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STRUCTURES IN TWO-DIMENSIONAL
MAGNETOFLUID TURBULENCE**

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STATISTICAL MECHANICS OF ORGANIZED STRUCTURES IN TWO-DIMENSIONAL MAGNETOFLUID TURBULENCE

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1. INTRODUCTION

A particularly fascinating feature of large Reynolds number two-dimensional (2d) magnetohydrodynamic (MHD) turbulence is the emergence of coherent structures. By a coherent structure, we mean a large-scale organized state that persists amidst the small-scale turbulent fluctuations of the magnetic field and the velocity field. The prominence of such macroscopic states in 2d MHD has been documented by numerous direct numerical simulations [1, 2, 3, 4], where the coherent structures typically appear as magnetic islands with flow.

In modeling these organized states in 2d MHD, it is natural to appeal to the methods of equilibrium statistical mechanics. There have been two popular approaches along these lines, which we would like to take some time to discuss. One is the k -space approach, which originated with Lee [5], and was subsequently developed by Fyfe and Montgomery [6]. This theory is based upon a canonical Gibbs ensemble for a truncated Fourier series approximation of the ideal MHD equations. While this model yields some important qualitative predictions about the long-time state of the magnetofluid, it accounts for only the purely quadratic conserved quantities of the ideal dynamics. It thereby ignores the infinitely many other conserved flux and cross-helicity integrals, which are not so easily expressed in spectral form. One consequence of this simplification is that it yields an ensemble with a vanishing mean field and mean flow. Thus the model does not predict a nontrivial coherent structure. The theory also suffers from the well-known Rayleigh-Jeans catastrophe, in that the ensemble-averages of the conserved quantities diverge as the number of spectral modes increases to infinity. Furthermore, it turns out that the Gibbs ensemble

predicted by this theory is *not* equivalent with the microcanonical ensemble associated with the ideal dynamics in the continuum limit of an infinite number of spectral modes. This result vitiates the equivalence of ensembles hypothesis on which the theory rests (see [7] for a discussion of these issues). The other statistical theory for MHD is the x -space approach, developed by Montgomery, Turner and Vahala in [8]. This model uses a field line discretization of the vorticity and the current density, together with an information-theoretic entropy functional. Most probable states are obtained by maximizing the entropy subject to given constraints on the classical (quadratic) invariants. However, the current and vorticity profiles predicted by this theory do not necessarily correspond to a steady solution of the ideal MHD

equations. This difficulty, along with the lack of a rigorous justification for the maximum entropy principle, leads us to believe that the model is a bit too crude.

Putting the above criticisms aside, it must be recognized that these early theories laid the groundwork for more recent, more refined models of coherent structures in MHD and in ordinary hydrodynamics. One such example is the k -space theory of Gruzinov and Isichenko [9], who use a formal asymptotic analysis to build an ensemble which accounts for the entire list of conserved quantities and which predicts a

nontrivial steady mean field-flow. By appropriately rescaling with the number of spectral modes the inverse temperature parameters in their Gibbs ensemble, they are able to obtain a meaningful continuum limit, in which the ensemble-averaged ideal invariants remain finite. The recent continuum statistical model of Robert *et al.* [10, 11] for coherent vortex structures in 2d

hydrodynamics owes a great deal to the information-theoretic x -space model of Montgomery *et al.* [8]. It too is based upon the maximization of entropy subject to constraints dictated by the conserved integrals of the ideal dynamics (the Euler equations in this case). The major innovation of the Robert approach is the use of an x -parameterized probability measure to provide a macroscopic description of the coherent structure. Such a description captures the statistics of the fluctuating vorticity field in an infinitesimal neighborhood of each point in the flow domain, and allows for the inclusion of the complete family of conserved integrals of the 2d Euler dynamics.

Our continuum statistical model of organized states in 2d MHD [12, 13, 7] is inspired by the Robert theory of coherent vortices in 2d hydrodynamics. We introduce an x -parameterized probability measure to capture the fluctuations of the magnetic field and the velocity field at each point in the spatial domain. We call such an x -parameterized

measure a macrostate. The most probable macrostate is then determined by maximizing an appropriate entropy functional subject to constraints dictated by the conserved quantities of the ideal (nondissipative) 2d MHD equations. The entropy maximizer defines a statistical equilibrium state consisting of a mean field-flow, which is a steady solution of the ideal dynamics, coupled with Gaussian fluctuations of the field and the flow in an infinitesimal neighborhood of each point in the spatial domain. The predictions of the theory are shown to be in good quantitative and qualitative agreement with the results of

high-resolution numerical studies of the dynamics of slightly dissipative 2d magnetofluids.

The paper is organized as follows. In section 2, we review the equations

ideal 2d MHD, and list the dynamical invariants. In section 3, the features of the dynamics are discussed, and the important separation of scales postulated is stated. The macroscopic description of the ideal MHD system in terms of the parameterized probability measures is introduced in section 4, and the maximum entropy principle to determine the statistical equilibrium states is formulated in section 5. In section 6, we employ the Lagrange multiplier rule to calculate the equilibrium states, and we discuss some of the most important predictions of the model and compare them with results of direct numerical simulations. We conclude in section 7 with a cursory account of our recently developed lattice model of coherent structures in 2d MHD, the construction of which leads to a rigorous justification of the continuum statistical model [7].

2. IDEAL MAGNETOHYDRODYNAMICS

The equations of ideal, incompressible MHD in appropriately normalized variables are:

$$B_t = \nabla \times (V \times B), \quad (1)$$

$$V_t + V \cdot \nabla V = (\nabla \times B) \times B - \nabla p, \quad (2)$$

$$\nabla \cdot B = 0, \quad \nabla \cdot V = 0, \quad (3)$$

where $B(x, t)$ is the magnetic field, $V(x, t)$ is the velocity field, and $p(x, t)$ is the fluid pressure. Note that p is determined instantaneously in response to the incompressibility constraint on V , so that the state of the magnetofluid is completely described by the field-flow variable $Y = (B, V)$. These equations are assumed to hold in a regular bounded spatial domain D in R^2 , and $x = (x_1, x_2)$ denotes a generic point in D . The magnetic field and the velocity field take values in R^2 . We will assume the perfectly conducting boundary conditions

$$B \cdot n = 0, V \cdot n = 0 \quad \text{on } C, \quad (4)$$

where C is the boundary of D and n is the outward normal to C . The model described below also applies, with minor modifications, to the case of periodic boundary conditions on a rectangle D .

A 2d ideal magnetofluid conserves energy, flux, and cross-helicity. These quantities are given by, respectively [14],

$$E = \frac{1}{2} \int_D (B^2 + V^2) dx, \quad (5)$$

$$F_f = \int_D f(a) dx, \quad (6)$$

$$H_f = \int_D B \cdot V f'(a) dx. \quad (7)$$

Here a is the vector potential (or flux function), and is defined by the relation

$$B = (a_{x_2}, -a_{x_1}). \quad (8)$$

The vector potential satisfies the homogeneous boundary condition,

$$a = 0 \quad \text{on } C. \quad (9)$$

The function f in (6) and (7) must satisfy certain regularity (eg. smoothness) conditions, but is otherwise arbitrary. Thus, there are infinite families of conserved flux integrals and cross-helicity integrals. These conserved functionals, which give the dynamics of the 2D magnetofluid its special characteristics, will play a fundamental role in the model sketched below. The physical meaning of the flux and cross-helicity integrals is most readily grasped by choosing $f(a) = \chi_{\{a > \sigma\}}$, the unit step function on the interior of

the magnetic surface $a = \sigma$. Then the conservation of F_f (for any σ) implies that the mass within any given flux tube is a constant

of the motion. Similarly, the invariance of H_f means that the total vorticity within any flux tube is conserved by the dynamics.

3. FEATURES OF THE DYNAMICS: SEPARATION OF SCALES

High-resolution numerical simulations clearly display the turbulent behavior of a slightly dissipative 2D magnetofluid [1, 2, 3, 4]. As the field-flow state $Y = (B, V)$ evolves, it develops rapid fluctuations on very fine spatial scales. Fluctuations at nearby points appear to be only weakly correlated. After a certain period of time, however, large scale coherent structures emerge in the form of macroscopic magnetic islands, typically with flow. These structures persist for a relatively long time period amidst the turbulent fluctuations, and seem to approach a quasi-stationary state, before the dissipation causes them to decay. In the ideal limit of vanishing dissipation, we expect that the mixing would continue indefinitely, exciting arbitrarily small spatial scales, and that a final relaxed

state, consisting of a large-scale coherent structure and infinitesimal-scale local fluctuations, would be

approached. These considerations lead us to postulate the following *separation of scales* property for the ideal magnetofluid:

SEPARATION OF SCALES HYPOTHESIS *In the long-time limit, the fluctuations of the field and the flow occur on an infinitesimal scale at each point in the spatial domain, and the statistics of the fluctuations at distinct points are uncorrelated.*

Our statistical model of coherent structures is built upon this hypothesis. The rationale behind this postulated separation of scales has been explained in detail in [7].

4. MACROSCOPIC DESCRIPTION OF THE IDEAL SYSTEM

The field-flow state Y constitutes a *microscopic* description of the MHD system. Due to its highly intricate small-scale behavior, the microstate Y does not furnish a meaningful description of the long-time behavior of the magnetofluid. For this reason, we introduce a coarse-grained, or *macroscopic* description of the system. A macrostate $(\rho(x, y))_{x \in D}$ is a family of probability densities on the values $y \in R^4$ of the microstate Y at each point x in the domain D . That is, for each x in D , $\rho(x, y)$ represents a joint probability

density on the values $y = (b, v)$ of the fluctuating field-flow pair $(B(x), V(x))$. The macrostate is intended to represent a possible long-time average weak limit of the microstate $Y(x, t)$ in the sense that

$$\int_D \int_{R^4} G(x, y) \rho(x, y) dy dx = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_D G(x, Y(x, t)) dx dt,$$

for all bounded and continuous test functions G . An equivalent way of expressing this relationship is

$$\rho(x, y) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta(y - Y(x, t)) dt,$$

where δ is the Dirac mass concentrated at the origin in R^4 , and the convergence is understood in the weak sense of probability densities on $D \times R^4$ [15].

The conservation of energy, flux and cross-helicity under the ideal dynamics now translates into corresponding constraints on admissible macrostates. These constraints are formulated in a manner consistent with the above-defined weak convergence of $Y(x, t)$ to ρ . The requisite expressions are (see [12, 13, 7] for mathematical details):

$$E(\rho) \equiv \frac{1}{2} \int_D \int_{R^4} (b^2 + v^2) \rho(x, y) dy dx = E^0, \quad (10)$$

$$F_f(\rho) \equiv \int_D f(\bar{a}(x)) dx = F_f^0, \quad (11)$$

$$H_f(\rho) \equiv \int_D \int_{R^4} b \cdot v f'(\bar{a}(x)) \rho(x, y) dy dx = H_f^0, \quad (12)$$

where E^0 , F_f^0 , and H_f^0 are the values of energy, flux, and cross-helicity fixed by the initial state of the MHD system; the local mean magnetic field $\bar{B}(x)$ is defined by the relation

$$\bar{B}(x) = \int_{R^4} b \rho(x, y) dy, \quad (13)$$

and $\bar{a}(x)$ is the vector potential corresponding to $\bar{B}(x)$ (see eqs. (8)–(9)). For future reference, we also define the local mean velocity field

$$\bar{V}(x) = \int_{R^4} v \rho(x, y) dy. \quad (14)$$

Let us note that in deriving the expressions for the flux and cross-helicity constraints, we have anticipated that the fluctuations of the vector potential a are negligible in the long-time limit, i.e., that

$$\bar{a}(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a(x, t) dt, \quad \text{var } a(x) = 0,$$

where var denotes the variance of a random variable. Intuitively, this vanishing of fluctuations is expected because the vector potential is obtained by “integrating” the magnetic field, and the process of integration smooths out the fluctuations. Similarly, we expect the velocity stream function

φ , which is defined by $V = (\varphi_{x_1}, -\varphi_{x_2})$, to be insensitive to fluctuations in the limit. The mathematical basis for the

vanishing of fluctuations of the flux and the stream function is the compactness of the operator $\text{curl}^{-1} : B \rightarrow a, V \rightarrow \varphi$ [12].

5. THE MAXIMUM ENTROPY PRINCIPLE

From standard principles of statistical mechanics, we expect that the system approaches a most probable macrostate which respects the prescribed values of the conserved quantities [16].

This most probable state is determined as the maximizer of an appropriate entropy functional subject to the constraints on energy, flux and cross-helicity. The entropy functional that we use is the classical Gibbs-Boltzmann-Shannon entropy:

$$S(\rho) = - \int_D \int_{R^4} \rho(x, y) \log \rho(x, y) dy dx. \quad (15)$$

As such, S is a measure of (the logarithm) of the number of microstates

corresponding to the macrostate ρ . Implicit in its definition as an

integral over D is the assumption that fluctuations at two separated points in D are statistically independent. This property is demanded by the separation of scales hypothesis enunciated above. As an integral over the y -variable, S uniformly weights the entire range R^4 of the values of the microstate Y . While these features of S are quite natural, the real justification for the use of this particular entropy functional relies on the Liouville property of the underlying dynamics, as is discussed briefly below and in detail in [7].

The maximum entropy principle to determine the statistical equilibrium states can now be formulated:

$$\text{(MEP)} \quad S(\rho) \rightarrow \max, \text{ subject to } E(\rho) = E^0, F_f(\rho) = F_f^0, H_f(\rho) = H_f^0,$$

where f varies over all (sufficiently smooth) functions on the invariant range of the flux function a . This is an infinitely-constrained nonconvex variational principle in the

density variable ρ . Because of the analytical difficulties associated with the infinitely many constraints, we find it desirable to approximate

the continuously infinite families of flux and cross-helicity integrals by the linear combination of a finite number of such integrals. For this purpose, we could choose a finite basis of functions $f_i, i = 1, \dots, N$, having suitable growth and regularity properties, and consider only the corresponding flux and cross-helicity integrals $F_{f_i} \equiv F_i, H_{f_i} \equiv H_i$. Of course, an arbitrary f can be approximated by a linear combination of the f_i . That this discretization of the flux and cross-helicity constraints yields an accurate approximation of the infinite families of these constraints follows from the analysis of Turkington *et al.* in [17]. The resulting maximum entropy problem, which has been discussed in detail in [7], yields equilibrium states and corresponding mean field-flow equations that are too complex to be analyzed in any meaningful way in this brief note. Thus, for clarity and economy, we consider here the further simplified problem

$$\text{(SMEP)} \quad S(\rho) \rightarrow \max \text{ subject to } E(\rho) = E^0, \quad F_i(\rho) = F_i^0, \quad H(\rho) = H^0,$$

where

$$F_i(\rho) = \int_D f_i(\bar{a}) dx, \quad H(\rho) = \int_{R^4} \int_D b \cdot v \rho(x, y) dy dx,$$

and F_i^0, H^0 are prescribed values of the flux integrals and the quadratic cross-helicity, respectively. The index i ranges from 1 to N .

In taking into account only the quadratic cross-helicity constraint, we are simplifying considerably the full statistical equilibrium problem (MEP). However, this reduced problem does capture the essence of the correlation

effects between the field and the flow that result from the conservation of cross-helicity. For an analysis of the consequences of including the generalized (nonquadratic) cross-helicity integrals, the reader is referred to [7].

6. PREDICTIONS AND COMPARISONS WITH NUMERICAL STUDIES

The solution ρ of (SMEP) follows from the Lagrange multiplier rule:

$$S'(\rho) = \beta E'(\rho) + \sum \alpha_i F_i'(\rho) + \gamma H'(\rho), \quad (16)$$

where β , α_i , and γ are Lagrange multipliers corresponding to the constraints on energy, flux, and cross-helicity, respectively. The derivatives in (16) are, of course, functional derivatives. From (16) it follows that

$$\rho = Z^{-1} \exp(-\beta E'(\rho) - \sum \alpha_i F_i'(\rho) - \gamma H'(\rho)),$$

where $Z(x)$ is the partition function which enforces the normalization constraint $\int_{R^4} \rho(x, y) dy = 1$ for all $x \in D$. After algebraic manipulations, we arrive at the expression

$$\rho = \frac{\beta^2(1-\mu^2)}{4\pi^2} \exp\left(-\frac{\beta}{2}(1-\mu^2)(b - \bar{B}(x))^2 - \frac{\beta}{2}(v - \mu b)^2\right), \quad (17)$$

where $\mu = -\gamma/\beta$. We note that $-1 < \mu < 1$ (see [12, 7]).

A glance at equation (17) reveals that the most probable macrostate

ρ is for each x in D a Gaussian distribution on the field-flow pair $(B(x), V(x))$. The covariance matrix can be determined by straightforward calculations. We obtain that

$$\text{var } B_i(x) = \text{var } V_i(x) = \frac{1}{\beta(1-\mu^2)}, \quad \text{corr}(B_i(x), V_i(x)) = \mu,$$

for $i = 1, 2$ and for each $x \in D$. The other components are uncorrelated. The mean field-flow, which is calculated via (13)–(14), can be shown to satisfy the equations (see [12, 7])

$$\bar{V}(x) = \mu \bar{B}(x), \quad \bar{J}(x) = \sum \lambda_i f_i'(\bar{a}(x)), \quad (18)$$

where $\lambda_i = -\alpha_i/(\beta(1-\mu^2))$, and the current density \bar{J} is defined by

$$\bar{J}(x) = \nabla \times \bar{B}(x) = -\nabla^2 \bar{a}(x). \quad (19)$$

In particular, it follows from (18)–(19) that the mean field-flow is a stationary solution of the ideal MHD equations (1)–(3). We also see from (18)–(19) that the mean field \bar{B} is a critical point of the (deterministic) magnetic energy, $\frac{1}{2} \int_D B^2 dx$, subject to the flux constraints, $\int_D f_i(a) dx = F_i^0$,

and that the mean vector potential \bar{a} satisfies the celebrated Grad–Shafranov equation from plasma physics; that is, the mean vector potential is given by a nonlinear elliptic equation of the form $-\nabla^2 \bar{a} = \Lambda(\bar{a})$ [18]. The theory predicts, therefore, that the ideal magnetofluid will relax to a state consisting of a stationary mean field-flow (the coherent structure) and microscopic Gaussian fluctuations about this steady mean.

A particularly remarkable prediction of our model is that the ratio of kinetic to magnetic energy in statistical equilibrium is less than 1, regardless of the initial ratio. This follows from straightforward calculations and the fact that the correlation μ satisfies $-1 < \mu < 1$. Indeed, we have for the magnetic energy E_m and the kinetic energy E_k the following expressions

$$E_m = \frac{1}{2} \int_D \int_{R^4} b^2 \rho(x, y) dy dx = \frac{1}{2} \int_D \bar{B}^2 dx + \text{volume}(D)/(\beta(1 - \mu^2)),$$

$$E_k = \frac{1}{2} \int_D \int_{R^4} v^2 \rho(x, y) dy dx = \frac{\mu^2}{2} \int_D \bar{B}^2 dx + \text{volume}(D)/(\beta(1 - \mu^2)).$$

This prediction, which is true even when the generalized cross-helicities are accounted for [7], is in accord with the numerical studies of Biskamp *et al.* [1, 2, 3], in which there was observed the rapid relaxation of E_k/E_m to an almost constant value less than 1, even for initial ratios as large as 25. In general, the predictions of our maximum entropy model are in good agreement with the numerical simulations of Biskamp *et al.* [1, 2, 3]. They observe local Gaussian distributions on the magnetic field and velocity

field, and predict a cascade of flux to large scales, which is indicative of the formation of macroscopic magnetic structures. There is also good qualitative agreement of our theory with other predictions of Biskamp *et al.*, as well as with the computational studies of Politano *et al.* [4]. For more detailed discussions of the predictions of our model and for further comparisons with the numerical simulations, the reader is referred to [12, 13, 7].

7. MORE RECENT DEVELOPMENTS: THE LATTICE MODEL

The continuum statistical model outlined above is based upon the essential separation of scales hypothesis, which asserts that the fluctuations of the field and the flow occur on an infinitesimal scale at each point in the spatial domain, and that the statistics of the fluctuations are uncorrelated from point to point. In our recent work [7], this crucial hypothesis is clarified through the construction of a lattice model that converges to the continuum model in the fixed-volume, small-spacing limit. The appropriate lattice model is developed with the help of the discrete Fourier transform, which allows us to exploit the relationship between truncation in Fourier space and discretization in real space. It is through this natural correspondence

between finite k -space and finite x -space that the basis for and the consequences of the separation of scales hypothesis can be fully appreciated. The need for the synthesis of the x -space and the k -space methods arises from the particular forms of the conserved quantities of 2d ideal MHD. These invariants include energy, as well as the two infinite families of flux and cross-helicity integrals. The disparate weights that these different invariants place on the spectral modes makes a k -space analysis essential, while the x -space analysis is needed to incorporate the nonlinear and nonquadratic (generalized) flux and cross-helicity integrals into the theory. The lattice model is defined by what we call the *implicit canonical ensemble* on discrete phase-space, which maximizes entropy subject to the approximated dynamical constraints. The term “implicit” reflects the fact that the constraints on flux and cross-helicity depend nonlinearly on the ensemble, which is unlike the case of the usual canonical ensemble, where the ensemble appears linearly in the constraints [6]. The implicit canonical ensemble is consistent with the Liouville property of the discrete dynamics, and most

importantly, it agrees with the microcanonical ensemble associated with these dynamics in the continuum limit. This asymptotic equivalence of ensembles provides a strong theoretical justification for our theory. The interested reader should consult [7] for details.

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