(u)|$-negligible set N, $S(u) \subset N$, such that for all $x_0 \in \Omega \setminus N$ and for all $k \in \mathbb{N}$

$$\frac{d\mathcal{F}(u;\cdot)}{d|C(u)|}(x_0) = \lim_{\varepsilon\to 0} \frac{m(u; Q_\nu^{(k)}(x_0,\varepsilon))}{|Du|(Q_\nu^{(k)}(x_0,\varepsilon))}\ , \tag{3.26}$$

$$t_\varepsilon^{(k)} := \frac{|Du|(Q_\nu^{(k)}(x_0,\varepsilon))}{\varepsilon^N k^{N-1}}\ , \quad t_\varepsilon^{(k)} \to +\infty\ , \quad \varepsilon t_\varepsilon^{(k)} \to 0. \tag{3.27}$$

Condition $\varepsilon t_\varepsilon^{(k)} \to 0$ follows easily from the fact that $\mathcal{H}^{N-1}(B) < +\infty$ implies that $|C(u)|(B) = 0$ (see Prop. 3.1 in [Am1]).

Define, for each $\varepsilon > 0$ and $k \in \mathbb{N}$,

$$\theta_\varepsilon^{(k)} := \frac{1}{\varepsilon^N k^{N-1}} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} u(x)\, dx, \tag{3.28}$$

$$b_\varepsilon^{(k)} := \frac{1}{\varepsilon^{N-1} k^{N-1}} \int_{x_0+\varepsilon\Pi_\nu^{(k)}(\frac{1}{2})} u(x)\, d\mathcal{H}^{N-1}(x) - \varepsilon t_\varepsilon^{(k)} \frac{a}{2}, \tag{3.29}$$

$$v_\varepsilon^{(k)}(x) := b_\varepsilon^{(k)} + t_\varepsilon^{(k)}\, a \otimes \nu(x - x_0), \tag{3.30}$$

$$A_\varepsilon^{(k)} := \frac{m(u; Q_\nu^{(k)}(x_0,\varepsilon)) - m(v_\varepsilon^{(k)}; Q_\nu^{(k)}(x_0,\varepsilon))}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}} \tag{3.31}$$

where, for $t \in \mathbb{R}$, $\Pi_\nu^{(k)}(t) := \{y \in \mathbb{R}^N |\ y\cdot\nu = t\ ,\ |y - (y\cdot\nu)\nu| \le \frac{k}{2}\}$.

Using Alberti's result on the blow-up of the Cantor part (see [A] and also [ADM], Theorem 2.3) and Lemma 5.1 in [L], we can also choose N so that for all $x_0 \in \Omega\setminus N$ there exists a sequence (ε_n) tending to 0 and, for every k, a nondecreasing function $\Psi^{(k)} : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ such that the following conditions hold :

$$\Psi^{(k)}\left(\frac{1}{2} - 0\right) - \Psi^{(k)}\left(-\frac{1}{2} + 0\right) = 1\ , \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi^{(k)}(s)ds = 0, \tag{3.32}$$

$$u_n^{(k)}(y) := \frac{u(x_0 + \varepsilon_n y) - \theta_{\varepsilon_n}^{(k)}}{\varepsilon_n t_{\varepsilon_n}^{(k)}} \to u_0^{(k)}(y) := \Psi^{(k)}(y\cdot\nu)a \text{ in} L^1(Q_\nu^{(k)};\mathbb{R}^d), \tag{3.33}$$

$$\lim_n |Du_n^{(k)}|(Q_\nu^{(k)}) = |Du_0^{(k)}|(Q_\nu^{(k)}) = k^{N-1}|a|. \tag{3.34}$$

We notice that the negligible set N and the sequence (ε_n) were chosen independently of k. Fix $x_0 \in \Omega \setminus N$.

Owing to (3.26) and (3.31) we have that, for every $k \in \mathbb{N}$,

$$\limsup_{\varepsilon \to 0} \frac{m(v_\varepsilon^{(k)}; Q_\nu^{(k)}(x_0, \varepsilon))}{k^{N-1}\varepsilon^N t_\varepsilon^{(k)}} = \frac{d\mathcal{F}(u;\cdot)}{d|C(u)|}(x_0) - \liminf_{\varepsilon \to 0} A_\varepsilon^{(k)},$$

which, together with (3.27) and (3.28), yields the result of Lemma 3.7 provided we show that

$$\begin{aligned} &(i)\ \lim_{\varepsilon \to 0} b_\varepsilon^{(k)} = u(x_0), \\ &(ii)\ \lim_{k \to +\infty}\ \liminf_{\varepsilon \to 0} |A_\varepsilon^{(k)}| = 0. \end{aligned} \tag{3.35}$$

Step 1. We prove (3.35) (i). Since $x_0 \notin S(u)$, we have that $\lim_{\varepsilon \to 0} \theta_\varepsilon^{(k)} = u(x_0)$. Then, in view of definitions (3.27), (3.28) and (3.29), it is enough to show that

$$\begin{aligned} \Big|\frac{1}{\varepsilon^N k^{N-1}} \int_{Q_\nu^{(k)}(x_0,\varepsilon)} u(x)\, dx - \frac{1}{\varepsilon^{N-1}k^{N-1}} \int_{x_0+\varepsilon\Pi_\nu^{(k)}(\frac{1}{2})} u(x)\, d\mathcal{H}^{N-1}(x)\Big| \le \\ \le \frac{|Du|(Q_\nu^{(k)}(x_0,\varepsilon))}{\varepsilon^{N-1}k^{N-1}}. \end{aligned} \tag{3.36}$$

With no loss of generality we prove (3.36) assuming that u is smooth. This extends to a general $u \in BV(Q_\nu^{(k)}(x_0,\varepsilon); \mathbb{R}^d)$ by considering a sequence (u_n) in $C^\infty(Q_\nu^{(k)}(x_0,\varepsilon); \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(Q_\nu^{(k)}(x_0,\varepsilon); \mathbb{R}^d)$, $|\nabla u_n|(Q_\nu^{(k)}(x_0,\varepsilon)) \to |Du|(Q_\nu^{(k)}(x_0,\varepsilon))$ and passing to the limit as $n \to +\infty$ in the corresponding inequality (3.36).

Setting, for each $t \in (-1/2, 1/2)$,

$$\alpha(t) := \int_{\Pi_\nu^{(k)}(t)} u(x_0 + \varepsilon x) d\mathcal{H}^{N-1}(x),$$

changing variables and using Fubini's Theorem we have that

$$\begin{aligned} &\Big|\frac{1}{\varepsilon^N}\int_{Q_\nu^{(k)}(x_0,\varepsilon)} u(x)dx - \frac{1}{\varepsilon^{N-1}}\int_{x_0+\varepsilon\Pi_\nu^{(k)}(\frac{1}{2})} u(x) d\mathcal{H}^{N-1}(x)\Big| = \Big|\int_{-1/2}^{1/2} [\alpha(t) - \alpha(1/2)]dt\Big| \\ &= \Big|\int_{-1/2}^{1/2}\int_t^{1/2} \alpha'(s)\, ds\, dt\Big| = \Big|\int_{-1/2}^{1/2}\int_t^{1/2}\int_{\Pi_\nu^{(k)}(s)} \nabla u(x_0 + \varepsilon x)\, \varepsilon\, x\, d\mathcal{H}^{N-1}(x)\, ds\, dt\Big| \\ &\le \varepsilon \int_{-1/2}^{1/2}\int_{\Pi_\nu^{(k)}(t)} |\nabla u(x_0 + \varepsilon x)|\ |x|\, d\mathcal{H}^{N-1}(x)\, dt \le O(k^{N-1})\frac{|Du|(Q_\nu^{(k)}(x_0,\varepsilon))}{\varepsilon^{N-1}}. \end{aligned}$$

Step 2. We prove (3.35) (ii). By Lemma 3.1, and using the change of variables $y = \dfrac{x - x_0}{\varepsilon_n}$, we have

$$\begin{aligned} |A_{\varepsilon_n}^{(k)}| &\leq C\, \frac{\int_{\partial Q_\nu^{(k)}(x_0,\varepsilon_n)} |u(x) - v_{\varepsilon_n}^{(k)}(x)| d\mathcal{H}^{N-1}(x)}{k^{N-1}\varepsilon_n^N t_{\varepsilon_n}^{(k)}} \\ &\leq \frac{C}{k^{N-1}} \int_{\partial Q_\nu^{(k)}} |u_n^{(k)}(y) - a(y\cdot\nu) + c_n^{(k)}| d\mathcal{H}^{N-1}(y), \end{aligned} \tag{3.37}$$

where $c_n^{(k)} := \dfrac{\theta_{\varepsilon_n}^{(k)} - b_{\varepsilon_n}^{(k)}}{\varepsilon_n t_{\varepsilon_n}^{(k)}} = \dfrac{a}{2} - \dfrac{1}{k^{N-1}} \displaystyle\int_{\Pi_\nu^{(k)}(\frac{1}{2})} u_n^{(k)}\, d\mathcal{H}^{N-1}$.

By (3.33), (3.34) and Lemma 2.3, we have the strong convergence of the trace of $u_n^{(k)}$ to the trace of $\Psi^{(k)}(y\cdot\nu)a$ on $\Pi_\nu^{(k)}(\frac{1}{2})$ and so

$$\begin{aligned} \lim_{n\to+\infty} c_n^{(k)} &= \lim_{n\to+\infty} \left(\frac{a}{2} - \frac{1}{k^{N-1}} \int_{\Pi_\nu^{(k)}(\frac{1}{2})} u_n^{(k)}\, d\mathcal{H}^{N-1}(y) \right) \\ &= \frac{a}{2} - \frac{1}{k^{N-1}} \int_{\Pi_\nu^{(k)}(\frac{1}{2})} \Psi^{(k)}(y\cdot\nu)\, a\, d\mathcal{H}^{N-1}(y) \\ &= -a\left(\Psi^{(k)}\left(\frac{1}{2}\right) - \frac{1}{2} \right). \end{aligned}$$

Thus, from (3.37) we deduce

$$\limsup_{n\to+\infty} |A_{\varepsilon_n}^{(k)}| \leq \frac{C|a|}{k^{N-1}} \int_{\partial Q_\nu^{(k)}} \left| \Psi^{(k)}(y\cdot\nu) - y\cdot\nu - \Psi^{(k)}\left(\frac{1}{2}\right) + \frac{1}{2} \right| d\mathcal{H}^{N-1}(y),$$

and by (3.32) the function $\left|\Psi^{(k)}(y\cdot\nu) - y\cdot\nu - \Psi^{(k)}(\frac{1}{2}) + \frac{1}{2}\right|$ vanishes on the facets $\Pi_\nu^{(k)}(\pm\frac{1}{2})$ and is bounded. We conclude that

$$\limsup_{k\to+\infty} \liminf_{\varepsilon\to 0} |A_\varepsilon^{(k)}| \leq \lim_{k\to\infty} \limsup_{n\to+\infty} |A_{\varepsilon_n}^{(k)}| \leq \frac{C}{k^{N-1}} |a| k^{N-2} = 0.$$

□

In order to identify the right hand side of (3.25), we assume now that the continuity assumption (2.4) holds, that is (see Section 2) there exists a modulus of continuity $\Phi(t)$ such that for all $(u, A, b, z) \in BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \times \mathbb{R}^d \times \mathbb{R}^N$, with $z + A \subset \Omega$,

$$|\mathcal{F}(u(\cdot - z) + b; z + A) - \mathcal{F}(u; A)| \leq \Phi(|b| + |z|)\, (\mathcal{L}^N(A) + |Du|(A)).$$

Remark 3.8. An immediate consequence of (2.4) and of the growth condition (2.3) is that the integrands $f(x_0, u_0, \xi)$ and $g(x_0, \lambda, \theta, \nu)$ defined by (3.13) and (3.14) are continuous with respect to x_0 and u_0. In fact, applying (2.4) with $A = Q_\nu(x_0, \varepsilon)$ and u such that $u(x) = u_0 + \xi(x - x_0)$ on $\partial Q_\nu(x_0, \varepsilon)$, we obtain

$$|m(u_0 + b + \xi(\cdot - z - x_0); Q_\nu(z + x_0, \varepsilon)) - m(u_0 + \xi(\cdot - x_0); Q_\nu(x_0, \varepsilon))| \leq \Phi(|b| + |z|)\ (1 + |\xi|)\ \varepsilon^N .$$

Dividing by ε^N and passing to the limit as $\varepsilon \to 0$, we are led to

$$|f(x_0 + z, u_0 + b, \xi) - f(x_0, u_0, \xi)| \leq \Phi(|b| + |z|)(1 + |\xi|) . \qquad (3.38)$$

Similarly, we obtain

$$|g(x_0 + z, \lambda + b, \theta + b, \nu) - g(x_0, \lambda, \theta, \nu)| \leq \Phi(|b| + |z|)\ |\theta - \lambda|.$$

On the other hand, from (2.4) and the coercivity assumption (2.3′), we can also infer that

$$|m(u(\cdot - z) + b; z + A) - m(u; A)| \leq \Phi(|b| + |z|)\ (\mathcal{L}^N(A) + C\, m(u; A)) . \qquad (3.39)$$

We notice that, since $\mathcal{F}(\cdot; \Omega)$ is weakly lower semicontinuous on $W^{1,1}(A; \mathbb{R}^d)$ and coincides with the functional $u \in W^{1,1}(A; \mathbb{R}^d) \to \int_A f(x, u, \nabla u)\, dx$ (see Theorem 3.4), the integrand $f(x_0, u_0, \cdot)$ must be quasiconvex for every $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ (see, for example, [D]). Thus, defining the *recession function* f^∞ by

$$f^\infty(x_0, u_0, \xi) \ := \ \limsup_{t \to +\infty} \frac{f(x_0, u_0, t\xi)}{t}, \qquad (3.40)$$

the right hand side of (3.40) is actually a limit whenever ξ is a rank one tensor.

Lemma 3.9. *Let $(a, \nu) \in \mathbb{R}^d \times S^{N-1}$, $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ and let (ε_n, t_n) be a sequence such that $\varepsilon_n \to 0$, $t_n \to +\infty$ and $\varepsilon_n t_n \to 0$. If (2.4) holds and if f is defined by (3.13) then*

$$\liminf_{n \to +\infty} \frac{m(u_0 + t_n\, a \otimes \nu(\cdot - x_0); x_0 + \varepsilon_n Q_\nu^{(k)})}{t_n\, \varepsilon_n^N\, k^{N-1}} \geq f(x_0, u_0, a \otimes \nu) - f(x_0, u_0, 0).$$

We leave the proof of this lemma to the end of this section. Now we are able to present the full representation of $\mathcal{F}$ on $BV(\Omega; \mathbb{R}^d)$.

Theorem 3.10. *Under hypotheses (2.1), (2.2), (2.3′) and (2.4), we have for every $u \in BV(\Omega; \mathbb{R}^d)$*

$$\mathcal{F}(u; A) = \int_A f(x, u, \nabla u)\ dx + \int_{S(u) \cap A} g(x, u^+, u^-, \nu_u)\ d\mathcal{H}^{N-1} + \int_A f^\infty \left(x, u, \frac{dC(u)}{d|C(u)|} \right)\ d|C(u)|,$$

where f, g, f^∞ are defined by (3.13), (3.14) and (3.40), respectively.

Proof. By Theorem 3.4 it remains to prove that for $|C(u)|$ a.e. $x \in \Omega$

$$\frac{d\mathcal{F}(u;\cdot)}{d|C(u)|}(x) = f^\infty\left(x, u(x), \frac{dC(u)}{d|C(u)|}(x)\right) .$$

Let x_0 be a point of approximate continuity of u where $\frac{dC(u)}{d|C(u)|}(x_0)=a_u(x_0)\otimes\nu_u(x_0)$ and set $u_0 := u(x_0)$. By Lemma 3.7 and taking into account (3.39), it is enough to show that for every fixed $k \in \mathbb{N}$, $a = a_u(x_0)$ and $\nu = \nu_u(x_0)$ one has

$$\lim_{n\to+\infty} \frac{m(u_0 + t_n\, a\otimes\nu(\cdot - x_0); x_0 + \varepsilon_n Q_\nu^{(k)})}{t_n\,\varepsilon_n^N\, k^{N-1}} = f^\infty(x_0, u_0, a\otimes\nu) , \qquad (3.41)$$

where, for simplicity of notation, we have deleted the superscript (k) from $t_n^{(k)}$. One inequality is easy. Indeed, by Theorem 3.4 we can write

$$\begin{aligned} m(u_0+t_n a\otimes\nu(\cdot - x_0); x_0+\varepsilon_n Q_\nu^{(k)}) &\le \mathcal{F}(u_0+t_n\, a\otimes\nu(\cdot - x_0); x_0+\varepsilon_n Q_\nu^{(k)}) \\ &\le \int_{x_0+\varepsilon_n Q_\nu^{(k)}} f(x, u_0+t_n a\otimes\nu(x - x_0), t_n a\otimes\nu)dx \end{aligned}$$

so that, by (3.38),

$$\begin{aligned} \limsup_{n\to+\infty} \frac{m(u_0 + t_n\, a\otimes\nu(\cdot - x_0); x_0 + \varepsilon_n Q_\nu^{(k)})}{t_n\,\varepsilon_n^N\, k^{N-1}} &\le \limsup_{n\to+\infty} \frac{f(x_0, u_0, t_n a\otimes\nu)}{t_n} \\ &\le f^\infty(x_0, u_0, a\otimes\nu). \qquad (3.42) \end{aligned}$$

To prove the opposite inequality, we apply Lemma 3.9 after replacing (t_n, a) by $\left(\frac{t_n}{t}, ta\right)$ for any $t > 0$ fixed. We get

$$\liminf_{n\to\infty} \frac{m(u_0 + t_n\, a\otimes\nu(x - x_0); x_0 + \varepsilon_n Q_\nu^{(k)})}{t_n\,\varepsilon_n^N\, k^{N-1}} \ge \frac{f(x_0, u_0, ta\otimes\nu) - f(x_0, u_0, 0)}{t} .$$

Letting t tend to $+\infty$ and taking into account (3.42), we obtain (3.41).

□

Proof of Lemma 3.9. Set $\alpha_n := m(u_0 + t_n\, a\otimes\nu(\cdot - x_0); x_0 + \varepsilon_n Q_\nu^{(k)})$ and $Q_\nu^{(k)}(x_0, \varepsilon) := x_0 + \varepsilon Q_\nu^{(k)}$. By the coercivity hypothesis (2.3′), we have $\alpha_n \ge \frac{1}{C} t_n |a| \varepsilon_n^N k^{N-1}$, and by (2.3) and since t_n tends to $+\infty$, we have $\limsup_{n\to+\infty} \frac{\alpha_n}{t_n\varepsilon_n^N} < +\infty$. Choosing $C > 0$ large enough, we may assume that

$$0 < \alpha_n \le C\varepsilon_n^N t_n. \qquad (3.43)$$

Fix $\eta > 1$. By the definition of the set function m, there exists a function $z_n \in BV(Q_\nu^{(k)}(x_0, \varepsilon_n); \mathbb{R}^d)$ such that

$$\mathcal{F}(z_n; Q_\nu^{(k)}(x_0, \varepsilon_n)) < \eta\ \alpha_n\ ,\ z_n = u_0 + t_n a \otimes \nu(\cdot - x_0) \quad \text{on } \partial Q_\nu^{(k)}(x_0, \varepsilon_n). \tag{3.44}$$

Taking into account the continuity assumption (2.4) and the coercivity (2.3′), we can choose ρ_0 small enough so that

$$|b|+|\tau|<2\rho_0 \Rightarrow \mathcal{F}(z_n(\cdot-\tau)+b; Q_\nu^{(k)}(x_0+\tau, \varepsilon_n)) < \eta^2\, \alpha_n + \Phi(2\rho_0)\varepsilon_n^N k^{N-1}. \tag{3.45}$$

Without loss of generality, we suppose that $x_0 = 0$ and $\nu = e_N$. Let us extend z_n to $(-k\varepsilon_n/2, k\varepsilon_n/2)^{N-1} \times \mathbb{R}$ by setting

$$z_n := u_0 + \frac{a}{2}\varepsilon_n t_n \quad \text{if } x_N > \frac{\varepsilon_n}{2}, \qquad z_n := u_0 - \frac{a}{2}\varepsilon_n t_n \quad \text{if } x_N < -\frac{\varepsilon_n}{2},$$

and define a function w_n on the whole $\mathbb{R}^N$ by considering, for every $(i,j) \in \mathbb{Z}^{N-1} \times \mathbb{Z}$, the hypercube

$$Q_n^{i,j} := \left(\left(i - \frac{1}{2}\right) k\varepsilon_n, \left(i + \frac{1}{2}\right) k\varepsilon_n\right)^{N-1} \times \left(\left(j - \frac{1}{2}\right) \varepsilon_n t_n, \left(j + \frac{1}{2}\right) \varepsilon_n t_n\right)\ ,$$

and defining

$$w_n(x', x_N) := z_n(x' - ik\varepsilon_n, x_N - j\varepsilon_n t_n) + aj\varepsilon_n t_n\ ,\ (x', x_N) \in Q_n^{i,j}.$$

Also, we introduce a family of piecewise affine functions

$$v_n(x', x_N) := \varphi_n(x_N - j\varepsilon_n t_n) + aj\varepsilon_n t_n\ ,\ (x', x_N) \in Q_n^{i,j},$$

where

$$\varphi_n(s) := \begin{cases} u_0 + at_n\frac{\varepsilon_n}{2} & \text{if } s > \frac{\varepsilon_n}{2} \\ u_0 + at_n s & \text{if } |s| < \frac{\varepsilon_n}{2} \\ u_0 - at_n\frac{\varepsilon_n}{2} & \text{if } s < -\frac{\varepsilon_n}{2}. \end{cases}$$

Fix $\rho > 0$ such that $0 < \rho < \rho_0$ and denote

$$I_n^\rho := \{(i,j) \in \mathbb{Z}^{N-1} \times \mathbb{Z} \mid Q_n^{i,j} \cap Q_\rho \neq \emptyset\}, \qquad Q_\rho := (-\rho/2, \rho/2)^N.$$

If N_n denotes the cardinality of I_n^ρ, it is clear that

$$\lim_{n\to+\infty} N_n\, \varepsilon_n^N\, t_n\, k^{N-1} = \rho^N\ . \tag{3.46}$$

Since w_n agrees with $z_n(\cdot - \tau_n) + ajt_n\varepsilon_n$ on $Q_n^{i,j}$, with $\tau_n := (ik\varepsilon_n, jt_n\varepsilon_n)$, and it coincides with v_n on $Q_n^{i,j} \cap \{|x_N - j\varepsilon_n t_n| > \varepsilon_n/2\}$, we have, for all $n \geq n_0$,

$$\mathcal{F}(w_n; Q_n^{i,j}) < \eta^2\ \alpha_n + \Phi(2\rho_0)\varepsilon_n^N k^{N-1} + \mathcal{F}(v_n; Q_n^{i,j}) \quad \text{for all } (i,j) \in I_n^\rho, \tag{3.47}$$

where we have used (3.45) and the fact that $Q_\nu^{(k)}(x_0+\tau_n;\varepsilon_n)=Q_n^{i,j}\cap\{|x_N-j\varepsilon_n t_n|<\frac{\varepsilon_n}{2}\}$. Now v_n is continuous and piecewise affine on $Q_n^{i,j}$, thus by the integral representation of Theorem 3.4 we can write $\mathcal{F}(v_n;Q_n^{i,j}) = \int_{Q_n^{i,j}} f(x,v_n,0)dx$. Summing (3.47) with respect to $(i,j)\in I_n^\rho$ and using the additivity of $\mathcal{F}$, we get

$$\mathcal{F}(w_n,Q_\rho)\leq N_n\left(\eta^2\,\alpha_n+\Phi(2\rho_0)\,\varepsilon_n^N\,k^{N-1}\right)+\int_{Q_\rho} f(x,v_n,0)dx.$$

Passing to the limit, as $n\to+\infty$ and then as $\eta\to 1$, in the previous inequality, using (3.46) and recalling that $t_n\to+\infty$, we obtain

$$\liminf_{n\to+\infty}\frac{\alpha_n}{\varepsilon_n^N t_n k^{N-1}}\geq\liminf_{n\to+\infty}\frac{\mathcal{F}(w_n,Q_\rho)}{\rho^N}-\limsup_{n\to\infty}\frac{1}{\rho^N}\int_{Q_\rho} f(x,v_n,0)dx\ . \quad (3.48)$$

A simple computation shows that

$$\lim_{n\to+\infty}||v_n-v_0||_{L^1(Q_\rho;\mathbb{R}^d)}=0,\quad\text{where}\quad v_0(x',x_N):=u_0+a\,x_N. \quad (3.49)$$

Hence, by (3.38), we have

$$\lim_{n\to+\infty}\int_{Q_\rho} f(x,v_n,0)dx=\int_{Q_\rho} f(x,v_0,0)dx\ . \quad (3.50)$$

On the other hand, using Poincaré's inequality in each $Q_n^{i,j}\cap\{|x_N-j\varepsilon_n t_n|<\varepsilon_n/2\}$ (with Poincaré constant $C\varepsilon_n$), we obtain

$$\begin{aligned}
\int_{Q_\rho}|w_n(x)-v_n(x)|\,dx &\leq \sum_{(i,j)\in I_n^\rho}\int_{Q_n^{i,j}\cap\{|x_N-j\varepsilon_n t_n|<\varepsilon_n/2\}}|w_n(x)-v_n(x)|\,dx\\
&\leq \sum_{(i,j)\in I_n^\rho} C\,\varepsilon_n\int_{Q_n^{i,j}\cap\{|x_N-j\varepsilon_n t_n|<\varepsilon_n/2\}}|Dw_n-Dv_n|\\
&\leq \sum_{(i,j)\in I_n^\rho} C\,\varepsilon_n\left(\int_{Q_\nu^{(k)}(x_0,\varepsilon_n)}|Dz_n|+|a|t_n\,k^{N-1}\,\varepsilon_n^N\right)\\
&\leq C'\rho^N\varepsilon_n.
\end{aligned}$$

where we have used (3.43), (3.44), (3.46) and the coercivity hypothesis (2.3′). We conclude that

$$\lim_{n\to+\infty}\int_{Q_\rho}|w_n(x)-v_n(x)|\to 0,$$

which, together with (3.49), yields

$$\lim_{n\to+\infty}||w_n-v_0||_{L^1(Q_\rho)}=0.$$

Finally, by (3.48), (3.50) and by the lower semicontinuity property of $\mathcal{F}(\cdot;Q_\rho)$, we deduce that

$$\liminf_{n\to+\infty}\frac{\alpha_n}{\varepsilon_n^N t_n k^{N-1}} \geq \frac{\mathcal{F}(v_0,Q_\rho)}{\rho^N} - \frac{1}{\rho^N}\int_{Q_\rho} f(x,v_0(x),0)\,dx$$
$$= \frac{1}{\rho^N}\int_{Q_\rho}[f(x,v_0(x),a\otimes\nu) - f(x,v_0(x),0)]\,dx.$$

The conclusion follows by letting ρ tend to 0 and using (3.38). □

4. Applications.

We apply the characterization of the relaxed energy obtained in Section 3 to particular situations where we are able to obtain a more explicit formula for the relaxed energy densities.

4.1. Relaxed Energy for Discontinuous Integrands.

Here the functional $\mathcal{F}$ is the relaxed energy corresponding to an integrand f_0 satisfying the following hypotheses :

(H1)

$$f_0\ :\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\to[0,+\infty)\ \text{ is a Borel integrand;}$$

(H2) there exists $C>0$ such that

$$\frac{1}{C}|\xi| \leq f_0(x,u,\xi) \leq C(1+|\xi|)$$

for all $(x,u,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$;

(H3) for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|u-v|<\delta \Rightarrow |f_0(x,u,\xi)-f_0(x,v,\xi)| \leq C\varepsilon(1+|\xi|)$$

for all $(x,u,v,\xi)\in\Omega\times(\mathbb{R}^d)^2\times\mathbb{R}^{d\times N}$;

(H4) there exist $C>0$, $0<m<1$, $L>0$ such that

$$\left|f_0^\infty(x,u,\xi) - \frac{f_0(x,u,t\xi)}{t}\right| \leq \frac{C}{t^m}$$

for all $\xi\in\mathbb{R}^{d\times N}$, $\|\xi\|=1$, $t>L$, and for all $(x,u)\in\Omega\times\mathbb{R}^d$, where the *recession function* f_0^∞ is defined by

$$f_0^\infty(x,u,\xi) := \limsup_{t\to+\infty}\frac{f_0(x,u,t\,\xi)}{t}.$$

The functional $\mathcal{F} : BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty)$ is defined by

$$\mathcal{F}(u; A) := \inf \left\{ \liminf_{n \to +\infty} \int_A f_0(x, u_n(x), \nabla u_n(x)) \, dx \,|\, u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d), \right.$$
$$\left. u_n \in W^{1,1}(\Omega; \mathbb{R}^d) \right\}. \tag{4.1.1}$$

Lemma 4.1.1. *Under hypothesis (H1) and (H2), the functional $\mathcal{F}$ defined by (4.1.1) satisfies conditions (2.1), (2.2) and (2.3′).*

We omit the proof of this lemma since it is quite similar to the one presented in Section 4.3 for the more general case of the Γ-limit of a sequence of functionals.

Thus we may apply the representation Theorem 3.4 and Lemma 3.7 to our case. In order to obtain a more explicit characterization of the energies, we need to identify $m(u; A)$, as introduced in (3.1). Given $(u; A) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$ define

$$m_0(u; A) := \inf \left\{ \int_A f_0(x, v(x), \nabla v(x)) \, dx \,|\, v \in W^{1,1}(\Omega; \mathbb{R}^d), \; v|_{\partial A} = u|_{\partial A} \right\}.$$

Lemma 4.1.2. *Under hypotheses (H1) and (H2), for all $(u; A) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$*

$$m_0(u; A) = m(u; A).$$

Proof. The inequality $m_0(u; A) \geq m(u; A)$ is trivial since for every $v \in W^{1,1}(\Omega; \mathbb{R}^d)$ with $v = u$ on ∂A we have $\int_A f_0(x, v(x), \nabla v(x)) dx \geq \mathcal{F}(v; A) \geq m(u; A)$.

Conversely, given $\varepsilon > 0$ let $v \in BV(\Omega; \mathbb{R}^d)$ be such that $v|_{\partial A} = u|_{\partial A}$ and

$$m(u; A) \geq \mathcal{F}(v; A) - \varepsilon. \tag{4.1.2}$$

Let (v_n) be a sequence in $W^{1,1}(\Omega; \mathbb{R}^d)$ converging to v in $L^1(\Omega; \mathbb{R}^d)$ such that

$$\mathcal{F}(v; A) = \lim_{n \to +\infty} \int_A f_0(x, v_n(x), \nabla v_n(x)) \, dx. \tag{4.1.3}$$

Using Lemma 2.6 and Remark 2.7, consider $w_n \in W^{1,1}(A; \mathbb{R}^d)$ such that $w_n = v = u$ on ∂A, $\|w_n - v\|_{L^1(A; \mathbb{R}^d)} \to 0$ and

$$\limsup_{n \to +\infty} \int_A f_0(x, w_n(x), \nabla w_n(x)) \, dx \leq \lim_{n \to +\infty} \int_A f_0(x, v_n(x), \nabla v_n(x)) \, dx.$$

From (4.1.2) and (4.1.3) we conclude that

$$m(u; A) \geq \limsup_{n \to +\infty} \int_A f_0(x, w_n(x), \nabla w_n(x)) \, dx - \varepsilon \geq m_0(u; A) - \varepsilon.$$

Letting ε go to zero the result follows. □

We now prove the following representation theorem.

Theorem 4.1.3. *Under hypotheses (H1), (H2), (H3) and (H4), the functional $\mathcal{F}$, defined by (4.1.1) and evaluated at $(u, A) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$, is given by*

$$\begin{aligned} \mathcal{F}(u; A) = \int_A f(x, u, \nabla u)\, dx + \int_{S(u)\cap A} g(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1} \\ + \int_A h(x, u, a_u, \nu_u)\, d|C(u)|, \end{aligned} \tag{4.1.4}$$

where $\nu_u(x)$ agrees with the unit normal to $S(u)$ at x for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ and with the unit vector that, together with a_u, satisfies (3.24) for $C(u)$ a.e. $x \in \Omega \setminus S(u)$. The energy densities are defined as

$$f(x_0, u_0, \xi) := \limsup_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q;\mathbb{R}^d)\\ v(y)=\xi y \text{ on} \partial Q}} \left\{ \int_Q f_0(x_0 + \varepsilon y, u_0, \nabla v(y))\, dy \right\}, \tag{4.1.5}$$

$$g(x_0, \lambda, \theta, \nu) := \limsup_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_\nu;\mathbb{R}^d)\\ v=u_{\lambda,\theta,\nu} \text{ on} \partial Q_\nu}} \left\{ \int_{Q_\nu} f_0^\infty(x_0 + \varepsilon y, v(y), \nabla v(y))\, dy \right\}, \tag{4.1.6}$$

$$\begin{aligned} h(x_0, u_0, a, \nu) : = \limsup_{k\to +\infty} \limsup_{\varepsilon\to 0} \\ \inf_{\substack{v\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ v(y)=a(\nu\cdot y) \text{ on} \partial Q_\nu^{(k)}}} \left\{ \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} f_0^\infty(x_0 + \varepsilon y, u_0, \nabla v(y)) dy \right\}, \end{aligned} \tag{4.1.7}$$

with

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y\cdot\nu > 0, \\ \theta & \text{otherwise,} \end{cases}$$

for all $(x_0, u_0) \in \Omega \times \mathbb{R}^d$, $(\lambda, \theta) \in (\mathbb{R}^d)^2$, $a \in \mathbb{R}^d$ and $\nu \in S^{N-1}$.

Remark 4.1.4 In general $\mathcal{F}$ does not verify the continuity condition (2.4), and so we cannot apply Theorem 3.10 to identify the Cantor part. Instead, we use Lemma 3.7 together with hypotheses (H3) and (H4). Note, however, that if f_0 is continuous with respect to (x, u) then f will coincide with the quasiconvex envelope of f_0, h will agree with f^∞ and we will recover the representation theorem of [FM1] and [FM2] or [ADM], under coercivity hypotheses. We remark that it is not necessary to assume (H3) and (H4) to obtain the representation of $\mathcal{F}$ on $SBV(\Omega; \mathbb{R}^d)$ which will hold like in Theorem 3.4 with f defined by

$$f(x_0, u_0, \xi) := \limsup_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q;\mathbb{R}^d)\\ v(y)=\xi y \text{ on} \partial Q}} \left\{ \int_Q f_0(x_0 + \varepsilon y, u_0 + \varepsilon v(y), \nabla v(y))\, dy \right\}, \tag{4.1.5'}$$

in place of (4.1.5), and with g defined by

$$g(x_0, \lambda, \theta, \nu) := \limsup_{\varepsilon \to 0} \inf_{\substack{v \in W^{1,1}(Q_\nu;\mathbb{R}^d) \\ v = u_{\lambda,\theta,\nu} \text{ on } \partial Q_\nu}} \left\{ \int_{Q_\nu} \varepsilon f_0(x_0 + \varepsilon y, v(y), \frac{1}{\varepsilon} \nabla v(y))\, dy \right\}, \tag{4.1.6'}$$

instead of (4.1.6).

Proof of Theorem 4.1.3. Using Lemma 4.1.2 and (3.13), the density f is given by

$$f(x_0, u_0, \xi) = \limsup_{\varepsilon \to 0} \inf_{\substack{v \in W^{1,1}(Q(x_0,\varepsilon);\mathbb{R}^d) \\ v(x) = u_0 + \xi(x - x_0) \text{ on } \partial Q(x_0,\varepsilon)}} \left\{ \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f_0(x, v(x), \nabla v(x))\, dx \right\}.$$

Using the change of variables $x = x_0 + \varepsilon y$, and considering as test functions $w(y) := \frac{v(x_0+\varepsilon y) - u_0}{\varepsilon}$, we get

$$f(x_0, u_0, \xi) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in W^{1,1}(Q;\mathbb{R}^d) \\ w(y) = \xi y \text{ on } \partial Q}} \int_Q f_0(x_0 + \varepsilon y, u_0 + \varepsilon w(y), \nabla w(y))\, dy. \tag{4.1.8}$$

Hypotheses (H2) and (H3), combined with Lemma 2.8, allow us to obtain (4.1.5). In fact, due to the coercivity hypothesis (H2), both infima in the right hand sides of (4.1.5) and (4.1.8) are attained on

$$E_R = \left\{ w \in W^{1,1}(Q;\mathbb{R}^d),\ w = \xi\, y \text{ on } \partial Q,\ \|\nabla w\|_{L^1(Q;\mathbb{R}^d)} \le R \right\},$$

for a convenient R, independent of ε. In view of this, using Lemma 2.8 for each $n \in \mathbb{N}$ we can find M_n, independent of ε, such that

$$\begin{aligned} \Bigg| & \inf_{\substack{w \in W^{1,1}(Q;\mathbb{R}^d) \\ w(y) = \xi y \text{ on } \partial Q}} \int_Q f_0(x_0 + \varepsilon y, u_0 + \varepsilon w(y), \nabla w(y))\, dy - \\ & - \inf_{\substack{w \in W^{1,1}(Q;\mathbb{R}^d) \cap L^\infty(Q;\mathbb{R}^d) \\ w(y) = \xi y \text{ on } \partial Q,\ \|w\|_\infty \le M_n}} \int_Q f_0(x_0 + \varepsilon y, u_0 + \varepsilon w(y), \nabla w(y))\, dy \Bigg| \le \frac{1}{n} \end{aligned} \tag{4.1.9}$$

and

$$\begin{aligned} \Bigg| & \inf_{\substack{w \in W^{1,1}(Q;\mathbb{R}^d) \\ w(y) = \xi y \text{ on } \partial Q}} \int_Q f_0(x_0 + \varepsilon y, u_0, \nabla w(y))\, dy - \\ & - \inf_{\substack{w \in W^{1,1}(Q;\mathbb{R}^d) \cap L^\infty(Q;\mathbb{R}^d) \\ w(y) = \xi y \text{ on } \partial Q,\ \|w\|_\infty \le M_n}} \int_Q f_0(x_0 + \varepsilon y, u_0, \nabla w(y))\, dy \Bigg| \le \frac{1}{n}. \end{aligned} \tag{4.1.10}$$

On the other hand, for fixed n and using (H3) we get that

$$\limsup_{\varepsilon\to 0} \inf_{\substack{w\in W^{1,1}(Q;\mathbb{R}^d)\cap L^\infty(Q;\mathbb{R}^d)\\ w(y)=\xi y \text{ on}\partial Q,\ \|w\|_\infty\le M_n}} \int_Q f_0(x_0+\varepsilon y, u_0+\varepsilon w(y), \nabla w(y))\, dy = \\ = \limsup_{\varepsilon\to 0} \inf_{\substack{w\in W^{1,1}(Q;\mathbb{R}^d)\cap L^\infty(Q;\mathbb{R}^d)\\ w(y)=\xi y \text{ on}\partial Q,\ \|w\|_\infty\le M_n}} \int_Q f_0(x_0+\varepsilon y, u_0, \nabla w(y))\, dy. \tag{4.1.11}$$

By (4.1.8), (4.1.9), (4.1.10) and (4.1.11) we obtain (4.1.5) up to an error of order $\frac{2}{n}$. It suffices to let $n\to+\infty$.

Using (3.14) and Lemma 4.1.2, the density g is given by

$$g(x_0,\lambda,\theta,\nu) = \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \inf_{\substack{v\in W^{1,1}(Q_\nu(x_0,\varepsilon);\mathbb{R}^d)\\ v(x)=u_{\lambda,\theta,\nu}(x-x_0)\text{on}\partial Q_\nu(x_0,\varepsilon)}} \int_{Q_\nu(x_0,\varepsilon)} f_0(x,v(x),\nabla v(x))dx.$$

For $y\in Q_\nu$, define $\tilde v_\varepsilon(y) := v(x_0+\varepsilon y)$. Thus $\tilde v_\varepsilon(y) = u_{\lambda,\theta,\nu}(y)$ for $\mathcal{H}^{N-1}$ a.e. $y\in\partial Q_\nu$, and

$$\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)} f_0(x,v(x),\nabla v(x))\,dx = \int_{Q_\nu} \varepsilon\, f_0\left(x_0+\varepsilon y, \tilde v_\varepsilon(y), \frac{1}{\varepsilon}\nabla\tilde v_\varepsilon(y)\right)\, dy,$$

consequently,

$$g(x_0,\lambda,\theta,\nu) = \limsup_{\varepsilon\to 0} \inf_{\substack{v\in W^{1,1}(Q_\nu;\mathbb{R}^d)\\ v=u_{\lambda,\theta,\nu}\text{on}\partial Q_\nu}} \int_{Q_\nu} \varepsilon\, f_0\Big(x_0+\varepsilon y, v(y), \frac{1}{\varepsilon}\nabla v(y)\Big)\, dy. \tag{4.1.12}$$

Hypothesis (H4) yields

$$\int_{Q_\nu} \varepsilon f_0\left(x_0+\varepsilon y, v(y), \frac{1}{\varepsilon}\nabla v(y)\right) dy = \zeta_\varepsilon(v) + \int_{Q_\nu} f_0^\infty(x_0+\varepsilon y, v(y), \nabla v(y))dy, \\ |\zeta_\varepsilon(v)| \le C\varepsilon^m \|\nabla v\|^{1-m}_{L^1(Q_\nu;\mathbb{R}^d)}. \tag{4.1.13}$$

Since the function f_0^∞ also satisfies hypotheses (H2) (with the same constant C), one sees easily that both infima in the right hand sides of (4.1.6) and (4.1.12) are attained on

$$E_R = \Big\{v\in W^{1,1}(Q_\nu;\mathbb{R}^d),\ v|_{\partial Q_\nu} = u_{\lambda,\theta,\nu}|_{\partial Q_\nu},\ \|\nabla v\|_{L^1(Q_\nu;\mathbb{R}^d)} \le R\Big\},$$

for a convenient R, independent of ε. Thus, taking the infima in (4.1.13), one obtains that

$$\Big| \inf_{v\in E_R} \int_{Q_\nu} \varepsilon f_0\left(x_0+\varepsilon y, v, \frac{1}{\varepsilon}\nabla v\right) dy - \inf_{v\in E_R}\int_{Q_\nu} f_0^\infty(x_0+\varepsilon y, v, \nabla v)dy\Big| \\ \le \sup_{v\in E_R} |\zeta_\varepsilon(v)| \\ \le C\varepsilon^m R^{1-m}.$$

Passing to the limit, as ε goes to zero, (4.1.6) follows.

Finally, we show (4.1.7). In view of Lemma 3.7 and by Lemma 4.1.2, we have, for $|C(u)|$ almost all $x_0 \in \Omega$ and for a suitable sequence $(b_\varepsilon^{(k)}, t_\varepsilon^{(k)})$ converging to $(u_0, +\infty)$,

$$h(x_0, u_0, a, \nu) = \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{m_0(b_\varepsilon^{(k)} + t_\varepsilon^{(k)}\,(a\otimes\nu)(\cdot - x_0); x_0 + \varepsilon Q_\nu^{(k)})}{t_\varepsilon^{(k)}\,\varepsilon^N\,k^{N-1}}$$

$$= \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{k^{N-1}}$$

$$\inf_{\substack{w\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ w(y)=a(\nu\cdot y) \text{ on } \partial Q_\nu^{(k)}}} \int_{Q_\nu^{(k)}} \frac{1}{t_\varepsilon^{(k)}}\, f_0(x_0 + \varepsilon y, b_\varepsilon^{(k)} + \varepsilon t_\varepsilon^{(k)} w(y), t_\varepsilon^{(k)}\nabla w(y))dy.$$

Using as before hypotheses (H2), (H3) and Lemma 2.8, we are led to

$$h(x_0, u_0, a, \nu) = \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{k^{N-1}}$$

$$\inf_{\substack{w\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ w(y)=a(\nu\cdot y) \text{ on } \partial Q_\nu^{(k)}}} \int_{Q_\nu^{(k)}} \frac{1}{t_\varepsilon^{(k)}}\, f_0(x_0 + \varepsilon y, u_0, t_\varepsilon^{(k)}\nabla v(y))dy.$$

Then (4.1.7) follows from (H2) and (H4). □

4.2 Relaxation of bulk and interfacial energies.

We consider the functional defined for each $A \in \mathcal{A}(\Omega)$ by

$$F(u;A) := \begin{cases} \displaystyle\int_A f_0\,(x, u, \nabla u)\,dx \\ \displaystyle + \int_{S(u)\cap A} g_0(x, u^+, u^-, \nu_u)\,d\mathcal{H}^{N-1} & \text{if } u \in SBV(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise ,} \end{cases} \tag{4.2.1}$$

where the densities f_0 and g_0 are continuous integrands satisfying the following hypotheses :

(H1) $f_0\ : \Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N} \to\ [0, +\infty)$ is a continuous function, and

$$\frac{1}{C}|\xi|\ \le f_0(x, u, \xi)\ \le C(1 + |\xi|)$$

for all $(x, u, \xi) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$ and for some $C > 0$;

(H2) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| + |u - v| < \delta \Rightarrow |f_0(x, u, \xi) - f(y, v, \xi)| \le C\,\varepsilon\,(1 + |\xi|)$$

for all $(x, y, u, v, \xi) \in \Omega^2 \times (\mathbb{R}^d)^2 \times \mathbb{R}^{d\times N}$;

(H3) $g_0 : \Omega \times (\mathbb{R}^d)^2 \times S^{N-1} \to [0, +\infty)$ is a continuous function, and

$$\frac{1}{C}|\lambda - \theta| \leq g_0(x, \lambda, \theta, \nu) \leq C(1 + |\lambda - \theta|)$$

for all $(x, \lambda, \theta, \nu) \in \Omega \times (\mathbb{R}^d)^2 \times S^{N-1}$;

(H4) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| + |z| < \delta \Rightarrow |g_0(x, \lambda + z, \theta + z, \nu) - g_0(y, \lambda, \theta, \nu)| \leq C\varepsilon\,|\lambda - \theta|$$

for all $(x, y, \lambda, \theta, z, \nu) \in \Omega^2 \times (\mathbb{R}^d)^3 \times S^{N-1}$.

Our aim is to identify the relaxation of F defined for each open subset $A \in \mathcal{A}(\Omega)$ as

$$\mathcal{F}(u, A) := \inf\left\{\liminf_{n\to+\infty} F(u_n; A)|\; u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d)\right\}. \tag{4.2.2}$$

Theorem 4.2.1. *Under hypotheses (H1), (H2), (H3) and (H4), the functional $\mathcal{F}$, defined by (4.2.2), is given by*

$$\mathcal{F}(u; A) = \int_A f(x, u, \nabla u)dx + \int_{S(u)\cap A} g(x, u^+, u^-, \nu_u)d\mathcal{H}^{N-1} + \int_A f^\infty\left(x, u, \frac{dC(u)}{d|C(u)|}\right) d|C(u)|,$$

where, for all $x_0 \in \Omega$, for all $(u_0, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$ and for all $(\lambda, \theta, \nu) \in (\mathbb{R}^d)^2 \times S^{N-1}$,

$$f(x_0, u_0, \xi) := \limsup_{\varepsilon\to 0} \inf_{\substack{v\in SBV(Q;\mathbb{R}^d)\\ v|_{\partial Q} = \xi y}} \left\{\int_Q f_0(x_0, u_0, \nabla v(y))dy + \int_{Q\cap S(v)} \frac{g_0(x_0, u_0 + \varepsilon v^+(y), u_0 + \varepsilon v^-(y), \nu_v(y))}{\varepsilon} d\mathcal{H}^{N-1}(y)\right\}, \tag{4.2.3}$$

$$g(x_0, \lambda, \theta, \nu) := \limsup_{\varepsilon\to 0} \inf_{\substack{v\in SBV(Q_\nu;\mathbb{R}^d)\\ v|_{\partial Q_\nu} = u_{\lambda,\nu}|_{\partial Q_\nu}}} \left\{\int_{Q_\nu} \varepsilon\, f_0\left(x_0, v(y), \frac{1}{\varepsilon}\nabla v(y)\right) dy + g_0\left(x_0, v^+(y), v^-(y), \nu_v(y)\right) d\mathcal{H}^{N-1}(y)\right\} \tag{4.2.4}$$

where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases} \lambda & \text{if } y \cdot \nu > 0, \\ \theta & \text{otherwise.} \end{cases}$$

Remark 4.2.2 Let us define

$$f_0^\infty(x_0,u_0,\xi) := \lim_{\varepsilon\to 0} \varepsilon f_0\left(x_0,u_0,\frac{1}{\varepsilon}\xi\right),$$

$$\overline{g}_0(x_0,u_0,\lambda,\theta,\nu) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\, g_0(x_0,u_0+\varepsilon\lambda,u_0+\varepsilon\theta,\nu)\,.$$

Using hypothesis (H4), one can easily see that $\overline{g}_0$ satisfies the invariance property $\overline{g}_0(x_0,u_0,\lambda+z,\theta+z,\nu) = \overline{g}_0(x_0,u_0,\lambda,\theta,\nu)$ for every $z\in\mathbb{R}^d$, therefore it can be written as

$$\overline{g}_0(x_0,u_0,\lambda,\theta,\nu) =: \widehat{g}_0(x_0,u_0,\lambda-\theta,\nu),$$

for a suitable function $\widehat{g}_0 : \Omega\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$.

Let us assume, in addition, that the following estimates hold

$$\left|f_0^\infty(x_0,u_0,\xi) - \varepsilon f_0\left(x_0,u_0,\frac{1}{\varepsilon}\xi\right)\right| \le C\,\varepsilon^m\,|\xi|^{1-m},$$

$$\left|\widehat{g}_0(x,u_0,\lambda-\theta,\nu) - \frac{1}{\varepsilon}g_0(x,u_0,\varepsilon\lambda,\varepsilon\theta,\nu)\right| \le C\varepsilon^\alpha|\lambda-\theta|^{1+\alpha},$$

for suitable $\alpha,\ m\in(0,1)$ and $\varepsilon<\varepsilon_0$, and for all $x_0\in\mathbb{R}^N,\ \xi\in\mathbb{R}^{d\times N},\ u_0,\ \lambda,\ \theta\in\mathbb{R}^d,\ \nu\in S^{N-1}$.

Then, as in Section 4.1, it is possible to verify that formulas (4.2.3) and (4.2.4) can be rewritten as

$$f(x_0,u_0,\xi) = \inf_{\substack{v\in SBV(Q;\mathbb{R}^d)\\ v|_{\partial Q}=\xi y}} \left\{ \int_Q f_0(x_0,u_0,\nabla v(y))dy + \int_{Q\cap S(v)} \widehat{g}_0(x_0,u_0,[v](y),\nu_v(y))dH^{N-1}(y)\right\}, \tag{4.2.3'}$$

$$g(x_0,\lambda,\theta,\nu) = \inf_{\substack{v\in SBV(Q_\nu;\mathbb{R}^d)\\ v|_{\partial Q_\nu}=u_{\lambda,\nu}|_{\partial Q_\nu}}} \left\{ \int_{Q_\nu} f_0^\infty(x_0,v(y),\nabla v(y))\,dy + g_0(x_0,v^+(y),v^-(y),\nu_v(y))d\mathcal{H}^{N-1}(y)\right\}. \tag{4.2.4'}$$

As a particular case we recover the characterizations of bulk and jump densities obtained in [BBBF] where it is assumed that $f_0 = f_0(x_0,\xi)$ and $g_0 = g_0(x_0,\lambda-\theta,\nu)$.

Proof. As in the proof of Lemma 4.3.4 of Section 4.3 it can be shown that the functional $\mathcal{F}$ defined by (4.2.2) satisfies conditions (2.1),(2.2) and (2.3′). In addition, assumptions (H2) and (H4) yield condition (2.4). Therefore, we may use Theorem 3.10 to obtain the integral representation of $\mathcal{F}$ on all $BV(\Omega;\mathbb{R}^d)$, and it remains to indentify the integrands f and g given by (3.13) and (3.14), respectively.

By Lemma 2.6 and Remark 2.7 1), we obtain that for every $(u, A) \in BV(\Omega) \times \mathcal{A}(\Omega)$ the function $m(u; A)$ defined in (3.1) agrees with

$$m_0(u;A) := \inf \{F(v;A) \mid v|_{\partial A} = u|_{\partial A}\} .$$

Replacing m by m_0 in (3.13) we have

$$\begin{aligned} f(x_0, a, \xi) = \limsup_{\varepsilon \to 0} \inf \Bigg\{ & \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f_0(x, w(x), \nabla w(x))\, dx \\ & + \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\cap S(w)} g_0(x, w^+(x), w^-(x), \nu_w(x)) d\mathcal{H}^{N-1}(x) \mid \\ & w \in SBV(Q(x_0,\varepsilon);\mathbb{R}^d),\ v(x) = a + \xi(x - x_0) \text{ on } \partial Q(x_0,\varepsilon) \Bigg\}, \end{aligned}$$

Using the change of variables $y = \frac{x-x_0}{\varepsilon}$, and setting $v(y) := \varepsilon\, w(\frac{x-x_0}{\varepsilon}) - a$, we are led to

$$\begin{aligned} f(x_0, a, \xi) = \limsup_{\varepsilon \to 0} \inf \Bigg\{ & \int_Q f_0(x_0 + \varepsilon y, a + \varepsilon v(y), \nabla v(y))\, dy \\ & + \frac{1}{\varepsilon} \int_{Q\cap S(v)} g_0(x_0 + \varepsilon y, a + v^+(y), a + v^-(y), \nu_v(y)) d\mathcal{H}^{N-1}(y) \mid \\ & v \in SBV(Q;\mathbb{R}^d),\ v(y) = \xi\, y \text{ on } \partial Q \Bigg\}. \end{aligned} \tag{4.2.5}$$

Similarly, replacing m by m_0 in (3.14), changing variables and setting now $v(y) := w(\frac{x-x_0}{\varepsilon})$, we get

$$\begin{aligned} g(x_0, \lambda, \theta, \nu) = \limsup_{\varepsilon \to 0} \inf \Bigg\{ & \int_{Q_\nu} \varepsilon\, f_0\left(x_0 + \varepsilon y, v(y), \frac{\nabla v(y)}{\varepsilon}\right) dy \\ & + \int_{Q_\nu \cap S(v)} g_0(x_0 + \varepsilon y, v^+(y), v^-(y), \nu_v(y)) d\mathcal{H}^{N-1}(y) \mid \\ & v \in SBV(Q_\nu;\mathbb{R}^d),\ v(y) = u_{\lambda,\nu}(y) \text{ on } \partial Q_\nu \Bigg\}. \end{aligned} \tag{4.2.6}$$

By the coercivity condition (2.3′), it turns out that sequences (v_ε) approaching the minimum in the right hand sides of (4.2.5) and (4.2.6) are uniformly bounded in $BV(\Omega;\mathbb{R}^d)$. Thus with the help of the continuity assumptions (H2) and (H4), we can replace $f_0(x_0 + \varepsilon y, \cdot, \cdot)$ by $f(x_0, \cdot, \cdot)$ in (4.2.5), and $g_0(x_0 + \varepsilon y, \cdot, \cdot, \cdot)$ by $g_0(x_0, \cdot, \cdot, \cdot)$ in (4.2.6). This concludes the proof of Theorem 4.4. □

4.3 Homogenization.

In what follows δ will stand for a positive parameter, converging to zero. For each $A \in \mathcal{A}(\Omega)$ consider the functionals $F_\delta(\cdot; A)$ defined in $BV(\Omega;\mathbb{R}^d)$ by

$$F_\delta(u;A) := \begin{cases} \displaystyle\int_A f_0\left(\frac{x}{\delta}, \nabla u(x)\right) dx \\ \displaystyle\quad + \int_{S(u)\cap A} g_0\left(\frac{x}{\delta}, [u](x), \nu_u(x)\right) d\mathcal{H}^{N-1}(x) & \text{if } u \in SBV(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise}, \end{cases} \tag{4.3.1}$$

where the densities f_0 and g_0 satisfy the following hypotheses :

(H1) $f_0 : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is a Borel function, Q-periodic in the first argument, and

$$\frac{1}{C}|\xi| \le f_0(x,\xi) \le C(1+|\xi|)$$

for all $\xi \in \mathbb{R}^{d\times N}$, for all $x \in \mathbb{R}^N$, and for some $C > 0$;

(H2) there exist m, L, $0 < m < 1$, $L > 0$, such that

$$\left| f_0^\infty(x,\xi) - \frac{f_0(x,t\xi)}{t} \right| \le \frac{C}{t^m}$$

for all $\xi \in \mathbb{R}^{d\times N}$, $\|\xi\| = 1$, $t > L$, and for all $x \in \mathbb{R}^N$, where the *recession function* f_0^∞ is defined by

$$f_0^\infty(x,\xi) := \limsup_{t\to+\infty} \frac{f_0(x,t\xi)}{t};$$

(H3) $g_0 : \mathbb{R}^N \times \mathbb{R}^d \times S^{N-1} \to [0,+\infty)$ is a Borel function, Q-periodic in the first argument, satisfying

$$\frac{1}{C}|\lambda| \le g_0(x,\lambda,\nu) \le C|\lambda|$$

for all $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^d$ and $\nu \in S^{N-1}$;

(H4) there exist α, l, $0 < \alpha < 1$, $l > 0$, such that

$$\left| \overline{g}_0(x,\lambda,\nu) - \frac{g_0(x,t\lambda,\nu)}{t} \right| \le Ct^\alpha$$

for all $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^d$, $\|\lambda\| = 1$, $\nu \in S^{N-1}$, $t < l$, where $\overline{g}_0$ is defined by

$$\overline{g}_0(x,\lambda,\nu) := \limsup_{t\to 0} \frac{g_0(x,t\lambda,\nu)}{t}.$$

We recall the following definitions (see [DM]) :

We say that a functional $F : BV(\Omega;\mathbb{R}^d) \to [0,+\infty]$ is the Γ-*lower limit* (respectively Γ-*upper limit*) of a sequence of functionals $F_n : BV(\Omega;\mathbb{R}^d) \to [0,+\infty]$ for the $L^1(\Omega;\mathbb{R}^d)$ topology if

i) given $u \in BV(\Omega;\mathbb{R}^d)$ and (u_n) in $BV(\Omega;\mathbb{R}^d)$, $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, then

$$F(u) \leq \liminf_{n\to+\infty} F_n(u_n) \quad (\text{respectively } F(u) \leq \limsup_{n\to+\infty} F_n(u_n));$$

ii) for each $u \in BV(\Omega;\mathbb{R}^d)$ there exists $(\bar{u}_n)$ in $BV(\Omega;\mathbb{R}^d)$ such that $\bar{u}_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and

$$F(u) = \liminf_{n\to+\infty} F_n(\bar{u}_n) \quad (\text{respectively } F(u) = \limsup_{n\to+\infty} F_n(u_n)).$$

We write

$$F = \Gamma - \liminf_{n\to+\infty} F_n \quad (\text{respectively } F = \Gamma - \limsup_{n\to+\infty} F_n).$$

We say that (F_n) Γ-converges to F if the Γ- lower limit and Γ- upper limit coincide, or, equivalently, if condition i) for the Γ- lower limit and the following condition iii) are both satisfied,

iii) for each $u \in BV(\Omega;\mathbb{R}^d)$ there exists $(\bar{u}_n)$ in $BV(\Omega;\mathbb{R}^d)$ such that $\bar{u}_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and

$$F(u) = \lim_{n\to+\infty} F_n(\bar{u}_n).$$

We write

$$F = \Gamma - \lim_{n\to+\infty} F_n.$$

Remark 4.3.1. Since $L^1(\Omega;\mathbb{R}^d)$ is a separable metric space, we can deduce from Kuratowski's Compactness Theorem (see [DM]) that a sequence (F_n) Γ-converges to F if and only if $F = \Gamma - \liminf_{k\to+\infty} F_{n_k}$, for any sequence of indexes $n_k \to +\infty$.

Given $A \in \mathcal{A}(\Omega)$, we define

$$\mathcal{F}^-(\cdot;A) = \Gamma - \liminf_{\delta\to0} F_\delta(\cdot;A) \quad \text{and} \quad \mathcal{F}^+(\cdot;A) = \Gamma - \limsup_{\delta\to0} F_\delta(\cdot;A).$$

Theorem 4.3.2. *Under hypotheses (H1) - (H4) we have $\mathcal{F}^- = \mathcal{F}^+ = \mathcal{F}$ where, for each $u \in BV(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}(\Omega)$, $\mathcal{F}$ is defined by*

$$\mathcal{F}(u;A) := \int_A f(\nabla u)dx + \int_{S(u)\cap A} g([u],\nu_u)d\mathcal{H}^{N-1} + \int_A f^\infty\Big(\frac{dC(u)}{d|C(u)|}\Big)d|C(u)|, \tag{4.3.2}$$

$$f(\xi):=\lim_{T\to+\infty}\frac{1}{T^N}\inf_{\substack{u\in SBV(TQ;\mathbb{R}^d)\\ u=\xi x \text{ on } \partial(TQ)}}\Big\{\int_{TQ}f_0(x,\nabla u)dx+\int_{S(u)\cap TQ}\overline{g}_0(x,[u],\nu_u)d\mathcal{H}^{N-1}\Big\}, \tag{4.3.3}$$

$$g(\lambda,\nu):=\lim_{T\to+\infty}\frac{1}{T^{N-1}}\inf_{\substack{u\in SBV(TQ_\nu;\mathbb{R}^d)\\ u=u_{\lambda,\nu} \text{ on } \partial(TQ_\nu)}}\Big\{\int_{TQ_\nu}f_0^\infty(x,\nabla u)dx + \int_{S(u)\cap TQ_\nu}g_0(x,[u],\nu_u)d\mathcal{H}^{N-1}\Big\}, \tag{4.3.4}$$

where $u_{\lambda,\nu}(y):=\begin{cases}\lambda & \text{if } y\cdot\nu>0\\ 0 & \text{otherwise.}\end{cases}$

According to Remark 4.3.1, in order to prove Theorem 4.3.2 it is enough to show that for any given sequence $\delta_n\to 0$ the Γ-lower limit of $(F_{\delta_n}(\cdot;A))$ agrees, for every $A\in\mathcal{A}(\Omega)$, with the functional $\mathcal{F}(u;\cdot)$ defined in Theorem 4.3.2. Having this in mind, and in order to simplify the notations, we will represent the sequence (δ_n) by the parameter δ.

Lemma 4.3.3. *The functional $\mathcal{F}^-$ satisfies*

$$\mathcal{F}^-(u(\cdot-h);A+h)=\mathcal{F}^-(u;A) \quad \text{and} \quad \mathcal{F}^-(u+a;A)=\mathcal{F}^-(u;A),$$

for all $u\in BV(\Omega;\mathbb{R}^d)$, $A\in\mathcal{A}(\Omega)$, $h\in\mathbb{R}^N$, and $a\in\mathbb{R}^d$.

For the proof of this lemma we refer to [BDV], Lemma 3.7.

Lemma 4.3.4. *Under hypotheses (H1) - (H4), $\mathcal{F}^-$ satisfies conditions (2.1), (2.2), (2.3′) and (2.4).*

Proof. Condition (2.4) is an immediate consequence of Lemma 4.3.3.

We prove (2.2). Since the Γ-lower limit of a sequence of the functionals is lower semicontinuous (c.f. [DM]), $\mathcal{F}^-(\cdot;A)$ is $L^1(\Omega;\mathbb{R}^d)$ lower semicontinuous. In view of the local character of $\mathcal{F}^-$, easily deduced from its definition, we conclude that $\mathcal{F}^-(\cdot;A)$ is also $L^1(A;\mathbb{R}^d)$ lower semicontinuous.

In order to prove (2.3′), and by (H1) and (H3), we consider the double inequality

$$\frac{1}{C}|Du|(A)\le F_\delta(u;A)\le C(\mathcal{L}^N(A)+|Du|(A)),$$

for all $(u,A)\in SBV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$, and we pass to the Γ - lower limit in each member.

Finally we prove (2.1). We claim that for every $u\in BV(\Omega;\mathbb{R}^d)$ and for every A,B,C in $\mathcal{A}(\Omega)$, the following implication holds :

$$C\subset\subset B\subset\subset A \;\Rightarrow\; \mathcal{F}^-(u;A)\le\mathcal{F}^-(u;B)+\mathcal{F}^-(u;A\setminus\overline{C}). \tag{4.3.5}$$

In fact, let (v_δ) and (w_δ) be two sequences converging to u in $L^1(\Omega;\mathbb{R}^d)$ and such that

$$\liminf_{\delta\to 0} F_\delta(v_\delta;B) = \mathcal{F}^-(u;B) \text{ and } \liminf_{\delta\to 0} F_\delta(w_\delta;A\setminus\overline{C}) = \mathcal{F}^-(u;A\setminus\overline{C}).$$

By means of hypotheses (H1) and (H3) we can apply Lemma 2.6 and Remark 2.7 to the sequence (F_δ), and find two other sequences (w'_δ) and (v'_δ) in $SBV(\Omega;\mathbb{R}^d)$, both converging to u in $L^1(\Omega;\mathbb{R}^d)$, $w'_\delta = v'_\delta = u$ on Σ and

$$\limsup_{\delta\to 0} F_\delta(w'_\delta;A\setminus\overline{B}_\rho) \le \liminf_{\delta\to 0} F_\delta(w_\delta;A\setminus\overline{B}_\rho),$$
$$\limsup_{\delta\to 0} F_\delta(v'_\delta;B_\rho) \le \liminf_{\delta\to 0} F_\delta(v_\delta;B_\rho),$$

where

$$\Sigma := \{x\in B\setminus\overline{C} \mid \text{dist}\,(x,\partial B)=\rho\} \text{ and } B_\rho := \{x\in B \mid \text{dist}\,(x,\partial B)>\rho\},$$

for some $0<\rho<\text{dist}\,(\partial B, C)$ and such that $|D_s u|(\Sigma)=0$. Defining $\overline{v_\delta} = w'_\delta$ in $\Omega\setminus\overline{B}_\rho$ and $\overline{v_\delta} = v'_\delta$ in B_ρ, we get $\overline{v_\delta}\to u$ in $L^1(\Omega;\mathbb{R}^d)$. Since $B_\rho\subset B$ and $A\setminus\overline{B}_\rho\subset A\setminus\overline{C}$, we also obtain

$$\begin{aligned}
\mathcal{F}^-(u;A) &\le \liminf_{\delta\to 0} F_\delta(\overline{v}_\delta;A) = \liminf_{\delta\to 0}\left[F_\delta(v'_\delta;B_\rho) + F_\delta(w'_\delta;A\setminus\overline{B}_\rho)\right]\\
&\le \liminf_{\delta\to 0} F_\delta(v'_\delta;B_\rho) + \limsup_{\delta\to 0} F_\delta(w'_\delta;A\setminus\overline{B}_\rho)\\
&\le \liminf_{\delta\to 0} F_\delta(v_\delta;B) + \liminf_{\delta\to 0} F_\delta(w_\delta;A\setminus\overline{C})\\
&= \mathcal{F}^-(u;B) + \mathcal{F}^-(u;A\setminus\overline{C}),
\end{aligned}$$

which proves (4.3.5).

Now consider (u_δ) in $SBV(\Omega;\mathbb{R}^d)$ such that

$$\mathcal{F}^-(u;\Omega) = \liminf_{\delta\to 0} F_\delta(u_\delta;\Omega).$$

Let μ be the Radon measure on the compact $\overline{\Omega}$ defined as the weak limit, up to a subsequence, of $\left(f_0\left(\frac{\cdot}{\delta},\nabla u_\delta\right)\mathcal{L}^N\lfloor\Omega + g_0\left(\frac{\cdot}{\delta},[u_\delta],\nu_{u_\delta}\right)\mathcal{H}^{N-1}\lfloor(S(u_\delta)\cap\Omega)\right)$ as $\delta\to 0$. We have

$$\mathcal{F}^-(u;\Omega) = \mu(\overline{\Omega}) \tag{4.3.6}$$

and, by definition of $\mathcal{F}^-$, for all $A\in\mathcal{A}(\Omega)$,

$$\mathcal{F}^-(u;A) \le \liminf_{\delta\to 0} F_\delta(u_\delta;A) \le \mu(\overline{A}). \tag{4.3.7}$$

Let $B\in\mathcal{A}(\Omega)$ and $\varepsilon>0$ be fixed and consider $C\in\mathcal{A}(\Omega)$, $C\subset\subset B$ such that $\mu(B\setminus C)<\varepsilon$. We get

$$\mu(B) \le \mu(C)+\varepsilon = \mu(\overline{\Omega}) - \mu(\overline{\Omega}\setminus C) + \varepsilon\,.$$

In view of (4.3.6), applying (4.3.7) with $A = \Omega \setminus \overline{C}$ and (4.3.5) with $A = \Omega$, it follows that

$$\mu(B) \leq \mathcal{F}^-(u;\Omega) - \mathcal{F}^-(u;\Omega \setminus \overline{C}) + \varepsilon \leq \mathcal{F}^-(u;B) + \varepsilon.$$

Letting $\varepsilon \to 0$, we conclude that

$$\mu(B) \leq \mathcal{F}^-(u;B) \leq \mu(\overline{B}),$$

for all $B \in \mathcal{A}(\Omega)$.

In order to prove that $\mathcal{F}^-(u;A) = \mu(A)$ for all $A \in \mathcal{A}(\Omega)$, we fix again $\varepsilon > 0$ and choose $C, B \in \mathcal{A}(\Omega)$ such that $C \subset\subset B \subset\subset A$ and $\mathcal{L}^N(A\setminus\overline{C}) + |Du|(A\setminus\overline{C}) < \varepsilon/C$. By (4.3.5), (4.3.7), and since $\mathcal{F}^-$ satisfies (2.3),

$$\mathcal{F}^-(u;A) \leq \mathcal{F}^-(u;A \setminus \overline{C}) + \mathcal{F}^-(u;B) \leq \varepsilon + \mu(\overline{B}) \leq \varepsilon + \mu(A).$$

We complete the proof by letting $\varepsilon \to 0$. □

Lemma 4.3.4 enables us to apply Theorems 3.4 and 3.9, which, together with Remark 3.5, yield

$$\mathcal{F}^-(u;A) = \int_A f^-(\nabla u)dx + \int_{S(u)\cap A} g^-([u],\nu_u)d\mathcal{H}^{N-1} + \int_A (f^-)^\infty\Big(\frac{dC(u)}{d|C(u)|}\Big)d|C(u)|,$$

where $f^- : \mathbb{R}^{d\times N} \to [0,+\infty)$ and $g^- : \mathbb{R}^d \times S^{N-1} \to [0,+\infty)$ are given by (3.13) and (3.14). In order to prove that $f^- = f$ and $g^- = g$, as defined in (4.3.3) and (4.3.4), respectively, we introduce (c.f. (3.1))

$$m(u;A) := \inf\Big\{\mathcal{F}^-(v;A)|\ v|_{\partial A} = u|_{\partial A}, v \in BV(\Omega;\mathbb{R}^d)\Big\}$$

and, for each $\delta > 0$,

$$m_\delta(u;A) := \inf\Big\{F_\delta(v;A)|\ v|_{\partial A} = u|_{\partial A}, v \in BV(\Omega;\mathbb{R}^d)\Big\}.$$

Lemma 4.3.5. *For each $u \in BV(\Omega;\mathbb{R}^N)$, $x_0 \in \Omega$, $\nu \in S^{N-1}$, we have*

$$\liminf_{\delta\to 0} m_\delta(u;Q_\nu(x_0,t)) = m(u;Q_\nu(x_0,t)),$$

for almost all $t > 0$ such that $Q_\nu(x_0,t) \subset \Omega$.

Proof. We divide the proof into two steps.

Step 1. We show that

$$\liminf_{\delta\to 0} m_\delta(u;Q_\nu(x_0,t)) \leq m(u;Q_\nu(x_0,t)) \tag{4.3.8}$$

for all $t > 0$ such that $Q_\nu(x_0, t) \subset \Omega$. Fix $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$, and let $v \in BV(\Omega; \mathbb{R}^d)$ be such that

$$m(u; A) > \mathcal{F}^-(v; A) - \varepsilon \text{ and } v = u \text{ on } \partial A.$$

Let (v_δ) be such that $\mathcal{F}^-(v; A) = \liminf_{\delta \to 0} F_\delta(v_\delta; A)$. Using Lemma 2.6 and Remark 2.7 2), we can find another sequence $(\tilde{v}_\delta)$ satisfying $\tilde{v}_\delta = u$ on ∂A and such that

$$\mathcal{F}^-(v; A) = \liminf_{\delta \to 0} F_\delta(\tilde{v}_\delta; A).$$

Since $m_\delta(u; A) \leq F_\delta(\tilde{v}_\delta; A)$, we have

$$\liminf_{\delta \to 0} m_\delta(u; A) - \varepsilon \leq \mathcal{F}^-(v; A) - \varepsilon < m(u; A).$$

Letting ε go to zero we conclude (4.3.8).

Step 2. We prove that for almost all $t \in (0, T)$ such that $Q_\nu(x_0, T) \subset \Omega$,

$$\liminf_{\delta \to 0} m_\delta(u; Q_\nu(x_0, t)) \geq m(u; Q_\nu(x_0, t)). \tag{4.3.9}$$

We claim that $t \mapsto m(u; Q_\nu(x_0, t))$ is a measurable function. Indeed,

$$m(u; Q_\nu(x_0, t')) \leq m(u; Q_\nu(x_0, t)) + C\Big(1 + |Du|\Big)\Big(Q_\nu(x_0, t') \setminus Q_\nu(x_0, t)\Big),$$

for $t > t'$, and so $\limsup_{t' \searrow t} m(u; Q_\nu(x_0, t')) \leq m(u; Q_\nu(x_0, t))$. This implies the measurability of $t \mapsto m(u; Q_\nu(x_0, t))$. Define

$$E := \Big\{t_0 \in (0, T) \mid t \mapsto m(u; Q_\nu(x_0, t)) \text{ is approximately continuous at } t_0\Big\}.$$

Recalling that a measurable finite function is approximately continuous almost everywhere (see [EG]), we have that $\mathcal{L}^1((0, T) \setminus E) = 0$. The conclusion of step 2 follows from the two following claims.

Claim 1. For each $t \in E$,

$$\limsup_{t' \searrow t} m(u; Q_\nu(x_0, t')) \geq m(u; Q_\nu(x_0, t)).$$

This is a consequence of the approximate continuity of the function $m(u, Q_\nu(x_0, \cdot))$ at t, which implies that, for every $\varepsilon > 0$, the set

$$\{t' \in (t, T) \mid m(u; Q_\nu(x_0, t')) < m(u; Q_\nu(x_0, t)) - \varepsilon\}$$

has Lebesgue density at t equal to 0, i.e.

$$\lim_{\delta\to 0}\frac{1}{\delta}\mathcal{L}^1(\{t'\in(t,t+\delta)|\ m(u;Q_\nu(x_0,t'))<m(u;Q_\nu(x_0,t))-\varepsilon\}=0).$$

Therefore, there exists a sequence $t_n\searrow t$ such that

$$m(u;Q_\nu(x_0,t_n))\geq m(u;Q_\nu(x_0,t))-\varepsilon.$$

The claim 1 follows by letting first $n\to+\infty$ and then $\varepsilon\to 0$.

Claim 2. For every $t>0$,

$$\liminf_{\delta\to 0} m_\delta(u;Q_\nu(x_0,t))\geq \limsup_{t'\searrow t} m(u;Q_\nu(x_0,t')).$$

For each $\delta>0$ choose u_δ satisfying $u_\delta=u$ on $\partial Q_\nu(x_0,t)$ and

$$m_\delta(u;Q_\nu(x_0,t))>F_\delta(u_\delta;Q_\nu(x_0,t))-\delta.$$

For $t'>t$ consider the extension $\tilde u_\delta$ of u_δ, $\tilde u_\delta:=u_\delta$ in $Q_\nu(x_0,t)$ and $\tilde u_\delta:=\tilde u$ in $Q_\nu(x_0,t')\setminus Q_\nu(x_0,t)$, where $\tilde u\in W^{1,1}(Q_\nu(x_0,t')\setminus Q_\nu(x_0,t);\mathbb{R}^d)$ and $\tilde u=u$ on $\partial Q_\nu(x_0,t')\cup\partial Q_\nu(x_0,t)$ (see Lemma 2.5). We have

$$m_\delta(u;Q_\nu(x_0,t))>F_\delta(\tilde u_\delta;Q_\nu(x_0,t'))-C\int_{Q_\nu(x_0,t')\setminus\overline{Q_\nu(x_0,t)}}(1+|\nabla\tilde u\ |)\,dx-\delta.$$

Using the coercivity conditions (H1) and (H3), together with Poincaré's inequality, we infer that the sequence $(\tilde u_\delta)$ is bounded in $BV((Q_\nu(x_0,t');\mathbb{R}^d)$. Let v be defined as the limit, up to a subsequence, of $\tilde u_\delta$ in $L^1(Q_\nu(x_0,t');\mathbb{R}^d)$. Since by construction $v=\tilde u=u$ on $\partial Q_\nu(x_0,t')$, we obtain

$$\begin{aligned}\liminf_{\delta\to 0} m_\delta(u;Q_\nu(x_0,t))&\geq\mathcal{F}^-(v;Q_\nu(x_0,t'))-C\int_{Q_\nu(x_0,t')\setminus\overline{Q_\nu(x_0,t)}}(1+|\nabla\tilde u\ |)\,dx\\&\geq m(u;Q_\nu(x_0,t'))-C\int_{Q_\nu(x_0,t')\setminus\overline{Q_\nu(x_0,t)}}(1+|\nabla\tilde u\ |)\,dx,\end{aligned}$$

The claim is proved by letting $t'\searrow t$.

□

The following lemma is due to C. Licht and G. Michaille (see [LM], Theorem 3.1 and its proof). We refer to Section 2 for the definition of the class $\mathcal{A}$.

Lemma 4.3.6. *Let $p\geq 1$ and let $S:\mathcal{A}(\mathbb{R}^p)\to\mathbb{R}^+$ be such that*

i) there exists $C>0$ such that $S(A)\leq C\,\mathcal{L}^p(A)$,

ii) $S(C)\leq S(A)+S(B)$, for all $A,B,C\in\mathcal{A}(\mathbb{R}^p)$, $A\cap B=\emptyset$, $\overline{C}=\overline{A}\cup\overline{B}$,

iii) there exist $\mathcal{T}\subset\mathbb{R}^p$ and $M>0$ such that $\mathcal{T}+[0,M)^p=\mathbb{R}^p$ and $S(A+\tau)=S(A)$ for all $A\in\mathcal{A}(\mathbb{R}^p)$ and $\tau\in\mathcal{T}$.

Then, for any cube A of the form $[a,b)^p$ there exists the limit of the sequence $\left(\dfrac{S(sA)}{\mathcal{L}^p(sA)}\right)$*, as $s \to +\infty$, and*

$$\lim_{s\to+\infty} \frac{S(sA)}{\mathcal{L}^p(sA)} = \lim_{s\to+\infty} \frac{S([0,s)^p)}{s^p}.$$

Futhermore, if $\{S_L\}_L$ is a family of set functions satisfying i) - iii) for C, $\mathcal{T}$ and M independent of L, the above limits are attained uniformly in L.

Lemma 4.3.7. *The limits in the right hand side of (4.3.3) and (4.3.4) exist and*

$$f^-(\xi) = f(\xi), \tag{4.3.10}$$
$$g^-(\lambda,\nu) = g(\lambda,\nu). \tag{4.3.11}$$

Proof.

Part 1. First we prove the existence of the limit in the right hand side of (4.3.3) and then we prove (4.3.10).

Let us define, for $\varepsilon, T > 0$ and $(w,A) \in BV(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$,

$$F_{\varepsilon,T}(w;A) := \int_A f_0(yT,\nabla w)dy + \frac{1}{\varepsilon}\int_{S(w)\cap A} g_0(yT,\varepsilon[w],\nu_w)d\mathcal{H}^{N-1}, \tag{4.3.12}$$

$$F_{0,T}(w;A) := \int_A f_0(yT,\nabla w)dy + \int_{S(w)\cap A} \overline{g}_0(yT,[w],\nu_w)d\mathcal{H}^{N-1}, \tag{4.3.13}$$

and

$$m_{0,T}(\xi x;A) := \inf\{F_{0,T}(w;A)|\ w \in SBV(A;\mathbb{R}^d),\ w = \xi x \text{ on } \partial A\}. \tag{4.3.14}$$

For $A \in \mathcal{A}(\Omega)$ set

$$S(A) := m_{0,1}(\xi x;A).$$

In view of the periodicity hypotheses (H1) and (H3), we may apply Lemma 4.3.6 to obtain

$$\lim_{T\to+\infty} \frac{S(TQ)}{T^N} = \lim_{T\to+\infty} \frac{1}{T^N}\, m_{0,1}(\xi x;TQ) = f(\xi), \tag{4.3.15}$$

which proves the existence of the limit in the right hand side of (4.3.3).

From (3.13) and in view of Remark 3.5, proving (4.3.10) is equivalent to asserting that

$$\lim_{\varepsilon\to0} \frac{m(\xi x;\varepsilon Q)}{\varepsilon^N} = f(\xi),$$

or, by virtue of Lemma 4.3.5, it suffices to prove that

$$\lim_{\varepsilon\to0}\liminf_{\delta\to0} \frac{m_\delta(\xi x;\varepsilon Q)}{\varepsilon^N} = f(\xi),$$

for a suitable subsequence still denoted by ε.

Step 1. We show that

$$\alpha := \lim_{\varepsilon \to 0} \liminf_{\delta \to 0} \frac{m_\delta(\xi x; \varepsilon Q)}{\varepsilon^N} \geq f(\xi).$$

We have

$$\alpha = \lim_{\varepsilon \to 0} \lim_{n} \frac{m_{\delta_{n,\varepsilon}}(\xi x; \varepsilon Q)}{\varepsilon^N},$$

where, for each $\varepsilon > 0$, $\delta_{n,\varepsilon} \underset{n \to +\infty}{\longrightarrow} 0$. We extract a diagonal subsequence $\delta(\varepsilon)$ such that $T_\varepsilon := \varepsilon/\delta(\varepsilon) \longrightarrow +\infty$ and

$$\alpha = \lim_{\varepsilon \to 0} \frac{m_{\delta(\varepsilon)}(\xi x; \varepsilon Q)}{\varepsilon^N} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} F_{\delta(\varepsilon)}(v_\varepsilon; \varepsilon Q)$$

for suitable $v_\varepsilon \in SBV(\Omega; \mathbb{R}^d)$, $v_\varepsilon = \xi x$ on $\partial(\varepsilon Q)$. Changing variables and writing $\overline{v}_\varepsilon(y) := \frac{1}{\varepsilon} v_\varepsilon(\varepsilon y)$, we have

$$\alpha = \lim_{\varepsilon \to 0} F_{\varepsilon, T_\varepsilon}(\overline{v}_\varepsilon; Q). \tag{4.3.16}$$

Due to the coercivity hypotheses (H1) and (H3), together with (4.3.16), we have $\sup_\varepsilon \|\overline{v}_\varepsilon\|_{BV(Q;\mathbb{R}^d)} = \overline{C} < +\infty$. Since $\overline{v}_\varepsilon(y) = \xi y$ on ∂Q, using Lemma 2.8 with $u_0 = \xi y$, for fixed $\eta > 0$ we may find $M_\eta = M\left(\eta, \overline{C}, C, \|\xi y\|_{L^\infty(Q;\mathbb{R}^d)}\right)$ and for each ε we may find $w_\varepsilon \in BV(Q; \mathbb{R}^d) \cap L^\infty(Q; \mathbb{R}^d)$ such that

$$\|w_\varepsilon\|_{L^\infty(Q;\mathbb{R}^d)} \leq M_\eta, \quad w_\varepsilon(y) = \xi y \text{ on } \partial Q, \quad |Dw_\varepsilon|(Q) \leq \overline{C},$$

and

$$\lim_{\varepsilon \to 0} F_{\varepsilon, T_\varepsilon}(\overline{v}_\varepsilon; Q) \geq \limsup_{\varepsilon \to 0} F_{\varepsilon, T_\varepsilon}(w_\varepsilon; Q) - \eta. \tag{4.3.17}$$

By (H4) we have

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \int_{S(w_\varepsilon) \cap Q} \left| \frac{1}{\varepsilon}\, g_0(yT_\varepsilon, \varepsilon\, [w_\varepsilon], \nu_{w_\varepsilon}) - \overline{g}_0(yT_\varepsilon, [w_\varepsilon], \nu_{w_\varepsilon}) \right| d\mathcal{H}^{N-1} \\
&\leq \limsup_{\varepsilon \to 0}\; C\, \varepsilon^\alpha \int_{S(w_\varepsilon) \cap Q} |[w_\varepsilon]|^{\alpha+1} d\mathcal{H}^{N-1} \\
&\leq \limsup_{\varepsilon \to 0}\; C\, \overline{C}\, \varepsilon^\alpha\, M_\eta^\alpha = 0.
\end{aligned}$$

Setting $\overline{w}_\varepsilon(y) := T_\varepsilon w_\varepsilon\,(y/T_\varepsilon)$, we deduce from (4.3.15), (4.3.16) and (4.3.17) that

$$\begin{aligned}
\alpha &\geq \limsup_{\varepsilon \to 0} F_{0,T_\varepsilon}(w_\varepsilon; Q) - \eta = \limsup_{\varepsilon \to 0} \frac{1}{T_\varepsilon^{\,N}} F_{0,1}(\overline{w}_\varepsilon; T_\varepsilon Q) - \eta \\
&\geq \limsup_{\varepsilon \to 0} \frac{1}{T_\varepsilon^{\,N}} m_{0,1}(\xi x; T_\varepsilon Q) - \eta = \lim_{T \to +\infty} \frac{1}{T^N} m_{0,1}(\xi x; TQ) - \eta = f(\xi) - \eta.
\end{aligned}$$

To conclude the proof of the first step it suffices to let η tend to zero.

Step 2. We prove that

$$\lim_{\varepsilon\to 0}\liminf_{\delta\to 0}\frac{m_\delta(\xi x;\varepsilon Q)}{\varepsilon^N}\leq f(\xi).$$

Let $u_T\in SBV(\Omega;\mathbb{R}^d), u_T=\xi x$ on $\partial(TQ)$ be such that

$$f(\xi)=\lim_{T\to+\infty}\frac{1}{T^N}\,F_{0,1}(u_T;TQ).$$

Setting $\overline{u}_T(y):=\frac{1}{T}u_T(yT)$, we obtain

$$f(\xi)=\lim_{T\to+\infty}\,F_{0,T}(\overline{u}_T;Q)$$

and so, just as in Step 1, given $\eta>0$ we may replace $\overline{u}_T$ by w_T such that $w_T=\xi y$ on ∂Q and $\sup_T\|w_T\|_{L^\infty(Q;\mathbb{R}^d)}=\overline{C}<+\infty$. We have

$$\begin{aligned}
f(\xi)&\geq\liminf_{T\to+\infty}\,F_{0,T}(w_T;Q)-\eta=\limsup_{\varepsilon\to 0}\,\liminf_{T\to+\infty}\,F_{\varepsilon,T}(w_T;Q)-\eta\\
&=\limsup_{\varepsilon\to 0}\,\liminf_{T\to+\infty}\frac{1}{\varepsilon^N}\,F_{\varepsilon/T}\,(w_{T,\varepsilon};\varepsilon Q)-\eta\\
&\geq\limsup_{\varepsilon\to 0}\,\liminf_{\delta\to 0}\frac{1}{\varepsilon^N}\,m_\delta\,(\xi x;\varepsilon Q)-\eta,
\end{aligned}$$

where $w_{T,\varepsilon}(y):=\varepsilon\,w_T(y/\varepsilon)$.

Part 2. We prove the existence of the limit in the right hand side of (4.3.4) and we prove (4.3.11).

For $\varepsilon,T>0$ and $(u,A)\in BV(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$ define

$$G_{\varepsilon,T}(w;A):=\varepsilon\int_A f_0\left(yT,\frac{1}{\varepsilon}\nabla w\right)dy+\int_{S(w)\cap A}g_0(yT,[w],\nu_w)d\mathcal{H}^{N-1},\quad(4.3.18)$$

$$G_{0,T}(w;A):=\int_A f_0^\infty(yT,\nabla w)dy+\int_{S(w)\cap A}g_0(yT,[w],\nu_w)d\mathcal{H}^{N-1}\qquad(4.3.19)$$

and

$$m_{0,T}(u_{\lambda,\nu};A):=\inf\{G_{0,T}(w;A)|\;w\in SBV(A;\mathbb{R}^d),\;w=u_{\lambda,\nu}\text{ on }\partial A\}.\quad(4.3.20)$$

From (3.14) and in view of Remark 3.5, to prove (4.3.11) is equivalent to assert that

$$\lim_{\varepsilon\to 0}\frac{m(u_{\lambda,\nu};\varepsilon Q_\nu)}{\varepsilon^{N-1}}=g(\lambda,\nu)\;.$$

Then, by virtue of Lemma 4.3.5, it suffices to show that

$$\lim_{\varepsilon\to 0}\ \liminf_{\delta\to 0} \frac{m_\delta(u_{\lambda,\nu};\varepsilon Q_\nu)}{\varepsilon^{N-1}} = g(\lambda,\nu), \tag{4.3.21}$$

for a suitable subsequence still denoted by ε. Provided we establish the existence of

$$\lim_{T\to+\infty} \frac{1}{T^{N-1}}\, m_{0,1}(u_{\lambda,\nu};TQ_\nu), \tag{4.3.22}$$

the proof of (4.3.21) is quite similar to the one presented in Steps 1 and 2 of Part 1. Indeed, it is enough to replace the functional $F_{\varepsilon,T}$ by $G_{\varepsilon,T}$, $F_{0,T}$ by $G_{0,T}$, and to use hypothesis (H2) instead of (H4).

We prove the existence of the limit (4.3.22) in three steps.

Step 1. We recall that, for $\nu \in S^{N-1}$, R_ν denotes a rotation satisfying $R_\nu(e_N) = \nu$ and $\nu \mapsto R_\nu(e_i)$ is continuous in $S^{N-1}\setminus\{e_N\}$, for all $i = 1,\cdots,N-1$ (see Section 2). As in [BDV], define S^* to be the set of all $\nu \in S^{N-1}$ such that $R_\nu(e_i) = \gamma_i z_i$, for some $\gamma_i \in \mathbb{R}\setminus\{0\}$, $z_i \in \mathbb{Z}^N$, $i = 1,\cdots,N-1$. The set S^* is dense in S^{N-1}. Let

$$Q_\nu^{T,L} := R_\nu\left(\{x \in \mathbb{R}^N \mid |x_N| < L/2 \text{ and } |x_i| < T/2, \text{ for } i = 1,\cdots,N-1\}\right).$$

Fix $\nu \in S^*$, $L > 0$ and define

$$\mathcal{T}(\nu) := \left\{\sum_{i=1}^{N-1} \frac{\lambda_i}{\gamma_i} e_i \mid \lambda_i \in \mathbb{Z},\ R_\nu(e_i) = \gamma_i z_i,\ \gamma_i \in \mathbb{R}\setminus\{0\},\ z_i \in \mathbb{Z}^N\right\}.$$

For each open subset $A \subset \mathbb{R}^{N-1}$ with Lipschitz boundary, set

$$S_L(A,\nu) := m_{0,1}(u_{\lambda,\nu}; R_\nu(A\times I_L)),$$

where $I_L = (-L/2, L/2)$. In view of the periodicity hypotheses (H1) and (H3) we have that, for C independent of L,

$$S_L(A+\tau,\nu) = S_L(A,\nu) \quad\text{and}\quad S_L(A,\nu) \le C\mathcal{L}^{N-1}(A), \tag{4.3.23}$$

for all $A \in \mathcal{A}(\Omega)$, $\tau \in \mathcal{T}(\nu)$, and also $\mathcal{T}(\nu) + [0,M)^{N-1} = \mathbb{R}^{N-1}$, where $M := \max_{1\le i\le N-1} \gamma_i$. Applying Lemma 4.3.6, with $p = N-1$, we conclude that

$$\lim_{T\to+\infty} \frac{1}{T^{N-1}} m_{0,1}(u_{\lambda,\nu}; Q_\nu^{T,L}) \tag{4.3.24}$$

exists and is finite.

Step 2. We prove that, for all $\nu \in S^*$,

$$\lim_{T\to+\infty} \frac{1}{T^{N-1}} m_{0,1}(u_{\lambda,\nu};TQ_\nu) = g(\lambda,\nu). \tag{4.3.25}$$

In fact,

$$\begin{aligned}\liminf_{T\to+\infty}\frac{1}{T^{N-1}}m_{0,1}(u_{\lambda,\nu};TQ_\nu) &\geq \liminf_{T\to+\infty}\ \inf_{L>0}\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};Q_\nu^{T,L})\\ &= \inf_{L>0}\ \lim_{T\to+\infty}\ \frac{1}{T^{N-1}}m_{0,1}(u_{\lambda,\nu};Q_\nu^{T,L}),\end{aligned}\tag{4.3.26}$$

having in mind that, since the limit in (4.3.24) is uniform in L, we can interchange the infimum in L with the limit as T goes to $+\infty$.

Conversely, fix L and let $T > L$. Using again, for each test function in $Q_\nu^{T,L}$, the extension by $u_{\lambda,\nu}$ to the whole TQ_ν, we obtain

$$\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};Q_\nu^{T,L}) \geq \frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};TQ_\nu)$$

and, consequently,

$$\inf_{L>0}\ \lim_{T\to+\infty}\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};Q_\nu^{T,L}) \geq \limsup_{T\to+\infty}\ \frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};TQ_\nu). \tag{4.3.27}$$

From (4.3.26) and (4.3.27) we conclude the proof of Step 2.

Step 3. We extend the proof of existence of the limit (4.3.22) to all $\nu \in S^{N-1}$. In view of the continuity of $\nu \mapsto R_\nu$, for each $\nu \in S^{N-1}\setminus\{e_N\}$ and $\varepsilon > 0$ we may find $\nu_\varepsilon \in S^*$ and η satisfying $1 < \eta < \varepsilon + 1$, such that

$$\begin{aligned}&(1/\eta)Q_{\nu_\varepsilon} \subset Q_\nu \subset \eta Q_{\nu_\varepsilon},\\ &\mathcal{H}^{N-1}\Big(\partial\Big(\frac{1}{\eta}Q_{\nu_\varepsilon}\Big)\cap\{x\in\mathbb{R}^N|\ x\cdot\nu_\varepsilon<0\}\cap\{x\in\mathbb{R}^N|\ x\cdot\nu>0\}\Big)+\\ &\mathcal{H}^{N-1}\Big(\partial\Big(\frac{1}{\eta}Q_{\nu_\varepsilon}\Big)\cap\{x\in\mathbb{R}^N|\ x\cdot\nu_\varepsilon>0\}\cap\{x\in\mathbb{R}^N|\ x\cdot\nu<0\}\Big)+\\ &\qquad\mathcal{H}^{N-1}\Big(\Big[Q_\nu\setminus\Big(\frac{1}{\eta}Q_{\nu_\varepsilon}\Big)\Big]\cap\{x\in\mathbb{R}^N|\ x\cdot\nu=0\}\Big)<\varepsilon,\end{aligned}\tag{4.3.28}$$

and analogous estimates hold with ηQ_{ν_ε} in place of Q_ν and Q_ν in place of $\frac{1}{\eta}Q_{\nu_\varepsilon}$.

Given $T > 0$, extending each test function defined in $(T/\eta)Q_{\nu_\varepsilon}$ to TQ_ν by $u_{\lambda,\nu}$, and taking into account estimates (4.3.28) and hypothesis (H3), it follows that

$$\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};TQ_\nu) \leq \frac{1}{(T/\eta)^{N-1}}\ m_{0,1}(u_{\lambda,\nu_\varepsilon};(T/\eta)Q_{\nu_\varepsilon}) + C|\lambda|\varepsilon.$$

Therefore, using Step 2 to justify the existence of the limit as T tends to $+\infty$ in the right hand side of the previous inequality, we get

$$\begin{aligned}\limsup_{T\to+\infty}\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu};TQ_\nu) &\leq \lim_{T\to+\infty}\frac{1}{(T/\eta)^{N-1}}\ m_{0,1}(u_{\lambda,\nu_\varepsilon};(T/\eta)Q_{\nu_\varepsilon}) + O(\varepsilon)\\ &= \lim_{T\to+\infty}\frac{1}{T^{N-1}}\ m_{0,1}(u_{\lambda,\nu_\varepsilon};TQ_{\nu_\varepsilon}) + O(\varepsilon).\end{aligned}\tag{4.3.29}$$

Similar reasoning concerning the inclusion $Q_\nu \subset \eta Q_{\nu_\varepsilon}$ leads to

$$\liminf_{T\to+\infty} \frac{1}{T^{N-1}}\, m_{0,1}(u_{\lambda,\nu};TQ_\nu) \geq \lim_{T\to+\infty} \frac{1}{(\eta T)^{N-1}}\, m_{0,1}(u_{\lambda,\nu_\varepsilon};\eta TQ_{\nu_\varepsilon}) - O(\varepsilon)$$
$$= \lim_{T\to+\infty} \frac{1}{T^{N-1}}\, m_{0,1}(u_{\lambda,\nu_\varepsilon};TQ_{\nu_\varepsilon}) - O(\varepsilon). \tag{4.3.30}$$

Letting ε go to zero in (4.3.29) and (4.3.30), we conclude the proof of Step 3. □

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