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Vlasov-Poisson-Fokker-Planck
System**

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A Difference Scheme for the Vlasov-Poisson-Fokker-Planck System

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Abstract

A finite difference scheme is proposed for a relatively simple one dimensional collisional plasma model (Vlasov-Poisson-Fokker-Planck). The convergence of the scheme was established in [10]. In this article the scheme is implemented and compared with a random particle method. The difference scheme is found to be more effective, particularly at resolving the plasma density, $f(t, x, v)$.

1 Introduction

In this article we consider a relatively simple one dimensional collisional plasma model. A finite difference scheme for this model is proposed and implemented. This scheme handles the convective part of the problem by a splitting procedure proposed by Cheng and Knorr [4]. The convergence of the difference scheme used in this article has been established in [10]. In this article the results of implementing this method are described and comparison with a random particle method is made. The result of the comparison is that (at least for this model problem) the finite difference scheme produces better results for comparable work in approximating the electric field and the total kinetic energy. In approximating the underlying density of ions in phase space the difference scheme is vastly superior.

The use of particle methods in simulating collisionless plasma is well developed ([2], [7]). Random particle methods have been found to be effective for collisional models such as that considered in this article [1]. However, there are also deterministic particle methods that may be used in collisional problems also ([5], [6], [9]).

We will consider the following model problem:

$$\partial_t f + v \partial_x f - E(t, x) \partial_v f = \sigma \partial_v^2 f + \beta \partial_v (vf) \quad (1)$$

$$\partial_x E(t, x) = b(x) - \int f dv. \quad (2)$$

Here $t \geq 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$ and f gives the number density of mobile ions in phase space (with mass one and charge minus one). The ions described by f collide with a secondary population. These collisions produce a diffusion with parameter $\sigma > 0$ and exert a viscous drag with parameter $\beta > 0$. See [3] for a more complete discussion of these parameters. $b(x)$ is a given background charge density. We will consider a periodic situation in which

$$f(t, x + 1, v) = f(t, x, v)$$

and

$$\int_0^1 E(t, x) dx = 0.$$

Correspondingly a periodic initial condition

$$f(0, x, v) \geq 0$$

is given.

2 Description of Methods

To motivate our finite difference scheme we point out that if (1) were simply

$$\partial_t f = \sigma \partial_v^2 f, \quad (3)$$

that is the heat equation, the backward Euler scheme is known to give good results. Let $\Delta t > 0$, $\Delta x > 0$, $\Delta v > 0$, $t^n = n\Delta t$, $x_k = k\Delta x$, $v_\ell = \ell\Delta v$, and

$$f_{k\ell}^n \approx f(t^n, x_k, v_\ell),$$

then the backward Euler scheme for (3) is

$$\frac{f_{k\ell}^{n+1} - f_{k\ell}^n}{\Delta t} = \sigma \frac{f_{k,\ell+1}^{n+1} - 2f_{k\ell}^{n+1} + f_{k,\ell-1}^{n+1}}{(\Delta v)^2}. \quad (4)$$

An implicit scheme, such as (4) is preferred here because of the infinite speed of propagation in (3). Note that to solve (4) we must solve a tridiagonal system for each k , which may be done as efficiently as implementing an explicit scheme. Our scheme for (1) is to be this way also.

We comment that although the Crank Nicolson scheme is second order in time and backward Euler is only first order, the second order accuracy is lost when the transport terms of (1) are included. Hence we opt for the simpler backward Euler methodology.

The main issue now is the transport effect represented by the left hand side of (1). Let us define the characteristics $(X(s, t, x, v), V(s, t, x, v))$ associated with (the left hand side of) (1) by

$$\begin{aligned} \frac{dX}{ds} &= V & X(t, t, x, v) &= x \\ \frac{dV}{ds} &= -E(s, X) & V(t, t, x, v) &= v, \end{aligned}$$

then

$$\frac{d}{ds} f(s, X, V) = (\partial_t f + V \partial_x f - E(s, X) \partial_v f) |_{(s, X, V)}.$$

Let us abbreviate

$$X(s) = X(s, t^{n+1}, x_k, v_\ell), V(s) = V(s, t^{n+1}, x_k, v_\ell)$$

then by (1)

$$\begin{aligned} & \frac{f(t^{n+1}, x_k, v_\ell) - f(t^n, X(t^n), V(t^n))}{\Delta t} \\ & \approx \frac{d}{ds} [f(s, X(s), V(s))] |_{s=t^{n+1}} \\ & = [\sigma \partial_v^2 f + \beta \partial_v (v f)] |_{(t^{n+1}, x_k, v_\ell)}. \end{aligned} \tag{5}$$

To derive a difference scheme from this we must approximate $f(t^n, X(t^n), V(t^n))$ using values of f at grid locations. We will develop this approximation shortly, but for now let us denote it by

$$(Sf^n)_{k\ell} \approx f(t^n, X(t^n, t^{n+1}, x_k, v_\ell), V(t^n, t^{n+1}, x_k, v_\ell)). \tag{6}$$

Then using the abbreviation

$$v_{\ell \pm \frac{1}{2}} = \left(\ell \pm \frac{1}{2} \right) \Delta v$$

our scheme derived from (5) is

$$\begin{aligned} \frac{f_{k\ell}^{n+1} - (Sf^n)_{k\ell}}{\Delta t} & = \sigma \frac{f_{k,\ell+1}^{n+1} - 2f_{k\ell}^{n+1} + f_{k,\ell-1}^{n+1}}{(\Delta v)^2} \\ & + \beta \frac{v_{\ell+\frac{1}{2}} \frac{1}{2} (f_{k,\ell+1}^{n+1} + f_{k,\ell}^{n+1}) - v_{\ell-\frac{1}{2}} \frac{1}{2} (f_{k\ell}^{n+1} + f_{k,\ell-1}^{n+1})}{\Delta v}. \end{aligned} \tag{7}$$

We require (7) to hold for $|\ell| < L$ and impose

$$f_{k\ell}^{n+1} = 0 \quad \text{if } |\ell| \geq L.$$

The last term of (7) was constructed in such a way that multiplying (7) by $f_{k\ell}^{n+1}$ and summing over $k = 1, \dots, K$ (where $\Delta x = \frac{1}{K}$) and $|\ell| < L$ yields

(after summation by parts, see [10], Proposition 4)

$$\begin{aligned} & \frac{1}{\Delta t} \sum_{k=1}^K \sum_{|\ell| < L} \left((f_{k\ell}^{n+1})^2 - f_{k\ell}^{n+1} (Sf^n)_{k\ell} \right) \\ &= -\sigma \sum_{k=1}^K \sum_{\ell=-L}^{L-1} \left(\frac{f_{k,\ell+1}^{n+1} - f_{k,\ell}^{n+1}}{\Delta v} \right)^2 + \frac{1}{2} \beta \sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^{n+1})^2. \end{aligned}$$

The approximation $(Sf^n)_{k\ell}$ is constructed so that

$$\sum_{k=1}^K \sum_{|\ell| < L} ((Sf^n)_{k\ell})^2 \leq \sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^n)^2 \quad (8)$$

(see [10], Proposition 3) so by the Schwarz inequality

$$\begin{aligned} \sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^{n+1})^2 &\leq \sqrt{\sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^{n+1})^2} \sqrt{\sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^n)^2} \\ &\quad + \frac{1}{2} \beta \Delta t \sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^{n+1})^2 \end{aligned}$$

and hence for $\frac{1}{2} \beta \Delta t < 1$

$$\sqrt{\sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^{n+1})^2} \leq \frac{1}{1 - \frac{1}{2} \beta \Delta t} \sqrt{\sum_{k=1}^K \sum_{|\ell| < L} (f_{k\ell}^n)^2}.$$

This ensures the stability of the scheme when $\beta \Delta t < 2$, and we point out the crucial use of (8) in showing the stability.

Now we consider (6) more fully. Sf^n is computed from f^n in three steps. The first will be denoted with S_1 . If we dealt with functions defined at all points, S_1 would simply be given by

$$S_1 f(x, v) = f\left(x - \frac{1}{2} \Delta t v, v\right).$$

Since we must draw on values located at grid points we perform an interpolation: For any ℓ choose $m \in \mathbb{Z}$ and $\theta \in [0, 1)$ such that

$$-\frac{1}{2}\Delta t v_\ell = (m + \theta) \Delta x$$

so that

$$x_k - \frac{1}{2}\Delta t v_\ell = x_{m+k} + \theta \Delta x$$

for all k . Then we define

$$(S_1 f^n)_{k\ell} = (1 - \theta) f_{k+m,\ell}^n + \theta f_{k+m+1,\ell}^n.$$

For the second step we'll need to approximate $E\left(t^{n+\frac{1}{2}}, x_k\right)$. We define $E^{n+\frac{1}{2}}$ by

$$\frac{\bar{E}_{k+1}^{n+\frac{1}{2}} - \bar{E}_k^{n+\frac{1}{2}}}{\Delta x} = \frac{1}{2} \left[b(x_{k+1}) + b(x_k) - \sum_{\ell} (S_1 f^n)_{k+1,\ell} - \sum_{\ell} (S_1 f^n)_{k\ell} \right]$$

and

$$\frac{1}{2} \left(\bar{E}_0^{n+\frac{1}{2}} + \bar{E}_K^{n+\frac{1}{2}} \right) + \sum_{k=1}^{K-1} \bar{E}_k^{n+\frac{1}{2}} = 0.$$

In order to ensure periodicity we take

$$E_k^{n+\frac{1}{2}} = \bar{E}_k^{n+\frac{1}{2}} \text{ if } k = 1, \dots, K$$

and

$$E_{k\pm K}^{n+\frac{1}{2}} = E_k^{n+\frac{1}{2}} \text{ for all } k.$$

Now S_2 would be given by

$$S_2 g(x, v) = g\left(x, v + \Delta t E\left(t^{n+\frac{1}{2}}, x\right)\right)$$

if interpolation were not needed. For any k choose $m \in \mathbb{Z}$ and $\theta \in [0, 1)$ such that

$$\Delta t E_k^{n+\frac{1}{2}} = (m + \theta) \Delta v$$

so that

$$v_\ell + \Delta t E_k^{n+\frac{1}{2}} = v_{m+\ell} + \theta \Delta v$$

for all ℓ . Then we define

$$(S_2 S_1 f^n)_{k\ell} = (1 - \theta) (S_1 f^n)_{k,\ell+m} + \theta (S_1 f^n)_{k,\ell+m+1}.$$

Finally the first and third steps are the same so

$$(S f^n)_{k\ell} = (S_1 S_2 S_1 f^n)_{k\ell}.$$

We point out that once f^n is known, $S f^n$ may be computed explicitly. Then the scheme (7) is a tridiagonal system in ℓ for each k . The operation count per time step is linear in the number of grid points. It is shown in [10] (comment following Proposition 4) that the linear system involved is nonsingular as long as $\beta \Delta t < 2$.

We will compare the difference scheme with the following random particle method. The underlying approximation is

$$f(t, x, v) \approx \sum_{k,\ell} w_{k\ell} \delta(x - X_{k\ell}(t)) \delta(v - V_{k\ell}(t))$$

where

$$X_{k\ell}(0) = \left(k - \frac{1}{2}\right) \Delta x, \quad V_{k\ell}(0) = \left(\ell - \frac{1}{2}\right) \Delta v,$$

and

$$w_{k\ell} = f(0, X_{k\ell}(0), V_{k\ell}(0)) \Delta x \Delta v.$$

The particle motion satisfies the system of stochastic differential equations

$$\begin{cases} dX_{k\ell} = V_{k\ell} dt \\ dV_{k\ell} = (-E(t, X_{k\ell}) - \beta V_{k\ell}) dt + \sqrt{2} \sigma dB(t) \end{cases} \quad (9)$$

where $B(t)$ is a Brownian motion. Let

$$\delta_{\Delta x}(x) = \frac{1}{\Delta x} \left(1 - \left|\frac{x}{\Delta x}\right|\right) \quad \text{for } |x| < \Delta x,$$

and zero for $|x| \geq \Delta x$. We define \tilde{E} at grid points by

$$\frac{\tilde{E}(t, x_{k+1}) - \tilde{E}(t, x_k)}{\Delta x} = b\left(x_{k+\frac{1}{2}}\right) - \sum_{j,\ell} w_{j\ell} \delta_{\Delta x}\left(x_{k+\frac{1}{2}} - X_{j\ell}(t)\right)$$

and

$$\frac{1}{2} \left(\tilde{E}(t, 0) + \tilde{E}(t, x_K) \right) + \sum_{k=1}^{K-1} \tilde{E}(t, x_k) = 0.$$

For other values of x , $\tilde{E}(t, x)$ is defined by linear interpolation. Now we use the explicit order 1.5 strong scheme (page 383 of [8]) to track the solutions (9). For a system of stochastic differential equations of the form

$$d\bar{Y} = \bar{a}(t, \bar{Y}) dt + \bar{b} dB(t)$$

with \bar{b} constant, this scheme is

$$\begin{aligned} \bar{Y}^{\bar{n}+1} &= \bar{Y}^{\bar{n}} + \Delta t \bar{a}(t^{\bar{n}}, \bar{Y}^{\bar{n}}) + \Delta B \bar{b} \\ &+ \frac{1}{4} \left(\bar{a}(t^{\bar{n}+1}, \bar{\gamma}_+) - 2 \bar{a}(t^{\bar{n}}, \bar{Y}^{\bar{n}}) + \bar{a}(t^{\bar{n}+1}, \bar{\gamma}_-) \right) \Delta t \\ &+ \frac{1}{2\sqrt{\Delta t}} \left(\bar{a}(t^{\bar{n}+1}, \bar{\gamma}_+) - \bar{a}(t^{\bar{n}+1}, \bar{\gamma}_-) \right) \Delta Z \end{aligned}$$

where

$$\bar{\gamma}_{\pm} = \bar{Y}^{\bar{n}} + \Delta t \bar{a}(t^{\bar{n}}, \bar{Y}^{\bar{n}}) \pm \sqrt{\Delta t} \bar{b}$$

$$\Delta B = u_1 \sqrt{\Delta t}$$

$$\Delta Z = \left(u_1 + \frac{1}{\sqrt{3}} u_2 \right) \frac{1}{2} (\Delta t)^{\frac{3}{2}}$$

and u_1 and u_2 are normally distributed random numbers with mean zero and unit variance.

Note that each time step of this procedure requires the field E to be evaluated twice. The operation count per time step is linear in the number of particles.

3 A Steady State Solution

In this section we consider the following steady state solution of (1) and (2):

$$f(x, v) = \exp \left(\sin(2\pi x) - \frac{\beta}{\sigma} \frac{1}{2} v^2 \right), \quad (10)$$

$$E(x) = \frac{-\sigma}{\beta} 2\pi \cos 2\pi x, \quad (11)$$

where

$$b(x) = \frac{\sigma}{\beta} (2\pi)^2 \sin(2\pi x) + \sqrt{\frac{2\pi\sigma}{\beta}} \exp(\sin(2\pi x)). \quad (12)$$

All computations were performed in Fortran 77 on a SUN Sparc 5 workstation. The difference scheme was implemented in double precision.

In the first set of runs both σ and β were set equal to one and the number of grid points/particles was increased systematically. As described in the previous section the grid points for the difference scheme are closely associated with the initial states of the particles. In both cases $|v| > 6$ was truncated as the resulting error of order

$$\exp\left(-\frac{1}{2}(6)^2\right) \approx 1.5(10^{-8})$$

is comparable with round off. All runs in this section display results at the final time $t = 1$. Table I contains the results. The numbers in the column headed "E" are the relative errors in E in $L2$ norm, i.e.,

$$\frac{\sqrt{\sum_{k=1}^N (E(x_k) - E_k^n)^2}}{\sqrt{\sum_1^N E^2(x_k)}}$$

Similarly those under the heading "kinetic energy" are the relative errors in the total kinetic energy. Those under "f" are relative errors in f in $L2$ norm, which was computed only for the difference scheme. The top line pertains to the difference scheme, the bottom to the particle method. The run times for the two methods are comparable, the difference scheme being about 20% faster.

Table I: Comparison of Errors for Steady State Solution with $\sigma = \beta = 1$

Δx	Δv	Δt	E	kinetic energy	f
.040	.240	.111	1.90E-03	5.34E-02	3.65E-02
			3.59E-02	1.31E-01	
.020	.120	.071	1.16E-03	2.33E-02	1.52E-02
			1.59E-02	6.86E-02	
.010	.060	.045	6.02E-04	9.51E-03	5.97E-03
			7.23E-03	2.44E-02	
.005	.030	.029	2.73E-04	3.64E-03	2.27E-03
			1.23E-03	6.28E-04	
.0025	.015	.019	1.21E-04	1.55E-03	9.27E-04
			8.38E-04	1.66E-05	

(Top entry is for the difference scheme.
Bottom entry is for the particle method.)

In [10], it is shown that for L suitably chosen the difference scheme is nearly first order convergent in Δx , Δv , and Δt . The results exhibited in Table I appear to be much better than first order in that moving down the table the errors usually drop faster than by factors of $\frac{1}{2}$ while $\Delta x + \Delta v + \Delta t$ drops more slowly than by factors of $\frac{1}{2}$.

Comparing the results of the two methods in Table I we see that the difference scheme gave better results for the field. For the kinetic energy the difference scheme gave better results when Δx was larger, but poorer results on the runs with finer resolution.

In the next set of runs we again take the final time to be one and truncate $|v| > 6\sqrt{\frac{\sigma}{\beta}}$ (same steady solution (10), (11), (12)). Now we will take

$$\Delta x = .01, \quad \Delta v = .06, \quad \text{and} \quad \Delta t = .045$$

and vary σ and β . The results appear in Table II. The headings in Table II have the same meanings as those in Table I.

**Table II: Comparison of Errors for Steady State Solution
with $\Delta x = .01$, $\Delta v = .06$, $\Delta t = .045$**

σ	β	E	kinetic energy	f
.01	.01	1.07E-03	1.94E-02	1.36E-02
		5.05E-03	2.41E-02	
.1	.1	1.01E-03	1.80E-02	1.23E-02
		4.73E-03	3.17E-02	
1	1	6.02E-04	9.51E-03	5.97E-03
		7.23E-03	2.44E-02	
10	10	1.33E-04	4.73E-04	4.39E-03
		6.80E-03	-4.13E-02	
100	100	3.61E-04	-7.68E-04	3.23E-03
		OVERFLOW	OVERFLOW	
1	.01	1.55E-02	-9.64E-01	4.61E-01
		8.88E-03	3.78E+01	
1	.1	1.37E-02	3.64E-02	2.88E-01
		2.82E-02	2.76E 00	
1	1	6.02E-04	9.51E-03	5.97E-03
		7.23E-03	2.44E-02	
1	10	2.33E-03	-1.55E-03	6.31E-03
		7.35E-03	-3.25E-02	
1	100	7.09E-03	-8.89E-02	2.43E-02
		OVERFLOW	OVERFLOW	

*(Top entry is for the difference scheme.
Bottom entry is for the particle method.)*

The difference scheme appears more robust than the particle method in that the particle method never gave better results, but failed badly in two cases (when $\beta = 100$, $\sigma = 100$ or 1) where the difference scheme gave good results. Both methods gave poor resolution of the kinetic energy in the case $\sigma = 1$, $\beta = .01$. In all cases decreasing Δt recovered good results.

4 A Dynamic Solution

In this section we consider the solution of (1) and (2) with

$$\sigma = \beta = 1,$$

$$f(0, x, v) = \sqrt{\frac{10}{\pi}} \exp(-10[v - 2.5]^2),$$

and

$$b(x) = 1 + \sin(2\pi x).$$

This initial condition for f represents a beam of negative ions traveling at average speed 2.5 that is cold relative to the background plasma. Since a closed form analytic solution is not known for this initial condition, both methods were implemented with 400 by 400 grid points/particles yielding very good agreement. The resulting data was saved and used as a “nearly exact” reference set.

In Figures 1 and 2 the electric field at time $t = 1$ is plotted. The solid curve is the reference data. In Figure 1 the squares are data points generated by the difference scheme with

$$\Delta x = 0.067, \quad \Delta v = 0.4, \quad \Delta t = 0.2. \quad (13)$$

In Figure 2 the squares are data points generated by the particle method with

$$\Delta x = 0.02, \quad \Delta v = 0.12, \quad \Delta t = 0.071. \quad (14)$$

In both cases $|v| > 6$ was truncated. Despite the smaller mesh values (and hence more particles) used with the particle method, the difference scheme gives significantly better results.

Figures 3 and 4 display the approximations of $f(t = 1, x = 0, v)$ resulting from the runs described above. The squares in Figure 3 are data points from the difference scheme with (13), the squares in Figure 4 are data points from the particle method with (14). The “size” of the particles was taken to be $10\Delta x$ by Δv in generating Figure 4. Clearly the particle method gives only a rough indication of the velocity distribution, while the difference scheme reproduces it well for less computation.

The squares in Figure 5 are the approximation of $f(t = 4, x = 0, v)$ generated by the difference scheme with (13). Comparing Figures 3 and 5 we can see that from $t = 1$ to $t = 4$ the negative ions have slowed from an average

speed of around one to nearly zero. Examining the width of these two graphs reveals that the plasma has warmed very little in this time. Thus the ability to resolve the distribution f can reveal physically relevant features.

5 Conclusions.

The above comparisons (particularly the time dependent solution) indicate that the difference scheme is superior to the random particle method used, particularly in reproducing local information such as pointwise values of f . It is to be expected that a random particle method will work most poorly when computing a quantity such as $f(t, x, v)$ which is not an average over all the particles; this is not the case for a difference scheme. Although further study is required to see if this advantage can be retained in higher dimension, it appears that working on a grid may hold advantages for the simulation of this collisional plasma model.

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Figure Legend

Figure 1: Electric field at time 1, $E(1, x)$. Squares are data points from the difference scheme with (13).

Figure 2: Electric field at time 1, $E(1, x)$. Squares are data points from the particle method with (14).

Figure 3: Density at $x = 0$ and time 1, $f(1, 0, v)$. Squares are data points from the difference scheme with (13).

Figure 4: Density at $x = 0$ and time 1, $f(1, 0, v)$. Squares are data points from the particle method with (14).

Figure 5: Density at $x = 0$ and time 4, $f(4, 0, v)$. Squares are data points from the difference scheme with (13).

FIGURE 1

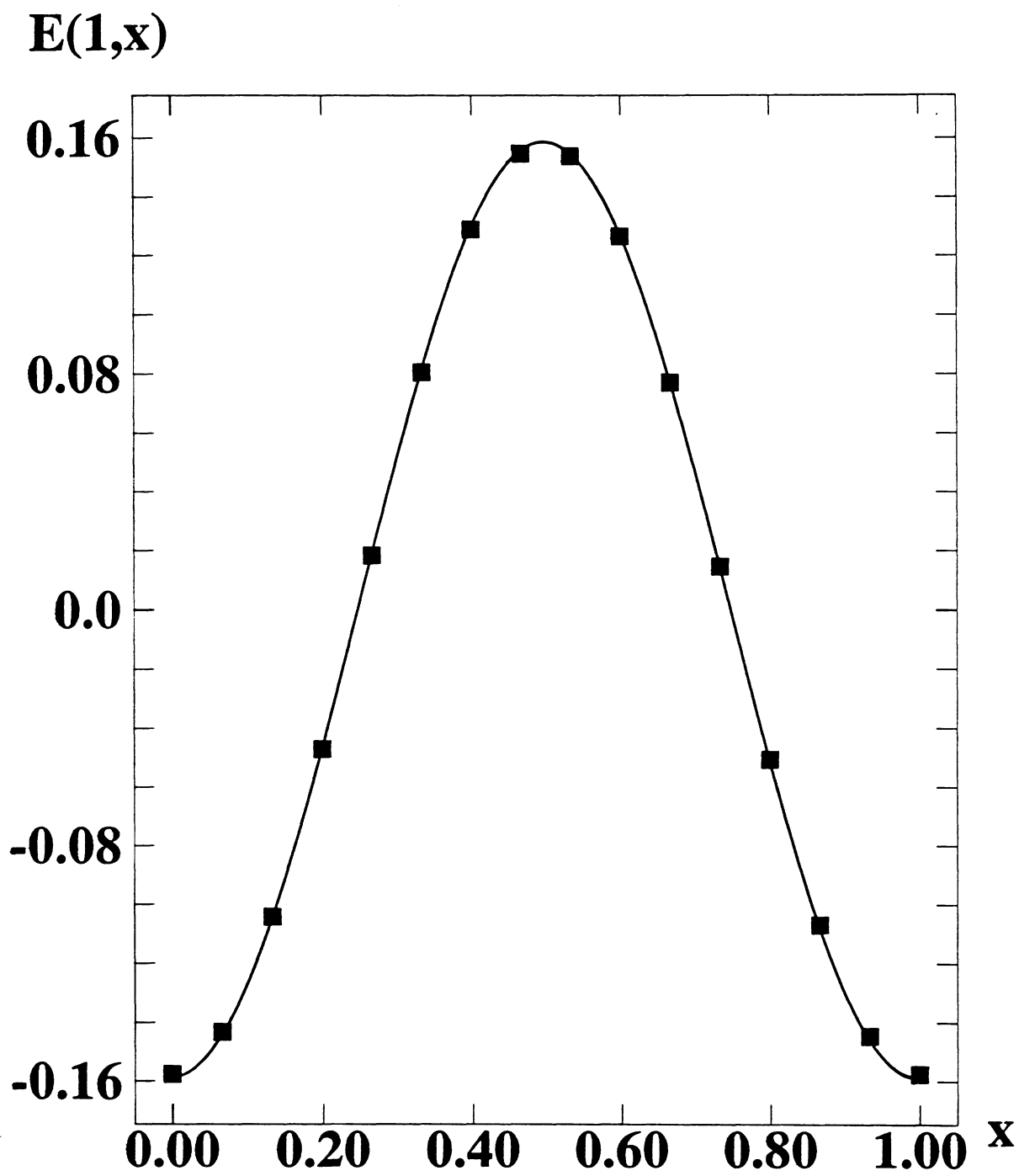


FIGURE 2

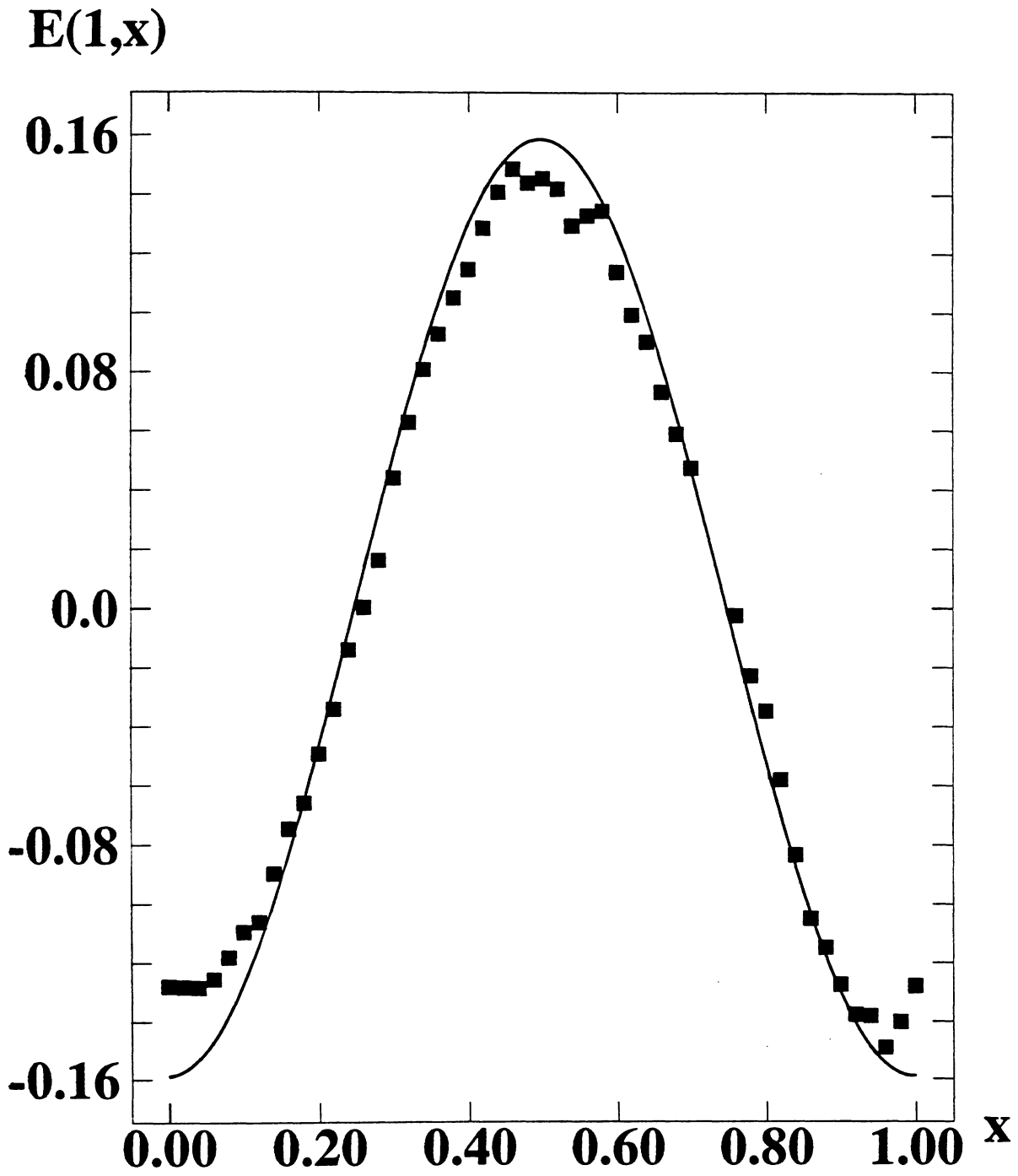


FIGURE 3

$f(1,0,v)$

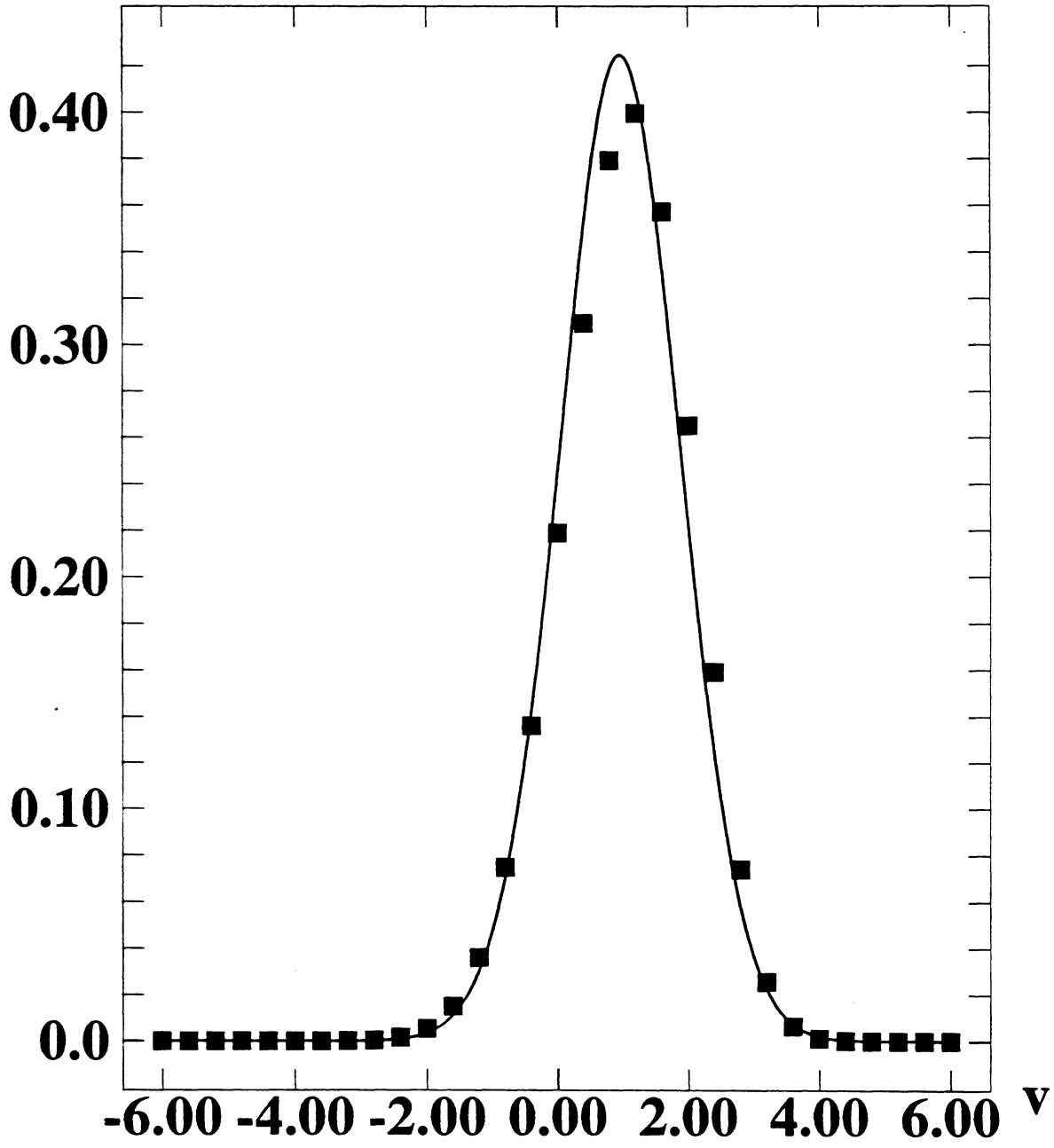


FIGURE 4

$f(1,0,v)$

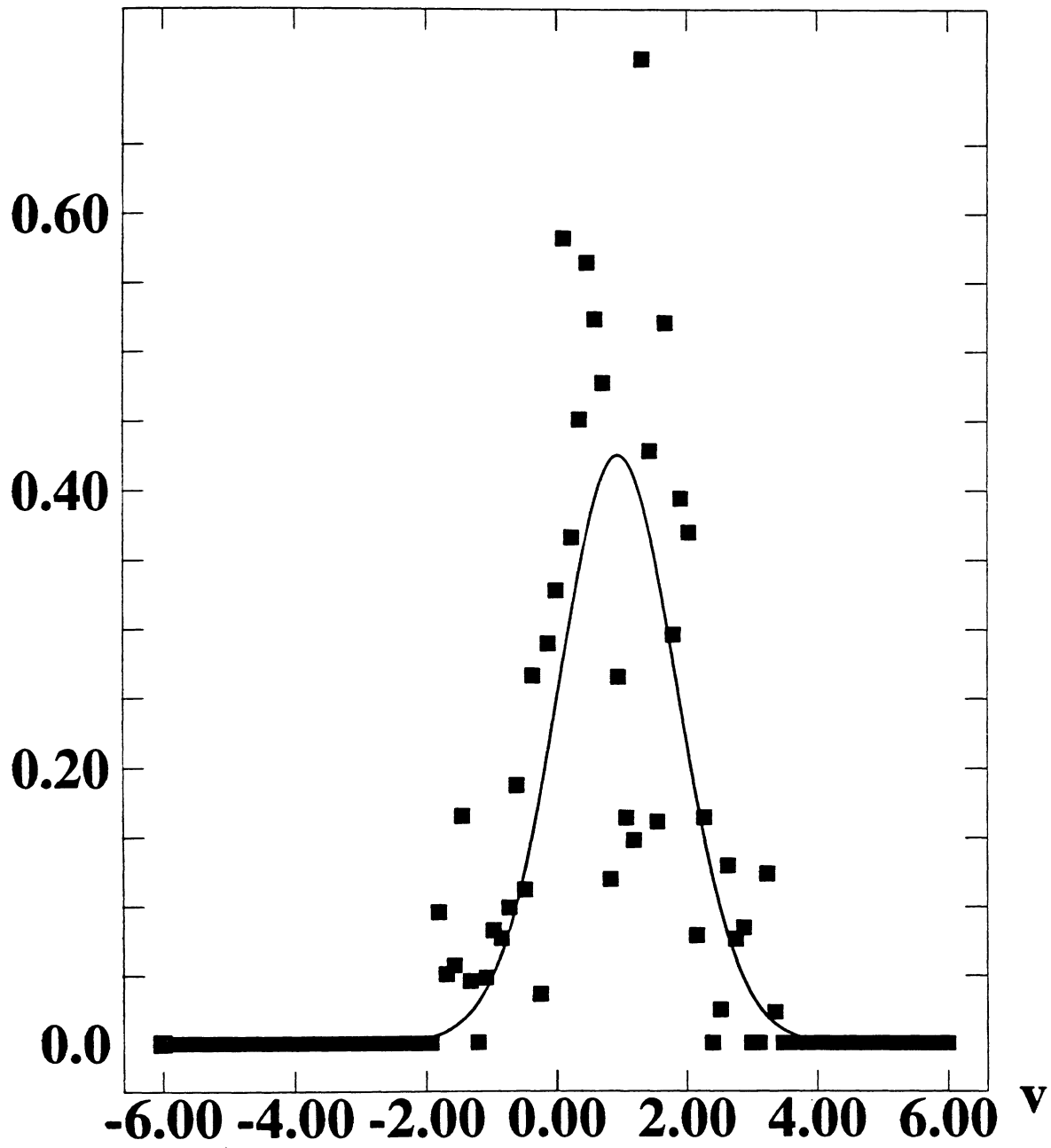
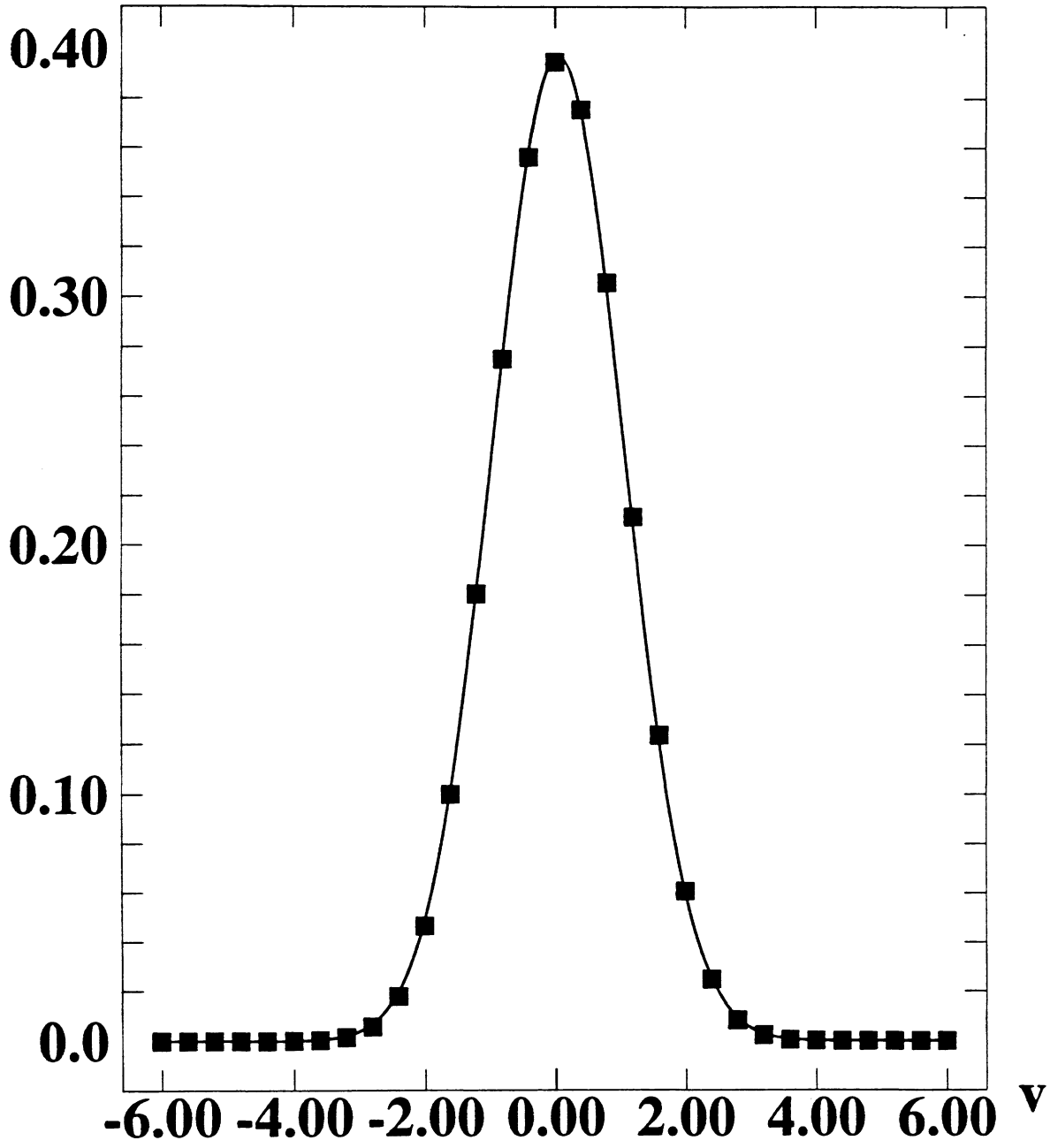


FIGURE 5

$f(4,0,v)$



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