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Oscillations in One-Dimensional
Elasticity with Surface Energy

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where

$$\sigma(p) := W'(p) = p^3 - p \tag{1.2}$$

is the stress, $\beta > 0, \gamma > 0, \alpha \geq 0$ corresponds to the stiffness of the elastic matrix, and where u_{xxt}, u, and u_{xxxx} account for, respectively, viscoelastic damping, forcing term, and local surface energy penalization. The associated potential energy is given by

$$\mathcal{I}(u)(t) := \frac{1}{4} \int_0^1 (u_x^2(x,t) - 1)^2 \, dx + \frac{\alpha}{2} \int_0^1 u^2(x,t) \, dx + \frac{\gamma}{2} \int_0^1 u_{xx}^2(x,t) \, dx.$$

Suppose, in addition, that u satisfies the initial conditions

$$\begin{cases} u(x,0) = a(x), & 0 < x < 1, \\ u_t(x,0) = b(x), & 0 < x < 1. \end{cases} \tag{1.3}$$

Given a sequence of initial data $\{a^\varepsilon, b^\varepsilon\}$, with corresponding family of solutions $\{u^\varepsilon\}$, we seek to understand how the initial oscillations propagate.

The creation and propagation of oscillations of solutions of (1.1) were studied by BRANDON, FONSECA, & SWART [BFS] in the case where local penalization of surface energy is not taken into account, i.e. when $\gamma = 0$. It was shown that if initial velocities have oscillations then these are lost immediatly as time increases, while oscillatory strains $\{u_x^\varepsilon\}$ (hence miscrostructure) cannot be created, but persist for all times if initially present. Here we prove that there is no propagation of oscillations in the velocities and strains; precisely, if $\{a^\varepsilon, b^\varepsilon\}$ remains bounded in $H_0^2(0,1) \times L^2(0,1)$ then for every $T > 0$ $\{u^\varepsilon\}$ and $\{u_t^\varepsilon\}$ converge strongly as $\varepsilon \to 0$ in $L^2((0,1) \times (0,T))$, and $\{u_x^\varepsilon\}, \{u_{xx}^\varepsilon\}$ converge strongly as $\varepsilon \to 0$ in $L_{\mathrm{loc}}^2((0,1) \times (0,T))$ (see Proposition 3.4 and Theorems 3.6, 3.7). Moreover, we show that there is no creation of oscillations in $\{u_{xt}^\varepsilon\}$, in that if $\{a^\varepsilon, b^\varepsilon\}$ is bounded in $(H_0^2(0,1) \cap H^4(0,1)) \times H_0^2(0,1)$ then $\{u_{xt}^\varepsilon\}$ converges strongly in $L_{\mathrm{loc}}^2((0,1) \times (0,T))$. These results rely heavily on the theory of compensated compactness of Murat and Tartar(see [M], [T1]), and on Aubin's interpolation theorem (see [A], [L] and [T2]).

In Section 2 we establish existence and uniqueness of strong and weak solutions for (1.1)-(1.3). In Section 3 we study the creation and propagation of oscillations. We end this section remarking that if the Dirichlet boundary conditions were placed on u_{xx} rather than on the strains u_x, and assuming that $\beta^2 > 4\gamma$ and that $\sup_\varepsilon \|a^\varepsilon\|_{H^3} + \|b^\varepsilon\|_{H^1} < +\infty$, then the strong convergence of $\{u^\varepsilon\}, \{u_x^\varepsilon\}, \{u_t^\varepsilon\}, \{u_{xx}^\varepsilon\}$ in $L^2((0,1) \times (0,T))$ and of $\{u_{xt}^\varepsilon\}, \{u_{xxx}^\varepsilon\}$ in $L_{\mathrm{loc}}^2((0,1) \times (0,T))$ for all $T > 0$ could be easily obtained using energy apriori estimates (see Proposition 3.11). In Section 4 we use a finite difference scheme to illustrate the behavior of solutions in the case where $a^\varepsilon = 0$ and $b^\varepsilon = \sin^6\left(\frac{1}{\varepsilon}\pi x\right)$. In particular, we show $\{u_{xt}^\varepsilon\}$ does not oscillate although the unboudedness of $\{b^\varepsilon\}$ in H^2 prevents us from applying Theorem 3.10.

2. FORMULATION OF THE PROBLEM. EXISTENCE THEOREM.

In this section we will obtain existence of solution for the equation

$$u_{tt} = [u_x^3 - u_x + \beta u_{xt} - \gamma u_{xxx}]_x - \alpha u \tag{2.1}$$

where $(x,t) \in (0,1) \times (0,\infty)$, $\beta, \gamma > 0, \alpha \geq 0$, with boundary conditions

$$\begin{cases} u(0,t) = u(1,t) = 0, & t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \end{cases} \tag{2.2}$$

and initial conditions

$$\begin{cases} u(x,0) = a(x), & 0 < x < 1, \\ u_t(x,0) = b(x), & 0 < x < 1. \end{cases} \tag{2.3}$$

The total energy corresponding to the system is

$$E(t) := \mathcal{K}(t) + \mathcal{I}(t),$$

where the kinetic energy is given by

$$\mathcal{K}(t) := \frac{1}{2} \int_0^1 u_t^2(x,t)\, dx,$$

and the potential energy is defined as

$$\mathcal{I}(t) := \frac{1}{4} \int_0^1 (u_x^2(x,t) - 1)^2\, dx + \frac{\alpha}{2} \int_0^1 u^2(x,t)\, dx + \frac{\gamma}{2} \int_0^1 u_{xx}^2(x,t)\, dx.$$

Differentiating the total energy

$$\begin{aligned} E(t) :=& \frac{1}{2} \int_0^1 u_t^2(x,t)\, dx + \frac{1}{4} \int_0^1 (u_x^2(x,t) - 1)^2\, dx \\ &+ \frac{\alpha}{2} \int_0^1 u^2(x,t)\, dx + \frac{\gamma}{2} \int_0^1 u_{xx}^2(x,t)\, dx \end{aligned}$$

and using (2.1)–(2.3), we obtain

$$\frac{dE}{dt}(t) = - \int_0^1 \beta u_{xt}^2(x,t)\, dx, \tag{2.4}$$

hence $E(\cdot)$ is non-increasing.

We prove existence and uniqueness of a classical solution for problem (2.1)–(2.3).

Theorem 2.1. *Let $a \in H^4(0,1) \cap H_0^2(0,1)$ and $b \in H_0^2(0,1)$. There exists a unique global solution of (2.1)–(2.3) with $u(t) \in H^4(0,1)$ for all $t \geq 0$ and such that*

$$u \in C^1([0,\infty); H_0^2(0,1)) \cap C^2([0,\infty); L^2(0,1)).$$

Proof. We introduce the new variables

$$\begin{cases} v := u_{xx} \\ w := u_t. \end{cases}$$

Then (2.1) and (2.3) can be rewritten as the initial-value problem

$$\begin{cases} z_t = Az + f(z), & t > 0, \\ z(0) = z_0 \end{cases} \tag{2.5}$$

in the Banach space $X := L^2(0,1) \times L^2(0,1)$, where $z = (v,w)$, $z_0 = (a_{xx}, b)$,

$$A := \begin{bmatrix} 0 & \Delta \\ -\gamma\Delta & \beta\Delta \end{bmatrix}, \qquad f(z) := \begin{bmatrix} 0 \\ G(v) \end{bmatrix},$$

$\Delta := \partial^2/\partial x^2$, and

$$G(v(x,t)) := (3V^2(x,t) - 1)v(x,t) - \alpha \int_0^x V(s,t)\, ds, \qquad V(x,t) := \int_0^x v(s,t)\, ds.$$

The boundary conditions (2.2) imply

$$\begin{cases} w(0,t) = w_x(0,t) = 0, \\ w(1,t) = w_x(1,t) = 0. \end{cases} \tag{2.6}$$

We define $D(A) := H^2(0,1) \times H_0^2(0,1)$, and we consider in X the inner product

$$< z_1, z_2 > := \int_0^1 (\gamma v_1 v_2 + w_1 w_2)\, dx,$$

where $z_i = (v_i, w_i)$, $i = 1, 2$.

By virtue of (2.4), if u is a solution of (2.1)–(2.3) then

$$\|z(t)\|_X^2 \le 2E(t) \le 2E(0) =: M^2, \tag{2.7}$$

and so it is natural to consider the truncated problem

$$\begin{cases} z_t = Az + \bar{f}(z), & t > 0, \\ z(0) = z_0 \end{cases} \tag{2.8}$$

where

$$\bar{f}(z) := \kappa(\|z\|)f(z),$$

and $\kappa(\cdot) \in C^1([0,\infty))$ is a decreasing function such that $\kappa(s) = 1$ for $0 \le s \le \frac{3M}{2}$, and $\kappa(s) = 0$ for $s > 2M$. By Theorems 6.1.2 and 6.1.5 in PAZY [P], the initial value problem (2.8) admits a unique solution

$$(v, w) \in C^1([0,\infty); L^2(0,1) \times L^2(0,1)),$$

$$(v(t), w(t)) \in H^2(0,1) \times H_0^2(0,1) \quad \text{for all} \quad t \ge 0,$$

provided A generates a C^0 semigroup on X, $\bar{f}$ is globally Lipschitz and continuously differentiable from X to X, and $z_0 \in D(A)$. We start by showing that A generates a C^0 semigroup on X. By the Hille-Yosida Theorem (see [P], Theorem 1.3.1), it suffices to prove that

$$\|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda} \quad \text{for all } \lambda > 0. \tag{2.9}$$

Fix $\lambda > 0$, let $z \in D(A)$, and set $y := (\lambda I - A)z$, $y =: (k, h)$. Thus

$$k = \lambda v - w_{xx}, \quad h = \lambda w + \gamma v_{xx} - \beta w_{xx},$$

and since $w \in H_0^2(0,1)$ we have

$$\|y\|_X \|z\|_X \ge\ < y, z > = \int_0^1 (\gamma k v + h w)\, dx$$

$$= \int_0^1 [\gamma v(\lambda v - w_{xx}) + w(\lambda w + \gamma v_{xx} - \beta w_{xx})]\, dx$$

$$= \lambda \int_0^1 (\gamma v^2 + w^2)\, dx + \gamma \int_0^1 (-v w_{xx} + w v_{xx})\, dx - \beta \int_0^1 w w_{xx}\, dx$$

$$= \lambda \|z\|_X^2 + \gamma(-v w_x + w v_x)|_0^1 - \beta w w_x|_0^1 + \beta \int_0^1 w_x{}^2\, dx$$

$$\ge \lambda \|z\|_X^2.$$

Hence

$$\|y\|_X \ge \lambda \|z\|_X$$

proving (2.9). In order to show that $\bar{f}$ is globally Lipschitz, we assert that

$$\|f(z_1) - f(z_2)\|_X \le C(1 + \|z_1\|_X^2 + \|z_2\|_X^2)\|z_1 - z_2\|_X. \qquad (2.10)$$

Indeed

$$\|f(z_1) - f(z_2)\|_X^2$$

$$= \int_0^1 \left\{ 3(V_1^2 v_1 - V_2^2 v_2) - (v_1 - v_2) - \alpha \int_0^x (V_1(s) - V_2(s))\,ds \right\}^2 dx$$

$$\le C \left(\int_0^1 (V_1^2 v_1 - V_2^2 v_2)^2 dx + \int_0^1 (v_1 - v_2)^2 dx + \int_0^1 \left[\int_0^x (V_1(s) - V_2(s))ds \right]^2 dx \right)$$

$$\le C \left(\|V_1^2 v_1 - V_2^2 v_2\|_{L^2}^2 + \|v_1 - v_2\|_{L^2}^2 + \|V_1 - V_2\|_{L^2}^2 \right).$$

Since

$$\|V_i\|_\infty \le \|v_i\|_{L^2} \qquad i = 1, 2, \qquad (2.11)$$

we conclude (2.10), and the Lipschitz property for $\bar{f}$ follows immediatly from the fact that κ is Lipschitz and has compact support. Continuous differentiability of $\bar{f}$ may be easily obtained.

We define

$$u(x, t) := \int_0^x \int_0^y v(z, t)\,dz\,dy. \qquad (2.12)$$

Clearly $u_{xx} = v$, and (2.8) implies that

$$u_t(x, t) = \int_0^x \int_0^y v_t(z, t)\,dz\,dy = \int_0^x \int_0^y w_{xx}(z, t)\,dz\,dy = w(x, t) \qquad (2.13)$$

because $w \in H_0^2(0, 1)$. Thus

$$u_t(1, t) = w(1, t) = 0, \qquad u_{xt}(1, t) = w_x(1, t) = 0,$$

hence $u(1, \cdot)$ and $u_x(1, \cdot)$ are constant functions. Since $a \in H_0^2(0, 1)$ we have

$$u(1, t) = u(1, 0) = \int_0^1 \int_0^x v(y, 0)\,dy\,dx = \int_0^1 \int_0^x a_{xx}(y)\,dy\,dx = 0$$

and

$$u_x(1, t) = u_x(1, 0) = \int_0^1 a_{xx}(x)\,dx = 0.$$

Note that the conditions $u(0, t) = u_x(0, t) = 0$ follow trivially from (2.12). Moreover, (2.12) and (2.13) yield

$$u(x, 0) = \int_0^x \int_0^y a_{xx}(z)\,dz\,dy = a(x), \quad u_t(x, 0) = w(x, 0) = b(x).$$

By (2.8)

$$u_{tt} = w_t = \beta w_{xx} - \gamma v_{xx} + \kappa(\|z\|_X)G(v)$$
$$= \beta u_{xxt} - \gamma u_{xxxx} + \kappa(\|z\|_X)[(3u_x^2 - 1)u_{xx} - \alpha u]. \qquad (2.14)$$

Now

$$\|z(t)\|_X^2 = \int_0^1 (\gamma v^2(x, t) + w^2(x, t))\,dx = \int_0^1 (u_t^2(x, t) + \gamma u_{xx}^2(x, t))\,dx$$

is a continuous function of t with

$$\|z(0)\|_X \leq M.$$

Hence

$$\mathcal{A} := \left\{ \tau > 0 : \|z(t)\|_X \leq \frac{3}{2} M \text{ for all } t \in [0, \tau) \right\}$$

is nonempty. Let T be the supremum of $\mathcal{A}$. Clearly $\kappa(\|z(t)\|_X) = 1$ for all $t \in [0, T)$, therefore z is a solution of (2.5) (that is u is a solution of (2.1)–(2.3)) in $[0, T)$ and (2.7) implies the energy bounds obtained earlier, i.e.

$$\|z(t)\|_X^2 \leq 2E(t) \leq 2E(0) \leq M^2$$

for all $t < T$. If T was finite then, by continuity, $\|z(T)\|_X \leq M$ and so $\|z(t)\|_X \leq \frac{3M}{2}$ for $t \in [0, T^*)$ for some $T^* > T$, contradicting the fact that T is the supremum of $\mathcal{A}$. The regularity properties of u stated in the theorem follow from (2.12) and (2.13). $\qquad\square$

Existence of weak solutions under milder initial regularity assumptions is guaranteed by Theorem 6.1.2 in PAZY [P].

Theorem 2.2. *Let $a \in H_0^2(0,1)$ and $b \in L^2(0,1)$. There exist unique weak solutions u of (2.1)–(2.3), and $z = (v, w)$ of (2.8), with*

$$u \in C([0, \infty); H_0^2(0,1)) \cap C^1([0, \infty); L^2(0,1)),$$

$$(v, w) \in C([0, \infty); L^2(0,1) \times L^2(0,1)).$$

3. PROPAGATION AND CREATION OF OSCILLATIONS.

Consider the parametrized family of problems

$$u_{tt}^\varepsilon = [(u_x^\varepsilon)^3 - u_x^\varepsilon + \beta u_{xt}^\varepsilon - \gamma u_{xxx}^\varepsilon]_x - \alpha u^\varepsilon$$

with boundary conditions (2.2) and initial conditions

$$\begin{cases} u^\varepsilon(x,0) = a^\varepsilon(x), & 0 < x < 1, \\ u_t^\varepsilon(x,0) = b^\varepsilon(x), & 0 < x < 1. \end{cases}$$

We assume that

$$\|a^\varepsilon\|_{H^2} + \|b^\varepsilon\|_{L^2} \leq C, \qquad (3.1)$$

where C does not depend on ε.

In the sequel C will denote a generic positive constant which may vary from formula to formula. If X is a Banach space and $0 < T \leq +\infty$ then a sequence $w^\varepsilon : (0, T) \to X$ of Borel measurable functions is said to belong to $BL^2(0, T; X)$ if

$$\sup_\varepsilon \int_0^T \|w^\varepsilon(t)\|_X^2 \, dt \leq C,$$

for some constant $C > 0$. Also, we write $\{w^\varepsilon\} \in BL^\infty(0, T; X)$ if

$$\sup_\varepsilon \operatorname*{ess\,sup}_{t \in (0,T)} \|w^\varepsilon(t)\|_X \leq C,$$

and $\{w^\varepsilon\} \in BH^1(0, T; X)$ if

$$\sup_\varepsilon \left\{ \int_0^T \|w^\varepsilon(t)\|_X^2 \, dt + \int_0^T \|w_t^\varepsilon(t)\|_X^2 \, dt \right\} \leq C.$$

By virtue of the bounds (3.1), the initial sequence of strains $\{a_x^\varepsilon\}$ does not oscillate. We start by showing that oscillations are not created in $\{u_x^\varepsilon\}$ for $t > 0$. To fix notation, if $0 < T < +\infty$ we define $\mathcal{K}_T := (0,1) \times (0,T)$ and $\mathcal{K}_\infty := (0,1) \times (0,\infty)$.

Proposition 3.1. *Assume that (3.1) holds. Then $\{u^\varepsilon\} \in BL^\infty(0,\infty; H_0^2(0,1))$, $\{u_t^\varepsilon\} \in BL^\infty(0,\infty; L^2(0,1))$, and $\{u_{xt}^\varepsilon\} \in BL^2(\mathcal{K}_T)$. In particular, there exists a subsequence and $U \in L^\infty(0,\infty; H_0^2(0,1))$ such that*

$$u^\varepsilon \rightharpoonup U \quad \textit{weakly* in } L^\infty(0,\infty; H_0^2(0,1)),$$

$$u_t^\varepsilon \rightharpoonup U_t \quad \textit{weakly* in } L^\infty(0,\infty; L^2(0,1)),$$

$$u_{tx}^\varepsilon \rightharpoonup U_{tx} \quad \textit{weakly in } L^2(\mathcal{K}_\infty).$$

Proof. We note that (2.4) implies that

$$E^\varepsilon(t) = \frac{1}{2} \int_0^1 (u_t^\varepsilon)^2(x,t)\,dx + \frac{1}{4} \int_0^1 ((u_x^\varepsilon)^2(x,t) - 1)^2\,dx$$

$$+ \frac{\alpha}{2} \int_0^1 (u^\varepsilon)^2(x,t)\,dx + \frac{\gamma}{2} \int_0^1 (u_{xx}^\varepsilon)^2(x,t)\,dx$$

$$\leq E^\varepsilon(0) \leq C.$$

Thus

$$\{u^\varepsilon\} \in BL^\infty(0,\infty; H_0^2(0,1)), \quad \{u_t^\varepsilon\} \in BL^\infty(0,\infty; L^2(0,1)).$$

Again, (2.4) implies that for all $\tau > 0$

$$C \geq E^\varepsilon(0) \geq E^\varepsilon(0) - E^\varepsilon(\tau) = \beta \int_0^\tau \int_0^1 (u_{xt}^\varepsilon)^2(x,t)\,dx\,dt$$

and so, letting $\tau \to \infty$ we conclude that $\{u_{xt}^\varepsilon\} \in BL^2(\mathcal{K}_\infty)$. $\qquad\square$

In what follows we assume that a subsequence has been extracted in accordance with the previous result. The strong convergence of $\{u_x^\varepsilon\}$ is provided by the method of compensated compactness ([M], [T1]), more precisely by the Div-Curl Lemma (see Theorem 1.1 in [T1]).

Lemma 3.2. *Let $\Omega \subset \mathrm{R}^N$ be an open, bounded domain, and let $E^\varepsilon, G^\varepsilon \in L^2(\Omega; \mathrm{R}^N)$ be such that*

$$\sum_{i=1}^N \frac{\partial E_i^\varepsilon}{\partial x_i}, \quad \frac{\partial G_i^\varepsilon}{\partial x_j} - \frac{\partial G_j^\varepsilon}{\partial x_i}$$

belong to a compact subset of H_{loc}^{-1}, for all $1 \leq i,j \leq N$. Suppose further that $E^\varepsilon \rightharpoonup E_0$ and $G^\varepsilon \rightharpoonup G_0$ in $L^2(\Omega; \mathrm{R}^N)$. Then

$$E^\varepsilon \cdot G^\varepsilon := \sum_{i=1}^N E_i^\varepsilon\, G_i^\varepsilon \rightharpoonup E_0 \cdot G_0$$

in the sense of distributions.

An immediate consequence is

Corollary 3.3. *Fix $T > 0$ and let $\{U^\epsilon\}$ be a bounded sequence in $H^1(\mathcal{K}_T)$ such that $\{U^\epsilon_{xx}\}$ is bounded in $L^2(\mathcal{K}_T)$. If $U^\epsilon \rightharpoonup U$ weakly in $H^1(\mathcal{K}_T)$ then $U^\epsilon_x \to U_x$ strongly in $L^2_{\mathrm{loc}}(\mathcal{K}_T)$.*

Proof. Using the notation

$$\mathrm{div}(f,g) := f_x + g_t, \quad \mathrm{curl}(f,g) := f_t - g_x,$$

it follows that $E^\epsilon := (U^\epsilon_x, 0) \rightharpoonup E_0 := (U_x, 0)$ and $G^\epsilon := (U^\epsilon_x, U^\epsilon_t) \rightharpoonup G_0 := (U_x, U_t)$ weakly in $L^2(\mathcal{K}_T)$, with

$$\mathrm{div}(U^\epsilon_x, 0) \in BL^2(\mathcal{K}_T) \text{ and } \mathrm{curl}(U^\epsilon_x, U^\epsilon_t) = 0.$$

Therefore, by the Div-Curl Lemma (see Lemma 3.2)

$$(U^\epsilon_x)^2 \rightharpoonup (U_x)^2$$

in the sense of distributions. Hence, for every $\phi \in D(\mathcal{K}_T)$ we have

$$\int_{\mathcal{K}_T} (U^\epsilon_x - U_x)^2 \phi = \int_{\mathcal{K}_T} (U^\epsilon_x)^2 \phi - 2 U^\epsilon_x U_x \phi + (U_x)^2 \phi$$

$$\to \int_{\mathcal{K}_T} (U_x)^2 \phi - 2(U_x)^2 \phi + (U_x)^2 \phi = 0,$$

and we conclude that

$$U^\epsilon_x \to U_x \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathcal{K}_T).$$

$\square$

Note that by Proposition 3.1 the sequence $U^\epsilon := u^\epsilon$ is bounded in $H^1(\mathcal{K}_\infty)$ and it satisfies the hypotheses of Corollary 3.3. Using Sobolev-Rellich-Kondrachov Compact Imbedding Theorem and Corollary 3.3 we obtain the following result.

Proposition 3.4. *Under the hypotheses of Proposition 3.1*

$$u^\epsilon \to U \text{ strongly in } L^p(\mathcal{K}_T) \text{ and } u^\epsilon_x \to U_x \text{ strongly in } L^p_{\mathrm{loc}}(\mathcal{K}_\infty) \text{ for all } 1 \le p < \infty.$$

Strong convergence of $\{u^\epsilon_t\}$ is an immediate consequence of an interpolation compactness result due to Aubin (see [A], [L], Theorem 5.1, and [T2], page 50).

Theorem 3.5. *Let $B_0 \subset B_1 \subset B_2$ be three Hilbert spaces with B_1 continuously imbedded in B_2 and B_0 compactly imbedded in B_1. Let $1 < p < \infty$ and consider a sequence $\{V^\epsilon\}$ such that*

$$V^\epsilon \in BL^p(0,T;B_0), \quad \frac{dV^\epsilon}{dt} \in BL^p(0,T;B_2).$$

Then $\{V^\epsilon\}$ is contained in a compact subset of $L^p(0,T;B_1)$.

We have

Theorem 3.6. *Under the hypotheses of Proposition 3.1*

$$u^\epsilon_t \to U_t \quad \text{strongly in } L^2(\mathcal{K}_T) \text{ for all } T > 0.$$

Proof. Set

$$V^\epsilon := u^\epsilon_t.$$

By Proposition 3.1 we have

$$V^\epsilon \in BL^2(0,T;H^1_0(0,1)),$$

and using (2.1) we obtain

$$\frac{dV^\epsilon}{dt} = u^\epsilon_{tt} \in BL^2(0,T; H^{-2}(0,1)).$$

Setting $B_0 := H^1_0(0,1)$, $B_1 := L^2(0,1)$ and $B_2 := H^{-2}(0,1)$, from Theorem 3.5 we conclude that every subsequence of $\{u^\epsilon_t\}$ admits a subsequence converging strongly in $L^2(\mathcal{K}_T)$, and so

$$u^\epsilon_t \to U_t \quad \text{strongly in } L^2(\mathcal{K}_T).$$

$\square$

Next we want to show that

Theorem 3.7. *Under the hypotheses of Proposition 3.1*

$$u^\epsilon_{xx} \to U_{xx} \quad \text{strongly in } L^2_{\text{loc}}(\mathcal{K}_\infty).$$

This result will follow from the partial differential equation satisfied by u^ϵ and from the compactness lemma below.

Lemma 3.8. *Fix $T > 0$ and let $\{R^\epsilon\}, \{U^\epsilon\}, \{V^\epsilon\}, \{W^\epsilon\}, \{Y^\epsilon\}, \{Z^\epsilon\} \in BL^2(\mathcal{K}_T)$. Suppose further that*

$$\begin{cases} U^\epsilon_{xx} &= R^\epsilon_x + V^\epsilon_t + W^\epsilon \\ U^\epsilon_t &= Z^\epsilon_x + Y^\epsilon. \end{cases}$$

Then $\{U^\epsilon\}$ remains in a bounded subset of $H^{1/3}(\mathcal{K}')$ for every compact $\mathcal{K}' \subset\subset \mathcal{K}_T$.

Before proving this result we show how to derive Theorem 3.7.

Proof of Theorem 3.7. Set $U^\epsilon := u^\epsilon_{xx}$. By Proposition 3.1 we have $\{U^\epsilon\} \in BL^2(\mathcal{K}_T)$,

$$U^\epsilon_t = Z^\epsilon_x, \quad \text{where } Z^\epsilon := u^\epsilon_{xt} \in BL^2(\mathcal{K}_T),$$

and

$$\begin{aligned} U^\epsilon_{xx} = u^\epsilon_{xxxx} &= \frac{1}{\gamma} \left[((u^\epsilon_x)^3 - u^\epsilon_x)_x + \beta u^\epsilon_{xxt} - \alpha u^\epsilon - u^\epsilon_{tt} \right] \\ &= R^\epsilon_x + V^\epsilon_t + W^\epsilon, \end{aligned}$$

where

$$R^\epsilon := \frac{1}{\gamma}((u^\epsilon_x)^3 - u^\epsilon_x), \quad V^\epsilon := \frac{1}{\gamma}(\beta u^\epsilon_{xx} - u^\epsilon_t), \quad W^\epsilon := -\frac{\alpha}{\gamma} u^\epsilon.$$

By Sobolev Imbedding Theorem and Proposition 3.1 it follows that $\{R^\epsilon\}, \{V^\epsilon\}$ and $\{W^\epsilon\} \in BL^2(\mathcal{K}_T)$. Hence, due to Lemma 3.8 we have that $\{u^\epsilon_{xx}\}$ is in a bounded subset of $H^{1/3}(\mathcal{K}')$ and, in particular, $\{u^\epsilon_{xx}\}$ is in a compact set of $L^2_{\text{loc}}(\mathcal{K}')$ for every compact $\mathcal{K}' \subset\subset \mathcal{K}_T$. $\square$

Proof of Lemma 3.8. Fix $\varphi \in C^\infty_0(0,T), \psi \in C^\infty_0(0,1)$ and define

$$\tilde{U}^\epsilon := \varphi(t)\psi(x)U^\epsilon(x,t).$$

Then

$$\begin{cases} \tilde{U}^\epsilon_{xx} &= A^\epsilon_x + B^\epsilon_t + C^\epsilon \\ \tilde{U}^\epsilon_t &= D^\epsilon_x + E^\epsilon \end{cases}$$

where

$$A^\epsilon := 2\varphi\psi_x U^\epsilon + \varphi\psi R^\epsilon, \quad B^\epsilon := \varphi\psi V^\epsilon, \quad C^\epsilon := -\varphi\psi_{xx} U^\epsilon + \varphi\psi W^\epsilon - \varphi_t\psi V^\epsilon - \varphi\psi_x R^\epsilon$$

and

$$D^\epsilon := \varphi\psi Z^\epsilon, \quad E^\epsilon := \varphi_t\psi U^\epsilon - \varphi\psi_x Z^\epsilon + \varphi\psi U^\epsilon$$

are uniformly bounded in $L^2(\mathcal{K}_T)$. Applying Fourier transform to the system of equations we have

$$\begin{cases} -\xi^2 \hat{U}^\epsilon &= \mathrm{i}\xi\,\hat{A}^\epsilon + \mathrm{i}\tau\,\hat{B}^\epsilon + \hat{C}^\epsilon \\ \mathrm{i}\tau \hat{U}^\epsilon &= \mathrm{i}\xi\,\hat{D}^\epsilon + \hat{E}^\epsilon. \end{cases}$$

Therefore

$$\hat{U}^\epsilon \frac{|\xi|^2}{|\xi| + |\tau| + 1}, \; \hat{U}^\epsilon \frac{|\tau|}{|\xi| + 1} \in BL^2(\mathrm{R}^2), \tag{3.2}$$

and as $\hat{U}^\epsilon \in BL^2(\mathrm{R}^2)$, from (3.2) we deduce that

$$\hat{U}^\epsilon \frac{|\xi|^2 + |\xi| + |\tau| + 1}{|\xi| + |\tau| + 1}, \; \hat{U}^\epsilon \frac{|\tau| + |\xi| + 1}{|\xi| + 1} \in BL^2(\mathrm{R}^2).$$

In particular, using Hölder inequality with $p = 3/2$ and $p' = 3$, we deduce that

$$|\hat{U}^\epsilon| \left(\frac{|\tau| + |\xi| + 1}{|\xi| + 1} \right)^{2/3} \left(\frac{|\xi|^2 + |\xi| + |\tau| + 1}{|\xi| + |\tau| + 1} \right)^{1/3} \in BL^2(\mathrm{R}^2),$$

so

$$|\hat{U}^\epsilon|(|\tau| + |\xi| + 1)^{1/3}$$

remains in a bounded set of $L^2(\mathrm{R}^2)$. We conclude that U^ϵ is in a bounded set of $H^{1/3}(\mathcal{K}')$ for all compact set $\mathcal{K}' \subset\subset \mathcal{K}_T$. $\square$

Next we show that $\{u^\epsilon_{xt}\}$ converges locally strongly provided the initial data $\{(a^\epsilon, b^\epsilon)\}$ admits uniform bounds on $H^4(0,1) \times H^2(0,1)$. We remark that this regularity was prescribed in Theorem 2.1 asserting existence of strong solutions, and that under this assumption $\{b^\epsilon_x\}$ does not oscillate. We will, therefore, establish that there is no creation of oscillations in $\{u^\epsilon_{xt}\}$. First we provide uniform bounds on $\{u^\epsilon_{tt}\}$ and $\{u^\epsilon_{xxt}\}$.

Proposition 3.9. *If $\{(a^\epsilon, b^\epsilon)\}$ is bounded in $(H^2_0(0,1) \cap H^4(0,1)) \times H^2_0(0,1)$ then*

$$\{u^\epsilon_{tt}\}, \quad \{u^\epsilon_{xxt}\} \in BL^\infty(0,\infty; L^2(0,1)). \tag{3.3}$$

Proof. Let u^ϵ be the solution provided by Theorem 2.1 with initial conditions (a^ϵ, b^ϵ), and define

$$\mathcal{E}^\epsilon(t) := \frac{1}{2} \int_0^1 [(u^\epsilon_{tt})^2(x,t) + \gamma(u^\epsilon_{xxt})^2(x,t) + \alpha(u^\epsilon_t)^2(x,t)]\,dx.$$

Note that $\mathcal{E}^\epsilon(t)$ is well defined for all $t \geq 0$ due to the fact that $u^\epsilon_{tt}, u^\epsilon_{xxt}, u^\epsilon_t \in C([0,\infty); L^2(0,1))$.

Define

$$\Delta_h f(x,t) := \frac{f(x, t+h) - f(x)}{h}.$$

By (2.1) we have

$$\Delta_h u^\epsilon_{tt}(x,t) = \Delta_h \left((u^\epsilon_x)^3 - u^\epsilon_x\right)_x + \beta\Delta_h u^\epsilon_{xxt} - \gamma\Delta_h u^\epsilon_{xxxx} - \alpha\Delta_h u^\epsilon,$$

and the boundary conditions (2.2) imply that for $t > 0$

$$\Delta_h u^\epsilon(x,t) = \Delta_h u^\epsilon_x(x,t) = \Delta_h u^\epsilon_t(x,t) = \Delta_h u^\epsilon_{xt}(x,t) = 0 \text{ for } x \in \{0,1\}.$$

Integrating by parts we have

$$\int_0^1 \Delta_h u_{tt}^\varepsilon(x,t) \Delta_h u_t^\varepsilon(x,t)\,dx = \int_0^1 \Delta_h u_t^\varepsilon(x,t) \Delta_h \left((u_x^\varepsilon)^3 - u_x^\varepsilon\right)_x (x,t)\,dx$$

$$- \beta \int_0^1 (\Delta_h u_{xt}^\varepsilon(x,t))^2\,dx - \gamma \int_0^1 \Delta_h u_{xxt}^\varepsilon(x,t) \Delta_h u_{xx}^\varepsilon(x,t)\,dx \qquad (3.4)$$

$$- \alpha \int_0^1 \Delta_h u_t^\varepsilon(x,t) \Delta_h u^\varepsilon(x,t)\,dx.$$

Setting

$$\mathcal{E}_h^\varepsilon(t) := \frac{1}{2} \int_0^1 \left(\Delta_h u_t^\varepsilon\right)^2 (x,t) + \gamma \left(\Delta_h u_{xx}^\varepsilon\right)^2 (x,t) + \alpha \left(\Delta_h u^\varepsilon\right)^2 (x,t)\,dx,$$

(3.4) implies that

$$\frac{d\mathcal{E}_h^\varepsilon(t)}{dt} = \int_0^1 \left[\Delta_h u_t^\varepsilon(x,t) \Delta_h \left((u_x^\varepsilon)^3 - u_x^\varepsilon\right)_x (x,t) - \beta \left(\Delta_h u_{xt}^\varepsilon(x,t)\right)^2 \right] dx$$

$$= -\int_0^1 \left[\Delta_h u_{xt}^\varepsilon(x,t) \Delta_h \left((u_x^\varepsilon)^3 - u_x^\varepsilon\right)(x,t) + \beta \left(\Delta_h u_{xt}^\varepsilon(x,t)\right)^2 \right] dx. \quad (3.5)$$

Hence

$$\mathcal{E}_h^\varepsilon(t) = \mathcal{E}_h^\varepsilon(0) - \int_0^t \int_0^1 \Delta_h u_{xt}^\varepsilon(x,s) \Delta_h \left((u_x^\varepsilon)^3 - u_x^\varepsilon\right)(x,s)\,dx\,ds$$

$$- \beta \int_0^t \int_0^1 \left(\Delta_h u_{xt}^\varepsilon(x,s)\right)^2 dx\,ds,$$

and since

$$ax - \beta x^2 \le \frac{1}{4\beta} a^2 \text{ for all } x,$$

we conclude that

$$\mathcal{E}_h^\varepsilon(t) \le \mathcal{E}_h^\varepsilon(0) + \frac{1}{4\beta} \int_0^t \int_0^1 \left(\Delta_h \left((u_x^\varepsilon)^3 - u_x^\varepsilon\right)\right)^2 dx\,ds. \qquad (3.6)$$

Given $f \in C^1((0,1) \times (0,t+h))$ we have

$$\left(\frac{f(x,s+h) - f(x,s)}{h}\right)^2 = \left(\frac{1}{h}\int_s^{s+h} f_t(x,r)\,dr\right)^2 \le \frac{1}{h}\int_s^{s+h} f_t^2(x,r)\,dr,$$

and so

$$\int_0^t \int_0^1 (\Delta_h f)^2(x,s)\,dx\,ds \le \frac{1}{h}\int_0^t \int_0^1 \int_s^{s+h} f_t^2(x,r)\,dr\,dx\,ds$$

$$= \frac{1}{h}\int_0^{t+h} \int_0^1 \int_{\max\{0,r-h\}}^{\min\{r,t\}} ds\, f_t^2(x,r)\,dx\,dr$$

$$\le \int_0^{t+h} \int_0^1 f_t^2(x,r)\,dx\,dr.$$

Using a density argument, we may apply this inequality to $f := (u_x^\varepsilon)^3 - u_x^\varepsilon$, and (3.6) yields

$$\mathcal{E}_h^\varepsilon(t) \le \mathcal{E}_h^\varepsilon(0) + \frac{1}{4\beta} \int_0^{t+h} \int_0^1 \left[((u_x^\varepsilon)^3 - u_x^\varepsilon)_t \right]^2 (x,r) dx \, dr \qquad (3.7)$$
$$\le \mathcal{E}_h^\varepsilon(0) + C\beta^{-1}$$

where C is independent of h, ε and t, and where we used Proposition 3.1 and Sobolev Imbedding Theorem to guarantee that $\{u_x^\varepsilon\} \in BL^\infty(\mathcal{K}_\infty)$. Finally, by Theorem 2.1

$$u_t^\varepsilon, u_{xx}^\varepsilon, u^\varepsilon \in C^1([0,\infty); L^2(0,1)),$$

hence

$$\|\Delta_h u_t^\varepsilon - u_{tt}^\varepsilon\|_{L^2(0,1)}, \|\Delta_h u_{xx}^\varepsilon - u_{xxt}^\varepsilon\|_{L^2(0,1)}, \|\Delta_h u^\varepsilon - u^\varepsilon\|_{L^2(0,1)}$$

tend to zero as $h \to 0$, and so

$$\lim_{h \to 0} \mathcal{E}_h^\varepsilon(t) = \mathcal{E}^\varepsilon(t).$$

Using the fact that the bounds on the initial conditions imply that $\mathcal{E}_h^\varepsilon(0)$ remain bounded, taking the limit in (3.7) as $h \to 0$ we deduce that

$$\mathcal{E}^\varepsilon(t) \le C\left(1 + \frac{1}{\beta}\right) \quad \text{for all } t > 0.$$

$\square$

Finally, we prove

Theorem 3.10. *If $\{(a^\varepsilon, b^\varepsilon)\}$ is bounded in $(H_0^2(0,1) \cap H^4(0,1)) \times H_0^2(0,1)$ then*

$$u_{xt}^\varepsilon \to U_{xt} \quad \text{strongly in } L_{\text{loc}}^2(\mathcal{K}_T) \text{ for all } T > 0.$$

Proof. By Propositions 3.1 and 3.9 the sequence $U^\varepsilon := u_t^\varepsilon$ satisfies the hypotheses of Corollary 3.3 and the proof is concluded. $\square$

We remark that the proofs leading to strong convergence of $\{u^\varepsilon\}$, $\{u_x^\varepsilon\}$, $\{u_t^\varepsilon\}$, $\{u_{xx}^\varepsilon\}$ and $\{u_{xt}^\varepsilon\}$ used different analytical arguments, including Sobolev-Rellich-Kondrachov Compact Imbedding Theorem, interpolation theorems, Fourier transforms, compensated compactness, and difference quotient regularity techniques. However, we note that if we prescribe Dirichlet boundary conditions on u_{xx} rather than on the strains u_x, and assuming that $\beta^2 > 4\gamma$ and that

$$\sup_\varepsilon \|a^\varepsilon\|_{H^3} + \|b^\varepsilon\|_{H^1} = C < +\infty, \qquad (3.8)$$

then strong convergence of $\{u^\varepsilon\}$, $\{u_x^\varepsilon\}$, $\{u_t^\varepsilon\}$, $\{u_{xx}^\varepsilon\}$ in $L^2(\mathcal{K}_T)$ and of $\{u_{xt}^\varepsilon\}$, $\{u_{xxx}^\varepsilon\}$ in $L_{\text{loc}}^2(\mathcal{K}_T)$ for all $T > 0$ may be be easily obtained using simple energy apriori estimates, compact imbedding theorems and compensated compactness. Indeed, suppose that

$$u^\varepsilon \in C([0,+\infty); H^3(0,1)) \cap C^1([0,+\infty); H^1(0,1)) \cap C^2([0,+\infty); H^{-1}(0,1))$$

is a solution of

$$u_{tt}^\varepsilon = [(u_x^\varepsilon)^3 - u_x^\varepsilon + \beta u_{xt}^\varepsilon - \gamma u_{xxx}^\varepsilon]_x - \alpha u^\varepsilon \qquad (3.9)$$

satisfying the boundary conditions

$$u^\varepsilon = u_{xx}^\varepsilon = 0 \text{ if } x \in \{0,1\}$$

and the initial conditions

$$u^\varepsilon(x,0) = a^\varepsilon(x), \ u_t^\varepsilon(x,0) = b^\varepsilon(x).$$

Proposition 3.11. *For $T > 0$ the sequences $\{u^\varepsilon\}, \{u_x^\varepsilon\}, \{u_t^\varepsilon\}$ and $\{u_{xx}^\varepsilon\}$ converge strongly in $L^2(\mathcal{K}_T)$, and $\{u_{xt}^\varepsilon\}, \{u_{xxx}^\varepsilon\}$ converge strongly in $L^2_{loc}(\mathcal{K}_T)$.*

As before, using Sobolev-Rellich-Kondrachov Compact Imbedding Theorem and Corollary 3.3, the above result follows immediatly if we prove that

$$\int_0^T \left(\|u_{xx}^\varepsilon(t)\|_{H^2(0,1)}^2 + \|u_{xxt}^\varepsilon(t)\|_{L^2(0,1)}^2 + \|u_t^\varepsilon(t)\|_{H^2(0,1)}^2 + \|u_{tt}^\varepsilon(t)\|_{L^2(0,1)}^2 \right) dt \le C. \tag{3.10}$$

To prove this inequality first we remark that by virtue of (2.4), which still holds, we have

$$E^\varepsilon(t) \le E^\varepsilon(0) \le C$$

and so

$$\|u_t^\varepsilon(t)\|_{L^2(0,1)} + \|u_{xx}^\varepsilon(t)\|_{L^2(0,1)} \le C. \tag{3.11}$$

We recall the following regularity result (see [E], page 326). If $z \in L^2(0,T; H_0^1(0,1))$, $z_t \in L^2(0,T; H^{-1}(0,1))$, and if

$$z_t = \mu z_{xx} + F$$

where $\mu > 0$, $F \in L^2(0,T; L^2(0,1))$, and $z(\cdot, 0) \in H_0^1(0,1)$, then

$$\int_0^T \left(\|z(t)\|_{H^2(0,1)}^2 + \|z_t(t)\|_{L^2(0,1)}^2 \right) dt$$
$$\le C \left(\|z(0)\|_{H_0^1(0,1)}^2 + \int_0^T \|F(t)\|_{L^2(0,1)}^2 dt \right). \tag{3.12}$$

We may write (3.9) as

$$(\partial_t - A\partial_x^2)(\partial_t - B\partial_x^2)u^\varepsilon = -\alpha u^\varepsilon + ((u_x^\varepsilon)^3 - u_x^\varepsilon)_x,$$

where $A, B > 0$, and setting $w^\varepsilon := (\partial_t - B\partial_x^2)u^\varepsilon$ and $f^\varepsilon := -\alpha u^\varepsilon + ((u_x^\varepsilon)^3 - u_x^\varepsilon)_x$ we deduce that

$$w_t^\varepsilon = Aw_{xx}^\varepsilon + f^\varepsilon, \quad \text{and} \ w^\varepsilon = 0 \ \text{if} \ x \in \{0,1\}.$$

By (3.8), (3.11) and (3.12) we have

$$\int_0^T \left(\|w^\varepsilon(t)\|_{H^2(0,1)}^2 + \|w_t^\varepsilon(t)\|_{L^2(0,1)}^2 \right) dt$$
$$\le C \left(\|b^\varepsilon - Ba_{xx}^\varepsilon\|_{H_0^1(0,1)}^2 + \int_0^T \|f^\varepsilon(t)\|_{L^2(0,1)}^2 dt \right) \tag{3.13}$$
$$\le C.$$

Since

$$u_t^\varepsilon = Bu_{xx}^\varepsilon + w^\varepsilon,$$

letting $z^\varepsilon := u_t^\varepsilon$ or $z^\varepsilon := u_{xx}^\varepsilon$ it follows that

$$\begin{cases} z_t^\varepsilon &= Bz_{xx}^\varepsilon + F^\varepsilon \\ z^\varepsilon &= 0 \ \text{if} \ x \in \{0,1\}, \end{cases}$$

where $F^\varepsilon := w_t^\varepsilon$ or $F^\varepsilon := w_{xx}^\varepsilon$. In both cases, by (3.8) and (3.13)

$$\|z^\varepsilon(0)\|_{H^1(0,1)} \le C, \quad \int_0^T \|F^\varepsilon(t)\|_{L^2(0,1)}^2 \, dt \le C,$$

and so, applying (3.12) to each one of the sequences $\{z^\varepsilon\}$ defined above we obtain (3.10).

4. NUMERICAL SIMULATION

In this section we use a finite difference scheme to illustrate the behavior of solutions of (2.1), (2.2) with initial conditions

$$\begin{cases} u^\varepsilon(x,0) = 0 & 0 < x < 1, \\ u_t^\varepsilon(x,0) = \sin^6\left(\frac{1}{\varepsilon}\pi x\right), & 0 < x < 1, \end{cases}$$

where ε is taken to be $1/m$ with m an integer. By Theorem 3.7 we know that $\{u_{xx}^\varepsilon\}$ converges strongly in $L^2_{\text{loc}}(\mathcal{K}_\infty)$, hence we ask whether $\{u_{xx}^\varepsilon\}$ converges strongly in $L^2(\mathcal{K}_T)$. The simulations carried out in Figures 1 through 4 seem to indicate that this is true, in spite the fact that a residual, bounded, boundary layer remains.

We also consider the sequence $\{u_{xt}^\varepsilon\}$. Note that the stronger assumptions on the initial conditions made in Theorem 3.10 do not hold here. However, Figure 5 through 9 lead us to believe that $\{u_{xt}^\varepsilon\}$ converges strongly under the present weaker conditions.

First we describe briefly the difference scheme used. Taking $\Delta x := \frac{1}{K-1}$ with K an integer, we set $x_k := (k - 1/2)\Delta x$, $t^n := n\Delta t$, and denote

$$u_k^n \approx u^\varepsilon(x_k, t^n)$$

(suppressing the dependence on ε). For the boundary conditions we require

$$u_0^n = u_1^n = u_K^n = u_{K-1}^n = 0,$$

and for the initial conditions

$$\begin{cases} u_k^0 = 0, \\ u_k^1 = \Delta t \sin^6\left(\frac{1}{\varepsilon}\pi x_k\right). \end{cases}$$

Then the difference scheme is

$$(\Delta t)^{-2}(u_k^{n+1} - 2u_k^n + u_k^{n-1}) =$$

$$\beta(\Delta t)^{-1}\left[\frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{(\Delta x)^2} - \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2}\right]$$

$$-\gamma\frac{u_{k+2}^{n+1} - 4u_{k+1}^{n+1} + 6u_k^{n+1} - 4u_{k-1}^{n+1} + u_{k-2}^{n+1}}{(\Delta x)^4}$$

$$-\alpha u_k^n + \frac{\left[(D_{k+1/2}^n)^3 - D_{k+1/2}^n\right] - \left[(D_{k-1/2}^n)^3 - D_{k-1/2}^n\right]}{\Delta x},$$

where

$$D_{k+1/2}^n := \frac{u_{k+1}^n - u_k^n}{\Delta x}.$$

This scheme is second order in space but only first order in time. The time stepping method was chosen because it is unconditionally stable and its Fourier analysis (on

the whole line with only the principle part of the equation) closely resembles that of the continuous equation; that is, high frequency modes decay rapidly in both.

Now we consider the solution to (2.1) with $\beta = 3$ and $\gamma = \alpha = 1$. Figures 1 through 4 show the graphs of

$$x \mapsto 10^3 u^\varepsilon_{xx}(x, 5 \times 10^{-5})$$

for $\varepsilon = 1/20, 1/40, 1/80, 1/160$, respectively. It appears that $\{u^\varepsilon_{xx}\}$ does converge strongly in $L^2(\mathcal{K}_T)$, but that the limit (as $\varepsilon \to 0$) has more interesting behavior near the boundary. Figures 5 through 9 display the graphs of

$$x \mapsto u^\varepsilon_{xt}(x, 10^{-5})$$

for $\varepsilon = 1/20, 1/40, 1/80, 1/160, 1/320$, respectively. These suggest that for fixed $t > 0$

$$x \mapsto u^\varepsilon_{xt}(x, t)$$

converges strongly in $L^\infty(0,1)$. Note that

$$x \mapsto u^\varepsilon_{xt}(x, 0)$$

is unbounded in $L^\infty(0,1)$.
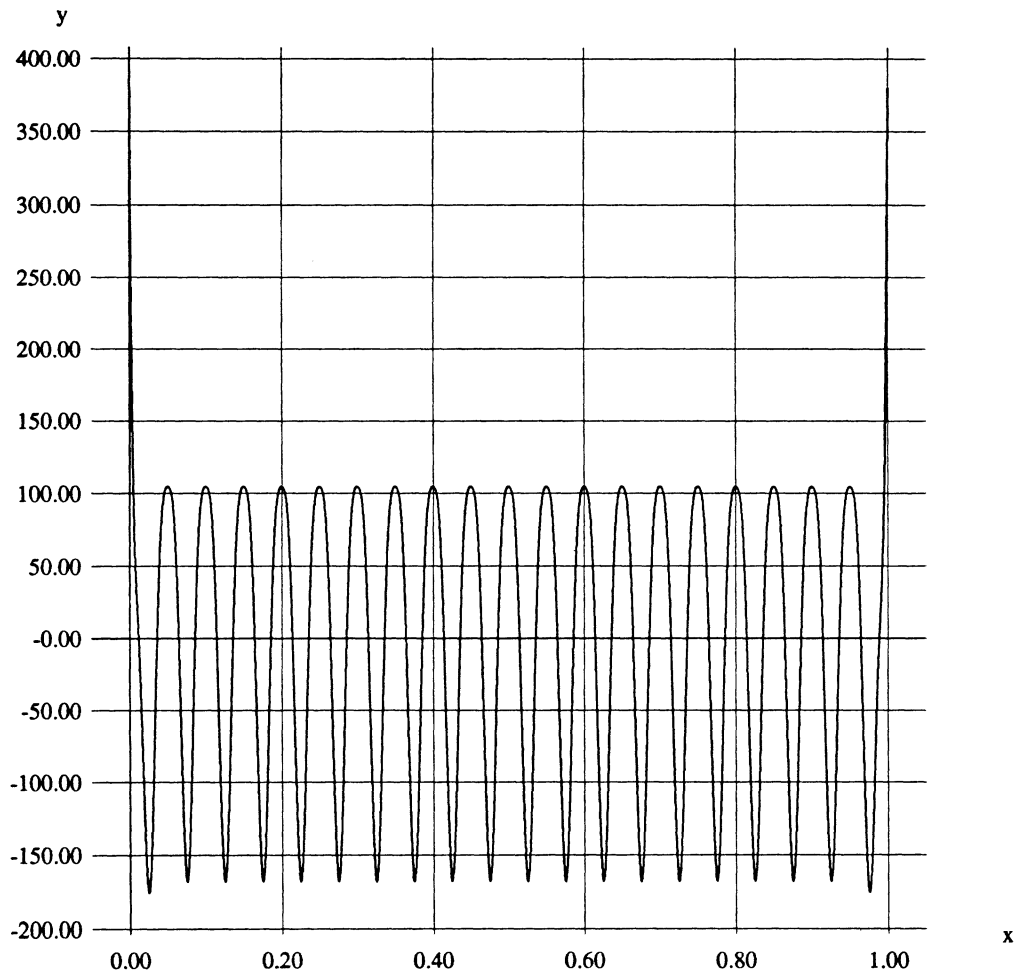
FIGURE 1. $u(x,0) = 0$, $u_t(x,0) = \sin^6(20\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$,
$\Delta t = 1.25 \times 10^{-6}$, $\Delta x = 1/4000$, $y \times 10^{-3} = u_{xx}(x, 5 \times 10^{-5})$.
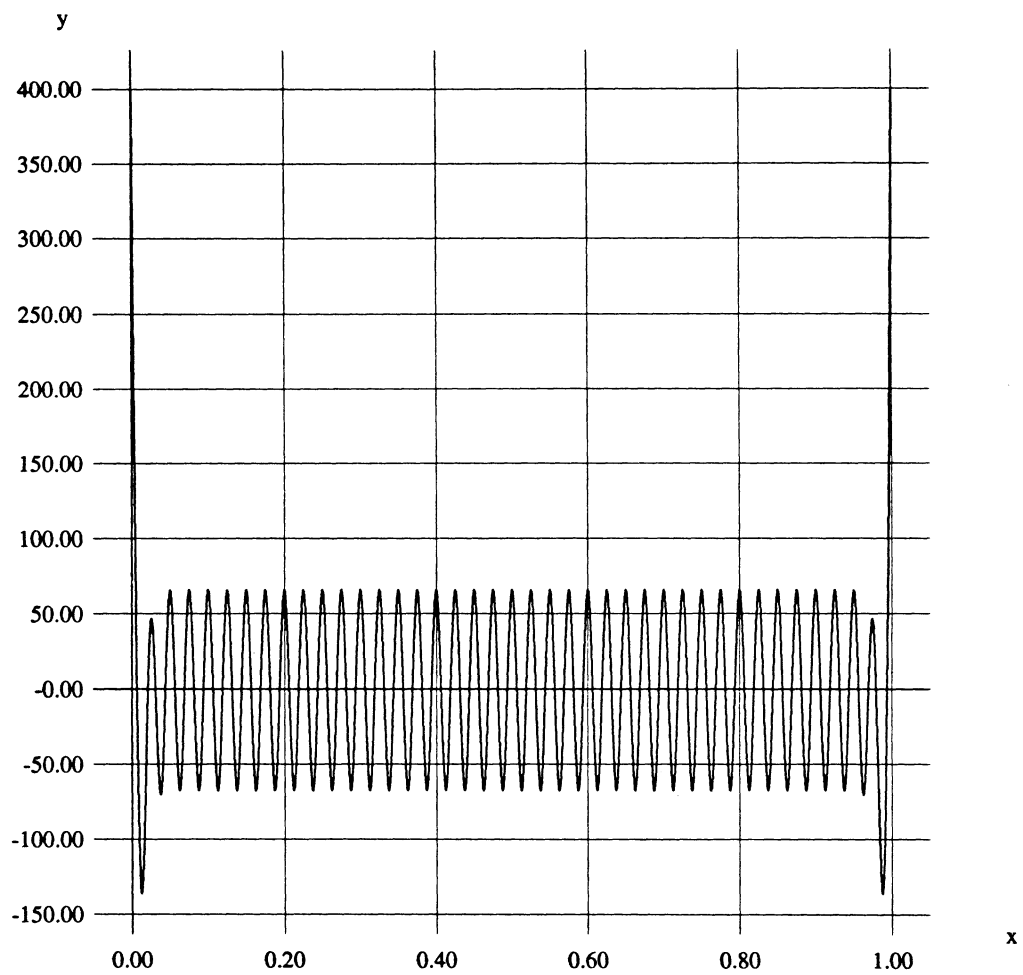
FIGURE 2. $u(x,0) = 0$, $u_t(x,0) = \sin^6(40\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 3.125 \times 10^{-7}$, $\Delta x = 1/8000$, $y \times 10^{-3} = u_{xx}(x, 5 \times 10^{-5})$.
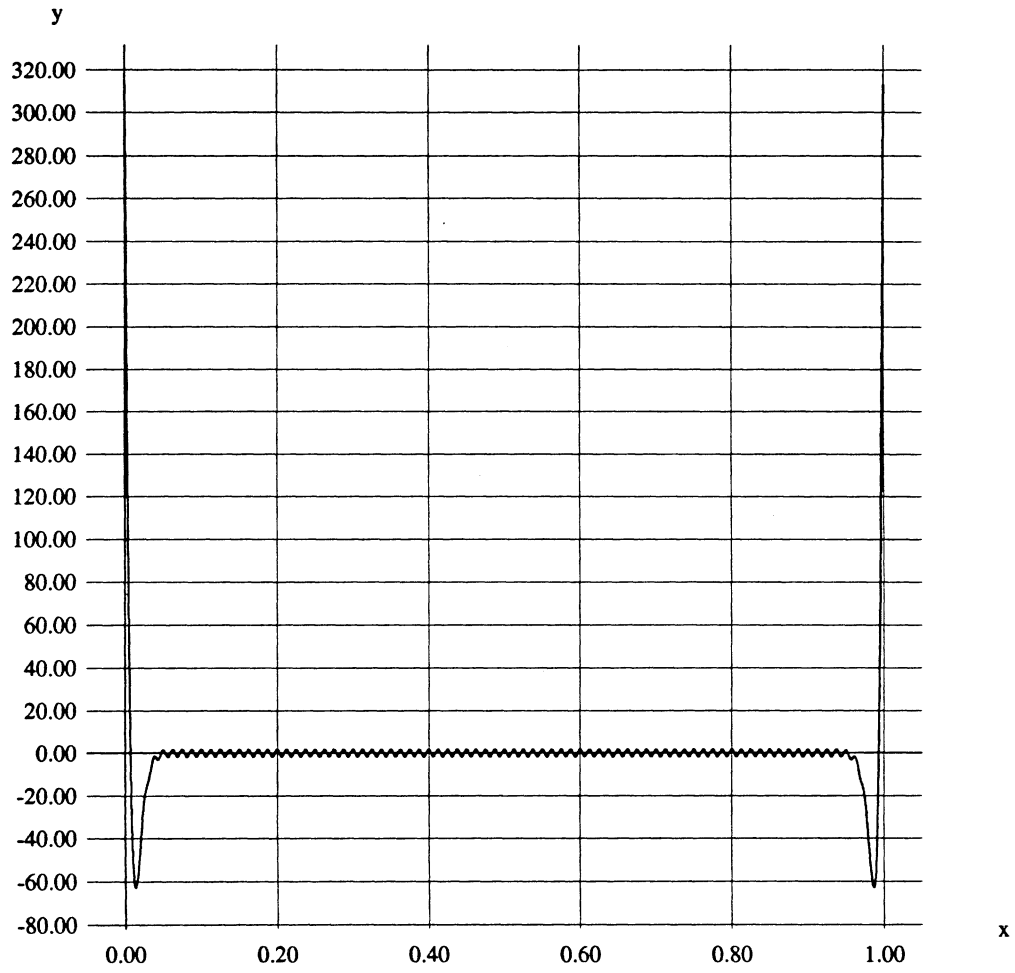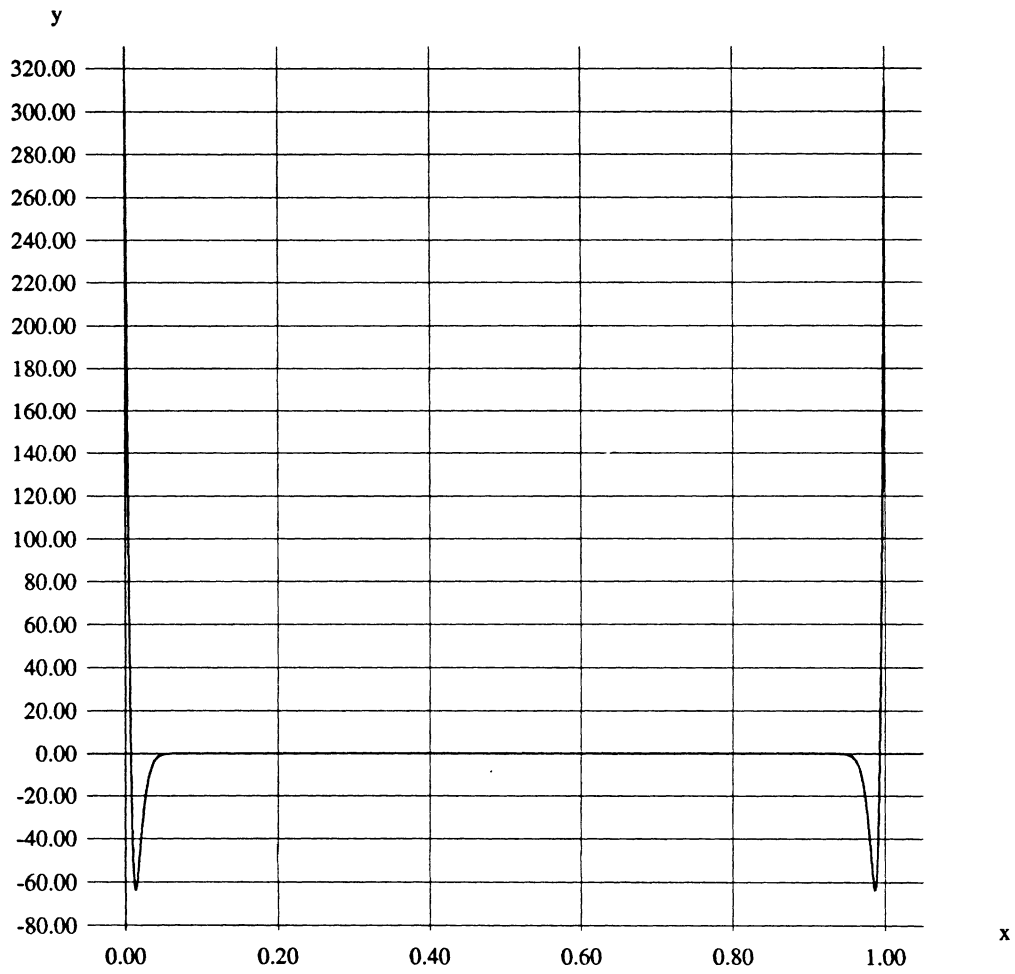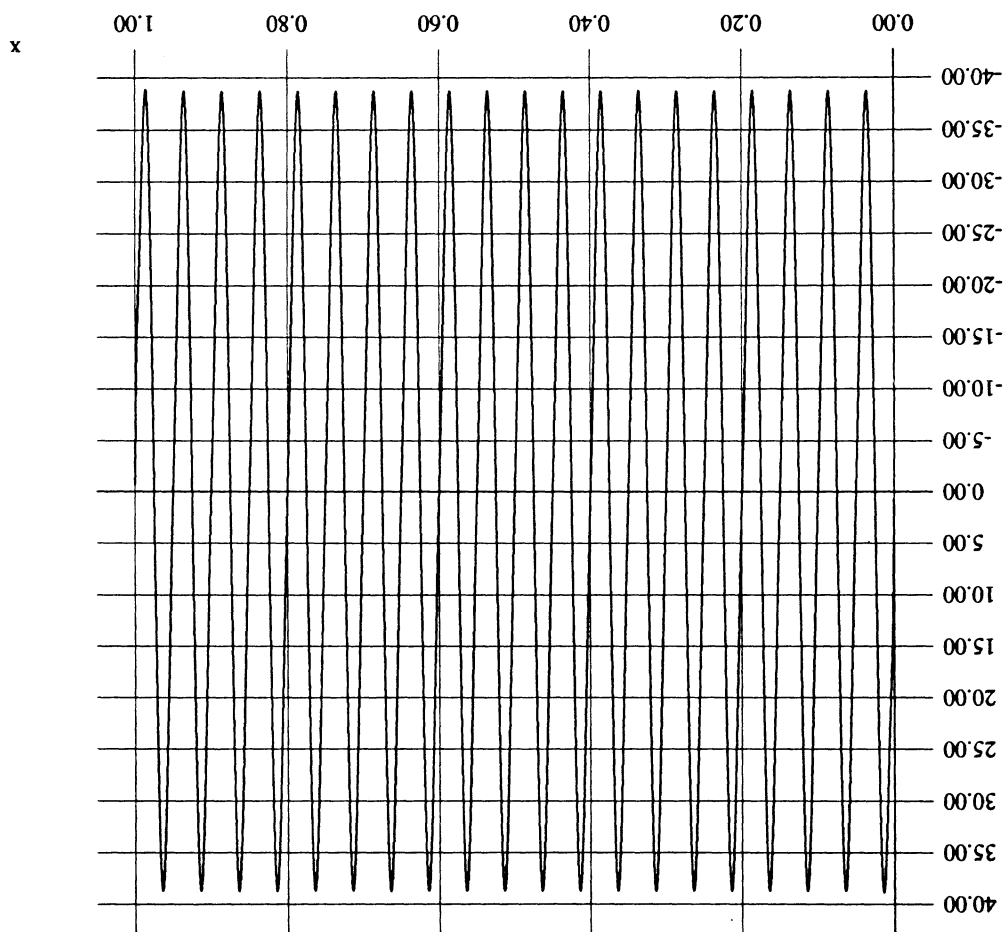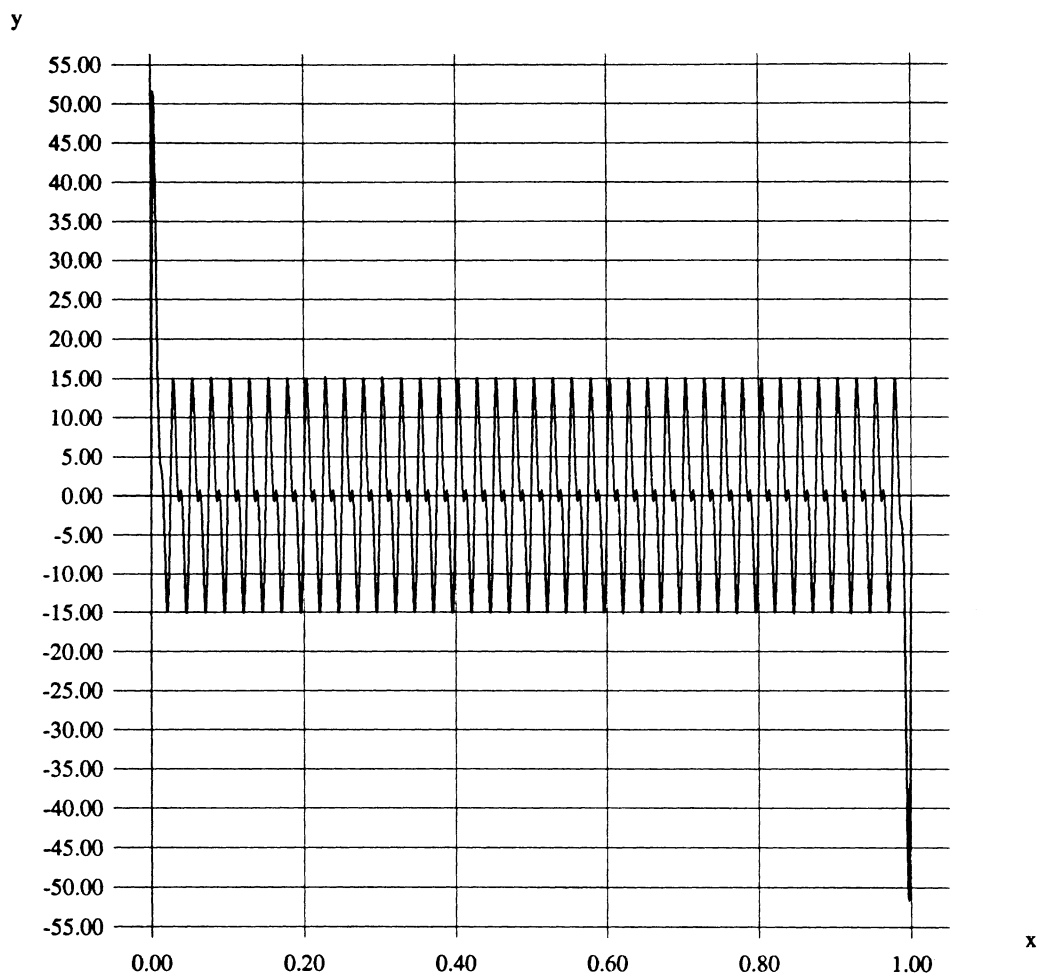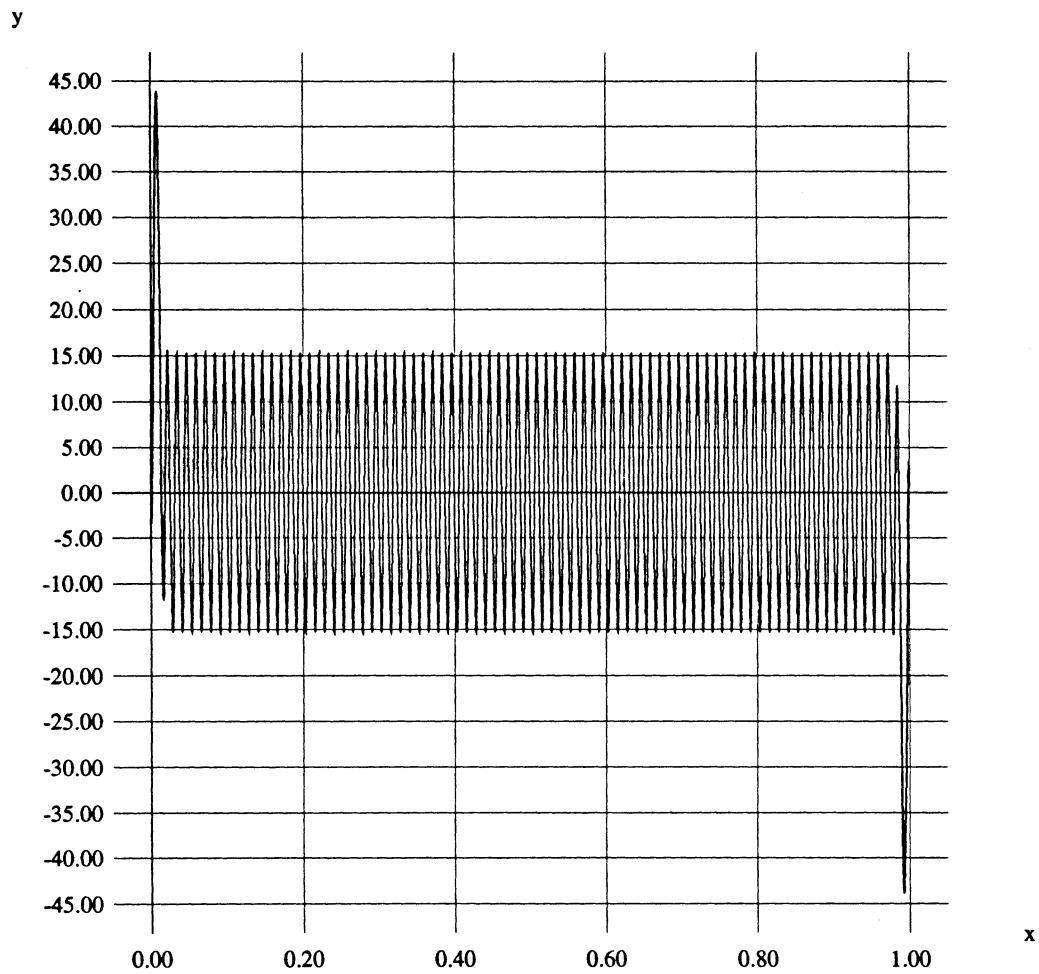
FIGURE 3. $u(x,0) = 0$, $u_t(x,0) = \sin^6(80\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 7.8125 \times 10^{-8}$, $\Delta x = 1/16000$, $y \times 10^{-3} = u_{xx}(x, 5 \times 10^{-5})$.

FIGURE 4. $u(x,0) = 0$, $u_t(x,0) = \sin^6(160\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 1.953125$, $\Delta x = 1/32000$, $y \times 10^{-3} = u_{xx}(x, 5 \times 10^{-5})$.

FIGURE 5. $u(x,0) = 0$, $u_t(x,0) = \sin^6(20\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 3.125 \times 10^{-7}$, $\Delta x = 1/8000$, $y = u_{xt}(x,10^{-5})$.

FIGURE 6. $u(x,0) = 0$, $u_t(x,0) = \sin^6(40\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 7.8125 \times 10^{-8}$, $\Delta x = 1/16000$, $y = u_{xt}(x, 10^{-5})$.
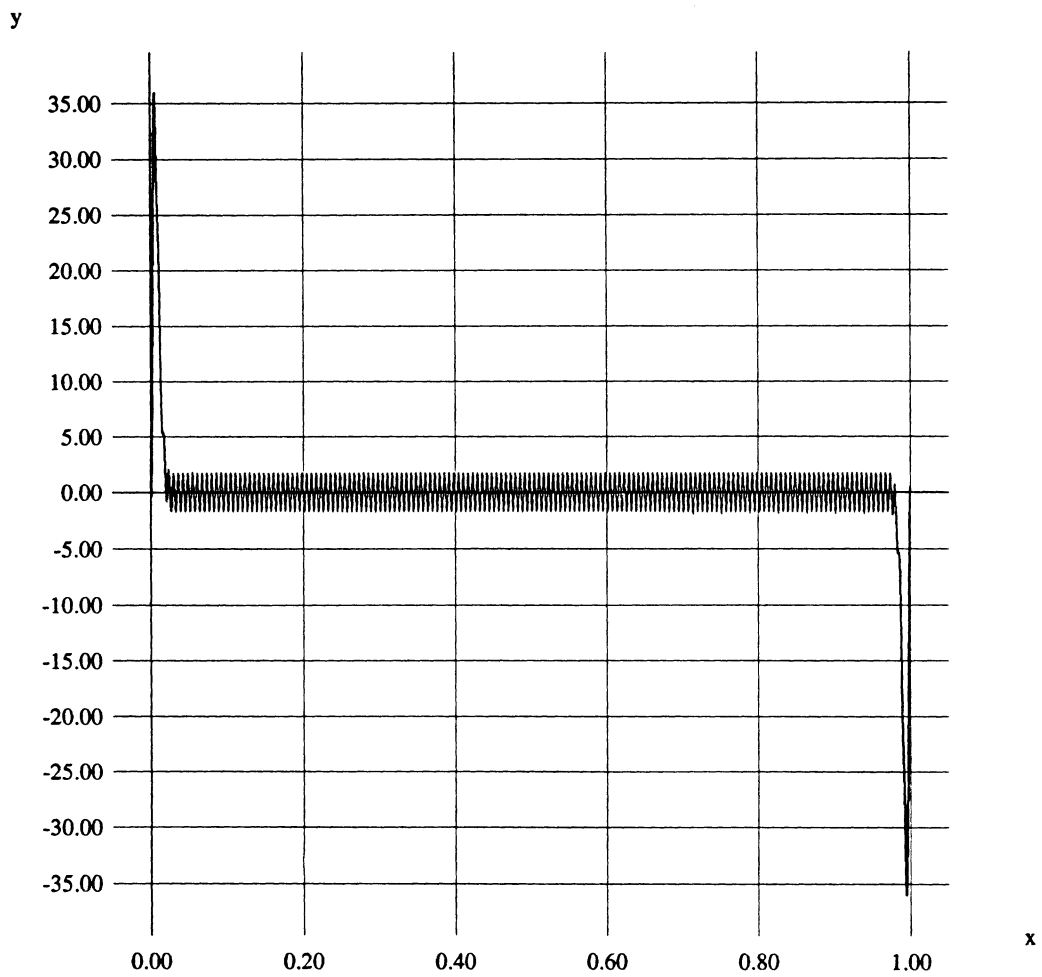
FIGURE 7. $u(x,0) = 0$, $u_t(x,0) = \sin^6(80\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 1.953125 \times 10^{-8}$, $\Delta x = 1/32000$, $y = u_{xt}(x, 10^{-5})$.

FIGURE 8. $u(x,0) = 0$, $u_t(x,0) = \sin^6(160\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 4.8828125 \times 10^{-9}$, $\Delta x = 1/64000$, $y = u_{xt}(x, 10^{-5})$.
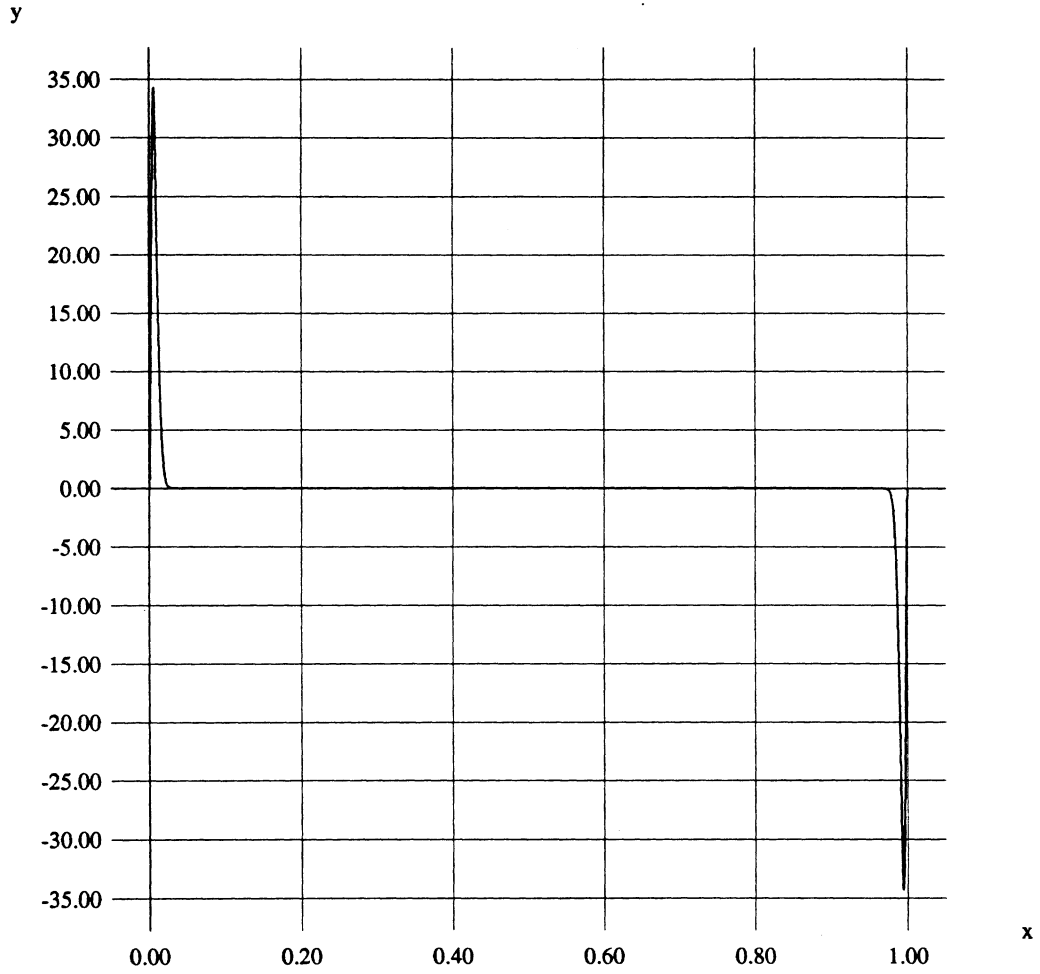
FIGURE 9. $u(x,0) = 0$, $u_t(x,0) = \sin^6(320\pi x)$, $\beta = 3$, $\gamma = \alpha = 1$, $\Delta t = 1.220703125 \times 10^{-9}$, $\Delta x = 1/128000$, $y = u_{xt}(x, 10^{-5})$.

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