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IRENE FONSECA † & GIOVANNI LEONI ‡

Abstract An integral representation formula in $BV(\Omega;\mathbf{R}^p)$ for the relaxation $\mathcal{H}(u,\Omega)$ with respect to the L^1 topology of functionals of the general form

$$H(u,\Omega) := \int_\Omega h(x,u(x),\nabla u(x))\,dx + \int_{\partial\Omega} \theta(x,T\,u(x))\,dH_{N-1}(x), \quad u \in W^{1,1}(\Omega;\mathbf{R}^p),$$

is obtained. Here $\Omega \subset \mathbf{R}^N$ is an open, bounded set of class C^2, T is the trace operator on $\partial\Omega$ and H_{N-1} is the $N-1$ dimensional Hausdorff measure. The main hypotheses on the functions h and θ are that $h(x,u,\cdot)$ is quasiconvex and has linear growth, and that $\theta(x,\cdot)$ is Lipschitz. The understanding of nucleation phenomena for materials undergoing phase transitions leads to the study of constrained minimization problems of the type

$$\inf\left\{\mathcal{H}(u,\Omega) + \int_\Omega \tau(x,u(x))\,dx : u \in BV(\Omega;K)\right\},$$

where K is a nonempty compact subset of $\mathbf{R}^p$, and $\tau : \Omega \times K \to \mathbf{R}$ is a continuous function. It is shown that if $\tau(x,\cdot)$ is a double well potential vanishing only at α and β, then minimizers u of the total energy are given by pure phases, that is, there exists $\Omega_u \subset \Omega$ such that $u(x) = \alpha$ for $\mathcal{L}^N$ a.e. $x \in \Omega_u$ (liquid) and $u(x) = \beta$ for $\mathcal{L}^N$ a.e. $x \in \Omega\backslash\Omega_u$ (solid). This conclusion is closely related to results previously obtained by Visintin, and where the interfacial energy is assumed to satisfy a *generalized co-area formula*. Here this condition is replaced by a property which may be verified by energies for which the co-area formula might not hold.

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§1. Introduction.

This paper is divided into two parts. In the first part we obtain an integral representation formula in $BV(\Omega;\mathbb{R}^p)$ for the relaxation $\mathcal{H}(u,\Omega)$ with respect to the L^1 topology of functionals of the general form

$$H(u,\Omega) := \int_\Omega h(x,u(x),\nabla u(x))\,dx + \int_{\partial\Omega} \theta(x,T\,u(x))\,dH_{N-1}(x), \quad u \in W^{1,1}(\Omega;\mathbb{R}^p), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded set of class C^2, T is the trace operator on $\partial\Omega$ and H_{N-1} is the $N-1$ dimensional Hausdorff measure. The main hypotheses on the functions h and θ are that $h(x,u,\cdot)$ is quasiconvex and has linear growth, and that $\theta(x,\cdot)$ is Lipschitz.

Under a *degenerate* coercivity assumption on $h(x,u,\cdot)$ we obtain the following integral representation for $u \in BV(\Omega;\mathbb{R}^p)$

$$\begin{aligned}\mathcal{H}(u,\Omega) = &\int_\Omega h(x,u(x),\nabla u(x))\,dx + \int_\Omega h^\infty(x,u(x),dC(u))\\ &+ \int_{S(u)\cap\Omega} K_h(x,u^-(x),u^+(x),\nu_u(x))\,dH_{N-1}(x) + \int_{\partial\Omega}\theta(x,T\,u(x))\,dH_{N-1}(x),\end{aligned} \tag{1.2}$$

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where ∇u is the density of the absolutely continuous part of the distributional derivative Du with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$, $(u^+ - u^-)$ is the jump across the interface $S(u)$, and $C(u)$ is the Cantor part of Du. For the canonical model $h(x,u,\nabla u) := \sigma|\nabla u|$, where $\sigma > 0$, the relaxed energy $\mathcal{H}(u,\Omega)$ reduces to

$$\mathcal{H}(u,\Omega) = \sigma \int_\Omega |Du| + \int_{\partial\Omega} \theta(x, Tu)\, dH_{N-1}, \qquad u \in BV(\Omega;\mathbb{R}^p). \tag{1.3}$$

In the scalar case where $p = 1$ the lower semicontinuity of the functional (1.3) was proved by Massari and Pepe [MP] when $\theta(x,u) := \hat{\sigma}|u|$, with $|\hat{\sigma}| \leq \sigma$, and by Modica [Mo2] under the assumption that

$$|\theta(x,u) - \theta(x,u_1)| \leq \sigma|u - u_1| \tag{1.4}$$

for all $x \in \partial\Omega$ and all u, $u_1 \in \mathbb{R}$.

One of the motivations for the introduction of a relaxed energy is that nonconvex variational problems may not have a minimizer in the space of smooth functions – this fact was first pointed out by Weirstrass in 1869, when he published his celebrated counterexample to Dirichlet's principle. Therefore, to apply the direct method of Calculus of Variations one has to extend the original functional. Although Sobolev spaces are considered to be the natural extension to the space of smooth functions, in recent years the theory of phase transitions, and the need to determine effective energies for materials exhibiting instabilities such as fractures and defects, have led us to further extend the domain of functionals of the form (1.1) in order to include functions u which present discontinuities along surfaces. Motivated somewhat by Lebesgue's definition of surface area, Serrin in [Se1, Se2] proposed the following notion for the relaxed energy of $H(u,\Omega)$ (in the case where $\theta \equiv 0$)

$$\mathcal{H}(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} H(u_n,\Omega) : \, u_n \in W^{1,1}(\Omega;\mathbb{R}^p),\, u_n \to u \quad \text{in} \quad L^1(\Omega;\mathbb{R}^p) \right\}.$$

One of the main issues in the Calculus of Variations concerns the search and characterization of an integral representation for $\mathcal{H}(u,\Omega)$ in the space $BV(\Omega;\mathbb{R}^p)$.

In the scalar case where $p = 1$ and $h(x,u,\cdot)$ is convex, the integral representation (1.2) was first obtained by Goffman and Serrin [GSe] when $h = h(\nabla u)$ (see also [Re]), and by Giaquinta, G. Modica and Souček [GMS] for $h = h(x,\nabla u)$. These results were then extended by Dal Maso [DM] who considered the general case where $h = h(x,u,\nabla u)$, and emphasized the important role of the coercivity condition in establishing (1.2). Indeed, Dal Maso showed that, while (1.2) holds for nonnegative functions $h = h(u,\nabla u)$ without any lower bound on h, when $h = h(x,\nabla u)$, or, more generally, when $h = h(x,u,\nabla u)$, the representation (1.2) may fail unless one requires a weak coercivity assumption of the form

$$h(x,u,\nabla u) \geq g(x,u)|\nabla u|. \tag{1.5}$$

In the vectorial case where $p > 1$ and $h(x,u,\cdot)$ is quasiconvex, Ambrosio and Dal Maso [ADM2] proved (1.2) when $h = h(\nabla u)$ and without (1.5). Independently, Fonseca and Müller [FM2] obtained this result for general functions $h(x,u,\nabla u)$ which verify (1.5).

In all the works mentioned above $\theta \equiv 0$, and one of the purposes of this paper is to extend these results to the new case where possibly $\theta \not\equiv 0$. The relaxation of functionals of the type (1.1) arises in the van der Waals–Cahn–Hilliard theory of phase transitions for fluids (cf. [CH1, CH 2, vdW]). In this context the boundary term $\int_{\partial\Omega} \theta(x,Tu)\, dH_{N-1}$ represents the contact energy between the fluid and the container walls, where $\theta(x,u)$ is the contact energy per unit area when the density is u (see [C, G]).

We present here two relaxation results. In Theorem 2.5 we show that, without any a priori coercivity on the function h, the functional on the right hand side of (1.2) actually gives the integral representation for the following relaxed energy

$$\mathcal{H}_b(u,\Omega) = \inf_{\{u_n\}} \{ \liminf_{n\to\infty} H(u_n,\Omega) : \, u_n \in W^{1,1}(\Omega;\mathbb{R}^p),\ u_n \to u \,\text{in}\, L^1(\Omega;\mathbb{R}^p),\ \sup_n ||u_n||_{W^{1,1}} < \infty \},$$

while in Theorem 2.2 we prove that $\mathcal{H}_b(u,\Omega) = \mathcal{H}(u,\Omega)$ if h satisfies a condition of the type (1.5). Therefore we may conclude that the right hand side of (1.2) always coincides with $\mathcal{H}_b(u,\Omega)$, and we restate all the results mentioned above by saying that $\mathcal{H}_b(u,\Omega) = \mathcal{H}(u,\Omega)$ in the scalar case if either $h = h(u,\nabla u)$ or if $h = h(x,u,\nabla u)$ satisfies (1.5), and in the vectorial case if either $h = h(\nabla u)$ or if $h = h(x,u,\nabla u)$ satisfies (1.5). In the remaining cases it may happen that $\mathcal{H}(u,\Omega) < \mathcal{H}_b(u,\Omega)$.

It is worth mentioning that the fact that the relaxation $\mathcal{H}(u,\Omega)$ is simply given by the decoupled sum of the relaxation of the functional $\int_\Omega h(x,u,\nabla u)\,dx$ and the contact energy may be somewhat deceiving, since it hides the competition between the bulk energy and the contact energy. A more insightful way to look at (1.1), and consequently, at (1.2), is perhaps to consider the equivalent form

$$H(u,\Omega) = \int_\Omega \left\{h(x,u(x),\nabla u(x)) + \varphi(x)\cdot\nabla u^T(x)\nabla_u\theta(x,u(x))\right\}\,dx$$
$$+\int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot\nabla_x\theta(x,u(x))\,dx,$$

where $\varphi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ depends only on Ω and $|\varphi(x)| < 1$ in Ω (see Lemma 3.2). In particular, in the isotropic case where $h(x,u,\nabla u) := \sigma|\nabla u|$, $\sigma > 0$, we obtain

$$H(u,\Omega) = \int_\Omega \left\{\sigma\,|\nabla u(x)| + \varphi(x)\cdot\nabla u^T(x)\nabla_u\theta(x,u(x))\right\}\,dx$$
$$+\int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot\nabla_x\theta(x,u(x))\,dx,$$

and it is clear that the functional H is not bounded from below in general, unless one assumes a condition of the type

$$|\nabla_u\theta(x,u)| \le \sigma \qquad \text{for a. e. } x\in\Omega \quad \text{and for all } u\in\mathbb{R}^p,$$

which is essentially the condition found by Massari and Pepe [MP] and by Modica [Mo2].

In the second part of the paper we are concerned with constrained minimization problems of the type

$$\inf\left\{\mathcal{H}(u,\Omega) + \int_\Omega \tau(x,u(x))\,dx \,:\, u\in BV(\Omega;K)\right\},$$

where K is a nonempty compact set of $\mathbb{R}^p$, and $\tau : \Omega\times K \to \mathbb{R}$ is a continuous function. This kind of problems has important applications in the study of phase transformations and in nucleation phenomena (cf. [V1, V2]). According to the van der Waals–Cahn–Hilliard theory of phase transitions (cf. [CH1, CH2, vdW]), the total energy of a fluid of total mass m and density $u(x)$ confined in a bounded container $\Omega\subset\mathbb{R}^N$, is given by

$$(1.6)\qquad E_\varepsilon(u) := \varepsilon^2\int_\Omega |\nabla u|^2\,dx + \int_\Omega W_1(u)\,dx + \varepsilon\int_{\partial\Omega} W_2(Tu)\,dH_{N-1}, \qquad u\in W^{1,1}(\Omega;\mathbb{R}),$$

where the coarse–grain energy $W_1(u)$ is a double well potential vanishing only at α and β and corresponding to the stable two–phase configuration of the fluid, the gradient term $\varepsilon^2|\nabla u|^2$ models the interfacial energy across a smooth transition layer, with ε a small parameter, and W_2 represents the contact energy between the fluid and the container walls. The stable configurations of the fluid correspond to solutions of the problem (see [C])

$$\inf\left\{E_\varepsilon(u) \,:\, u\in W^{1,1}(\Omega;\mathbb{R}),\ \int_\Omega u\,dx = m\right\}.$$

Confirming a conjecture of Gurtin [G], Modica in [Mo2] was able to show that if a sequence of minimizers $\{u_\varepsilon\}$ converges in L^1 to a function u_0, then u_0 solves the *liquid-drop problem*

$$\inf\left\{\mathcal{H}(u,\Omega) \,:\, u\in BV(\Omega;\{\alpha,\,\beta\})\right\},$$

where $\mathcal{H}(u,\Omega)$ is the relaxed energy of

$$H(u,\Omega) = \int_\Omega |\nabla u|\,dx + \hat{\sigma}\int_{\partial\Omega} T\,u\,dH_{N-1}, \qquad u \in W^{1,1}(\Omega;\mathbb{R}).$$

Here $\hat{\sigma}$ depends only on W_1 and W_2. The liquid–drop problem admits a solution if and only if $|\hat{\sigma}| \le 1$. An analogous result is due to Alberti, Bouchitté and Seppecher [ABS] who recently showed that if the parameter ε in front of the contact energy in (1.6) is replaced by λ_ε, where

$$\lim_{\varepsilon\to 0} \varepsilon \log \lambda_\varepsilon = K \in (0,\infty)$$

and W_2 is a double well potential which vanishes only at α_1 and β_1, then the limit problem is given by a different model for capillarity with line tension. It is worth noting that in this case the effective energy takes the form

$$\mathcal{H}(u,\Omega) = \int_\Omega |D(G(u))| + \inf\left\{\int_{\partial\Omega} |G(T\,u) - G(v)|\,dH_{N-1} + \frac{K}{\pi}\int_{\partial\Omega} |Dv|^2 : v \in BV(\partial\Omega;\{\alpha_1,\beta_1\})\right\}$$

for $u \in BV(\Omega;\{\alpha,\beta\})$ and $\mathcal{H}(u,\Omega) = \infty$ otherwise. Here G is a primitive of $2\sqrt{W_1}$. It can be seen immediately that in this capillarity model the contact energy is strongly nonlinear, which leads us to consider functions θ other than $\theta(x,u) = \hat{\sigma}u$ (see [V1, V2]).

In the last section of the paper we prove some minimization results which are related to solid nucleation. For a complete description of this phenomenon we refer to the recent monograph of Visintin [V1] and to the bibliography contained therein. By *solid nucleation* we mean the formation of a new solid phase, that is of a connected component of solid in a liquid. If the new solid phase is formed in the interior of the liquid, the nucleation is called *homogeneous*, while if it is also in contact with other substances, such as the container, impurities dispersed in the liquid or nucleants, then we name it *heterogeneous* nucleation (cf. [V1, Ch. VII.2]). By thinking of these impurities or particles as holes in the domain Ω, we can represent the contact energy by an integral term over the boundary of Ω. Furthermore, since the new solid phase is formed through crystallization, and crystals are *anisotropic*, the classical isotropic interfacial energy $\sigma\int_\Omega |Du|$ is now replaced by $\int_\Omega h(x,Du)$. In the applications one sees often $h(x,Du) = |A(x)Du|$, where $A(x)$ is a nonnegative definite $N\times N$ tensor (cf. [V1, p. 157]).

The main results of this part are Theorems 5.1 and 5.4, where we show that minimizers u of the total energy are given by pure phases, that is, there exists $\Omega_u \subset \Omega$ such that $u(x) = \alpha$ for $\mathcal{L}^N$ a.e. $x \in \Omega_u$ (liquid) and $u(x) = \beta$ for $\mathcal{L}^N$ a.e. $x \in \Omega\backslash\Omega_u$ (solid). This result is closely related to Theorem 2 in [V2], where the interfacial energy is assumed to satisfy a *generalized co-area formula*. We replace here this condition by some hypotheses which are easy to verify and allow us to include interfacial energies of the form $\int_\Omega h(x,Du)$, where $h(x,\cdot)$ is convex and positively homogeneous of degree one, and for which the co-area formula might not hold.

§2. Relaxation.

We consider the functional

$$H(u,\Omega) := \int_\Omega h(x,u(x),\nabla u(x))\,dx + \int_{\partial\Omega} \theta(x,T\,u(x))\,dH_{N-1}(x)$$

defined on the Sobolev space $W^{1,1}(\Omega;\mathbb{R}^p)$, where $\Omega \subset \mathbb{R}^N$ is an open, bounded set of class C^2, T is the trace operator on $\partial\Omega$, H_{N-1} is the $N-1$ dimensional Hausdorff measure and the functions

$$h : \Omega \times \mathbb{R}^p \times \mathbb{M}^{p\times N} \to [0,\infty), \qquad \theta : \partial\Omega \times \mathbb{R}^p \to \mathbb{R}$$

satisfy the following hypotheses:
(H_1) h is continuous;

(H_2) $h(x,u,\cdot)$ is quasiconvex for all $(x,u) \in \Omega \times \mathbb{R}^p$;
(H_3) there exist a nonnegative, bounded, continuous function $g : \Omega \times \mathbb{R}^p \to [0,\infty)$ and a constant $C > 0$ such that

$$g(x,u)|\xi| \le h(x,u,\xi) \le C\,g(x,u)(1+|\xi|) \tag{2.1}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^p \times \mathbb{M}^{p\times N}$, where $\mathbb{M}^{p\times N}$ is the vector space of $p \times N$ matrices;
(H_4) for every compact set $K \Subset \Omega \times \mathbb{R}^p$ there exists a continuous function $\omega : [0,\infty) \to [0,\infty)$, with $\omega(0) = 0$, such that

$$|h(x,u,\xi) - h(x_1,u_1,\xi)| \le \omega(|x-x_1| + |u-u_1|)(1+|\xi|) \tag{2.2}$$

for all (x,u,ξ), $(x_1,u_1,\xi) \in K \times \mathbb{M}^{p\times N}$. In addition, for every $x_0 \in \Omega$ and $\delta > 0$ there exists $\varepsilon > 0$ such that

$$h(x_0,u,\xi) - h(x,u,\xi) \le \delta(1 + g(x,u)\,|\xi|) \tag{2.3}$$

for all $x \in \Omega$ with $|x - x_0| \le \varepsilon$ and for all $(u,\xi) \in \mathbb{R}^p \times \mathbb{M}^{p\times N}$;
(H_5) there exist $C' > 0$ and $m \in (0,1)$ such that

$$|h^\infty(x,u,\xi) - h(x,u,\xi)| \le C' g(x,u)(1+|\xi|^{1-m})$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^p \times \mathbb{M}^{p\times N}$, where the *recession function* h^∞ of h is defined as

$$h^\infty(x,u,\xi) := \limsup_{t\to\infty} \frac{h(x,u,t\,\xi)}{t};$$

(H_6) θ admits an extension $\theta \in C(\overline{\Omega} \times \mathbb{R}^p;\mathbb{R}) \cap C^1(\Omega \times \mathbb{R}^p;\mathbb{R})$ such that

$$|\nabla_x \theta(x,u)| \le a_1(x) + C_1(1+|u|^{q_c})$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $u \in \mathbb{R}^p$, where $a_1 \in L^1(\Omega,\mathbb{R})$, $C_1 > 0$ and q_c is the Sobolev exponent $q_c := N/(N-1)$ if $N > 1$ and $q_c < \infty$ if $N = 1$. Moreover, for every $x_0 \in \Omega$ and $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|\nabla_u \theta(x_0,u) - \nabla_u \theta(x,u)| \le \delta\, g(x,u) \tag{2.4}$$

for all $x \in \Omega$ with $|x - x_0| \le \varepsilon$ and for all $u \in \mathbb{R}^p$;
(H_7) $g(x,u) \ge |\nabla_u \theta(x,u)|$ for all $(x,u) \in \Omega \times \mathbb{R}^p$.

Remark 2.1. (i) Conditions $(H_1) - (H_5)$ were considered by Fonseca and Müller (see [FM2]), who treated the case where $\theta \equiv 0$. It can be shown that the *recession function* h^∞ of h is still quasiconvex and is positively homogeneous of degree one in the ξ variable (see [FM2, M]).

(ii) By the Mean Value Theorem and conditions (H_3) and (H_7) we have

$$|\theta(x,u) - \theta(x,u_1)| \le |\nabla_u \theta(x,\hat{u})|\,|u-u_1| \le ||g||_{L^\infty}|u-u_1| \tag{2.5}$$

for all $x \in \Omega$ and all u, $u_1 \in \mathbb{R}^p$. Taking $u_1 = 0$ it follows by (H_6) and (2.5) that

$$|\theta(x,u)| \le ||g||_{L^\infty}\,|u| + ||\theta(x,0)||_{L^\infty} \tag{2.6}$$

for all $(x,u) \in \Omega \times \mathbb{R}^p$. This growth condition, together with (2.1), implies in particular that the functional $H(u,\Omega)$ is well defined and finite for $u \in W^{1,1}(\Omega;\mathbb{R}^p)$.

(iii) A typical example of the energy densities is (see Visintin [V1 , V2])

$$h(x,u,\xi) := \sigma\,|\xi|, \qquad \theta(x,u) := \hat{\sigma}\,u, \tag{2.7}$$

where $\sigma > 0$ and $\hat{\sigma} \in \mathbb{R}$. It is easy to see that conditions $(H_1) - (H_6)$ hold with $g(x,u) := \sigma$, while assumption (H_7) reduces to the inequality $|\hat{\sigma}| \le \sigma$. More generally, (H_6) is trivially satisfied if $\theta = \theta(u)$.

(iv) If in (1.3) we take $\theta(x,u) := \hat{\sigma}|u|$ for $(x,u) \in \partial\Omega \times \mathbb{R}^p$ (cf. [MP]), then it is possible to extend θ to $\overline{\Omega} \times \mathbb{R}^p$ as follows

$$\theta(x,u) := \hat{\sigma}\sqrt{|u|^2 + \psi^2(x)},$$

where $\psi \in C^1(\overline{\Omega};\mathbb{R})$ is such that $\psi(x) > 0$ for $x \in \Omega$ and $\psi(x) = 0$ for $x \in \partial\Omega$. Conditions $(H_1) - (H_7)$ are then verified with $g(x,u) := \sigma$, provided $|\hat{\sigma}| \le \sigma$. The problem of finding an extension of $\theta : \partial\Omega \times \mathbb{R}^p \to \mathbb{R}$ to $\overline{\Omega} \times \mathbb{R}^p$ which satisfies $(H_6) - (H_7)$ for the functional (1.3), and when (1.4) holds, will be addressed in a forthcoming paper.

Our first goal in this paper is to obtain an integral representation for the relaxation of $H(u,\Omega)$ in $BV(\Omega;\mathbb{R}^p)$ with respect to the L^1 topology, that is

$$\mathcal{H}(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} H(u_n,\Omega) : \, u_n \in W^{1,1}(\Omega;\mathbb{R}^p),\, u_n \to u \quad \text{in} \quad L^1(\Omega;\mathbb{R}^p) \right\}.$$

From the definition of $\mathcal{H}(u,\Omega)$ it follows immediately that the functional $\mathcal{H}(u,\Omega)$ is lower semicontinuous in $L^1(\Omega;\mathbb{R}^p)$.

Before stating the main theorems of the section we introduce the *surface energy* associated to the function h. For any $\nu \in S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ let $\{\nu_1, \cdots, \nu_{N-1}, \nu\}$ be an orthonormal basis of $\mathbb{R}^N$ varying continuously with ν. For fixed a, $b \in \mathbb{R}^p$ we define $\mathcal{A}(a,b,\nu)$ as the class of all functions $\psi \in W^{1,1}(Q_\nu;\mathbb{R}^p)$ such that

$$T\,\psi(y) = \begin{cases} a & \text{if } y\cdot\nu = -1/2 \\ b & \text{if } y\cdot\nu = \;\;1/2 \end{cases}$$

and which are periodic of period one in the remaining directions $\nu_1, \cdots, \nu_{N-1}$. Here $Q_\nu := \{x \in \mathbb{R}^N : |x\cdot\nu_i| < 1/2,\, |x\cdot\nu| < 1/2,\, i = 1, \cdots, N-1\}$. The surface energy $K_h(x,a,b,\nu)$ associated to the function h is defined by

$$K_h(x,a,b,\nu) := \inf\left\{ \int_{Q_\nu} h^\infty(x,\psi(y),\nabla\psi(y))\,dy : \, \psi \in \mathcal{A}(a,b,\nu) \right\}.$$

For a detailed study of the properties of the function $K_h(x,a,b,\nu)$ we refer to [FR].

We recall briefly some facts about functions of bounded variation which will be useful in the sequel. A function $u \in L^1(\Omega;\mathbb{R}^p)$ is said to be of *bounded variation* if for all $i = 1, \cdots p$, and $j = 1, \cdots N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx = -\int_\Omega \varphi(x)\,d\mu_{ij}$$

for every $\varphi \in C_0^1(\Omega;\mathbb{R})$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}. Given $u \in BV(\Omega;\mathbb{R}^p)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1, \cdots p$, are given by

$$u_i^+(x) := \inf\left\{ t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\,\mathcal{L}^N\left(\{y \in \Omega\cap B(x,\varepsilon) : \, u_i(y) > t\}\right) = 0 \right\}$$

and

$$u_i^-(x) := \sup\left\{ t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\,\mathcal{L}^N\left(\{y \in \Omega\cap B(x,\varepsilon) : \, u_i(y) < t\}\right) = 0 \right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \cup_{i=1}^p \{x \in \Omega : \, u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \cup_{n=1}^{\infty} K_n \cup E,$$

where $H_{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega \backslash S(u)$ then $u(x)$ is taken as the common value of $(u_1^+(x), \cdots, u_p^+(x))$ and $(u_1^-(x), \cdots, u_p^-(x))$. It can be shown that $u(x) \in \mathbb{R}^p$ for H_{N-1} a.e. $x \in \Omega \backslash S(u)$. Furthermore, for H_{N-1} a.e. $x \in S(u)$ there exist a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at x, and two vectors $u^-(x)$, $u^+(x) \in \mathbb{R}^p$ (the traces of u on $S(u)$ at the point x) such that

$$\lim_{r \to 0} \frac{1}{r^N} \int_{\{y \in B(x_0,r):\,(y-x)\cdot\nu_u(x) > 0\}} |u(y) - u^+(x)|^{N/(N-1)} dy = 0$$

and

$$\lim_{r \to 0} \frac{1}{r^N} \int_{\{y \in B(x_0,r):\,(y-x)\cdot\nu_u(x) < 0\}} |u(y) - u^-(x)|^{N/(N-1)} dy = 0.$$

Note that in general $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. Moreover, the Sobolev inequality

$$\left(\int_\Omega |u(x)|^{N/(N-1)} dx \right)^{(N-1)/N} \leq C(N) ||u||_{BV}$$

holds in $BV(\Omega; \mathbb{R}^p)$ when $N > 1$. Finally, Du may be represented as

$$Du = \nabla u \, \mathcal{L}^N + (u^+ - u^-) \otimes H_{N-1} \lfloor S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$. These three measures are mutually singular.

We are now ready to state the main results of this section. For $u \in BV(\Omega; \mathbb{R}^p)$ we define the functional

$$\begin{aligned}\mathcal{L}(u, \Omega) := & \int_\Omega h(x, u(x), \nabla u(x))\, dx + \int_\Omega h^\infty(x, u(x), dC(u)) \\ & + \int_{S(u)\cap\Omega} K_h(x, u^-(x), u^+(x), \nu_u(x))\, dH_{N-1}(x) + \int_{\partial\Omega} \theta(x, T\, u(x))\, dH_{N-1}(x).\end{aligned}$$

Here, and in what follows, if g is a positively homogeneous function of degree one and if μ is a $\mathbb{R}^m$–valued measure then we define

$$\int_\Omega g(d\mu) := \int_\Omega g(\alpha(x))\, d|\mu(x)|,$$

where $|\mu|$ is the nonnegative total variation measure of μ, and $\alpha : \Omega \to S^{m-1}$ is the Radon–Nikodym derivative of μ with respect to $|\mu|$.

Theorem 2.2. *Let $(H_1) - (H_7)$ hold. If $u \in BV(\Omega; \mathbb{R}^p)$ then*

$$\mathcal{H}(u, \Omega) = \mathcal{L}(u, \Omega).$$

Corollary 2.3. *If $h = h(x, \xi)$ then*

$$\begin{aligned}\mathcal{H}(u, \Omega) = & \int_\Omega h(x, \nabla u(x))\, dx + \int_\Omega h^\infty(x, dC(u)) \\ & + \int_{S(u)\cap\Omega} h^\infty(x, (u^+(x) - u^-(x)) \otimes \nu_u(x))\, dH_{N-1}(x) + \int_{\partial\Omega} \theta(x, T\, u(x))\, dH_{N-1}(x).\end{aligned}$$

The proof of Corollary 2.3 follows from Remark 2.17 in [FM2].

Remark 2.4. (i) Rather surprisingly, in general the functional $\mathcal{L}(u,\Omega)$ is not lower semicontinuous in L^1 if the domain Ω is only Lipschitz. This fact was first pointed out by Modica in [Mo2] who gave the following simple example. Let $\Omega := (0,1)\times(0,1)\subset\mathbb{R}^2$ and take h and θ as in (2.7), with $-\sigma\le\hat\sigma<-\sigma\sqrt{2}/2$. Then (H_1)–(H_7) are satisfied (see Remark 2.1 (iii)), and

$$\mathcal{L}(u,\Omega)=\sigma\int_\Omega|Du|+\hat\sigma\int_{\partial\Omega}Tu\,dH_{N-1},\quad u\in BV(\Omega;\mathbb{R}).$$

Consider the sequence

$$u_n(x_1,x_2):=\begin{cases}0 & \text{if } x_1+x_2\ge 1/n\\ n & \text{if } x_1+x_2<1/n.\end{cases}$$

Then $u_n(x)\to 0$ in $L^1(\Omega;\mathbb{R})$ but $\mathcal{L}(u_n,\Omega)=\sigma\sqrt{2}+2\hat\sigma<\mathcal{L}(0,\Omega)=0$, and this shows that $\mathcal{H}(u,\Omega)\neq\mathcal{L}(u,\Omega)$ since $\mathcal{H}(u,\Omega)$ is lower semicontinuous in L^1.

It is worth noting that in the special case where $\theta(x,u)=|u-\psi(x)|$ in (1.3), with $\psi\in L^1(\partial\Omega;\mathbb{R})$, one can still prove lower semicontinuity of $\mathcal{L}$ for Lipschitz domains. The first result in this direction is due to Massari and Pepe [MP] who treated the case where $\psi\equiv 0$. Modica [Mo2] then extended it to include $\psi\in L^1(\partial\Omega;\mathbb{R})$. The idea in [MP, Mo2] is to find a function $\hat\psi\in BV(\mathbb{R}^N\setminus\overline{\Omega};\mathbb{R})$ whose trace is ψ and then use an extension theorem (see [EG, Th. 5.4.1]) to rewrite the integral $\int_{\partial\Omega}|Tu-\psi|\,dH_{N-1}$ as

$$\int_{\partial\Omega}|Tu-T\hat\psi|\,dH_{N-1}=\int_{\mathbb{R}^N}|D\hat u|-\int_\Omega|Du|-\int_{\mathbb{R}^N\setminus\overline{\Omega}}|D\hat\psi|,$$

where

$$\hat u(x):=\begin{cases}u(x) & \text{if } x\in\Omega\\ \hat\psi(x) & \text{if } x\in\mathbb{R}^N\setminus\overline{\Omega}.\end{cases}$$

(ii) Without condition (H_7) Theorem 2.2 may fail. As an example, let $\Omega:=(0,1)\subset\mathbb{R}$ and take h and θ as in (2.7). In this case condition (H_7) is equivalent to the inequality $|\hat\sigma|\le\sigma$. Assume that $\sigma<\hat\sigma$ and consider the sequence

$$u_n(x):=\begin{cases}-n^3(x-1)-n & \text{if } 1-1/n^2\le x\le 1\\ 0 & \text{otherwise.}\end{cases}$$

Then $u_n(x)\to 0$ in $L^1(\Omega;\mathbb{R})$ but $\mathcal{L}(u_n,\Omega)=(\sigma-\hat\sigma)n<\mathcal{L}(0,\Omega)=0$.

Theorem 2.5. *Let $(H_1)-(H_6)$ hold, with (2.1) and (H_7) replaced by the weaker hypothesis*

$$|\nabla_u\theta(x,u)|\,|\xi|\le h(x,u,\xi)\le C\,g(x,u)(1+|\xi|)\tag{2.8}$$

for all $(x,u,\xi)\in\Omega\times\mathbb{R}^p\times\mathbb{M}^{p\times N}$, and some $C>0$. Then the relaxation of $H(u,\Omega)$

$$\mathcal{H}_b(u,\Omega)=\inf_{\{u_n\}}\left\{\liminf_{n\to\infty}H(u_n,\Omega):\,u_n\in W^{1,1}(\Omega;\mathbb{R}^p),\ u_n\to u\ in\,L^1(\Omega;\mathbb{R}^p),\ \sup_n\|u_n\|_{W^{1,1}}<\infty\right\}$$

in $BV(\Omega;\mathbb{R}^p)$ with respect to the L^1 topology has the integral representation

$$\mathcal{H}_b(u,\Omega)=\mathcal{L}(u,\Omega).$$

Remark 2.6. Under the assumptions of Theorem 2.5, the functional $\mathcal{L}(u,\Omega)$ provides the correct integral representation for $\mathcal{H}_b(u,\Omega)$ but not necessarily for $\mathcal{H}(u,\Omega)$. Indeed, in the scalar case where $p=1$ and when $\theta\equiv 0$, Dal Maso has shown in [DM] that $\mathcal{H}(u,\Omega)=\mathcal{L}(u,\Omega)$ when $h=h(u,\xi)$ satisfies only (2.8), while possibly $\mathcal{H}(u,\Omega)<\mathcal{L}(u,\Omega)$ for $h=h(x,\xi)$ unless one assumes a condition of the type (2.1).

In the vectorial case where $p>1$ and when $\theta\equiv 0$, Ambrosio and Dal Maso [ADM2] proved that $\mathcal{H}(u,\Omega)=\mathcal{L}(u,\Omega)$ when $h=h(\xi)$ satisfies only (2.8). Independently, Fonseca and Müller [FM2] have obtained this result for general functions $h(x,u,\xi)$ which verify (2.1), still in the case where $\theta\equiv 0$.

§3. Proof of Theorems 2.2 and 2.5.

In this section we give the proofs of Theorems 2.2 and 2.5. We start with some preliminary results. In what follows, and unless otherwise specified, we always assume that conditions $(H_1)-(H_7)$ hold.

Lemma 3.1. *If $u \in BV(\Omega;\mathbb{R}^p)$ then the function $v(x) := \theta(x,u(x)) \in BV(\Omega;\mathbb{R})$ and*

$$Dv = \begin{cases} \nabla_x\theta(x,u)\,\mathcal{L}^N + Du^T\nabla_u\theta(x,u) & \text{on } \Omega\backslash S(u) \\ (\theta(x,u^+) - \theta(x,u^-)) \otimes \nu_u\, H_{N-1}\lfloor S(u) & \text{on } S(u). \end{cases}$$

Moreover

$$T\,v(x) = \theta(x, T\,u(x)).$$

The proof of Lemma 3.1 is straightforward in light of related results on the chain rule for BV functions (see [ADM1] and the references contained therein).

Lemma 3.2. *There exists $\varphi \in C_0^1(\mathbb{R}^N;\mathbb{R}^N)$ with $|\varphi(x)| < 1$ in Ω such that for any $u \in BV(\Omega;\mathbb{R}^p)$*

$$\int_{\partial\Omega} \theta(x, T\,u(x))\,dH_{N-1}(x) = \int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot d(D(\theta(x,u(x)))).$$

Proof. Since $\partial\Omega$ is compact and of class C^2, we can find a finite open covering $\{U_j\}_j$ of $\partial\Omega$, where U_j are balls centered at points of $\partial\Omega$, $j = 1,\cdots,P$, and for each U_j there is a C^2 diffeomorphism $\Phi_j : U_j \to \Phi_j(U_j)$ such that $\Phi_j(U_j) \subset B(0,R_j) \subset \mathbb{R}^N$ for some $R_j > 0$,

$$\Omega \cap U_j = \{x \in U_j : (\Phi_j(x))_N < 0\} \tag{3.1}$$

and for $x \in \partial\Omega \cap U_j$ the exterior normal to $\partial\Omega$ at x is given by

$$n(x,\Omega) = \frac{\nabla\Phi_j^T(x)\,e_N}{|\nabla\Phi_j^T(x)\,e_N|}.$$

Let Ψ be a partition of the unity for $\cup_{j=1}^P U_j$ subordinate to $\{U_j\}_j$. For any $\psi \in \Psi$ there exists $j \in \{1,\cdots,P\}$ such that $\psi \in C_0^\infty(U_j)$, and we define

$$\varphi_\psi(x) := \frac{\nabla\Phi_j^T(x)\,e_N}{|\nabla\Phi_j^T(x)\,e_N|}\left(1 + \frac{(\Phi_j(x))_N}{R_j}\right)\psi(x); \tag{3.2}$$

then $\varphi_\psi(x) \in C_0^1(U_j;\mathbb{R}^N)$ and $|\varphi_\psi(x)| < 1$ for $x \in \Omega \cap U_j$. If we set φ_ψ to be zero outside U_j we obtain that $\varphi_\psi(x) \in C_0^1(\mathbb{R}^N;\mathbb{R}^N)$, and thus we can apply the Trace Theorem (cf. [EG,Th. 5.3.1]) to the BV function $v(x) = \theta(x,u(x))$ to obtain

$$\int_{\partial\Omega} \varphi_\psi(x)\cdot n(x,\Omega)\,\theta(x,T\,u(x))\,dH_{N-1}(x) = \int_\Omega \theta(x,u(x))\operatorname{div}\varphi_\psi(x)\,dx + \int_\Omega \varphi_\psi(x)\cdot d(D(\theta(x,u(x)))),$$

where we have used Lemma 3.1. On the other hand, since by (3.1) $\varphi_\psi(x) = n(x,\Omega)\,\psi(x)$ if $x \in \partial\Omega \cap U_j$, while $\varphi_\psi(x) = 0$ if $x \in \partial\Omega\backslash U_j$, we get

$$\int_{\partial\Omega} \varphi_\psi(x)\cdot n(x,\Omega)\,\theta(x,T\,u(x))\,dH_{N-1}(x) = \int_{\partial\Omega} \psi(x)\theta(x,T\,u(x))\,dH_{N-1}(x).$$

Hence

$$\int_{\partial\Omega} \theta(x,T\,u(x))\,dH_{N-1}(x) = \sum_{\psi\in\Psi}\int_{\partial\Omega} \psi(x)\theta(x,T\,u(x))\,dH_{N-1}(x)$$
$$= \int_\Omega \theta(x,u(x))\operatorname{div}\left(\sum_{\psi\in\Psi}\varphi_\psi(x)\right)dx + \int_\Omega\left(\sum_{\psi\in\Psi}\varphi_\psi(x)\right)\cdot d(D(\theta(x,u(x)))).$$

The proof of Lemma 3.2 is complete if we show that $\varphi(x) := \sum_{\psi\in\Psi} \varphi_\psi(x)$ satisfies $|\varphi(x)| < 1$ in Ω. Fix $x \in \Omega$. If $x \notin \cup_{j=1}^P U_j$ then $\varphi(x) = 0$. If $x \in \cup_{j=1}^P U_j$ then $\sum_{\psi\in\Psi} \psi(x) = 1$, and so there exists at least one $\psi_0 \in \Psi$ such that $\psi_0(x) > 0$. Let $j \in \{1, \cdots, P\}$ be such that $\psi_0 \in C_0^\infty(U_j)$. Then by (3.1) and (3.2)

$$|\varphi_{\psi_0}(x)| = \left(1 + \frac{(\Phi_j(x))_N}{R_j}\right) \psi_0(x) < \psi_0(x),$$

and consequently, since $\psi(x) = 0$ for all but finitely many $\psi \in \Psi$,

$$|\varphi(x)| \le \sum_{\psi\in\Psi} |\varphi_\psi(x)| < \sum_{\psi\in\Psi} \psi(x) = 1.$$

■

By Lemmas 3.1 and 3.2, for any $u \in W^{1,1}(\Omega; \mathbb{R}^p)$ we can rewrite the functional $H(u, \Omega)$ as

$$\begin{aligned} H(u,\Omega) = &\int_\Omega \{h(x,u(x),\nabla u(x)) + \varphi(x)\cdot\nabla u^T(x)\nabla_u\theta(x,u(x))\}\,dx \\ &+ \int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot\nabla_x\theta(x,u(x))\,dx. \end{aligned} \tag{3.3}$$

This equivalent form gives us a better insight into the competing roles played by the two energy integrals $\int_\Omega h(x,u,\nabla u)\,dx$ and $\int_{\partial\Omega} \theta(x, T\,u)\,dH_{N-1}$. In particular, it is now clear that without a condition of the type

$$h(x,u,\xi) \ge |\nabla_u\theta(x,u)|\,|\xi|$$

one may have $\mathcal{H}(u,\Omega) = -\infty$, as in the example in Remark 2.4(ii).

Define $f(x,u,\xi) := h(x,u,\xi) + \varphi(x)\cdot\xi^T\nabla_u\theta(x,u)$ for $(x,u,\xi) \in \Omega \times \mathbb{R}^p \times \mathbb{M}^{p\times N}$, set

$$F(u,\Omega) := \int_\Omega f(x,u(x),\nabla u(x))\,dx, \quad u \in W^{1,1}(\Omega;\mathbb{R}^p),$$

and let

$$\mathcal{F}(u,\Omega) := \inf_{\{u_n\}} \left\{\liminf_{n\to\infty} F(u_n,\Omega) : u_n \in W^{1,1}(\Omega;\mathbb{R}^p),\ u_n \to u \quad \text{in} \quad L^1(\Omega;\mathbb{R}^p)\right\}.$$

Lemma 3.3. *If $u \in BV(\Omega; \mathbb{R}^p)$ then*

$$\mathcal{H}(u,\Omega) = \mathcal{F}(u,\Omega) + \int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot\nabla_x\theta(x,u(x))\,dx.$$

Proof. Clearly it is enough to show that

$$\liminf_{n\to\infty} H(u_n,\Omega) = \liminf_{n\to\infty} F(u_n,\Omega) + \int_\Omega \theta(x,u(x))\operatorname{div}\varphi(x)\,dx + \int_\Omega \varphi(x)\cdot\nabla_x\theta(x,u(x))\,dx$$

for any sequence $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^p)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^p)$. We first observe that, since $\varphi \in C_0^1(\mathbb{R}^N;\mathbb{R}^N)$, the functions φ and $\operatorname{div}\varphi$ are bounded in Ω. Moreover, by (2.5)

$$|\theta(x,u_n(x)) - \theta(x,u(x))| \le ||g||_{L^\infty}\,|u_n(x) - u(x)| \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega.$$

Hence

$$\lim_{n\to\infty} \int_\Omega \theta(x,u_n)\operatorname{div}\varphi\,dx = \int_\Omega \theta(x,u)\operatorname{div}\varphi\,dx.$$

By (H_6), by virtue of the Sobolev inequality, and due to the fact that φ is bounded, the functional $u \mapsto \int_\Omega \varphi \cdot \nabla_x \theta(x, u(x))\, dx$ is continuous in $L^1(\Omega; \mathbb{R}^p)$ (see [K, Th. 2.1]) and thus

$$\lim_{n\to\infty} \int_\Omega \varphi \cdot \nabla_x \theta(x, u_n)\, dx = \int_\Omega \varphi(x) \cdot \nabla_x \theta(x, u)\, dx.$$

■

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Lemma 3.3, in order to find an integral representation for $\mathcal{H}(u, \Omega)$ in $BV(\Omega; \mathbb{R}^p)$ it is sufficient to determine one for $\mathcal{F}(u, \Omega)$. The idea is to apply Theorem 2.16 of [FM2]. In order to do so we need to show that the function

$$f(x, u, \xi) = h(x, u, \xi) + \varphi(x) \cdot \xi^T \nabla_u \theta(x, u)$$

satisfies conditions $(H_1) - (H_5)$ which are essentially the same of [FM2].

Condition (H_1) is trivially verified since the functions θ and φ are of class C^1. As f is the sum of a quasiconvex function and a function linear in ξ, it is clear that $f(x, u, \cdot)$ is still quasiconvex and that

$$f^\infty(x, u, \xi) = h^\infty(x, u, \xi) + \varphi(x) \cdot \xi^T \nabla_u \theta(x, u),$$

which, in turn, implies that

$$|f^\infty(x, u, \xi) - f(x, u, \xi)| = |h^\infty(x, u, \xi) - h(x, u, \xi)| \le C' g(x, u)(1 + |\xi|^{1-m})$$

by (H_5). Thus f verifies also (H_2) and (H_5).

To prove (2.2) for f, consider a compact set let $K \Subset \Omega \times \mathbb{R}^p$. Applying (2.2) to the function h, we have

$$\begin{aligned} |f(x, u, \xi) - f(x_1, u_1, \xi)| &\le \omega(|x - x_1| + |u - u_1|)(1 + |\xi|) \\ &\quad + |\varphi(x) \otimes \nabla_u \theta(x, u) - \varphi(x_1) \otimes \nabla_u \theta(x_1, u_1)|\, |\xi| \end{aligned}$$

for all (x, u, ξ), $(x_1, u_1, \xi) \in K \times \mathbb{M}^{p\times N}$. There exist a compact set $K_1 \Subset \Omega$ and a ball $B_p(0, R) \subset \mathbb{R}^p$ such that $K \subset K_1 \times B_p(0, R)$, and, without loss of generality, we may assume that $\Omega \subset B_N(0, R) \subset \mathbb{R}^N$. Take

$$\begin{aligned} \tilde{\omega}(s) := \max\bigl\{ |\varphi(x) \otimes \nabla_u \theta(x, u) - \varphi(x_1) \otimes \nabla_u \theta(x_1, u_1)| : \ &|x - x_1| + |u - u_1| \le s, \\ &x,\ x_1 \in K_1,\ |u|,\ |u_1| \le R \bigr\}. \end{aligned}$$

The function $\tilde{\omega}(s)$ is non decreasing, with $\lim_{s\to 0} \tilde{\omega}(s) = 0 = \tilde{\omega}(0)$ and $\tilde{\omega}(4R) < \infty$. Therefore we can find ω_1, continuous and non decreasing, such that $\omega_1(s) \ge \tilde{\omega}(s)$ for all $s \in [0, 4R]$ and $\omega_1(0) = 0$. Condition (2.2) for f now follows by taking $\omega_0 := \omega + \omega_1$.

Next we prove (2.3). Fix $x_0 \in \Omega$ and $\delta > 0$. There exists $\varepsilon > 0$ such that for $x \in \Omega$ with $|x - x_0| \le \varepsilon$ and $(u, \xi) \in \mathbb{R}^p \times \mathbb{M}^{p\times N}$

$$\begin{aligned} h(x_0, u, \xi) - h(x, u, \xi) \le \tfrac{1}{3}\delta(1 + g(x, u)|\xi|), \qquad &|\varphi(x) - \varphi(x_0)| \le \tfrac{1}{3}\delta, \\ |\nabla_u \theta(x_0, u) - \nabla_u \theta(x, u)| \le \tfrac{1}{3}\delta\, g(x, u) \end{aligned}$$

by the continuity of φ, (2.3) and (2.4). Hence

$$\begin{aligned} f(x_0, u, \xi) &= h(x_0, u, \xi) + \varphi(x_0) \cdot \xi^T \nabla_u \theta(x_0, u) \\ &= f(x, u, \xi) + h(x_0, u, \xi) - h(x, u, \xi) \\ &\quad + [\varphi(x_0) - \varphi(x)] \cdot \xi^T \nabla_u \theta(x, u) + \varphi(x_0) \cdot \xi^T [\nabla_u \theta(x_0, u) - \nabla_u \theta(x, u)] \\ &\le f(x, u, \xi) + \tfrac{1}{3}\delta(1 + g(x, u)|\xi|) + \tfrac{2}{3}\delta\, g(x, u)|\xi| \end{aligned}$$

which is (2.3), and where we have used (H7) and the fact that $|\varphi(x_0)| \leq 1$.

Finally, condition (H_3) is replaced by the condition

$$g(x,u)|\xi|(1-|\varphi(x)|) \leq f(x,u,\xi) \leq 2C\,g(x,u)(1+|\xi|) \tag{3.4}$$

which follows from (2.1) and (H_7). Although (3.4) is weaker than condition (H_3) in [FM2], the proof there carries out even with (3.4). Indeed, condition (H_3) was used in [FM2] only to show that

$$\mathcal{F}(u,\Omega) \geq \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,u,dC(u)) + \int_{S(u)\cap\Omega} K_f(x,u^-,u^+,\nu_u)\,dH_{N-1}(x).$$

The proof of this inequality relies on the blow-up argument introduced in [FM1] which is a *local* argument, in the sense that in order to prove the three main pointwise inequalities (2.10)–(2.11) in [FM2] at points $x_0 \in \Omega$, one is only interested in what happens in a ball $B(x_0,\varepsilon)$. Since in our case $|\varphi(x_0)| < \epsilon_0 < 1$ for some $\epsilon_0 > 0$, if we take ε sufficiently small we can assume that $|\varphi(x)| \leq \epsilon_0$ for all $x \in B(x_0,\varepsilon)$ and thus (3.4) reduces to

$$g(x,u)|\xi|(1-\epsilon_0) \leq f(x,u,\xi) \leq 2C\,g(x,u)(1+|\xi|)$$

for all $(x,u,\xi) \in B(x_0,\varepsilon) \times \mathbb{R}^p \times \mathbb{M}^{p\times N}$, which is the *local* version of (H_3) in [FM2].

In conclusion, we may apply Theorem 2.16 of [FM2] (see Remark 3.5 below) to obtain that for $u \in BV(\Omega;\mathbb{R}^p)$

$$\begin{aligned}\mathcal{F}(u,\Omega) = &\int_\Omega \left\{h(x,u,\nabla u) + \varphi\cdot\nabla u^T\nabla_u\theta(x,u)\right\}\,dx + \int_\Omega h^\infty(x,u,dC(u))\\ &+ \int_\Omega (\varphi\otimes\nabla_u\theta(x,u))\cdot dC^T(u) + \int_{S(u)\cap\Omega} K_f(x,u^-,u^+,\nu_u)\,dH_{N-1},\end{aligned}$$

where

$$K_f(x,a,b,\nu) = \inf\left\{\int_{Q_\nu}\left[h^\infty(x,\psi(y),\nabla\psi(y)) + \varphi(x)\cdot\nabla\psi^T(y)\nabla_u\theta(x,\psi(y))\right]dy : \psi\in\mathcal{A}(a,b,\nu)\right\}.$$

Given any $\psi \in \mathcal{A}(a,b,\nu)$ we have

$$\begin{aligned}\int_{Q_\nu}\varphi(x)\cdot\nabla\psi^T(y)\nabla_u\theta(x,\psi(y))\,dy &= \int_{\partial Q_\nu}\varphi(x)\cdot n(y,Q_\nu)\,\theta(x,T\psi(y))\,dH_{N-1}(y)\\ &= \varphi(x)\cdot(\theta(x,b)-\theta(x,a))\,\nu,\end{aligned}$$

and so

$$K_f(x,a,b,\nu) = K_h(x,a,b,\nu) + \varphi(x)\cdot(\theta(x,b)-\theta(x,a))\,\nu.$$

If we now use Lemmas 3.1, 3.2 and 3.3, we finally obtain that $\mathcal{H}(u,\Omega) = \mathcal{L}(u,\Omega)$. This concludes the proof of Theorem 2.2.

■

Remark 3.4. The continuity hypotheses (H_1), (H_4) and (2.4) may be replaced by

$(H_1)'$ h is Carathéodory;

$(H_4)'$ for all $(x_0,u_0) \in \Omega\times\mathbb{R}^p$ and for all $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|h(x,u_1,\xi) - h(x,u_2,\xi)| \leq \varepsilon(1+|\xi|)$$

for all $x \in \Omega$ with $|x-x_0| \leq \varepsilon$, u_1, $u_2 \in B(u_0,\varepsilon)$ and $\xi \in \mathbb{M}^{p\times N}$;

provided $\xi \mapsto f(x,u,\cdot)$ is coercive (e.g. if $g(x,u) \geq \alpha > ||\nabla_u \theta||_{L^\infty}$ for some $\alpha > 0$). In this case, in Theorem 2.2 we would use the integral representation obtained by Bouchitté, Fonseca and Mascarenhas [BFM] in place of the corresponding result by Fonseca and Müller [FM2].

Proof of Theorem 2.5. By Lemma 3.3 it is enough to find an integral representation for the corresponding $\mathcal{F}_b(u,\Omega)$ in $BV(\Omega;\mathbb{R}^p)$. Let $f_\varepsilon(x,u,\xi) := f(x,u,\xi) + \varepsilon|\xi|$, for $\varepsilon \in (0,1)$, where, as before, $f(x,u,\xi) := h(x,u,\xi) + \varphi(x)\cdot \xi^T \nabla_u \theta(x,u)$, and define

$$\mathcal{F}_\varepsilon(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F_\varepsilon(u_n,\Omega) : \, u_n \in W^{1,1}(\Omega;\mathbb{R}^p),\, u_n \to u \quad \text{in} \quad L^1(\Omega;\mathbb{R}^p) \right\},$$

where

$$F_\varepsilon(u,\Omega) := \int_\Omega f_\varepsilon(x,u(x),\nabla u(x))\, dx, \qquad u \in W^{1,1}(\Omega;\mathbb{R}^p).$$

We claim that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,\Omega) = \mathcal{F}_b(u,\Omega).$$

Fix $u \in L^1(\Omega;\mathbb{R}^p)$. For any given $\delta > 0$ there exists a sequence $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^p)$, with $\sup_n ||u_n||_{W^{1,1}} = M < \infty$, such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^p)$ and

$$\mathcal{F}_b(u,\Omega) + \delta \geq \lim_{n\to\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

In turn, for all $\varepsilon > 0$

$$\mathcal{F}_b(u,\Omega) + \delta \geq \liminf_{n\to\infty} \int_\Omega f_\varepsilon(x,u_n(x),\nabla u_n(x))\, dx - \varepsilon M,$$

and using the definition of $\mathcal{F}_\varepsilon(u,\Omega)$ we obtain

$$\mathcal{F}_b(u,\Omega) + \delta \geq \mathcal{F}_\varepsilon(u,\Omega) - \varepsilon M.$$

Therefore

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,\Omega) \leq \mathcal{F}_b(u,\Omega) + \delta,$$

and it suffices to let $\delta \to 0$ to conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,\Omega) \leq \mathcal{F}_b(u,\Omega).$$

Conversely, fix $u \in L^1(\Omega;\mathbb{R}^p)$ and $\varepsilon > 0$. Then there exists a sequence $\{u_n^\varepsilon\} \subset W^{1,1}(\Omega;\mathbb{R}^p)$ such that $u_n^\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^p)$ as $n \to \infty$ and

$$\mathcal{F}_\varepsilon(u,\Omega) + \varepsilon \geq \lim_{n\to\infty} \int_\Omega [f(x,u_n^\varepsilon,\nabla u_n^\varepsilon) + \varepsilon|\nabla u_n^\varepsilon|]\, dx \geq \liminf_{n\to\infty} \int_\Omega f(x,u_n^\varepsilon,\nabla u_n^\varepsilon)\, dx. \tag{3.5}$$

Without loss of generality we can assume that $\mathcal{F}_\varepsilon(u,\Omega) < \infty$. Since $|\varphi(x)| < 1$ in Ω, by (2.8) we have

$$f(x,u,\xi) = h(x,u,\xi) + \varphi(x)\cdot \xi^T \nabla_u \theta(x,u) \geq 0,$$

hence by (3.5) it follows that $\sup_n ||u_n^\varepsilon||_{W^{1,1}} < \infty$ and so

$$\mathcal{F}_\varepsilon(u,\Omega) + \varepsilon \geq \mathcal{F}_b(u,\Omega).$$

We conclude that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,\Omega) \geq \mathcal{F}_b(u,\Omega)$$

and the claim is proven.

It is not difficult to show that the function $f_\varepsilon(x,u,\xi)$ satisfies conditions $(H_1)-(H_5)$. We omit the details since the proof is very similar to that of Theorem 2.2.

By Theorem 2.16 of [FM2] we obtain that for $u \in BV(\Omega;\mathbb{R}^p)$

$$\begin{aligned}\mathcal{F}_\varepsilon(u,\Omega) = \int_\Omega \{f(x,u,\nabla u)+\varepsilon|\nabla u|\}\,dx + \int_\Omega f^\infty(x,u,dC(u)) + \varepsilon\int_\Omega |dC(u)| \\ + \int_{S(u)\cap\Omega} K_{f_\varepsilon}(x,u^-,u^+,\nu_u)\,dH_{N-1}.\end{aligned} \tag{3.6}$$

If we let $\varepsilon \to 0$ in (3.6) we obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u,\Omega) = \int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,u,dC(u)) \\ + \lim_{\varepsilon\to 0}\int_{S(u)\cap\Omega} K_{f_\varepsilon}(x,u^-,u^+,\nu_u)\,dH_{N-1},\end{aligned}$$

and so the proof is completed provided we show that

$$\lim_{\varepsilon\to 0}\int_{S(u)\cap\Omega} K_{f_\varepsilon}(x,u^-,u^+,\nu_u)\,dH_{N-1} = \int_{S(u)\cap\Omega} K_f(x,u^-,u^+,\nu_u)\,dH_{N-1}. \tag{3.7}$$

We first prove the pointwise convergence

$$\lim_{\varepsilon\to 0} K_{f_\varepsilon}(x,a,b,\nu) = K_f(x,a,b,\nu) \tag{3.8}$$

for all $x\in\Omega$, a, $b\in\mathbb{R}^p$ and $\nu\in S^{N-1}$. For any fixed $\delta>0$ there exists $\psi_\delta\in\mathcal{A}(a,b,\nu)$ such that for all $\varepsilon>0$

$$K_f(x,a,b,\nu)+\delta \ge \int_{Q_\nu} f^\infty(x,\psi_\delta(y),\nabla\psi_\delta(y))\,dy \ge K_{f_\varepsilon}(x,a,b,\nu) - \varepsilon\int_{Q_\nu}|\nabla\psi_\delta(y)|\,dy.$$

If now we let $\varepsilon\to 0$ in the previous inequality we get

$$K_f(x,a,b,\nu)+\delta \ge \limsup_{\varepsilon\to 0} K_{f_\varepsilon}(x,a,b,\nu),$$

and by letting $\delta\to 0$ we deduce that

$$K_f(x,a,b,\nu) \ge \limsup_{\varepsilon\to 0} K_{f_\varepsilon}(x,a,b,\nu).$$

Conversely, let $\psi\in\mathcal{A}(a,b,\nu)$. Then

$$K_f(x,a,b,\nu) \le \int_{Q_\nu} f^\infty(x,\psi(y),\nabla\psi(y))\,dy \le \int_{Q_\nu} f_\varepsilon^\infty(x,\psi(y),\nabla\psi(y))\,dy,$$

where we used the fact that $f^\infty \le f_\varepsilon^\infty$. Taking the infimum over all $\psi\in\mathcal{A}(a,b,\nu)$, we get

$$K_f(x,a,b,\nu) \le K_{f_\varepsilon}(x,a,b,\nu).$$

Therefore (3.8) holds.

As it can be seen from the proof of Lemma 2.15 in [FM2], we may find a constant C_1 independent of ε such that

$$0 \le K_{f_\varepsilon}(x,a,b,\nu) \le C_1|a-b|$$

and hence (3.7) follows by Lebesgue Dominated Convergence Theorem and (3.8). This concludes the proof of the theorem.

■

Remark 3.5. In the proof of Theorem 2.16 of [FM2] the inequality

$$\mathcal{F}(u, S(u) \cap \Omega) \leq \int_{S(u)\cap\Omega} K_f(x, u^-, u^+, \nu_u)\, dH_{N-1}(x) \tag{3.9}$$

was derived by using a result of [FR] which requires the function $f(x, u, \cdot)$ to be coercive, that is to satisfy the inequality

$$f(x, u, \xi) \geq c_1|\xi| - c_2 \tag{3.10}$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^p \times \mathbb{M}^{p\times N}$, which is stronger than condition (H_3). To circumvent this difficulty consider the function $f_\varepsilon(x, u, \xi)$ defined as in the proof of Theorem 2.5. Since it satisfies conditions $(H_1) - (H_5)$ and (3.10), the inequality (3.9) holds for f_ε. Also the inequality $f \leq f_\varepsilon$ clearly implies that

$$\mathcal{F}(u, S(u) \cap \Omega) \leq \mathcal{F}_\varepsilon(u, S(u) \cap \Omega) \leq \int_{S(u)\cap\Omega} K_{f_\varepsilon}(x, u^-, u^+, \nu_u)\, dH_{N-1}(x).$$

If we now let $\varepsilon \to 0$ and use (3.7) we conclude that (3.9) holds also for f.

§4. Mesoscopic scale.

We are interested in the following constrained minimization problem

$$\inf\left\{\mathcal{H}(u, \Omega) + \int_\Omega \tau(x, u(x))\, dx \,:\, u \in BV(\Omega; \mathbb{R}^p),\ u(x) \in K \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega\right\},$$

where K is a nonempty compact set of $\mathbb{R}^p$, and $\tau : \Omega \times K \to \mathbb{R}$ is a Carathéodory function such that

$$|\tau(x, u)| \leq a_0(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \text{ and for all } u \in K, \tag{4.1}$$

for some function $a_0 \in L^1(\Omega; \mathbb{R})$. In applications in phase transitions, often $K = \{a, b\}$ or K is a convex set.

For $u \in L^1(\Omega; \mathbb{R}^p)$ we define the functional

$$\mathcal{I}(u, \Omega) := \mathcal{H}(u, \Omega) + I_K(u, \Omega),$$

where

$$I_K(u, \Omega) = \begin{cases} \int_\Omega \tau(x, u(x))\, dx & \text{if } u(x) \in K \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 4.1. *If $(H_1) - (H_7)$ hold then the functional $\mathcal{I}(u, \Omega)$ is lower semicontinuous in $L^1(\Omega; \mathbb{R}^p)$.*

Proof. Consider u_n, $u \in L^1(\Omega; \mathbb{R}^p)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$. If $\liminf_{n\to\infty} \mathcal{I}(u_n, \Omega) = \infty$ there is nothing to prove. Assume that $\liminf_{n\to\infty} \mathcal{I}(u_n, \Omega) < \infty$ and take a subsequence $\{u_{n_k}\}$ which converges pointwise to u for $\mathcal{L}^N$ a.e $x \in \Omega$, and such that

$$\lim_{k\to\infty} \mathcal{I}(u_{n_k}, \Omega) = \liminf_{n\to\infty} \mathcal{I}(u_n, \Omega) < \infty.$$

For k sufficiently large we can assume that $\mathcal{I}(u_{n_k}, \Omega) < \infty$, hence

$$\mathcal{I}(u_{n_k}, \Omega) = \mathcal{H}(u_{n_k}, \Omega) + \int_\Omega \tau(x, u_{n_k}(x))\, dx$$

and $u_{n_k}(x) \in K$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Since $\{u_{n_k}\}$ converges pointwise to u for $\mathcal{L}^N$ a.e $x \in \Omega$, we obtain that $u(x) \in K$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. In turn $\mathcal{I}(u,\Omega) = \mathcal{H}(u,\Omega) + \int_\Omega \tau(x,u(x))\,dx$. The assertion now follows from the lower semicontinuity of $\mathcal{H}(u,\Omega)$ in $L^1(\Omega;\mathbb{R}^p)$ and the fact that

$$\lim_{k\to\infty} \int_\Omega \tau(x,u_{n_k}(x))\,dx = \int_\Omega \tau(x,u(x))\,dx$$

by (4.1) and by Lebesgue Dominated Convergence Theorem.

■

In addition to conditions $(H_1) - (H_7)$ we now assume the following hypotheses:
(F_1) there exist a function $\rho \in C(\overline{\Omega} \times \mathbb{R}^p;\mathbb{R}^p) \cap C^1(\Omega \times \mathbb{R}^p;\mathbb{R}^p)$ and a function $b \in L^1(\Omega;\mathbb{R})$ such that

$$|\nabla_x \rho(x,u)| \le b(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \text{ and for all } u \in K, \tag{4.2}$$

and

$$h^\infty(x,u,\xi) \ge |\nabla_u \rho(x,u)|\,|\xi| \tag{4.3}$$

for all $(x,u,\xi) \in \Omega \times K \times \mathbb{M}^{p\times N}$;
(F_2) for $\mathcal{L}^N$ a.e. $x \in \Omega$ the function $\rho(x,\cdot) : K \Subset \mathbb{R}^p \to \rho(x,K) \Subset \mathbb{R}^p$ is invertible and $(x,y) \mapsto (\rho(x,\cdot))^{-1}(y)$ is Carathéodory. In addition, there exists a function $c \in L^1(\Omega;\mathbb{R})$ such that

$$|\rho(x,\cdot)^{-1}(v)| \le c(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \text{ and for all } v \in \rho(x,K). \tag{4.4}$$

Let

$$D(\mathcal{I}) := \{u \in L^1(\Omega;\mathbb{R}^p) : \mathcal{I}(u,\Omega) < \infty\}.$$

Then

$$D_1 := \{u \in BV(\Omega;\mathbb{R}^p) : u(x) \in K \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega\} \subset D(\mathcal{I})$$

but in general the two sets do not coincide, unless one assumes that $h(x,u,\cdot)$ is coercive.

Theorem 4.2. *There exists a function $u \in D(\mathcal{I})$ such that*

$$\mathcal{I}(u,\Omega) \le \inf\{\mathcal{I}(w,\Omega) : w \in D_1\}.$$

Proof. Let $\{u_n\} \subset D_1$ be a minimizing sequence, that is

$$\lim_{n\to\infty} \mathcal{I}(u_n,\Omega) = \inf\{\mathcal{I}(w,\Omega) : w \in D_1\} < M < \infty.$$

Then, for n sufficiently large,

$$\mathcal{I}(u_n,\Omega) = \mathcal{H}(u_n,\Omega) + \int_\Omega \tau(x,u_n(x))\,dx \le M. \tag{4.5}$$

We claim that $T\,u_n(x) \in K$ for H_{N-1} a.e. $x \in \partial\Omega$. Indeed let $E_n := \{x \in \partial\Omega : T\,u_n(x) \notin K\}$ and suppose for contradiction that $H_{N-1}(E_n) > 0$. Take $x_0 \in E_n$ for which (cf. [Z,Th. 5.14.4])

$$\lim_{r\to 0} \frac{1}{\text{meas}\,(B(x_0,r)\cap\Omega)} \int_{B(x_0,r)\cap\Omega} |u_n(x) - T\,u_n(x_0)|^{N/(N-1)} dx = 0.$$

Since K is compact we have $\text{dist}(T\,u_n(x_0),K) = \varepsilon_0 > 0$, while from the fact that $u_n(x) \in K$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, it follows that

$$\varepsilon_0^{N/(N-1)} \le |u_n(x) - T\,u_n(x_0)|^{N/(N-1)}$$

for $\mathcal{L}^N$ a.e. $x \in B(x_0, r) \cap \Omega$. Taking the average over $B(x_0, r) \cap \Omega$ and letting $r \to 0$, we get a contradiction. Therefore the claim holds, and by (4.5), Theorem 2.2, and (2.6) we have

$$\begin{aligned} &\int_\Omega h(x, u_n, \nabla u_n)\, dx + \int_\Omega h^\infty(x, u_n, dC(u_n)) \\ &\qquad + \int_{S(u_n)\cap\Omega} K_h(x, u_n^-, u_n^+, \nu_{u_n})\, dH_{N-1} \le M_1, \end{aligned} \tag{4.6}$$

for some constant M_1 independent of n. By (H_5), $(2.1)_1$ and(4.6)

$$\begin{aligned} \int_\Omega h^\infty(x, u_n, \nabla u_n)\, dx &\le \int_\Omega (h^\infty(x, u_n, \nabla u_n) - h(x, u_n, \nabla u_n))\, dx + M_1 \\ &\le C'\|g\|_{L^\infty} + C'\|g\|_{L^\infty}^m \int_\Omega h^{1-m}(x, u_n, \nabla u_n)\, dx + M_1. \end{aligned}$$

Using Hölder's inequality and (4.6) again, we conclude that there exists $M_2 \in (0, \infty)$ such that for all n

$$\begin{aligned} &\int_\Omega h^\infty(x, u_n, \nabla u_n)\, dx + \int_\Omega h^\infty(x, u_n, dC(u_n)) \\ &\qquad + \int_{S(u_n)\cap\Omega} K_h(x, u_n^-, u_n^+, \nu_{u_n})\, dH_{N-1} \le M_2. \end{aligned} \tag{4.7}$$

Define $v_n := \rho(x, u_n(x))$. As in Lemma 3.1 we can show that $v_n(x) \in BV(\Omega; \mathbb{R}^p)$ with

$$Dv_n = \begin{cases} \nabla_x \rho(x, u_n)\, \mathcal{L}^N + \nabla_u \rho(x, u_n) Du_n & \text{on } \Omega \backslash S(u_n) \\ (\rho(x, u_n^+) - \rho(x, u_n^-)) \otimes \nu_{u_n}\, H_{N-1} \lfloor S(u_n) & \text{on } S(u_n). \end{cases} \tag{4.8}$$

Furthermore

$$\int_\Omega |Dv_n| = \int_\Omega |\nabla v_n|\, dx + \int_\Omega |\nabla_u \rho(x, u_n)\, dC(u_n)| + \int_{S(v_n)\cap\Omega} |(v_n^+ - v_n^-) \otimes \nu_{v_n}|\, dH_{N-1}. \tag{4.9}$$

By Remark 2.17 in [FM2] and the fact that $S(v_n) = S(u_n)$ and $\nu_{v_n} = \nu_{u_n}$, we can rewrite the last integral as

$$\int_{S(u_n)\cap\Omega} K_{|\cdot|}(x, v_n^-, v_n^+, \nu_{u_n})\, dH_{N-1}$$

where

$$K_{|\cdot|}(x, v_n^-, v_n^+, \nu_{u_n}) := \inf\left\{ \int_{Q_{\nu_{u_n}}} |\nabla \psi(y)|\, dy : \psi \in \mathcal{A}(v_n^-(x), v_n^+(x), \nu_{u_n}) \right\}.$$

Given $\eta \in \mathcal{A}(u_n^-(x), u_n^+(x), \nu_{u_n})$, the function $\psi(y) := \rho(x, \eta(y)) \in \mathcal{A}(v_n^-(x), v_n^+(x), \nu_{u_n})$ and $\nabla \psi(y) = \nabla_u \rho(x, \eta(y)) \nabla \eta(y)$. By (4.3) this implies that

$$\begin{aligned} K_{|\cdot|}(x, v_n^-, v_n^+, \nu_{u_n}) &\le \inf\left\{ \int_{Q_{\nu_{u_n}}} |\nabla_u \rho(x, \eta(y))|\, |\nabla \eta(y)|\, dy : \eta \in \mathcal{A}(u_n^-(x), u_n^+(x), \nu_{u_n}) \right\} \\ &\le K_h(x, u_n^-, u_n^+, \nu_{u_n}). \end{aligned}$$

Therefore, also by (4.2), (4.3), (4.7), (4.8) and (4.9)

$$\begin{aligned} \int_\Omega |Dv_n| &\le \int_\Omega |\nabla_x \rho(x, u_n)|\, dx + \int_\Omega |\nabla_u \rho(x, u_n)|\, |\nabla u_n|\, dx + \int_\Omega |\nabla_u \rho(x, u_n)|\, |dC(u_n)| \\ &\quad + \int_{S(u_n)\cap\Omega} K_h(x, u_n^-, u_n^+, \nu_{u_n}) dH_{N-1} \\ &\le \|b\|_{L^1} + M_2. \end{aligned} \tag{4.10}$$

Finally, since $v_n(x) \in \rho(x,K)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, $\rho(x,K)$ is a compact set of $\mathbb{R}^p$, and by (4.10), there exists a subsequence, still denoted $\{v_n\}$, which converges strongly in $L^1(\Omega;\mathbb{R}^p)$ and pointwise almost everywhere to a function $v \in BV(\Omega;\mathbb{R}^p)$ (see [Z, Cor. 5.3.4]), with $v(x) \in \rho(x,K)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Define $u(x) := (\rho(x,\cdot))^{-1}(v(x))$. By (F_2) the function u is measurable. Since $u_n(x) = (\rho(x,\cdot))^{-1}(v_n(x))$ it follows that $u_n(x) \to u(x)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, thus $u(x) \in K$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Moreover, by (4.4) we have that $|u_n(x)| \le c(x)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, therefore by Lebesgue Dominated Convergence Theorem $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^p)$. By Lemma 4.1 we conclude that

$$\mathcal{I}(u,\Omega) \le \inf\{\mathcal{I}(u,\Omega) : u \in D_1\}.$$

■

Corollary 4.3. *Assume that conditions (F_1) and (F_2) in Theorem 4.2 are replaced by the assumption*

$$h^\infty(x,u,\xi) \ge \alpha|\xi| \qquad \text{for all } (x,u,\xi) \in \Omega \times K \times \mathbb{M}^{p\times N},$$

for some $\alpha > 0$. Then $D_1 = D(\mathcal{I})$ and there exists a function $u \in D_1$ such that

$$\mathcal{I}(u,\Omega) = \inf\{\mathcal{I}(w,\Omega) : w \in D_1\}.$$

Proof. It suffices to take $\rho(x,u) := \alpha\, u$ in Theorem 4.2. Then $v_n = \alpha\, u_n$ converge strongly in $L^1(\Omega;\mathbb{R}^p)$ to a function $v \in BV(\Omega;\mathbb{R}^p)$, and therefore $u := \frac{1}{\alpha} v$ is the desired minimizer.

■

§5. Nucleation: the scalar case.

In this section we study the constrained minimization problem introduced in Section 4, restricted to the scalar case $p = 1$, when K is a closed, connected subset of $\mathbb{R}$ (not necessarily bounded), and when the potential $\tau(x,u)$ is given by

$$\tau(x,u) := \tau_1(x,u) + \psi(x)\,\tau_2(u),$$

where $\tau_1(x,u)$ is a Carathéodory function, concave in the u variable, ψ is a nonnegative, measurable function, and τ_2 is a continuous function such that

(5.1) all the connected components of $S := \{u \in \operatorname{int} K : \tau_2^{**}(u) < \tau_2(u)\}$ are bounded,

where τ_2^{**} is the convex envelope of τ_2. As remarked in [V2], (5.1) holds if

$$\limsup_{|u|\to\infty} \frac{\tau_2(u)}{|u|} = \infty.$$

Furthermore we assume that

(5.2) $\tau(x,u) \ge -L_1 - L_2|u|$ for $\mathcal{L}^N$ a.e. $x \in \Omega$ and for all $u \in K$,

for some $L_1, L_2 > 0$.

Under appropriate assumptions on the functions h and θ, we prove that minimizers $u \in L^1(\Omega;\mathbb{R}^p)$ of

$$\mathcal{I} : v \in L^1(\Omega;K) \mapsto \mathcal{H}(v,\Omega) + \int_\Omega \tau(x,v(x))\,dx$$

have the phase structure

$$u(\Omega) \subset K\backslash S.$$

In particular, if $K = [a,b]$, if τ_2 is concave in $[a,b]$, and if u is a minimizer of $\mathcal{I}$, then u must have a 2-phase structure, i.e. there exists a set $\Omega_0 \subset \Omega$ such that $u(x) = a$ for $\mathcal{L}^N$ a.e. $x \in \Omega_0$ and $u(x) = b$ for $\mathcal{L}^N$ a.e. $x \in \Omega\backslash\Omega_0$. This result has important applications in *nucleation phenomena* which have been studied extensively by Visintin in [V1,V2], where usually K is bounded, $\tau_1(x,u) := -\xi(x)\,u$, $\xi \in L^\infty(\Omega;\mathbb{R})$ is proportional to the relative temperature, and $\psi(x)\,\tau_2(u)$ is the double well potential $\psi(x)(b-u)(u-a)$ (see Remark 5.2 below). Given a simple function $u \in L^1(\Omega; K)$ of the form

$$u(x) = \sum_{i=1}^{k} c_i\,\chi_{\omega_i}(x), \tag{5.3}$$

with $c_i \in K$, $\mathcal{L}^N(\omega_i) > 0$ for all $i = 1, \cdots, k$, and $\mathcal{L}^N(\Omega\backslash \cup_{i=1}^k \omega_i) = 0$, without loss of generality we may assume that

$$\inf K \le c_1 < c_2 < \cdots < c_k \le \sup K. \tag{5.4}$$

Theorem 5.1. *Let $\mathcal{E}$ be an algebra of measurable subsets of Ω, and consider a functional $\mathcal{V} : L^1(\Omega;\mathbb{R}) \to [0,\infty]$ such that*

$$\mathcal{S}_1 := \left\{u \in L^1(\Omega;\mathbb{R}) : u = \sum_{i=1}^{k} c_i\,\chi_{\omega_i},\ \omega_i \in \mathcal{E}, k \in \mathbb{N}\right\} \subset D(\mathcal{V}) := \{u \in L^1(\Omega;\mathbb{R}) : \mathcal{V}(u) < \infty\},$$

and

(i) *for any $u \in D(\mathcal{V}) \cap L^1(\Omega; K)$ there exists a sequence $\{u_n\} \subset \mathcal{S}_1 \cap L^1(\Omega;K)$ converging to u in $L^1(\Omega;K)$ and such that*

$$\limsup_{n\to\infty} \mathcal{V}(u_n) \le \mathcal{V}(u).$$

(ii) *For any $u \in \mathcal{S}_1$ of the form* (5.3) – (5.4), *with $k \ge 2$, there holds*

$$\mathcal{V}(u) = \sum_{i=1}^{k-1}(c_{i+1} - c_i)\mathcal{V}(\chi_{\cup_{r=i+1}^k \omega_r}). \tag{5.5}$$

(iii) *The function $c \mapsto \mathcal{V}(c)$ is concave in K.*

In addition, suppose that the functional $u \mapsto \int_\Omega \tau(x,u(x))\,dx$ is continuous in $D(\mathcal{V}) \cap L^1(\Omega;K)$. Then

$$\inf\{\mathcal{V}(u) + I_K(u,\Omega) : u \in D(\mathcal{V})\} = \inf\{\mathcal{V}(u) + I_K(u,\Omega) : u \in D(\mathcal{V}),\ u(x) \in K\backslash S \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega\}.$$

Remark 5.2. (i) The functional $\mathcal{V}(u) + I_K(u,\Omega)$ is well defined by (5.2).

(ii) Theorem 5.1 is closely related to Theorem 2 in [V2], where $K = \mathbb{R}$ and conditions (i) and (ii) are replaced by the assumption that $\mathcal{V}$ satisfies the *generalized co-area formula*

$$\mathcal{V}(u) = \int_{\mathbb{R}} \mathcal{V}(\chi_{\{x\in\Omega:\,u(x)\ge t\}})\,dt. \tag{5.6}$$

It is easy to see that (5.6) reduces to (5.5) for functions u of the form (5.3)-(5.4). Therefore (5.5) is weaker than (5.6). On the other hand, conditions (i) and (5.6) do not seem to be related. Indeed, consider the functional

$$\mathcal{V}(u) := \int_\Omega |Du| + \begin{cases} \int_\Omega \max\{u(x),0\}dx & \text{if } H_{N-1}(S(u)\cap\Omega) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

From the proof of Theorem 5.4 below it follows that $\mathcal{V}$ satisfies hypotheses (i)—(iii) of Theorem 5.1. Take $u(x) := 1$ in (5.6); then $\mathcal{V}(1) = \mathcal{L}^N(\Omega)$, while the right hand side of (5.6) is infinite. Therefore (5.6) fails. We note that $\mathcal{V}$ is not lower semicontinuous in L^1.

We remark that Theorem 5.1 may be applied to a large class of functionals of the form (1.1), for which the co-area formula might not hold.

(iii) If, in addition to hypotheses (i)–(iii) in Theorem 5.1, we assume that $\mathcal{V}$ is lower semicontinuous in $L^1(\Omega;\mathbb{R})$, that $K = \mathbb{R}$, and that there exist a set $\omega \in \mathcal{E}$ with $0 < \mathcal{L}^N(\omega) < \mathcal{L}^N(\Omega)$, then $\mathcal{V}$ satisfies the following properties:

1) $\mathcal{V}(c) = 0$ for all $c \in \mathbb{R}$;

2) $\mathcal{V}(\lambda u) = \lambda \mathcal{V}(u)$ for all $\lambda > 0$ and $u \in D(\mathcal{V})$;

3) $\mathcal{V}(u + c) = \mathcal{V}(u)$ for all $c \in \mathbb{R}$ and $u \in D(\mathcal{V})$;

4) $\mathcal{V}(u) \geq \int_{\mathbb{R}} \mathcal{V}(\chi_{\{x\in\Omega:\, u(x)\geq t\}})\, dt$ for all $u \in D(\mathcal{V})$.

In order to prove the first property, define

$$u_n(x) := \begin{cases} c + \varepsilon_n & \text{if } x \in \omega \\ c & \text{if } x \in \Omega\backslash\omega, \end{cases}$$

where $\varepsilon_n := \frac{1}{n}\min\{1, 1/\mathcal{V}(\chi_\omega)\}$ if $\mathcal{V}(\chi_\omega) > 0$, and $\varepsilon_n := \frac{1}{n}$ otherwise. Clearly $u_n \to c$ in $L^1(\Omega;\mathbb{R})$, therefore by the lower semicontinuity of $\mathcal{V}$ and (5.5)

$$0 \leq \mathcal{V}(c) \leq \liminf_{n\to\infty} \mathcal{V}(u_n) = \lim_{n\to\infty} \varepsilon_n \mathcal{V}(\chi_\omega) = 0,$$

where we have used the fact that $\mathcal{V}(\chi_\omega) < \infty$ because $S_1 \subset D(\mathcal{V})$.

We omit the proofs of properties 2) and 3) since they follow quite easily from hypotheses (i) and (ii) of Theorem 5.1 and from the lower semicontinuity of $\mathcal{V}$.

In order to show 4), fix $u \in D(\mathcal{V})$. By (i) there exists a sequence $\{u_n\} \subset S_1$ converging to u in $L^1(\Omega;\mathbb{R})$ and $\mathcal{L}^N$ a.e. $x \in \Omega$ such that

$$\mathcal{V}(u) \geq \lim_{n\to\infty} \mathcal{V}(u_n) = \lim_{n\to\infty} \int_{\mathbb{R}} \mathcal{V}(\chi_{\{x\in\Omega:\, u_n(x)\geq t\}})\, dt \geq \int_{\mathbb{R}} \liminf_{n\to\infty} \mathcal{V}(\chi_{\{x\in\Omega:\, u_n(x)\geq t\}})\, dt$$

by (5.5) and Fatou's Lemma. Since $\mathcal{L}^N(\{x \in \Omega :\, u(x) = t\}) = 0$ for all $t \in \mathbb{R}\backslash M$, where $\mathcal{L}^1(M) = 0$, we fix $t \in \mathbb{R}\backslash M$ and take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\liminf_{n\to\infty} \mathcal{V}(\chi_{\{x\in\Omega:\, u_n(x)\geq t\}}) = \lim_{k\to\infty} \mathcal{V}(\chi_{\{x\in\Omega:\, u_{n_k}(x)\geq t\}}).$$

Then $\{\chi_{\{x\in\Omega:\, u_{n_k}(x)\geq t\}}\}$ converges pointwise to $\chi_{\{x\in\Omega:\, u(x)\geq t\}}$ for $\mathcal{L}^N$ a.e. $x \in \Omega$ and, by Lebesgue Dominated Convergence Theorem, also strongly in $L^1(\Omega;\mathbb{R})$. Therefore by the lower semicontinuity of $\mathcal{V}$

$$\liminf_{n\to\infty} \mathcal{V}(\chi_{\{x\in\Omega:\, u_n(x)\geq t\}}) \geq \mathcal{V}(\chi_{\{x\in\Omega:\, u(x)\geq t\}})$$

for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, and we conclude that

$$\int_{\mathbb{R}} \liminf_{n\to\infty} \mathcal{V}(\chi_{\{x\in\Omega:\, u_n(x)\geq t\}})\, dt \geq \int_{\mathbb{R}} \mathcal{V}(\chi_{\{x\in\Omega:\, u(x)\geq t\}})\, dt.$$

We do not know if the reversed inequality of 4) holds, i.e. if the co-area formula (5.6) is satisfied.

Let

$$\beta := \inf\left\{\mathcal{V}(u) + I_K(u,\Omega) :\, u \in D(\mathcal{V}),\, u(x) \in K\backslash S \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega\right\}.$$

Lemma 5.3. *If $u \in \mathcal{S}_1$ then*

$$\mathcal{V}(u) + I_K(u,\Omega) \geq \beta.$$

Proof. As $I_K(u,\Omega) = \infty$ for $u \notin L^1(\Omega;K)$ it suffices to prove the result for $u \in \mathcal{S}_1 \cap L^1(\Omega;K)$. By (5.1) we can decompose the open set S as a disjoint union of bounded intervals

$$S = \cup_{r \in \mathcal{R}}(a_r, b_r).$$

Following Visintin [V2] we replace the function τ_2 by

$$\tilde{\tau}_2(u) := \begin{cases} \tau_2(u) & \text{if } u \in \mathbb{R}\backslash S \\ \dfrac{\tau_2(b_r) - \tau_2(a_r)}{b_r - a_r}(u - a_r) + \tau_2(a_r) & \text{if } u \in (a_r, b_r), \end{cases}$$

and denote by $\tilde{\tau}$ and $\tilde{I}_K(u,\Omega)$ the corresponding functionals. Define

$$\beta_i := \mathcal{V}(\chi_{\cup_{r=i+1}^k \omega_r}), \qquad \alpha_i(c) := \int_{\omega_i} \tilde{\tau}(x,c)\,dx.$$

Then by (5.3), (5.4), and (5.5)

$$\mathcal{V}(u) + \tilde{I}_K(u,\Omega) = \sum_{i=1}^{k-1}(c_{i+1} - c_i)\beta_i + \sum_{i=1}^{k}\alpha_i(c_i).$$

Let $r \in \mathcal{R}$ be such that $c_i \in (a_r, b_r)$ for some $i \in \{1, \cdots, k\}$. There can only be finitely many such r. Assume that $k \geq 2$, and suppose that $c_l \in (a_r, b_r)$, $l \in \{2, \cdots, k-1\}$, $c_i \leq a_r$ for all $i < l$ (the cases where $l = 1$ or $l = k$ can be treated analogously). Define the function

$$\Phi(t) := \sum_{i=1,\, i \neq l-1,\, l}^{k-1} \beta_{i+1}(c_{i+1} - c_i) + \beta_l(t - c_{l-1}) + \beta_{l+1}(c_{l+1} - t) + \sum_{i=1,\, i\neq l}^{k} \alpha_i(c_i) + \alpha_l(t)$$

for $t \in [a_r, d]$, where $d := c_{l+1}$ if $c_{l+1} \leq b_r$ and $d := b_r$ if $c_{l+1} > b_r$. Since $\tau_2(u) \geq \tilde{\tau}_2(u)$ by construction, then clearly $\mathcal{V}(u) + I_K(u,\Omega) \geq \mathcal{V}(u) + \tilde{I}_K(u,\Omega) = \Phi(c_l)$. Observe that since $\tilde{\tau}(x,\cdot) = \tau_1(x,\cdot) + \psi(x)\,\tilde{\tau}_2(\cdot)$ is concave in $[a_r, d]$, then the function $\alpha_l(\cdot)$ is also concave in $[a_r, d]$, and $\Phi(t)$, being the sum of a linear function and a concave function, attains its minimum at one of the endpoints Q of $[a_r, d]$. It follows that

$$\mathcal{V}(u) + I_K(u,\Omega) \geq \mathcal{V}(u) + \tilde{I}_K(u,\Omega) = \Phi(c_l) \geq \Phi(Q) = \mathcal{V}(\bar{u}) + \tilde{I}_K(\bar{u},\Omega)$$

where

$$\bar{u}(x) := \begin{cases} \sum_{i=1,\, i\neq l}^{k} c_i\,\chi_{\omega_i}(x) + a_r\chi_{\omega_l}(x) & \text{if } Q = a_r \\ \sum_{i=1,\, i\neq l}^{k} c_i\,\chi_{\omega_i}(x) + b_r\chi_{\omega_l}(x) & \text{if } Q = b_r \\ \sum_{i=1,\, i\neq l,l+1}^{k} c_i\,\chi_{\omega_i}(x) + c_{l+1}\chi_{\omega_l \cup \omega_{l+1}}(x) & \text{if } Q = c_{l+1}. \end{cases}$$

If $k = 1$, namely if $u(x) \equiv c$, then

$$\mathcal{V}(u) + I_K(u,\Omega) \geq \mathcal{V}(u) + \tilde{I}_K(u,\Omega) = \mathcal{V}(c) + \int_\Omega \tilde{\tau}(x,c)\,dx. \tag{5.7}$$

Assume that $c \in (a_r, b_r)$ for some $r \in \mathcal{R}$. Since by (iii) and by the construction of $\tilde{\tau}_2$ the right hand side of (5.7) is a concave function of $c \in [a_r, b_r]$, its infimum is attained at one of the endpoints, say at b_r, and thus we can replace $u(x)$ by $\bar{u}(x) := b_r \in K \setminus S$.

We conclude that it is energetically possible to reduce at least by one the number of values c_i between a_r and b_r. Repeating this procedure for the finite number of intervals (a_r, b_r) which contain at least one of the c_i, by means of a finite induction argument we can construct a simple function $\hat{u}$ of the form

$$\hat{u}(x) = \sum_{i=1}^{\hat{k}} \hat{c}_i \, \chi_{\hat{\omega}_i}(x),$$

where $\hat{k} \leq k$, such that $\hat{u}(\Omega) \subset K \backslash (a_r, b_r)$ for any $r \in \mathcal{R}$ and $\mathcal{V}(u) + I_K(u, \Omega) \geq \mathcal{V}(\hat{u}) + \tilde{I}_K(\hat{u}, \Omega)$. Since $\tau_2(u) = \tilde{\tau}_2(u)$ for $u \in K \backslash S$, it follows that $I_K(\hat{u}, \Omega) = \tilde{I}_K(\hat{u}, \Omega)$ and thus $\mathcal{V}(u) + I_K(u, \Omega) \geq \mathcal{V}(\hat{u}) + I_K(\hat{u}, \Omega) \geq \beta$. This concludes the proof of the lemma.

■

Proof of Theorem 5.1. Let $u \in D(\mathcal{V}) \cap L^1(\Omega; K)$. By (i) there exists a sequence $\{u_n\} \subset \mathcal{S}_1 \cap L^1(\Omega; K)$ converging to u in $L^1(\Omega; K)$ such that

$$\limsup_{n \to \infty} \mathcal{V}(u_n) \leq \mathcal{V}(u).$$

Moreover, by hypothesis

$$\lim_{n \to \infty} \int_\Omega \tau(x, u_n(x)) \, dx = \int_\Omega \tau(x, u(x)) \, dx,$$

and since by Lemma 5.3 $\mathcal{V}(u_n) + I_K(u_n, \Omega) \geq \beta$, it follows that

$$\mathcal{V}(u) + I_K(u, \Omega) \geq \beta,$$

and we conclude that

$$\inf \{\mathcal{V}(u) + I_K(u, \Omega) : u \in D(\mathcal{V})\} \geq \beta.$$

The reversed inequality is trivially satisfied.

■

In order to apply Theorem 5.1 to functionals of the form (1.1), we consider the special case where

$$h = h(x, \xi) \quad \text{is positively homogeneous of degree one in } \xi, \quad \theta(x, u) := \hat{\sigma} \, u, \quad \hat{\sigma} \neq 0,$$

and $(H_1) - (H_7)$ are satisfied. Clearly $h(x, \xi) = h^\infty(x, \xi)$; moreover, from (H_7) it follows that $g = g(x) \geq |\hat{\sigma}| > 0$ for all $x \in \Omega$, so $h(x, \cdot)$ is coercive. By Corollary 2.3, (3.3), and Lemma 3.3, for $u \in BV(\Omega; \mathbb{R})$ we have

$$\mathcal{H}(u, \Omega) = \int_\Omega f(x, \nabla u) \, dx + \int_\Omega f(x, dC(u))$$
$$+ \int_{S(u) \cap \Omega} f(x, (u^+ - u^-)\nu_u) \, dH_{N-1} + \hat{\sigma} \int_\Omega u \operatorname{div} \varphi \, dx,$$

where, we recall,

$$f(x, \xi) = h(x, \xi) + \hat{\sigma} \, \varphi(x) \cdot \xi \qquad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N.$$

Furthermore, we assume that the potential τ also satisfies the growth condition

$$\tau(x, u) \leq b_1(x) + M_1(1 + |u|^{q_c}) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \text{ and all } u \in K, \tag{5.8}$$

where $b_1 \in L^1(\Omega, \mathbb{R})$, $M_1 > 0$ and, as before, q_c is the Sobolev exponent $q_c := N/(N-1)$ if $N > 1$ and $q_c < \infty$ if $N = 1$. Define

$$\mathcal{V}(u) := \begin{cases} \int_\Omega f(x, Du) & \text{if } u \in BV(\Omega; \mathbb{R}) \\ \infty & \text{otherwise,} \end{cases}$$

where
$$\int_{\Omega} f(x, Du) := \int_{\Omega} f(x, \nabla u)\, dx + \int_{\Omega} f(x, dC(u)) + \int_{S(u)\cap\Omega} f(x, (u^+ - u^-)\nu_u)\, dH_{N-1},$$

and take $\mathcal{E}$ to be the algebra generated by the class of open polyhedral subsets of Ω. If $u \in S_1 \cap L^1(\Omega; K)$ has the form (5.3), then either $\omega_i = E_i$ or $\omega_i = \Omega\backslash E_i$, where E_i is an open polyhedral set of Ω. Therefore, in both cases $\partial\omega_i \cap \Omega = \partial E_i \cap \Omega$, which is given by the intersection of a finite union of hyperplanes. Consequently, if $i < j$ for H_{N-1} a.e. $x \in \partial\omega_j \cap \partial\omega_i$ the outward unit normal $\nu_j(x)$ to the set ω_j at the point x coincides with $-\nu_i(x)$. Moreover, for $j < i < l$ we have

$$H_{N-1}(\partial\omega_j \cap \partial\omega_i \cap \partial\omega_l \cap \Omega) = 0, \tag{5.9}$$

and

$$\begin{aligned} &\Omega = \cup_{i=1}^{k} \overline{\omega}_i \cap \Omega, \qquad \partial\omega_i \cap \Omega = \cup_{j\neq i} \partial\omega_i \cap \partial\omega_j \cap \Omega, \\ &\partial\left(\cup_{l=i}^{k} \overline{\omega}_l \cap \Omega\right) = \partial\left(\cup_{l=i+1}^{k} \overline{\omega}_l \cap \Omega\right) \setminus \left(\cup_{l=i+1}^{k} \partial\omega_l \cap \partial\omega_i \cap \Omega\right) \cup \left(\cup_{j=1}^{i-1} \partial\omega_j \cap \partial\omega_i \cap \Omega\right). \end{aligned} \tag{5.10}$$

These properties will be useful in the sequel.

The main result of the section is the following theorem

Theorem 5.4. *If $(H_1) - (H_7)$ and (5.8) are verified then*

$$\inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega; K)\} = \inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega; K\backslash S)\}.$$

Remark 5.5. Theorem 5.4 no longer holds in general if $\theta(x,\cdot)$ is non linear. Indeed, consider the simple case where $\Omega := (c,d)$, $K := [-1,1]$,

$$\mathcal{H}(u,\Omega) := \sigma \int_{\Omega} |Du(x)| - \int_{\partial\Omega} \sin(\pi\, T\, u(x))\, dH_{N-1}(x), \quad u \in BV(\Omega; \mathbb{R})$$

and $\tau(x,u) := a(1-u^2)$. Here $K \setminus S = \{-1, 1\}$, and if $\sigma > \pi$ then all conditions $(H_1) - (H_7)$ are satisfied. Let $u \in BV(\Omega; \mathbb{R})$, with $u(x) \in \{-1,1\}$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Then $\mathcal{I}(u,\Omega) = \sigma \int_{\Omega} |Du(x)| \geq 0$. On the other hand, if we take $\bar{u}(x) \equiv \frac{1}{2}$ then $\mathcal{I}(\bar{u},\Omega) = -2 + \frac{3}{4} a(d-c) < 0$ provided $a(d-c) < \frac{8}{3}$.

For the proof of the lemma below we refer to [F], [LM], and [Re].

Lemma 5.6. *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $\xi \in \mathbb{R}^N \mapsto f(x,\xi)$ is positively homogeneous of degree one for all $x \in \Omega$, and*

$$0 \leq f(x,\xi) \leq C(1 + |\xi|) \quad \textit{for some } C > 0, \textit{ all } x \in \Omega \textit{ and } \xi \in \mathbb{R}^N.$$

Let $\{\mu_n\}$ be a sequence of Radon measures converging weakly-$\star$ to a Radon measure μ and

$$\lim_{n\to\infty} |\mu_n|(\Omega) = |\mu|(\Omega).$$

Then
$$\lim_{n\to\infty} \int_{\Omega} f(x, d\mu_n) = \int_{\Omega} f(x, d\mu).$$

Proof of Theorem 5.4. We claim that $\mathcal{V}$ satisfies conditions (i)–(iii) of Theorem 5.1. To prove (i) fix $u \in BV(\Omega; K)$. We can find a sequence $\{u_n\}$ of the form

$$u_n(x) = \sum_{i=1}^{k_n} c_{n,i}\, \chi_{\omega_{n,i}}(x)$$

such that u_n converges strongly to u in $L^1(\Omega;\mathbb{R})$ and $\int_\Omega |Du_n| \to \int_\Omega |Du|$ (see [AMT]). Here $c_{n,i} \in \mathbb{R}$, the sets $\omega_{n,i}$ are open polyhedral set of Ω, $\cup_{i=1}^{k_n} \overline{\omega}_{n,i} \cap \Omega = \Omega$, and as in (5.4)

$$c_{n,1} < c_{n,2} < \cdots < c_{n,k_n}.$$

By (5.10) it is not difficult to see that

$$S(u_n) = \cup_{(i,j)\in J_n}(\partial\omega_{n,i} \cap \partial\omega_{n,j}),$$

where $J_n = \{(i,j) \in \mathbb{N}^2 : 1 \le i < j \le k_n\}$. Furthermore, if $x_0 \in \partial\omega_{n,i} \cap \partial\omega_{n,j}$ then $u_n^+(x_0) = c_{n,j}$, $u_n^-(x_0) = c_{n,i}$, and

$$\int_\Omega |Du_n| = \sum_{(i,j)\in J_n} (c_{n,j} - c_{n,i}) H_{N-1}(\partial\omega_i \cap \partial\omega_j \cap \Omega).$$

Let $\bar{u}_n(x) := \sum_{i=1}^{k_n} d_{n,i}\, \chi_{\omega_{n,i}}(x)$, where

$$d_{n,i} = \begin{cases} \sup K & \text{if } c_{n,i} \ge \sup K \\ c_{n,i} & \text{if } -\inf K < c_{n,i} < \sup K \\ \inf K & \text{if } c_{n,i} \le \inf K. \end{cases}$$

Since $0 \le (d_{n,j} - d_{n,i}) \le (c_{n,j} - c_{n,i})$, it follows that $\int_\Omega |D\bar{u}_n| \le \int_\Omega |Du_n|$. Consequently

$$\limsup_{n\to\infty} \int_\Omega |D\bar{u}_n| \le \int_\Omega |Du|.$$

On the other hand, since $u(x) \in K$ for $\mathcal{L}^N$ a.e. $x \in \Omega$, then $|u(x) - \bar{u}_n(x)| \le |u(x) - u_n(x)|$ by construction, and so $\{\bar{u}_n\}$ converges strongly to u in $L^1(\Omega;\mathbb{R})$. By the lower semicontinuity of the total variation we have that $\int_\Omega |Du| \le \liminf_{n\to\infty} \int_\Omega |D\bar{u}_n|$, thus

$$\lim_{n\to\infty} \int_\Omega |D\bar{u}_n| = \int_\Omega |Du|$$

and from Lemma 5.6 we conclude that

$$\lim_{n\to\infty} \int_\Omega f(x, D\bar{u}_n) = \int_\Omega f(x, Du).$$

In order to verify (ii), fix $u \in BV(\Omega;\mathbb{R})$ of the form (5.3)-(5.4), where $c_i \in K$, $\omega_i \in \mathcal{E}$, $\cup_{i=1}^k \overline{\omega}_i \cap \Omega = \Omega$, $k \ge 2$, and

$$c_1 < c_2 < \cdots < c_k.$$

Since by homogeneity $f(x,0) = 0$, we have

$$\mathcal{V}(u) = \sum_{(i,j)\in J} (c_j - c_i) \int_{\partial\omega_i \cap \partial\omega_j \cap \Omega} f(x, \nu_j)\, dH_{N-1}$$

with $J = \{(i,j) \in \mathbb{N}^2 : 1 \le i < j \le k\}$, or, equivalently,

$$\begin{aligned} \mathcal{V}(u) = {} & c_k \sum_{j=1}^{k-1} \int_{\partial\omega_k \cap \partial\omega_j \cap \Omega} f(x, \nu_k)\, dH_{N-1} - c_1 \sum_{l=2}^{k} \int_{\partial\omega_l \cap \partial\omega_1 \cap \Omega} f(x, \nu_l)\, dH_{N-1} \\ & + \sum_{i=2}^{k-1} c_i \left(\sum_{j=1}^{i-1} \int_{\partial\omega_i \cap \partial\omega_j \cap \Omega} f(x, \nu_i)\, dH_{N-1} - \sum_{l=i+1}^{k} \int_{\partial\omega_l \cap \partial\omega_i \cap \Omega} f(x, \nu_l)\, dH_{N-1} \right). \end{aligned} \tag{5.11}$$

By (5.9) and (5.10) we can rewrite the first two terms as, respectively,

$$c_k \int_{\partial\omega_k\cap\Omega} f(x,\nu_k)\,dH_{N-1} \quad \text{and} \quad -c_1 \int_{\partial(\cup_{l=2}^k \overline{\omega_l})\cap\Omega} f(x,-\nu_1)\,dH_{N-1},$$

and for $i \in \{2,\cdots,k-1\}$

$$\sum_{j=1}^{i-1} \int_{\partial\omega_i\cap\partial\omega_j\cap\Omega} f(x,\nu_i)\,dH_{N-1} - \sum_{l=i+1}^{k} \int_{\partial\omega_l\cap\partial\omega_i\cap\Omega} f(x,\nu_l)\,dH_{N-1}$$
$$= \int_{\partial(\cup_{l=i}^k \overline{\omega_l})\cap\Omega} f(x,\hat\nu_i)\,dH_{N-1} - \int_{\partial(\cup_{l=i+1}^k \overline{\omega_l})\cap\Omega} f(x,\hat\nu_{i+1})\,dH_{N-1}, \tag{5.12}$$

where $\hat\nu_i$ and $\hat\nu_{i+1}$ are, respectively, the outward unit normals to the sets $\cup_{l=i}^k \overline{\omega_l}$ and $\cup_{l=i+1}^k \overline{\omega_l}$. It now follows from (5.11)–(5.12) that

$$\mathcal{V}(u) = c_k\mathcal{V}(\chi_{\omega_k}) - c_1\mathcal{V}(\chi_{\cup_{l=2}^k\omega_l}) + \sum_{i=2}^{k-1} c_i\mathcal{V}(\chi_{\cup_{l=i}^k\omega_l}) - \sum_{i=2}^{k-1} c_i\mathcal{V}(\chi_{\cup_{l=i+1}^k\omega_l}),$$

which is (ii).

Finally, by (5.2), (5.8) and the Sobolev inequality, the functional $u \mapsto \int_\Omega \tau(x,u(x))\,dx$ is continuous in $BV(\Omega;K)$ (see [K, Th. 2.1]). Therefore we can now apply Theorem 5.1 (with $\tau(x,u)$ replaced by $\tau(x,u) + \hat\sigma\, u \operatorname{div}\varphi(x))$ to obtain that

$$\inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega;K)\} = \inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega;K\backslash S)\}.$$

■

Corollary 5.7. *Assume that $K = [a,b]$ in Theorem 5.4. Then there exists a function $u \in BV(\Omega;[a,b]\backslash S)$ such that*

$$\mathcal{I}(u,\Omega) = \inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega;[a,b])\}.$$

Proof. By Theorem 5.4

$$\inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega;[a,b])\} = \inf\{\mathcal{I}(u,\Omega) : u \in BV(\Omega;[a,b]\backslash S)\} = \beta.$$

To complete the proof it suffices to apply Corollary 4.3, with $K := [a,b]\backslash S$, to find $u \in BV(\Omega;[a,b]\backslash S)$ such that $\mathcal{I}(u,\Omega) = \beta$.

■

Remark 5.8. If we assume that τ_2 is concave in $[a,b]$, then $S = (a,b)$ and consequently the minimizer u in Corollary 5.7 has the property that $u(x) \in \{a,b\}$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

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