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CLASSIFICATION OF POSITIVE SOLUTIONS OF THE ELLIPTIC EQUATION $\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)+x\cdot\nabla\left(u^q\right)+au^q=0$ IN $\mathbb{R}^n$

GIOVANNI LEONI
Department of Mathematical Sciences
Carnegie Mellon Univ.
Pittsburgh, PA 15213

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CLASSIFICATION OF POSITIVE SOLUTIONS OF THE ELLIPTIC EQUATION $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + x \cdot \nabla(u^q) + \alpha\, u^q = 0$ IN $\mathbb{R}^n$

GIOVANNI LEONI

Center for Nonlinear Analysis, Department of Mathematics,
Carnegie Mellon University, Pittsburgh, PA 15213, USA

ABSTRACT. We prove the existence and the asymptotic behavior of nonnegative radial ground states of the semilinear elliptic equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + x \cdot \nabla(u^q) + \alpha\, u^q = 0$, which arises in the study of selfsimilar solutions of degenerate parabolic equations.

§1. Introduction.

In this paper we study the existence and the asymptotic behavior of *ground states* of the quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + x \cdot \nabla(u^q) + \alpha\, u^q = 0, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $p > 1$, $n \geq 1$, $q > 0$ and $\alpha > 0$. By a ground state we mean a nonnegative, nontrivial entire solution of (1.1) which approaches zero as $|x| \to \infty$.

We restrict our attention to *radial* ground states $u = u(|x|)$ of (1.1) and thus we are led to the ordinary differential equation

$$\left(|u'|^{p-2}u'\right)' + \frac{n-1}{r}\,|u'|^{p-2}u' + r(u^q)' + \alpha\, u^q = 0, \quad r = |x| > 0. \tag{1.2}$$

Equation (1.2) arises in the study of selfsimilar solutions of the degenerate parabolic equation

$$q\, v^{q-1} v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v), \tag{1.3}$$

which includes, as special cases, the *heat equation*

$$v_t = \Delta v, \tag{1.4}$$

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the *porous media equation*

$$v_t = \Delta(v^m) \tag{1.5}$$

and the *evolution p–Laplacian equation*

$$v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v). \tag{1.6}$$

Equation (1.3) has been studied by several authors in the one dimensional case $n = 1$, see in particular the work of Barenblatt [2], Esteban and Vazquez [10], Kalashnikov [16], and it arises in physical situations such that water filtration through porous media in the case of large velocities (see [10] and the references contained therein) and in the turbulence flow of a gas in a porous medium.

When $p = 2$ in (1.3), after a simple change of variables we obtain the celebrated porous media equation (1.5). This equation has received an enormous amount of attention in the last decades. It arises in the theory of gas flow in homogeneous porous media, but it also has other applications in the theory of ionized gases at high temperature, in plasma physics, lubrication theory and in populations dynamics. We refer to the paper of Aronson [1] for more details and for an extensive bibliography. Finally equation (1.6) has been studied in relation with the mechanics of non–Newtonian fluids (cf. [10]). See the work of Di Benedetto and Herrero [7, 8] and of Hulshof and Vazquez [14].

As a corollary of our main results on equation (1.2) we give a fairly complete picture of existence and non existence of selfsimilar solutions of the initial value problem

$$q\,v^{q-1}v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v),$$
$$v(x,0) = A\,|x|^{-a}$$

which turns out to be particularly important in the study of the large time behavior of solutions of (1.3) whose initial data $v(x,0)$ decays algebraically as $|x| \to \infty$; see the work of Kamin and Peletier [17] and Peletier and Junning [23].

It is worth mentioning that selfsimilar solutions only give a first approximation of the asymptotic behavior of solutions of (1.3). In [28] Tartar has constructed a larger class of explicit solutions for (1.4) and (1.5). These solutions give a better estimate of the asymptotic behavior of solutions for the heat equation (1.4), and it is natural to expect that the same holds for (1.5).

In addition to its connection with the parabolic equations (1.3)–(1.6), we believe that the elliptic equation (1.2) is of interest in itself and thus *we do not impose any restriction on the coefficient α and on the exponents p and q.* Indeed a simple change of scale transforms (1.2) into the equation

$$\left(|u'|^{p-2}u'\right)' + \frac{n-1}{r}\,|u'|^{p-2}u' + \varepsilon r(u^q)' + u^q = 0, \tag{1.7}$$

where $\varepsilon = 1/\alpha$, and thus we can regard equation (1.7) as a perturbation of the celebrated Emden–Fowler equation

$$\left(|u'|^{p-2}u'\right)' + \frac{n-1}{r}\,|u'|^{p-2}u' + u^q = 0. \tag{1.8}$$

It is well known that positive ground states of (1.8) exist if and only if the exponent q is greater or equal than the critical exponent

$$q_c = \frac{n(p-1)+p}{n-p}$$

and that they exhibit the asymptotic behavior (1.12) below. From a heuristic point of view we would expect, at least for α large (equivalently for ε small), solutions of (1.7) to behave somewhat as those of (1.8). This turns out to be the case, as shown in Theorems 3–5 below. On the other hand, when α is small, more precisely when $\alpha < n$ the gradient term has a dominant effect with regards to both existence and asymptotic behavior of solutions of (1.7), as it appears from Theorem 2 below. The case $\alpha = n$ represents somewhat the borderline and in this case we can actually determine explicitly the solutions of (1.2). To see this multiply the equation (1.2) by r^{n-1} and integrate by parts. We obtain

$$r^{n-1}|u'|^{p-2}u' + r^n u^q = (n-\alpha)\int_0^r t^{n-1}u^q dt, \tag{1.9}$$

where we have used the symmetry condition $u'(0) = 0$. Since $\alpha = n$ a simple integration of the identity $|u'|^{p-2}u' + r\,u^q = 0$ gives the famous Barenblatt solutions [2]

Theorem 1. *Assume that $\alpha = n$ in (1.2). Then the solution $u = u(r)$ of (1.2) is given by*

$$u(r) = \begin{cases} u(0)\exp\left(-\frac{p-1}{p}\, r^{p/(p-1)}\right) & \text{if } q = p-1 \\ \left(u(0)^{(p-q-1)/(p-1)} - \frac{p-q-1}{p} r^{p/(p-1)}\right)^{(p-1)/(p-q-1)} & \text{if } q \neq p-1 \end{cases}$$

for $0 \le r < R$, where

$$R = \begin{cases} \infty & \text{if } q \ge p-1 \\ u(0)^{(p-q-1)/p}\left(\frac{p}{p-q-1}\right)^{(p-1)/p} & \text{if } q < p-1. \end{cases}$$

Note that when $q < p-1$ then $u(R) = u'(R) = 0$ so the solution has compact support. We consider next the case $\alpha < n$. Let

$$\lambda_1 = \frac{\alpha}{q}, \qquad \lambda_2 = \frac{p}{q+1-p}, \qquad \lambda_3 = \frac{n-p}{p-1}, \tag{1.10}$$

where obviously λ_2 is defined only when $q \neq p-1$.

Theorem 2. *Assume that* $\alpha < n$ *in* (1.2). *Then all solutions* u *of* (1.2) *are positive. Moreover*

(i) *if* $q \leq p-1$ *or* $q > p-1$ *and* $\lambda_1 < \lambda_2$ *then*

$$\lim_{r\to\infty} r^{\lambda_1} u(r) = \ell \in (0,\infty); \tag{1.11}$$

(ii) *if* $q > p-1$ *and* $\lambda_1 = \lambda_2$ *then*

$$\lim_{r\to\infty} r^{\lambda_1} (\log r)^{-1/(q+1-p)} u(r) = \lambda_1^{\lambda_1} \left[\frac{p(p-1)}{q\lambda_1^2}(\lambda_3 - \lambda_1)\right]^{1/(q+1-p)};$$

(iii) *if* $q > p-1$ *and* $\lambda_1 > \lambda_2$ *then*

$$\lim_{r\to\infty} r^{\lambda_2} u(r) = \lambda_2^{\lambda_2} \left[\frac{p-1}{q\lambda_2} \cdot \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\right]^{1/(q+1-p)}. \tag{1.12}$$

When $\alpha > n$ the situation is more complicated and not complete.

Theorem 3. *Assume that* $\alpha > n$ *in* (1.2). *If*

$$q \leq p-1 \quad \text{or} \quad p \geq n \quad \text{or} \quad p < n \quad \text{and} \quad 1-p < q \leq \frac{n(p-1)}{n-p} \tag{1.13}$$

then (1.2) *has no positive solutions.*

From the proof of Theorem 3 it is easy to see that when (1.13) holds all solutions u of (1.2) change sign at least once. Changing sign solutions of (1.2) have been studied extensively by Hulshof in [13] when $p = 2$ and $q < 1$ and by Dohmen in [9] when $p = 2$ and $q \geq 1$.

Theorem 4. *Assume that* $\alpha > n > p$ *and* $\lambda_1 \leq \lambda_3$ *in* (1.2). *Then all solutions* u *of* (1.2) *are positive. Moreover if* $q \geq q_c$ *then* u *exhibits the asymptotic behavior* (1.12), *while if* $q < q_c$ *then*

$$c_1 r^{-\lambda_2} \leq u(r) \leq c_2 r^{-\lambda_2} \tag{1.14}$$

for r *sufficiently large and where* $c_1, c_2 > 0$.

To prove (1.12) and (1.14) in Theorem 4 we transform (1.2) into a dynamical system. This idea has been used by several authors, see in particular the work of Johnson, Pan and Yi [15] for elliptic equations and of Dohmen [9] and Hulshof [13].

In the remaining range the critical exponent q_c turns out to play an important role. We consider first the case in which q is either *supercritical or critical*, that is $q \geq q_c$.

Theorem 5. *Assume that* $\alpha > n > p$, $\lambda_1 > \lambda_3$ *and* $q \geq q_c$ *in* (1.2). *Then all solutions* u *of* (1.2) *are positive and satisfy* (1.12).

In the *subcritical case* $\frac{n(p-1)}{n-p} < q < q_c$ positive solutions do not exist when α is large.

Theorem 6. *Assume that* $\alpha > n > p$, $\lambda_1 > \lambda_3$ *and* $\frac{n(p-1)}{n-p} < q < q_c$ *in* (1.2). *If*

$$\lambda_1 > \lambda_* = \frac{n}{(p-1)(q_c - q)}$$

then (1.2) *has no positive solutions.*

The question of existence of positive solutions of (1.2) in the range

$$n > p, \quad \lambda_3 < \lambda_1 < \lambda_* \quad \text{and} \quad \frac{n(p-1)}{n-p} < q < q_c$$

is open, although from Theorems 4 and 6 it is reasonable to expect the existence of a critical exponent λ_c such that positive solutions exist when $\lambda \leq \lambda_c$, while they do not when $\lambda > \lambda_c$. A natural candidate seems to be the exponent λ given in (3.1) in Section 3 below, since as λ crosses the value (3.1) the nature of the equilibrium point $(\ell_1, -\lambda_2 \ell_1)$ of the autonomous system (3.8) changes from attractor to repeller.

This paper is organized as follows. In Section 2 we give some preliminary results and prove Theorem 2. In Sections 3 and 4 we prove Theorems 3, 4 and 5, 6 respectively. Finally in Section 5 we apply the results of the previous sections to the study of selfsimilar solutions of the degenerate parabolic equation (1.4).

§2. Proof of Theorem 2.

In this section we present some preliminary results and give the demonstration of Theorem 2. We consider the initial value problem

$$\begin{aligned} &\left(|u'|^{p-2}u'\right)' + \frac{n-1}{r}\,|u'|^{p-2}u' + r(u^q)' + \alpha u^q = 0, \\ &u(0) = u_0, \quad u'(0) = 0. \end{aligned} \tag{2.1}$$

Proposition 2.1. *For any* $u_0 > 0$ *the initial value problem* (2.1) *admits a positive solution* $u : [0, R) \to \mathbb{R}^+$, *with* R *possibly infinite.* (i) $u'(r) < 0$ *for* $r \in (0, R)$;

(ii) $\lim_{r \to R^-} u(r) = 0$;

(iii) $\lim_{r \to \infty} u'(r) = 0$ *when* $R = \infty$.

Proof. See [18, Theorem 1].

Lemma 2.2. *Let $u \in C^2(r_0, \infty)$ be a positive function tending to zero as $r \to \infty$ and such that*

$$-\left(r^{n-1}|u'|^{p-2}u'\right)' \geq C\, r^{n-1}u^q \quad \text{for } r \geq r_0.$$

Then $n > p$, $q > \frac{n(p-1)}{n-p}$ and

$$c_1 r^{-\lambda_3} \leq u(r) \leq c_2 r^{-\lambda_2} \quad \text{for } r \text{ large.}$$

Proof. See [22, Theorems 2.1, 2.2, 6.1, 6.2] and [4].

Lemma 2.3. *Let u be a positive solution of (2.1) in $[0, R)$. Consider the function $E_c(r) = cu + r\,u'$, where $c > 0$.* (i) *At any time $r > 0$ for which $E_c(r) = 0$ we have*

$$(p-1)|u'|^{p-2}E_c'(r) = -(p-1)[c-\lambda_3]c^{p-1}\left(\frac{u}{r}\right)^{p-1} + q(c-\lambda_1)r\,u^q.$$

(ii) *If $R = \infty$ then $E_c(r)$ cannot be ultimately nonnegative for*

$$0 < c < \begin{cases} \lambda_1 & \text{if } q \geq p-1 \\ \min\{\lambda_1, \lambda_2\} & \text{if } q < p-1. \end{cases}$$

(iii) *If $R = \infty$ and either $q \leq p-1$ or $q > p-1$ and $\lim_{r\to\infty} r^{\lambda_2}u(r) = \ell \in \{0, \infty\}$ then $E_c(r)$ is ultimately of the same sign for all $c \neq \lambda_1$, λ_3.*

(iv) *$E_{\lambda_1}(r)$ is always positive if $\lambda_3 \geq \lambda_1$ and can have at most one zero if $\lambda_3 < \lambda_1$.*

(v) *$E_{\lambda_3}(r)$ is always positive if $\lambda_3 \geq \lambda_1$ and can have at most one zero if $\lambda_3 < \lambda_1$. Moreover if $\lambda_3 < \lambda_1$ and $E_{\lambda_1}(r)$ is always positive then $E_{\lambda_3}(r)$ is always positive.*

Proof. (i) Direct calculation. To show (ii) assume for contradiction that $E_c(r) \geq 0$ for all $r \geq r_0$, for some $r_0 > 0$. From the identity

$$(r^c u(r))' = r^{c-1}E_c(r) \tag{2.2}$$

we obtain that

$$\lim_{r\to\infty} r^c u(r) > 0. \tag{2.3}$$

On the other hand by (2.1)

$$-\left(r^{n-1}|u'|^{p-2}u'\right)' = q(\lambda_1 - c)r^{n-1}u^q + qr^{n-1}u^{q-1}E_c(r) \geq q(\lambda_1 - c)r^{n-1}u^q \quad \text{for } r \geq r_0.$$

Therefore we can apply Lemma 2.2 to obtain that $n > p$, $q > \frac{n(p-1)}{n-p}$ and

$$c_3 r^{-c} \leq u(r) \leq c_2 r^{-\lambda_2} \quad \text{for } r \text{ large,}$$

where we have used (2.3). This is a contradiction since $c < \lambda_2$ and the proof of property (ii) is complete.

We prove (iii) only for $c < \lambda_1$ and $\ell = \infty$, the other cases being completely analogous. If $q \le p-1$ then since $u(r) \to 0$ as $r \to \infty$, by (i)

$$(p-1)|u'|^{p-2}E_c'(r) = \frac{u^q}{r^{p-1}}\left\{r^p q(c-\lambda_1) - (p-1)[c-\lambda_3]c^{p-1}u^{p-1-q}\right\} < 0$$

at any time r sufficiently large, say $r \ge r^*$, such that $E_c(r) = 0$. Therefore E_c cannot have more than two zeros after the time r^*, otherwise we would find $r_0 > r^*$ such that $E_c(r_0) = 0$ and $E_c'(r_0) \le 0$ which is a contradiction. Hence E_c is ultimately of the same sign. If $q > p-1$ and $\lim_{r\to\infty} r^{\lambda_2}u(r) = \infty$ then again by (i) we have

$$(p-1)|u'|^{p-2}E_c'(r) = \frac{u^{p-1}}{r^{p-1}}\left\{q(c-\lambda_1)r^p u^{q-p+1}(r) - (p-1)(c-\lambda_3)c^{p-1}\right\} < 0$$

at any time r sufficiently large. Therefore E_c is ultimately of the same sign and the proof of (iii) is complete.

At any time r for which $E_{\lambda_1}(r) = 0$ we have

$$(p-1)|u'|^{p-2}E_{\lambda_1}'(r) = (p-1)[\lambda_3-\lambda_1]\lambda_1^{p-1}\left(\frac{u}{r}\right)^{p-1}$$

therefore $E_{\lambda_1}(r)$ is always positive if $\lambda_3 > \lambda_1$ (since $E_{\lambda_1}(0) > 0$) and can have at most one zero if $\lambda_3 < \lambda_1$. If $\lambda_1 = \lambda_3$ then from the equation

$$(p-1)|u'|^{p-2}E_{\lambda_1}'(r) + q\,r\,u^{q-1}E_{\lambda_1}(r) = (p-1)[\lambda_3-\lambda_1]|u'|^{p-1} \tag{2.4}$$

we obtain

$$E_{\lambda_1}(r) = E_{\lambda_1}(r_0)\exp\left(-\frac{q}{p-1}\int_{r_0}^r u^{q-1}|u'|^{2-p}t\,dt\right)$$

which implies that $E_{\lambda_1}(r)$ is always positive, provided r_0 is taken so close to zero that $E_{\lambda_1}(r_0) > 0$.

The proof of the first part of (v) is very similar to (iv) and therefore we omit. To prove the second part we use an idea of Clément, de Figueiredo and Mitidieri [5]. We first observe that since $E_{\lambda_1}(r) > 0$ by hypothesis, then we have $R = \infty$ by (2.2). Assume for contradiction that $E_{\lambda_3}(r_1) = 0$ for some $r_1 > 0$. Then by the first part $E_{\lambda_3}(r) < 0$ for all $r > r_1$. Moreover by (2.1) and our assumption that $E_{\lambda_1}(r) > 0$

$$(p-1)r^{n-2}|u'|^{p-2}E_{\lambda_3}'(r) = -q\,r^{n-1}u^{q-1}E_{\lambda_1}(r) < 0,$$

so the function $E_{\lambda_3}(r)$ is negative and strictly decreasing for $r > r_1$, which implies that

$$r\,u'(r) < E_{\lambda_3}(r) < -\epsilon \qquad \text{for all } r \ge r_2 > r_1.$$

Integration of the previous inequality gives

$$u(r) - u(r_2) < \epsilon \log(r_2/r).$$

If we now let $r \to \infty$ we obtain a contradiction.

Proof of Theorem 2. Since $n > \alpha$ the function $r^{n-1}|u'|^{p-2}u' + r^n u^q$ is strictly increasing and positive for $r > 0$ by (1.9). This clearly implies that $R = \infty$ since $u'(r) < 0$ in $(0, R)$ by Prop. 2.1(i).

Furthermore when $q > p-1$ integration of the inequality $|u'|^{p-2}u' + r\, u^q > 0$ yields

$$u(r) \geq c_1 r^{-\lambda_2} \quad \text{for } r \text{ large.} \tag{2.5}$$

Multiplying the equation in (2.1) by $r^{\alpha-1}$ and integrating by parts from $r_1 > 0$ to $r \geq r_1$ we get

$$r^{\alpha-1}|u'|^{p-2}u' + r^\alpha u^q = r_1^{\alpha-1}|u'|^{p-2}u'(r_1) + r_1^\alpha u^q(r_1) + (n-\alpha)\int_{r_1}^r t^{\alpha-1}|u'|^{p-1}dt, \tag{2.6}$$

which implies that the function $r^{\alpha-1}|u'|^{p-2}u' + r^\alpha u^q$ is increasing and positive by the previous argument. Therefore

$$u(r) \geq c_2 r^{-\lambda_1} \quad \text{for } r \text{ large.} \tag{2.7}$$

By Lemma 2.3(iv) the function $E_{\lambda_1}(r)$ is ultimately of the same sign and thus by (2.2) there exists

$$\lim_{r\to\infty} r^{\lambda_1}u(r) = \ell \in (0, \infty] \tag{2.8}$$

by (2.7).

We now divide the proof according to the three cases (i)–(iii).

Proof of Theorem 2(i). To show Theorem 2(i) it remains to exclude the case $\ell = \infty$ in (2.8). Thus assume that $\ell = \infty$. In turn $E_{\lambda_1}(r) = -r|u'(r)| + \lambda_1 u(r)$ must be ultimately positive, say for $r \geq r_1$. Consequently, from (2.6)

$$r^\alpha u^q \leq C(r_1) + \lambda_1^{p-1} r^{\alpha-p} u^{p-1} + (n-\alpha)\lambda_1^{p-1}\int_{r_1}^r t^{\alpha-p-1}u^{p-1}dt. \tag{2.9}$$

We now use Lemma 2.3. Fix c in Lemma 2.3(ii) in such a way that $\alpha - p < c(p-1)$ (this can be done by the hypotheses of Theorem 1(i)). By Lemma 2.3(ii)–(iii) the function E_c is ultimately nonpositive. Therefore by (2.2) there exists $\lim_{r\to\infty} r^c u(r) = 0$ (if the limit were not zero for some c than by replacing c with $c+\epsilon$ we would have $\lim_{r\to\infty} r^{c+\epsilon}u(r) = \infty$ which would contradict the fact that $E_{c+\epsilon}$ is ultimately nonpositive) . For r sufficiently large we have $u(r) \leq r^{-c}$. Consequently from (2.9)

$$\left(r^{\lambda_1}u\right)^q \leq C(r_1) + \lambda_1^{p-1} r^{\alpha-p-c(p-1)} + (n-\alpha)\lambda_1^{p-1}\int_{r_1}^r t^{\alpha-p-1-c(p-1)}dt$$

provided r_1 is taken sufficiently large. From the fact $\alpha - p < c(p-1)$ we conclude that the right hand side of the previous inequality is bounded as $r \to \infty$ and thus $\ell < \infty$. This concludes the proof of Theorem 2(i).

Lemma 2.4. *Let u be a positive solution of (2.1) in $[0,R)$. Consider the function $F_c(r)=(c+1)u'+r\,u''$, where $c>0$.* (i) *At any time $r>0$ for which $F_c(r)=0$ we have*

$$(p-1)|u'|^{p-2}F_c'(r)=\frac{1}{r}\,|u'|^{p-1}\left\{A+qr^2u^{q-2}|u'|^{p-2}[(\alpha-c)u+(q-1)ru']\right\},$$

where $A=A(n,p,q,c,\alpha)$.

(ii) *If $q>p-1$, $\lambda_1=\lambda_2$ and $\lim_{r\to\infty}r^{\lambda_2}u(r)=\infty$ then $F_c(r)$ is ultimately of the same sign for all $c\neq\lambda_1$.*

Proof. (i) Direct calculation. To show (ii) we need to exclude the possibility that F_c changes sign infinitely many times as $r\to\infty$. Fix $c_1<\lambda_2$ and $c_2>\lambda_2$. By Lemma 2.3(ii)–(iii) the function $E_{c_1}(r)$ is ultimately nonpositive, while from the facts that $\lim_{r\to\infty}r^{\lambda_2}u(r)=\infty$ and $c_2>\lambda_2$ it follows from Lemma 2.3(iii) and (2.2) that $E_{c_2}(r)$ is ultimately nonnegative. Therefore

$$c_1u(r)\le r|u'(r)|\le c_2u(r)\quad\text{for all } r \text{ sufficiently large.}\tag{2.10}$$

Let $c\neq\lambda_1=\alpha/q$ and assume that $q>1$. Then $(\alpha-c)/(q-1)\neq\lambda_1$. Take $\gamma>0$ inside the segment of endpoints $(\alpha-c)/(q-1)$ and λ_1 and write

$$(\alpha-c)u+(q-1)ru'=[(\alpha-c)-\gamma(q-1)]u+(q-1)(\gamma u+ru').\tag{2.11}$$

There are various cases. If $c<\lambda_1$ then $\lambda_1<\gamma<(\alpha-c)/(q-1)$ and consequently the right hand side of (2.11) is greater than $[(\alpha-c)-\gamma(q-1)]u>0$, since E_γ is ultimately nonnegative. Therefore from (i)

$$(p-1)|u'|^{p-2}F_c'(r)\ge\frac{1}{r}\,|u'|^{p-1}\left\{A+qr^pu^{q+1-p}[(\alpha-c)-\gamma(q-1)]c_3^{p-2}\right\},\tag{2.12}$$

at any time r sufficiently large such that $F_c(r)=0$, where we have used (2.10) and $c_3=c_1$ if $p\ge2$ while $c_3=c_2$ otherwise. Since $\lim_{r\to\infty}r^pu^{q+1-p}(r)=\infty$ by hypothesis, we get that $(p-1)|u'|^{p-2}F_c'(r)>0$ at any time r sufficiently large such that $F_c(r)=0$. Consequently $F_c(r)$ is ultimately either nonnegative or nonpositive.

If $c>\lambda_1$ then $(\alpha-c)/(q-1)<\gamma<\lambda_1$ and consequently the right hand side of (2.10) is less than $[(\alpha-c)-\gamma(q-1)]u<0$, since E_γ is ultimately nonpositive. In this case we obtain (2.12) but with the reversed inequality sign and we conclude that $(p-1)|u'|^{p-2}F_c'(r)<0$ at any time r sufficiently large such that $F_c(r)=0$. Similar estimates can be obtained when $q\le1$.

Proof of Theorem 2(ii). We claim that $\ell=\infty$ in (2.8). Indeed, assume for contradiction that $\ell<\infty$. From (1.9)

$$r^{\alpha-1}|u'|^{p-1}=r^\alpha u^q-(n-\alpha)r^{\alpha-n}\int_0^r t^{n-1}u^q dt.\tag{2.13}$$

Since $\alpha < n$ and $\lambda_1 = \alpha/q$ the integral on the right hand side diverges as $r \to \infty$ by (2.8). Thus we can apply Hospital's rule to obtain

$$\lim_{r\to\infty} (n-\alpha) r^{\alpha-n} \int_0^r t^{n-1} u^q dt = \ell^q.$$

Hence from (2.13) and the fact that $(\alpha-1)/(p-1) = \lambda_1 + 1$ we get

$$\lim_{r\to\infty} \left(-r^{\lambda_1+1} u'\right)^{p-1} = 0$$

which implies, again by Hospital's rule, that $\ell = 0$ in (2.8), a contradiction by (2.7). Therefore $\ell = \infty$ and the claim is proved.

We now let $w = r^\lambda u(r)$. From (2.1) we obtain

$$\begin{aligned}(p-1)|r\,w' - \lambda\, w|^{p-2}[r^2 w'' + (\lambda_3 - 2\lambda + 1) r\, w' + \lambda\, w(\lambda - \lambda_3)]\\ + q(\lambda_1 - \lambda) r^{p-\lambda(q+1-p)} w^q + q\, r^{p+1-\lambda(q+1-p)} w^{q-1} w' = 0.\end{aligned} \tag{2.14}$$

Taking $\lambda = \lambda_1 = \lambda_2$ gives

$$\begin{aligned}w^{q-p} w' =& \frac{p-1}{q}(\lambda_3 - \lambda_1)\lambda_1^{p-1}\frac{1}{r} + \frac{p-1}{q}(\lambda_3 - \lambda_1)\lambda_1^{p-1}\frac{1}{r}\left(\left|1 - \frac{r\,w'}{\lambda_1 w}\right|^{p-2} - 1\right)\\ &- \frac{p-1}{q}\lambda_1^{p-2}\frac{1}{r}\left|1 - \frac{r\,w'}{\lambda_1 w}\right|^{p-2}\left[\frac{r^2 w''}{w} + (\lambda_3 - 2\lambda_1 + 1)\frac{r\,w'}{w}\right].\end{aligned} \tag{2.15}$$

Lemma 2.5. *Under the hypotheses of Theorem 2(ii)*

$$\lim_{r\to\infty} \frac{r\,w'}{w} = \lim_{r\to\infty} \frac{r^2 w''}{w} = 0.$$

Proof. Since $\alpha < n$ and $\lambda_1 = \lambda_2$ we have $\lambda_3 > \lambda_1$ and thus from Lemma 2.3(iv) the function $E_{\lambda_1}(r)$ is always positive. On the other hand from the facts that $\ell = \infty$ in (2.8) and that $\lambda_2 = \lambda_1$ we can apply Lemma 2.3(ii)–(iii) to conclude that $E_c(r) \leq 0$ ultimately for all $c < \lambda_1$. In conclusion

$$E_c(r) = r\,u'(r) + c\,u(r) = r^{-\lambda_1}(r\,w' + (c - \lambda_1) w)\begin{cases} \geq 0 & \text{for } c = \lambda_1 \\ \leq 0 & \text{for } c < \lambda_1 \text{ and } r \text{ large.}\end{cases} \tag{2.16}$$

Therefore

$$0 \leq r w'(r) \leq (\lambda_1 - c) w \quad \text{for } c < \lambda_1 \text{ and } r \geq r_c$$

which shows that that

$$\lim_{r\to\infty} \frac{r\,w'}{w} = 0. \tag{2.17}$$

Furthermore from (2.2), (2.8) and (2.16) we also obtain that

$$\lim_{r\to\infty} r^c u(r) = \begin{cases} \infty & \text{for } c > \lambda_1 \\ 0 & \text{for } c < \lambda_1 \end{cases}$$

and thus by Hospital's rule,

$$\lim_{r\to\infty} r^{c+1} u'(r) = \begin{cases} -\infty & \text{for } c > \lambda_1 \\ 0 & \text{for } c < \lambda_1, \end{cases}$$

where the limit exists by Lemma 2.4(ii) and the fact that

$$\left(r^{c+1}u'(r)\right)' = r^c F_c(r). \tag{2.18}$$

In turn

$$\begin{aligned} F_c(r) &= r\,u''(r) + (c+1)u'(r) = r^{-\lambda_1-2}[r^2 w'' \\ &\quad + (c+1-2\lambda_1) r\,w' + \lambda_1(\lambda_1 - c)w] \begin{cases} \le 0 & \text{for } c > \lambda_1 \text{ and } r \text{ large} \\ \ge 0 & \text{for } c < \lambda_1 \text{ and } r \text{ large.} \end{cases} \end{aligned} \tag{2.19}$$

Consequently for $c_1 < \lambda_1 < c_2$ and $r \ge r^* = r^*(c_1, c_2)$

$$-\lambda_1(\lambda_1 - c_1) - (c_1 + 1 - 2\lambda_1)\frac{r\,w'}{w} \le \frac{r^2 w''}{w} \le -(c_2 + 1 - 2\lambda_1)\frac{r\,w'}{w} + \lambda_1(c_2 - \lambda_1).$$

If we now use (2.17) and the fact that c_1 and c_2 can be taken arbitrarily close to λ_1 we obtain that $\lim_{r\to\infty} r^2 w''/w = 0$ and the proof is complete.

Proof of Theorem 2(ii) completed. By Lemma 2.5 and (2.15)

$$w^{q-p} w' = \frac{p-1}{q}(\lambda_3 - \lambda_1)\lambda_1^{p-1}\frac{1}{r} + o\left(\frac{1}{r}\right),$$

which after integration gives

$$\frac{w^{q-p+1}}{\log r} = \frac{p(p-1)}{q}(\lambda_3 - \lambda_1)\lambda_1^{p-2} + o(1)$$

and completes the proof of Theorem 2(ii).

Proof of Theorem 2(iii). From (1.9)

$$r^{\lambda_2 q}u^q = r^{\lambda_2 q-1}|u'|^{p-1} + (n-\alpha)r^{\lambda_2 q-n}\int_0^r t^{n-1}u^q dt. \tag{2.20}$$

Since $E_{\lambda_2}(0) = \lambda_2 u_0 > 0$, for r small $E_{\lambda_2}(r) = -r|u'(r)| + \lambda_2 u(r) > 0$ and the function $r^{\lambda_2}u(r)$ is increasing. Consequently, from (2.20)

$$\left(r^{\lambda_2}u(r)\right)^q < \lambda_2^{p-1}\left(r^{\lambda_2}u(r)\right)^{p-1} + (n-\alpha)r^{\lambda_2 q-n}\left(r^{\lambda_2}u(r)\right)^q\int_0^r t^{n-1-\lambda_2 q}dt$$

or equivalently

$$\left(r^{\lambda_2}u(r)\right)^q < \lambda_2^{p-1}\left(r^{\lambda_2}u(r)\right)^{p-1} + \frac{n-\alpha}{n-\lambda_2 q}\left(r^{\lambda_2}u(r)\right)^q$$

as long as $E_{\lambda_2}(r) > 0$, where we have used the fact that $n > \lambda_2 q$. In turn, since $n/q > \lambda_1 > \lambda_2$

$$\left(r^{\lambda_2}u(r)\right)^{q-p+1} < \frac{n-\lambda_2 q}{\alpha-\lambda_2 q}\lambda_2^{p-1} = \lambda_2^{p-1}\frac{p-1}{q}\cdot\frac{\lambda_3-\lambda_2}{\lambda_1-\lambda_2} = \ell_1^{q-p+1}. \tag{2.21}$$

Assume for contradiction that there exists $r_1 > 0$ such that $E_{\lambda_2}(r_1) = 0$. Then $E'_{\lambda_2}(r_1) \le 0$ and hence by Lemma 2.3(i) we have $r_1^{\lambda_2}u(r_1) \ge \ell_1$, which, together with (2.21) implies that

$$r_1^{\lambda_2}u(r_1) = \ell_1, \qquad r_1^{\lambda_2+1}u'(r_1) = -\lambda_2 r_1^{\lambda_2}u(r_1) = -\lambda_2\ell_1. \tag{2.22}$$

Making the change of variables

$$x = r^{\lambda_2}u(r), \qquad w = r^{\lambda_2+1}u'(r), \qquad t = \log r \tag{2.23}$$

we obtain the first order *autonomous* system

$$\begin{aligned} x'(t) &= \lambda_2 x(t) + w(t) \\ w'(t) &= (\lambda_2-\lambda_3)w(t) - \frac{q}{p-1}x^{q-1}(t)|w|^{2-p}w(t) - \frac{\alpha}{p-1}x^q(t)|w|^{2-p}(t). \end{aligned} \tag{2.24}$$

The equilibrium positions of (2.24) are $(0,0)$ and $(\ell_1, -\lambda_2\ell_1)$. Since at the time $t_1 = \log r_1$ we have $x(t_1) = r_1^{\lambda_2}u(r_1) = \ell_1$ and $w(t_1) = r_1^{\lambda_2+1}u'(r_1) = -\lambda_2\ell_1$ by standard uniqueness theory we get that $r^{\lambda_2}u(r) \equiv \ell_1$ and $r^{\lambda_2+1}u'(r) \equiv -\lambda_2\ell_1$ which is a contradiction. Therefore $E_{\lambda_2}(r) > 0$ for all $r \ge 0$ and thus, by (2.2) there exists

$$\lim_{r\to\infty} r^{\lambda_2}u(r) = \ell_2 \in (0, \ell_1]$$

by (2.5) and (2.21). To show that $\ell_2 = \ell_1$ we proceed somewhat as in the first part of Theorem 2(ii). We omit the details.

§3. Proof of Theorems 3 and 4.

In this section we give the proof of Theorems 3 and 4.

Proof of Theorem 3. By (1.9) and the fact that $\alpha > n$

$$\left(r^{n-1}|u'|^{p-2}u' + r^n u^q\right)' < 0 \qquad \text{for all } r \in (0,R) \tag{3.1}$$

and in turn

$$|u'|^{p-2}u' + r\,u^q < 0 \qquad \text{for all } r \in (0,R). \tag{3.2}$$

The remaining of the proof is now essentially as in Theorems 2.2, 2.3, 6.1 and 6.2 of [22] (see also [20, 21]) and therefore we omit it.

Proof of Theorem 4. When $R = \infty$, it is not difficult to see (cf. [22, Theorems 2.1 and 2.2]), using (3.1) and (3.2), that

$$c_1 r^{-\lambda_3} \le u(r) \le c_2 r^{-\lambda_2} \quad \text{for } r \text{ large.} \tag{3.3}$$

Since $\lambda_3 \ge \lambda_1$, by Lemma 2.3(iv) the function $E_{\lambda_1}(r) = \lambda_1 u + ru' > 0$ for all $r \in [0,R)$ and hence $R = \infty$ by (2.2). Moreover by (2.4) and Prop. 2.1(i)

$$\lambda_1 u(r) \ge E_{\lambda_1}(r) \ge E_{\lambda_1}(r_0)\exp\left(-\frac{q}{p-1}\int_{r_0}^{r} u^{q-1}|u'|^{2-p}t\,dt\right) \ge 0. \tag{3.4}$$

Assume now for contradiction that

$$\lim_{r\to\infty} r^{\lambda_2}u(r) = 0; \tag{3.5}$$

then by Lemma 2.3(ii)–(iii) the function $E_c(r)$ is ultimately negative for $c < \lambda_2$. Therefore

$$cu(r) \le r|u'(r)| \le \lambda_1 u(r) \quad \text{for all } r \ge r_c. \tag{3.6}$$

Fix $\epsilon < \lambda_2$, then for $r \ge r_\epsilon \ge r_c$

$$\frac{q}{p-1}\,ru^{q-1}|u'|^{2-p} \le \frac{q}{p-1}\,c_1^{2-p}r^{p-1}u^{q+1-p} \le \epsilon r^{-1}$$

by (3.5) and (3.6), where $c_1 = \lambda_1$ if $p < 2$ and $c_1 = c$ otherwise. Therefore from (3.4), with $r_0 = r_\epsilon$,

$$\lambda_1 u(r) \ge E_{\lambda_1}(r_\epsilon)\exp\left(-\epsilon\log(r/r_\epsilon)\right) = E_{\lambda_1}(r_\epsilon)\frac{r_\epsilon^\epsilon}{r^\epsilon},$$

which contradicts $(3.3)_2$ by the choice of ϵ. Hence, also by $(3.3)_2$,

$$\limsup_{r\to\infty} r^{\lambda_2}u(r) = \ell_1 \in (0,\infty). \tag{3.7}$$

If $E_{\lambda_2}(r)$ is ultimately of the sign then (3.7) is actually a limit by (2.2) and reasoning as in the first part of the proof of Theorem 2(ii) we can show that ℓ_1 is as in (1.12).

If $E_{\lambda_2}(r)$ changes sign infinitely many times as $r\to\infty$ the situation is more complicated. In this case we return to the autonomcus system (2.24). The linearization of (2.24) at the equilibrium point $(\ell_1, -\lambda_2\ell_1)$, where the value ℓ_1 is given in (2.21), leads to the system

$$\begin{aligned} x'(t) &= \lambda_2 x(t) + w(t) \\ w'(t) &= q\,\lambda_2\frac{\lambda_3-\lambda_2}{\lambda_2-\lambda_1}\left[\lambda_1 - \lambda_2\frac{(q-1)}{q}\right]x + \frac{\lambda_2-\lambda_3}{\lambda_2-\lambda_1}\left[\lambda_2(p-2)-\lambda_1(p-1)\right]w. \end{aligned} \tag{3.8}$$

The determinant and the trace of the associated matrix A are respectively

$$\det A = p(\lambda_3-\lambda_2), \quad \operatorname{trace} A = \frac{(\lambda_1-\lambda_2)[\lambda_2+(p-1)(\lambda_2-\lambda_3)] - \lambda_2(\lambda_3-\lambda_2)}{\lambda_1-\lambda_2}. \tag{3.9}$$

Since $\lambda_1,\ \lambda_3 > \lambda_2$ we have in particular that $\det A > 0$. A straightforward calculation shows that

$$\lambda_2 + (p-1)(\lambda_2-\lambda_3) > 0 \quad \text{iff } q \text{ is subcritical.} \tag{3.10}$$

Therefore when $q \geq q_c$ the trace of A is always negative, while when $q < q_c$ then the trace of A is negative iff

$$\lambda_1 < \lambda_c = \lambda_3 + \frac{(p-1)(\lambda_2-\lambda_3)^2}{\lambda_2+(p-1)(\lambda_2-\lambda_3)}. \tag{3.11}$$

By standard dynamical system theory it follows that when $\lambda_1,\ \lambda_3 > \lambda_2$ the equilibrium point $(\ell_1, -\lambda_2\ell_1)$ is an *attractor* when either $q \geq q_c$ or $q < q_c$ and $\lambda_1 < \lambda_c$, while it is a *repeller* when $q < q_c$ and $\lambda_1 > \lambda_c$.

Since under the hypotheses of Theorem 4 we have $\lambda_1 \leq \lambda_3$ then the equilibrium point $(\ell_1, -\lambda_2\ell_1)$ is an attractor.

By $(3.3)_1$ and the fact that $E_{\lambda_1}(r) = \lambda_1 u(r) - r|u'(r)| > 0$ we have

$$x(t) = r^{\lambda_2}u(r) \leq C, \qquad |w(t)| = r^{\lambda_2+1}|u'(r)| \leq C$$

and hence by Poincaré–Bendixson theory either (x,w) tends to the equilibrium point $(\ell_1, -\lambda_2\ell_1)$ or there exists a closed orbit γ around $(\ell_1, -\lambda_2\ell_1)$ which does not touch the axes and such that (x,w) spirals toward γ. When q is supercritical we can exclude the second possibility. Let $w = y|y|^{(2-p)/(p-1)}$, then (2.24) becomes

$$\begin{aligned} x'(t) &= \lambda_2 x + y|y|^{(2-p)/(p-1)} = P(x,y) \\ y'(t) &= (p-1)(\lambda_2-\lambda_3)y - q\,x^{q-1}y|y|^{(2-p)/(p-1)} - \alpha x^q = Q(x,y). \end{aligned} \tag{3.12}$$

Since

$$\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} = \lambda_2 + (p-1)(\lambda_2 - \lambda_3) - \frac{q}{p-1}\, x^{q-1}|y|^{(2-p)/(p-1)} < 0$$

in the simply connected region $E = \{(x,y) : x > 0, \quad y < 0\}$ by (3.10) we can apply Bendixon's criterion (see Theorem 3.9.1 in [24] or Theorem 1.10 in [29]) to conclude that when $q \geq q_c$ there are no closed orbits in E and thus (x,w) approaches the equilibrium point $(\ell_1, -\lambda_2\ell_1)$. If $q < q_c$ we cannot apply Bendixon's criterion, so we cannot exclude the existence of a limit circle and we can only conclude that (1.14) holds.

§4. Proof of Theorems 5 and 6.

In this section we give the proof of Theorems 5 and 6. The two main tools are the following Pohozaev–Pucci–Serrin differential identity [25]

$$(4.1)\qquad \begin{aligned} &\left\{r^k\left[\tfrac{p-1}{p}\,|u'|^p + \tfrac{\alpha}{q+1}\,u^{q+1}\right] + a\,r^{k-1}|u'|^{p-2}u'u\right\}' \\ &\quad = r^{k-1}|u'|^p\left[\tfrac{(p-1)k}{p} - (n-1) + a\right] + \alpha\,r^{k-1}u^{q+1}\left(\tfrac{k}{q+1} - a\right) \\ &\quad\quad + a(k-n)r^{k-2}|u'|^{p-2}u'u + q\,r^k u^{q-1}|u'|(a\,u + r\,u') \end{aligned}$$

and an extension of Poincaré inequality which is due to Serrin and Zou [27].

Proof of Theorem 5. We claim that $E_{\lambda_1}(r) > 0$ for all $r \geq 0$ when $R = \infty$. Indeed from $\lambda_1 > \lambda_3$ and $(3.3)_2$ we get $\lim_{r\to\infty} r^{\lambda_1}u(r) = \infty$. By (2.2) and Lemma 2.3(iv) it follows that $E_{\lambda_1}(r) > 0$ cannot have a zero. Therefore

$$(4.2)\qquad r|u'(r)| < \lambda_1 u(r) \qquad \text{for all } r \geq 0.$$

Property (4.2) allows to extend a result of Serrin and Zou [27] for $R < \infty$ to the case $R = \infty$.

Lemma 4.1 (Serrin and Zou). *Let u be a positive solution of* (2.1) *in* $[0,R)$ *and* μ, ν, $\gamma > 0$, *with* $\gamma > \nu$. *If* $R = \infty$ *assume that* (4.2) *holds and that*

$$(4.3)\qquad \lim_{r\to\infty} r^{\gamma-\nu}u^{\mu+\nu}(r) = 0, \qquad \int_1^\infty r^{\gamma-\nu-1}u^{\mu+\nu}(r) < \infty.$$

Then

$$(4.4)\qquad \int_0^R u^\mu |u'|^\nu\, r^{\gamma-1}dr \leq \frac{\mu+\nu}{\gamma-\nu}\int_0^R u^{\mu-1}|u'|^{\nu+1}\, r^\gamma dr.$$

Proof. The proof is exactly the same of Proposition 2 in [27], with the exception that when $\Omega = \mathbb{R}^n$ the condition $u = 0$ on $\partial\Omega$ should be replaced by $(4.3)_1$, while conditions (4.2) and $(4.3)_2$ are only used to guarantee that all the integrals in (4.4) are finite. Note also that in Proposition 2 in [27] $\mu = 1$, but the proof there carries out also for $\mu > 0$.

Lemma 4.2. *Let u be a positive solution of* (2.1) *in* $[0,\infty)$. *Assume that*

$$\lim_{r\to\infty} r^{\lambda_2}u(r)=0. \tag{4.5}$$

then

$$\lim_{r\to\infty} r^{\lambda_3}u(r)=\ell\in(0,\infty) \quad \textit{and} \quad \lim_{r\to\infty} r^{\lambda_3+1}u'(r)=-\ell\,\lambda_3. \tag{4.6}$$

Proof. We first claim that

$$r^{\lambda_3}u(r)\le C \qquad \text{for all } r\ge 0, \tag{4.7}$$

for some constant C. Property (4.7) has been proved by Serrin and Zou [26, Lemma 4.5] for the equation $\Delta u+u^p-|\nabla u|^q=0$. Their proof also works for (2.1) and we present it here for the convenience of the reader. We now apply identity (4.1) with $k=p(n-1-\delta)/(p-1)$ and $a=0$. We obtain

$$\left\{r^k\left[\tfrac{p-1}{p}\,|u'|^p+\tfrac{\alpha}{q+1}\,u^{q+1}\right]\right\}'=-\delta\, r^{k-1}|u'|^p+\tfrac{\alpha\, k}{q+1}\, r^{k-1}u^{q+1}-q\, r^{k+1}u^{q-1}|u'|^2 \tag{4.8}$$

By Lemma 2.3(iii), (2.2) and (4.6) the function $E_{\lambda_2}(r)=-r|u'(r)|+\lambda_2 u(r)\le 0$ for r sufficiently large, therefore

$$u^{q+1}(r)\le u^{q+1-p}r^p|u'|^p\lambda_2^{-p}.$$

In turn by (4.8)

$$\left\{r^k\left[\tfrac{p-1}{p}\,|u'|^p+\tfrac{\alpha}{q+1}\,u^{q+1}\right]\right\}'\le r^{k-1}|u'|^p\left(-\delta+\tfrac{\alpha\, k}{(q+1)\lambda_2^p}\, r^p u^{q+1-p}\right)<0$$

for r sufficiently large by (4.5), which shows that

$$r^{k/p}|u'(r)|\le C \qquad r^{(k-p)/p}u(r)\le C. \tag{4.9}$$

Another use of identity (4.1), this time with $k=p(n-1)/(p-1)$ and $a=0$ gives

$$\left\{r^k\left[\tfrac{p-1}{p}\,|u'|^p+\tfrac{\alpha}{q+1}\,u^{q+1}\right]\right\}'=\tfrac{\alpha\, k}{q+1}\, r^{k-1}u^{q+1}-q\, r^{k+1}u^{q-1}|u'|^2\le C_1 r^{k-1+(q+1)(p-k)/p}$$

by (4.9). Integration of the previous inequality, together with the fact that $k+(q+1)(p-k)/p<0$, gives the desired estimate (4.7).

By Lemma 2.3(v), (2.2), (3.3) and (4.7) we obtain $(4.6)_1$. To prove $(4.6)_2$ we proceed somewhat as in the first part of Theorem 2(ii). We omit the details.

Proof of Theorem 5 continued. We first prove that $R = \infty$. Thus assume for contradiction that $R < \infty$. By (3.1) and (3.2) the function u cannot have compact support. Applying identity (4.1) with $k = n$ and $a = n/(q+1)$, we obtain

$$\begin{aligned} &r^n\left[\tfrac{p-1}{p}|u'|^p + \tfrac{\alpha}{q+1}u^{q+1}\right] + \tfrac{n}{q+1}r^{n-1}|u'|^{p-2}u'u \\ &= \left(\tfrac{n}{q+1} - \tfrac{n-p}{p}\right)\int_0^r t^{n-1}|u'|^p dt + \tfrac{nq}{q+1}\int_0^r t^n u^q|u'|dt - q\int_0^r t^{n+1}u^{q-1}|u'|^2 dt. \end{aligned} \tag{4.10}$$

If we now take $r = R$ we obtain a contradiction, since the left hand side of (4.10) reduces to $R^n\frac{p-1}{p}|u'|^p(R) > 0$, while the right hand side is nonnegative by the fact that $q \geq q_c$ and by the inequality

$$\int_0^R r^n u^q|u'|\,dr \leq \frac{q+1}{n}\int_0^R r^{n+1}u^{q-1}|u'|^2 dr \tag{4.11}$$

which follows from Lemma 4.1 by taking

$$\mu = q, \qquad \nu = 1, \qquad \gamma = n+1. \tag{4.12}$$

This shows that all solutions are positive.

To prove (1.12) assume again for contradiction that (4.5) holds. By Lemma 4.2

$$r^n u^{q+1} = O\left(r^{-\lambda_3(q-q_c)-\frac{n}{(p-1)}}\right), \quad r^{n-1}|u'|^{p-2}u'u = O\left(r^{-\lambda_3-1}\right), \quad r^n|u'|^p = O\left(r^{-\lambda_3}\right).$$

Therefore condition (4.3) is satisfied when the parameters μ, ν and γ are chosen as in (4.12) so we can apply Lemma 4.1 to obtain (4.11) with $R = \infty$. Furthermore we can let $r \to \infty$ in (4.10) and we get

$$0 \leq \left(\tfrac{n}{q+1} - \tfrac{n-p}{p}\right)\int_0^\infty t^{n-1}|u'|^p dt + \tfrac{nq}{q+1}\int_0^\infty t^n u^q|u'|dt - q\int_0^\infty t^{n+1}u^{q-1}|u'|^2 dt \leq 0.$$

If $q > q_c$ we immediately get a contradiction, while if $q = q_c$ then we conclude that (4.11) is an equality, but since the only tool in the proof of Proposition 2 in [27] is Holder's inequality then we can have equality sign in (4.11) iff $r^n u^q|u'| = \text{Const.}\ r^{n+1}u^{q-1}|u'|^2$ which is clearly impossible. Consequently (3.7) holds and we can continue as in the proof of Theorem 4 to obtain (1.12).

Proof of Theorem 6. To prove Theorem 6 we use the following modified Pohozaev–Pucci–Serrin identity

$$\begin{aligned} &\left\{r^k\left[\tfrac{p-1}{p}|u'|^p + \tfrac{\alpha(1-\epsilon)}{q+1}u^{q+1}\right] + a\,r^{k-1}|u'|^{p-2}u'u\right\}' \\ &\qquad = r^{k-1}|u'|^p\left[\tfrac{(p-1)k}{p} - (n-1) + a\right] + \alpha\,r^{k-1}u^{q+1}\left(\tfrac{k(1-\epsilon)}{q+1} - a\right) \\ &\qquad\quad + a(k-n)r^{k-2}|u'|^{p-2}u'u + q\,r^k u^{q-1}|u'|[(a+\lambda_1\epsilon)\,u + r\,u']. \end{aligned} \tag{4.13}$$

Choosing

$$k = n, \qquad a = \frac{n-p}{n}, \qquad \epsilon = \frac{(n-p)(q_c - q)}{np} < 1$$

in (4.13) gives

$$\left\{ r^n \left[\tfrac{p-1}{p} \, |u'|^p + \tfrac{\alpha(1-\epsilon)}{q+1} \, u^{q+1} \right] + a\, r^{n-1} |u'|^{p-2} u' u \right\}' = q\, r^n u^{q-1} |u'| E_c(r), \tag{4.14}$$

where

$$c = \frac{n-p}{p} \left(1 + \lambda_1 \frac{q_c - q}{n} \right).$$

Since $\lambda_1 \geq \frac{n}{(p-1)(q_c-q)}$ it follows that $c \geq \lambda_3$. By (4.2) and Lemma 2.3(v) the function $E_c(r)$ is positive for all $r \geq 0$. Therefore the right hand side of (4.14) is positive and we can continue as in the proof of Theorem 3.1 of Ni and Serrin [22].

§5. Applications to parabolic equations.

We consider the parabolic equation (1.4)

$$q\, v^{q-1} v_t = \operatorname{div}(|\nabla v|^{p-2} \nabla v), \tag{5.1}$$

Due to the symmetry of the problem it is natural to study *selfsimilar* solutions of the form

$$v(x,t) = t^{-a/b} u\left(|x| t^{-1/b} \right), \qquad a,\, b > 0, \tag{5.2}$$

where $u(r)$ is a nonnegative solution of

$$\left(|u'|^{p-2} u' \right)' + \frac{n-1}{r} \, |u'|^{p-2} u' + \frac{1}{b} r (u^q)' + \frac{a}{b} \, u^q = 0, \quad r = |x| t^{-1/b} > 0 \tag{5.3}$$

with

$$a(q-p+1) + b = p. \tag{5.4}$$

Clearly, for (5.4) to hold we need to impose some restrictions on the exponent α, more precisely $(q-p+1)a > -p$. A simple change of scale transforms (5.3) in (2.1) where $\alpha = a\, q$.

Theorem A. *The initial value problem*

$$q\, v^{q-1} v_t = \operatorname{div}(|\nabla v|^{p-2} \nabla v),$$
$$v(x,0) = A\, |x|^{-a},$$

where A, $a > 0$,admits a selfsimilar solution v of the form (5.2) *if and only if*

$$a\, q < n \qquad \textit{and} \qquad a(q+1-p) < p.$$

References

1. D.G. Aronson, *The porous media equation*, Nonlinear diffusion problems (A. Fasano and M. Primicerio, eds.), Lecture Notes in Mathematics, vol. 1224, Springer–Verlag, New York, 1986.
2. G.I. Barenblatt, *On self-similar motions of compressible fluids in porous media*, (Russian), Prikl. Math. Mech. **16** (1952), 679–698.
3. P. Bénilan, M.G. Crandall and M. Pierre, *Solutions of the porous media equation in* $\mathbb{R}^N$ *under optimal conditions on initial values*, Indiana Univ. Math. J. **33** (1984), 51–87.
4. G. Caristi and E. Mitidieri, *Nonexistence of positive solutions of quasilinear equations* (to appear).
5. P. Clément, D.G. de Figueiredo and E. Mitidieri, *Quasilinear elliptic equations with critical exponents*, Topol. Methods Nonlinear Anal. **7** (1996), 133–170.
6. J.I. Diaz and J. E. Saa, *Uniqueness of very singular self-similar solution of a quasilinear degenerate parabolic equation with absorption*, Publ. Matem. **36** (1992), 19–38.
7. E. DiBenedetto and M.A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. Amer. Math. Soc. **314** (1991), 187–224.
8. E. DiBenedetto and M.A. Herrero, *Non negative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when* $1 < p < 2$, Arch. Rational Mech. Anal. **111** (1990), 985–1039.
9. C. Dohmen, *Selfsimilar solutions of the porous media equation without sign restriction*, Hokkaido Math. J. **23** (1994), 475–505.
10. J.R. Esteban and J.L. Vazquez, *Homogeneous diffusion in* $\mathbb{R}$ *with power-like diffusivity*, Arch. Rational Mech. Anal. **103** (1988), 39–88.
11. B.H. Gilding and L.A. Peletier, *On a class of similarity solutions of the porous media equation*, J. Math. Anal. Appl. **55** (1976), 351–364.
12. M.A. Herrero and M. Pierre, *The Cauchy problem for* $u_t = \triangle u^m$ *when* $0 < m < 1$, Trans. Amer. Math. Soc. **291** (1985), 145–158.
13. J. Hulshof, *Similarity solutions of the porous media equation with sign changes*, J. Math. Anal. Appl. **157** (1991), 75–111.
14. J. Hulshof and J.L. Vazquez, *Self-similar solutions of the second kind for the modified porous media equation*, European J. Appl. Math. **5** (1994), 391–403.
15. R.A. Johnson, X. Pan and Y. Yi, *Singular ground states of semilinear elliptic equations via invariant manifold theory*, Nonlinear Anal. **20** (1993), 1279–1302.
16. A.S. Kalashnikov, *On a nonlinear equation appearing in the theory of nonstationary filtration*, (Russian), Trud. Sem. I. G. Petrovsky **4** (1978), 137–146.
17. S. Kamin and L.A. Peletier, *Large time behaviour of solutions of the porous media equation with absorption*, Israel. J. Math. **55** (1986), 129–146.
18. G. Leoni, *Existence of solutions for strongly degenerate differential systems*, Calc. Var. Partial Differential Equations (to appear).
19. G. Leoni, *On very singular selfsimilar solutions for the porous media equation with absorption*, Differential Integral Equations (to appear).
20. G. Leoni, *A very singular solution for the porous media equation* $u_t = \triangle(u^m) - u^p$ *when* $0 < m < 1$, J. Differential Equations **132** (1996), 353–376.
21. G. Leoni, *On very singular solutions for a degenerate parabolic equation with absorption* (to appear).
22. W.M. Ni and J. Serrin, *Existence and non-existence theorems for ground states of quasilinear partial differential equations. The anomalous case*, Accad. Naz. Lincei, Convegni dei Lincei **77** (1986), 231-257.
23. L.A. Peletier and Z. Junning, *Large time behaviour of solutions of the porous media equation with absorption: the fast diffusion case*, Nonlinear Anal. **17** (1991), 99–109.
24. L. Perko, *Differential equations and dynamical systems*, Texts in Applied Mathematics, vol. 7, Springer–Verlag, New York, 1996.
25. P. Pucci and J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), 681–703.

26. J. Serrin and H. Zou, *Classification of positive solutions of quasilinear elliptic equations*, Topol. Methods Nonlinear Anal. **3** (1994), 1–25.
27. J. Serrin and H. Zou, *Existence and non-existence results for ground states of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **121** (1992), 101–130.
28. L. Tartar, *Special solutions of* $u_t = \Delta(u^m)$ *and asymptotic behaviour*, unpublished (1986).
29. Y. Yan–Qian and Others, *Theory of limit cycles*, Translations of Mathematical Monographs, vol. 66, Amer. Math. Soc., Providence, R.I., 1986.

Center for Nonlinear Analysis, Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213, USA

Current address: Department of Mathematics, University of Perugia, Via Vanvitelli 1, Perugia, Italy 06123

E-mail address: leoni@dipmat.unipg.it