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THE EFFECTIVE BULK ENERGY OF THE RELAXED ENERGY OF MULTIPLE INTEGRALS BELOW THE GROWTH EXPONENT

GUY BOUCHITTÉ,[1] IRENE FONSECA[2] AND JAN MALÝ[3]

ABSTRACT

The characterization of the bulk energy density of the relaxation in $W^{1,p}(\Omega;\mathbb{R}^d)$ of a functional

$$F(u,\Omega) := \int_\Omega f(\nabla u)\,dx$$

is obtained for $p > q - q/N$, where $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, and f is a continuous function on the set of $d \times N$ matrices verifying

$$0 \le f(\xi) \le C(1 + |\xi|^q)$$

for some constant $C > 0$ and $1 \le q < +\infty$. Typical examples may be found in cavitation and related theories. Standard techniques cannot be used due to the gap between the exponent q of the growth condition and the exponent p of integrability of the macroscopic strain ∇u. A recently introduced global method for relaxation and fine Sobolev trace and extension theorems are applied.

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1 INTRODUCTION

In this paper we identify the bulk energy density of the relaxed energy when the class of admissible fields strictly contains the Sobolev space where the functional is known to be continuous. Precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and consider a functional

$$F(u,U) := \int_U f(\nabla u)\,dx$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $U \subset \Omega$ is an open set, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, f is a continuous function on the set of $d \times N$ matrices, $\mathbb{M}^{d\times N}$, verifying

$$0 \leq f(\xi) \leq C(1+|\xi|^q)$$

for some constant $C > 0$ and $1 \leq p, q < +\infty$. This growth condition guarantees continuity of F on $W^{1,q}$.

We introduce the relaxed energies

$$\mathcal{F}(u,U) := \inf_{\{u_n\}} \left\{ \liminf F(u_n,U) \colon u_n \in W^{1,q}(U;\mathbb{R}^d),\ u_n \rightharpoonup u \text{ in } W^{1,p}(U;\mathbb{R}^d) \right\},$$

$$\mathcal{F}_{\mathrm{loc}}(u,U) := \inf_{\{u_n\}} \left\{ \liminf F(u_n,U) \colon u_n \in W^{1,q}_{\mathrm{loc}}(U;\mathbb{R}^d),\ u_n \rightharpoonup u \text{ in } W^{1,p}(U;\mathbb{R}^d) \right\}.$$

In the case where $p \geq q$ one has (see [AF], [B], [D], [M])

$$\mathcal{F}(u,U) = \mathcal{F}_{\mathrm{loc}}(u,U) = \int_U Qf(\nabla u)\,dx,$$

where the *quasiconvex envelope* of f is defined by

$$Qf(\xi) := \inf \left\{ \int_{(0,1)^N} f(\xi + \nabla\varphi(x))\,dx \;:\; \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d) \right\}.$$

It is clear that $QF \leq F$, and F is said to be *quasiconvex* if $QF = F$.

Here we treat the case where there is a gap between the space of admissible macroscopic fields, $W^{1,p}$, and the space where continuity of the energy follows immediatly from growth hypotheses, $W^{1,q}$. As a prototype example, often occurring in models related to elastic cavitation, let $d = N$ and

$$f(\xi) := |\xi|^{N-1} + |\det \xi|, \qquad \xi \in \mathbb{M}^{d\times N}.$$

Clearly sequences of deformations in $W^{1,N}$ with bounded energy will be weakly compact in $W^{1,N-1}$ but not necessarily in $W^{1,N}$. Here $q = N$ and $p = N-1$. This example has been studied at length, and in particular we refer to [ADM], [CDM], [FMar].

If $p < q - q/N$ then one may have $\mathcal{F}(u,\Omega) = 0$ (see [BM], [H]), and in the case where $p = q - q/N$ it may happen that $\mathcal{F}(u,\cdot)$ is not even subbaditive (see [CDM]). These degeneracies cannot occur if $1 < p < q$ and $p > q - q/N$. Within this range it was proven in [FMý] (see Theorems 3.1 and 3.2) that $\mathcal{F}(u,\cdot)$ is subbaditive, and $\mathcal{F}_{\mathrm{loc}}(u,\cdot)$ is a Radon measure if finite, i.e. if $\mathcal{F}_{\mathrm{loc}}(u,\Omega) < \infty$ then there exists a finite, Radon measure $\mathcal{R}(u,\cdot)$ such that

$$\mathcal{F}_{\mathrm{loc}}(u,U) = \mathcal{R}(u,U) \quad \text{and} \quad \mathcal{R}(u,U) \leq \mathcal{F}(u,U) \leq \mathcal{R}(u,\overline{U})$$

for all open sets $U \subset \Omega$. In addition, it can be shown easily that

$$\mathcal{F}_{\mathrm{loc}}(u,U) = \sup\{\mathcal{F}(u,V) \colon V \subset\subset U, V \text{open}\}.$$

A lower bound for the effective bulk energy density was obtained in [FMý] (see Theorem 4.1 and Corollary 4.2), precisely

$$\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) \geq Qf(\nabla u(x_0)) \tag{1.1}$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and for almost every $x_0 \in \Omega$. In this paper we obtain equality in (1.1). This result is achieved by using the global method for relaxation introduced by Bouchitté, Fonseca and Mascarenhas (see [BFM]) together with an extension operator $\mathbf{P}$ from $W^{1,p}$ into $W^{1,q}$ obtained by Fonseca and Malý in [FMý], Lemma 2.2.

Earlier results on lower semicontinuity for certain ranges $p < q$ and with quasiconvex integrands were obtained by [Mar1], [Mar2], and in the case of polyconvex energy densities and $p \geq N-1$, $q = N$, we refer to [ADM], [CDA], [CDM], [DM], [DMS], [FH], [G], [Mý1], [Mý2], [Mý3].

2 Preliminaries

In this section we introduce some notation and we recall some trace and extension theorems for Sobolev spaces. Also, throughout this work constants are designated by C and may vary from line to line, and $B(x_0, r)$ denotes the open ball $\{x \in \mathbb{R}^N \colon |x - x_0| < r\}$.

Given $x_0 \in \mathbb{R}^N$ and two radii $0 < r_1 < r_2$ we set

$$A(x_0, r_1, r_2) := \{x \colon r_1 < |x - x_0| < r_2\} = B(x_0, r_2) \setminus \overline{B}(x_0, r_1).$$

We denote by $\mathbf{T}$ the *trace operator*; if $u \in W^{1,p}(U;\mathbb{R}^d)$ and $\partial B(x_0,r) \subset \overline{U}$, then $\mathbf{T}[\partial B(x_0,r)]\, u$ is the trace of u on $\partial B(x_0,r)$. We write simply $\mathbf{T}$ if the center and radius of the sphere are clearly understood.

Let $x_0 \in \mathbb{R}^N$ and $0 < r_0 < r_1 < r_2 < 2r_0$. We consider a linear, compact operator

$$\mathbf{E} = \mathbf{E}[x_0, r_1, r_2] : v \in W^{1,p}(\partial A(x_0, r_1, r_2);\mathbb{R}^d) \mapsto \mathbf{E}v \in W^{1,q}(A(x_0, r_1, r_2);\mathbb{R}^d)$$

such that v is a trace of $\mathbf{E}v$. Since $p > q - q/N$ the existence of $\mathbf{E}$ follows from standard Sobolev trace and compact embedding theorems.

Furthermore (see [FMý], Lemma 2.2), for $p > q - q/N$ there exists a linear, continuous extension operator

$$\mathbf{P} = \mathbf{P}[x_0, r_1, r_2] : u \in W^{1,p}(A(x_0, r_1, r_2);\mathbb{R}^d) \mapsto \mathbf{P}u \in W^{1,p}(A(x_0, r_1, r_2);\mathbb{R}^d)$$

such that u and $\mathbf{P}u$ have the same traces on $\partial A(x_0, r_1, r_2)$, and

$$\begin{aligned} \|\mathbf{P}u\|_{W^{1,q}(A(x_0,r_1,r_2);\mathbb{R}^d)} \leq C(r_2 - r_1)^{\tau}\Big(&\sup_{t \in (r_1,r_2)} \frac{\|u\|_{W^{1,p}(A(x_0,r_1,t);\mathbb{R}^d)}}{(t-r_1)^{1/p}} \\ &+ \sup_{t \in (r_1,r_2)} \frac{\|u\|_{W^{1,p}(A(x_0,t,r_2);\mathbb{R}^d)}}{(r_2-t)^{1/p}}\Big), \end{aligned} \tag{2.1}$$

where $C = C(N,p,q,r_0)$ and $\tau = \tau(N,p,q) > 0$.

The following properties of maximal functions may be found in [S]. Given $\phi \in L^1(\mathbb{R}^N)$ its *maximal function* is defined by

$$M(\phi)(x) := \sup_{\varepsilon>0} \frac{1}{\mathcal{L}^N(B(x,\varepsilon))} \int_{B(x,\varepsilon)} |\phi(y)|\, dy,$$

where $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure. It can be shown that $M(\phi)$ is Lebesgue measurable and that for every $\alpha > 0$

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : M(\phi)(x) > \alpha\}) \leq \frac{C_1}{\alpha} \int_{\mathbb{R}^N} |\phi(y)|\, dy. \tag{2.2}$$

If $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ then we set

$$\phi[u,x_0](r) := \int_{\partial B(x_0,r)} (|\mathbf{T}u|^p + |\nabla \mathbf{T}u|^p)\, dH^{N-1}, \tag{2.3}$$

where H^{N-1} is the $N-1$-dimensional Hausdorff measure. Clearly $\phi[u,x_0] \in L^1(0,R)$ whenever $B(x_0,R) \subset \Omega$.

In ligth of the definitions of maximal function and of (2.3), it follows that (2.1) can be written as

$$\begin{aligned} \|\mathbf{P}u\|_{W^{1,q}(A(x_0,r_1,r_2);\mathbb{R}^d)} \leq C(r_2-r_1)^\tau \Big(& M(\phi[u,x_0])(r_1)^{1/p} \\ & + M(\phi[u,x_0])(r_2)^{1/p}\Big). \end{aligned} \tag{2.4}$$

2.2. Definition. A function $\phi : \mathbb{R}^N \to \mathbb{R}$ is said to be *approximately upper semicontinuous* at x if $\phi(x) \geq \operatorname{ap}\limsup_{y\to x} \phi$, where

$$\operatorname{ap}\limsup_{y\to x} \phi := \inf\left\{ t \in \mathbb{R} : \lim_{\varepsilon\to 0} \frac{\mathcal{L}^N(B(x,\varepsilon)\cap\{\phi > t\})}{\mathcal{L}^N(B(x,\varepsilon))} = 0 \right\}.$$

Similarly, we say that $\phi : \mathbb{R}^N \to \mathbb{R}$ is *approximately lower semicontinuous* at x if $\phi(x) \leq \operatorname{ap}\liminf_{y\to x} \phi$, where

$$\operatorname{ap}\liminf_{y\to x} \phi := \sup\left\{ t \in \mathbb{R} : \lim_{\varepsilon\to 0} \frac{\mathcal{L}^N(B(x,\varepsilon)\cap\{\phi < t\})}{\mathcal{L}^N(B(x,\varepsilon))} = 0 \right\}.$$

The function ϕ is *approximately continuous* at x if it is approximately upper semicontinuous and lower semicontinuous at that point.

We note that in Definition 2.2 $\mathcal{L}^N$ stands for the N-dimensional Lebesgue outer measure. Also, it follows easily that

$$\operatorname{ap}\limsup_{y\to x} \phi \geq \operatorname{ap}\liminf_{y\to x} \phi$$

and if ϕ is approximately continuous at x then

$$\phi(x) = \operatorname{ap}\limsup_{y\to x} \phi = \operatorname{ap}\liminf_{y\to x} \phi.$$

It was shown by Denjoy and Stepanoff that Lebesgue measurability is equivalent to approximate continuity (see [F], Theorem 2.9.13).

2.3. Theorem. *$\phi : \mathbb{R}^N \to \mathbb{R}$ is Lebesgue measurable if and only if it is approximately continuous at $\mathcal{L}^N$ almost every point.*

In the case where $N = 1$ this result may be improved as follows.

2.4. Theorem. *$\phi : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if it is right approximately upper semicontinuous at $\mathcal{L}^1$ almost every point, i.e. for $\mathcal{L}^1$ a. e. x*

$$\phi(x) \geq \operatorname{ap}\limsup_{y \to x^+} \phi := \inf \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0} \frac{\mathcal{L}^1((x, x+\varepsilon) \cap \{\phi > t\})}{\mathcal{L}^1(B(x,\varepsilon))} = 0 \right\}.$$

Proof. Suppose that ϕ is right approximately upper semicontinuous at $\mathcal{L}^1$ a. e. x, where here $\mathcal{L}^1$ stands for the one-dimensional Lebesgue outer measure. Fix $\alpha \geq 0$. We want to prove that $E := \{x \in \mathbb{R} : \phi(x) \geq \alpha\}$ is a measurable set.

Let $\tilde{E}$ be a Borel set such that $E \subset \tilde{E}$ and

$$\mathcal{L}^1(E \cap I) = \mathcal{L}^1(\tilde{E} \cap I)$$

for all intervals $I \subset \mathbb{R}$. Define

$$\mathcal{N} := \{x \in \mathbb{R} : \phi \text{ is not right approximately upper semicontinuous at } x\}$$

and

$$E^* := \left\{ x \in \mathbb{R} : \lim_{\varepsilon \to 0} \frac{\mathcal{L}^1(B(x,\varepsilon) \cap \tilde{E})}{\mathcal{L}^1(B(x,\varepsilon))} = 1 \right\}.$$

We claim that

$$(E^* \setminus \mathcal{N}) \subset E \subset \tilde{E}. \tag{2.5}$$

Since $\mathcal{L}^1(\mathcal{N}) = 0$, by Lebesgue's Density Theorem

$$\mathcal{L}^1(\tilde{E} \setminus (E^* \setminus \mathcal{N})) = 0,$$

and so (2.5) entails the Lebesgue measurability of E. Clearly (2.5) is equivalent to showing that

$$E^c \cap \mathcal{N}^c \subset (E^*)^c.$$

Fix $x \in E^c \cap \mathcal{N}^c$. Since $\phi(x) < \alpha$ and ϕ is right approximately upper semicontinuous at x, we have

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}^1((x, x+\varepsilon) \cap E)}{\mathcal{L}^1(B(x,\varepsilon))} = 0,$$

and so

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \frac{\mathcal{L}^1(B(x,\varepsilon) \cap \tilde{E})}{\mathcal{L}^1(B(x,\varepsilon))} &= \limsup_{\varepsilon \to 0} \frac{\mathcal{L}^1(B(x,\varepsilon) \cap E)}{\mathcal{L}^1(B(x,\varepsilon))} \\
&= \limsup_{\varepsilon \to 0} \frac{\mathcal{L}^1((x-\varepsilon, x) \cap E)}{\mathcal{L}^1(B(x,\varepsilon))} \\
&\leq \frac{1}{2},
\end{aligned}$$

thus proving that $x \notin E^*$. □

3 Characterization of the bulk effective energy density

The main result of this paper is the following.

3.1. Theorem. *Let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function verifying*

$$0 \le f(\xi) \le C(1+|\xi|^q),$$

for some constant $C > 0$, $1 \le q < +\infty$, and all $\xi \in \mathbb{M}^{d\times N}$. Let $1 \le p \le q$, $p > q - q/N$. If $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $\mathcal{F}_{\mathrm{loc}}(u,\Omega) < +\infty$ then

$$\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) = Qf(\nabla u(x_0))$$

for $\mathcal{L}^N$ almost every $x_0 \in \Omega$, where $\mathcal{F}_{\mathrm{loc}}(u,\cdot)$ and the finite, Radon measure $\mathcal{R}(u,\cdot)$ are as in the introduction.

As in [BFM], given $u \in W^{1,p}(\partial B(x_0,r);\mathbb{R}^d)$ we define

$$\mathbf{m}(u,B(x_0,r)) := \inf\{F(v,B(x_0,r)) : v \in W^{1,q}(B(x_0,r);\mathbb{R}^d),\ \mathbf{T}v = u\}.$$

If $u \in W^{1,p}(B(x_0,r);\mathbb{R}^d)$ then we write $\mathbf{m}(u,B(x_0,r))$ in place of $\mathbf{m}(\mathbf{T}u,B(x_0,r))$. Note that if $u(x) = \xi x$, $\xi \in \mathbb{M}^{d\times N}$, then

$$\frac{\mathbf{m}(u,B(x_0,r))}{\mathcal{L}^N(B(x_0,r))} = Qf(\xi)$$

whenever $B(x_0,r) \subset \Omega$. The theorem below asserts that $\mathcal{F}_{\mathrm{loc}}(u,\cdot)$ and $\mathbf{m}(u,\cdot)$ have the same behavior on small balls, and this will entail Theorem 3.1.

3.2. Theorem. *Let $1 \le p \le q$, $p > q - q/N$, and let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function verifying*

$$\frac{1}{C}|\xi|^p \le f(\xi) \le C(1+|\xi|^q),$$

for some constant $C > 0$ and for all $\xi \in \mathbb{M}^{d\times N}$. If $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and if $\mathcal{F}_{\mathrm{loc}}(u,\Omega) < +\infty$ then

$$\lim_{\substack{r\to 0 \\ r\in\mathcal{E}(u,x_0)}} \frac{\mathcal{F}_{\mathrm{loc}}(u,B(x_0,r))}{\mathbf{m}(u,B(x_0,r))} = 1$$

for $\mathcal{R}(u,\cdot)$ a. e. $x_0 \in \Omega$, where $\mathcal{E}(u,x_0)$ is a subset of $(0,+\infty)$ such that $\mathcal{L}^1((0,r_0)\setminus\mathcal{E}(u,x_0)) = 0$ for some $r_0 > 0$.

In the sequel we fix f, p and q satisfying the hypotheses of Theorem 3.1, and we consider a function $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\mathcal{F}_{\mathrm{loc}}(u,\Omega) < +\infty$.

The proof of Theorem 3.2 is divided into a series of lemmas.

3.3. Lemma. *The function* $\mathbf{m}(u, B(x_0, \cdot))$ *is measurable.*

Proof. We will prove that $\psi := \mathbf{m}(u, B(x_0, \cdot))$ is almost everywhere right approximately upper semicontinuous, which, in light of Theorem 2.4, entails measurability. Setting

$$\phi := \phi[u, x_0],$$

fix $R > 0$ such that $M(\phi)$ is finite and approximately continuous at R. By (2.2) and Theorem 2.3 the complement set to this set of numbers R has measure zero. Let $\varepsilon > 0$ and choose t such that

$$t \leq M(\phi)(R) + \varepsilon, \qquad \lim_{\delta \to 0} \frac{\mathcal{L}^1(B(R,\delta) \cap \{M(\phi) > t\})}{\mathcal{L}^1(B(R,\delta))} = 0.$$

If $r > R$ and $M(\phi)(r) \leq t$ then

$$\psi(r) \leq F(v, B(x_0, R)) + F(\mathbf{P}[x_0, R, r]u, A(x_0, R, r))$$

for every $v \in W^{1,q}(B(x_0, R); \mathbb{R}^d)$ with $\mathbf{T}v = \mathbf{T}u$, hence, by (2.4)

$$\begin{aligned}
\psi(r) &\leq \psi(R) + C\ (M(\phi)(R) + M(\phi)(r))^{q/p}(r-R)^{q\tau} \\
&\leq \psi(R) + C\ (2M(\phi)(R) + \varepsilon)^{q/p}(r-R)^{q\tau}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{\mathcal{L}^1((R, R+\delta) \cap \{\psi > \psi(R) + \varepsilon\})}{\mathcal{L}^1(B(R,\delta))} \\
&\qquad \leq \lim_{\delta \to 0} \frac{\mathcal{L}^1((R, R+\delta) \cap \{M(\phi) > t\})}{\mathcal{L}^1(B(R,\delta))} \\
&\qquad + \lim_{\delta \to 0} \frac{\mathcal{L}^1((R, R+\delta) \cap \{M(\phi) \leq t\} \cap \{\psi > \psi(R) + \varepsilon\})}{\mathcal{L}^1(B(R,\delta))} \\
&\qquad \leq \lim_{\delta \to 0} \frac{\mathcal{L}^1((R, R+\delta) \cap \{r : \psi(R) + \varepsilon < \psi(R) + C(r-R)^{q\tau}\})}{\mathcal{L}^1(B(R,\delta))} = 0.
\end{aligned}$$

We conclude that $\operatorname{ap}\limsup_{r \to R^+} \psi \leq \psi(R)$. □

3.4. Good radii. Let $x_0 \in \Omega$. A radius $R > 0$ is said to be *good radius* (for u at x_0), if $B(x_0, R) \subset\subset \Omega$ and if the following conditions are satisfied:

(i) $M(\phi[u, x_0])$ is finite and approximately continuous at R,

(ii) $\mathbf{m}(u, B(x_0, .))$ is approximately continuous at R,

(iii) $\mathcal{R}(u, \partial B(x_0, R)) = 0$.

The set of all good radii for u at x_0 is denoted by $\mathcal{E}(u, x_0)$. By (2.2), Theorem 2.3 and Lemma 3.3, we have that $\mathcal{L}^1$ almost all radii in $\{r \colon B(x_0, r) \subset \Omega\}$ are good.

3.5. Lemma. *Suppose that $R \in \mathcal{E}(u, x_0)$ and $B = B(x_0, R)$. If $u_n \rightharpoonup \mathbf{T}u$ weakly in $W^{1,p}(\partial B(x_0, R); \mathbb{R}^d)$ then $\mathbf{m}(u, B) = \lim_{n\to\infty} \mathbf{m}(u_n, B)$.*

Proof. STEP 1. We prove that

$$\mathbf{m}(u, B) \leq \liminf_{n\to\infty} \mathbf{m}(u_n, B).$$

Fix $\varepsilon > 0$ and let $v_n \in W^{1,q}(B(x_0, R); \mathbb{R}^d)$ be such that $\mathbf{T}v_n = u_n$ and

$$F(v_n, B(x_0, R)) \leq \mathbf{m}(u_n, B(x_0, R)) + \varepsilon.$$

Set

$$\phi := \phi[u, x_0], \quad \psi(r) := \mathbf{m}(u, B(x_0, r)).$$

Since R is a good radius for u at x_0, $M(\phi)$ and ψ are finite and approximately continuous at R. It follows that for any $\delta > 0$ there exists $r \in (R, R + \delta) \cap \mathcal{E}(u, x_0)$ such that $B(x_0, r) \subset \Omega$ and

$$\psi(R) - \varepsilon \leq \psi(r) \leq \psi(R) + \varepsilon < \infty \quad \text{and} \quad M(\phi)(r) \leq M(\phi)(R) + \varepsilon < \infty. \tag{3.1}$$

Abbreviating

$$\mathbf{P} := \mathbf{P}[x_0, R, r], \quad \mathbf{E} := \mathbf{E}[x_0, R, r],$$

and setting

$$\theta_n := \begin{cases} \mathbf{T}(u_n - u) & \text{on } \partial B(x_0, R), \\ 0 & \text{on } \partial B(x_0, r), \end{cases}$$

$$w_n := \begin{cases} v_n & \text{in } B(x_0, R), \\ \mathbf{P}u + \mathbf{E}\theta_n & \text{on } B(x_0, r) \setminus B(x_0, R), \end{cases}$$

then

$$\mathbf{m}(u, B(x_0, r)) \leq F(w_n, B(x_0, r)) = F(v_n, B(x_0, R)) + F(w_n, A(x_0, R, r)). \tag{3.2}$$

Since $\theta_n \rightharpoonup 0$ weakly in $W^{1,p}(\partial A(x_0, R, r); \mathbb{R}^d)$ we have

$$w_n \to \mathbf{P}u \quad \text{strongly in } W^{1,q}(A(x_0, R, r); \mathbb{R}^d).$$

Thus, the continuity of F on $W^{1,q}(A(x_0, R, r); \mathbb{R}^d)$, (3.1), (2.4), and (3.2), yield

$$\begin{aligned} \psi(r) &\leq \inf_n [F(v_n, B(x_0, R)) + F(w_n, A(x_0, R, r))] \\ &\leq \inf_n [\mathbf{m}(u_n, B(x_0, R)) + F(w_n, A(x_0, R, r))] + \varepsilon \\ &\leq \liminf_{n\to\infty} \mathbf{m}(u_n, B(x_0, R)) + F(\mathbf{P}u, A(x_0, R, r)) + \varepsilon \\ &\leq \liminf_{n\to\infty} \mathbf{m}(u_n, B(x_0, R)) + C\delta^{q\tau}(2\, M(\phi)(R) + \varepsilon)^{q/p} + \varepsilon. \end{aligned} \tag{3.3}$$

Choosing δ so that

$$C\delta^{q\tau}(2\, M(\phi)(R) + \varepsilon)^{q/p} \leq \varepsilon, \tag{3.4}$$

by (3.1) and (3.3) we conclude that

$$\mathbf{m}(u, B(x_0, R)) \leq \mathbf{m}(u, B(x_0, r)) + \varepsilon \leq \liminf_{n\to\infty} \mathbf{m}(u_n, B(x_0, R)) + 3\varepsilon.$$

STEP 2. Now we will prove that

$$\mathbf{m}(u, B) \geq \limsup_{n\to\infty} \mathbf{m}(u_n, B).$$

Fix $\varepsilon > 0$. Since R is a good radius for u at x_0, for any $\delta \in (0, R/2)$ there exists $r \in (R-\delta, R) \cap \mathcal{E}(u, x_0)$ such that (3.1) holds. For a fixed δ we find such an r, we write

$$\mathbf{P} := \mathbf{P}[x_0, r, R], \quad \mathbf{E} := \mathbf{E}[x_0, r, R],$$

and set

$$\theta_n := \begin{cases} \mathbf{T}(u_n - u) & \text{on } \partial B(x_0, R), \\ 0 & \text{on } \partial B(x_0, r), \end{cases}$$

$$w_n := \begin{cases} v & \text{in } B(x_0, r), \\ \mathbf{P}u + \mathbf{E}\theta_n & \text{on } B(x_0, R) \setminus B(x_0, r), \end{cases}$$

where $v \in W^{1,q}(B(x_0, r); \mathbb{R}^d)$, $\mathbf{T}v = \mathbf{T}u$,

$$F(v, B(x_0, r)) \leq \mathbf{m}(u, B(x_0, r)) + \varepsilon.$$

Then, just as in the first step of this proof, using (3.1) and (3.4) we have

$$\begin{aligned} \limsup_{n\to\infty} \mathbf{m}(u_n, B(x_0, R)) &\leq \limsup_{n\to\infty} F(w_n, B(x_0, R)) \\ &\leq \limsup_{n\to\infty} [F(v, B(x_0, r)) + F(w_n, A(x_0, r, R))] \\ &\leq \mathbf{m}(u, B(x_0, r)) + F(\mathbf{P}u, A(x_0, r, R)) + \varepsilon \\ &\leq \mathbf{m}(u, B(x_0, R)) + 3\varepsilon. \end{aligned}$$

It suffices to let $\varepsilon \to 0^+$. □

3.6. Lemma. *Let $x_0 \in \Omega$ and $R \in \mathcal{E}(u, x_0)$. Then*

$$\mathbf{m}(u, B(x_0, R)) \leq \mathcal{F}_{\text{loc}}(u, B(x_0, R)).$$

Proof. Fix $\varepsilon > 0$ and let $u_n \in W^{1,q}_{\text{loc}}(B(x_0, R); \mathbb{R}^d)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(B(x_0, R); \mathbb{R}^d)$ and

$$\lim F(u_n, B(x_0, R)) < \mathcal{F}_{\text{loc}}(u, B(x_0, R)) + \varepsilon.$$

Let us tacitly assume that u_n and u are represented in such a way that

$$u_n = \mathbf{T}[x_0, r]u_n, \quad u = \mathbf{T}[x_0, r]u$$

for every $r \in (0,R)$. By Rellich's compact imbedding theorem we have$||u_n - u||_p \to 0$, hence there are $\alpha_n \to \infty$ and C such that

$$\sup_n \int_{B(x_0,R)} (|\nabla u_n|^p + \alpha_n |u - u_n|^p)\, dx \le C.$$

Setting

$$\phi_n(r) := \int_{\partial B(x_0,r)} (|\nabla u_n|^p + \alpha_n |u - u_n|^p)\, dH^{N-1},$$

by Fatou's Lemma

$$\int_0^R \liminf_{n\to\infty} \phi_n(r)\, dr \le \liminf_{n\to\infty} \int_0^R \phi_n(r)\, dr \le C$$

and so $\mathcal{L}^1(\mathcal{N}) = 0$, where

$$\mathcal{N} := \{r \in (0,R) \colon \liminf_n \phi_n(r) = \infty\}.$$

Let $\delta \in (0,R)$ and define

$$E := \{r \in (R-\delta, R) \cap \mathcal{E}(u,x_0) \colon \mathbf{m}(u, B(x_0,r)) > \mathbf{m}(u, B(x_0,R)) - \varepsilon\}.$$

Then E is a set of positive measure, and if $r \in E \setminus \mathcal{N}$ then there is a subsequence u_{n_k} such that

$$u_{n_k} \rightharpoonup u \quad W^{1,p}(\partial B(x_0,r); \mathbb{R}^d). \tag{3.5}$$

Using Lemma 3.5 we conclude that

$$\begin{aligned}\mathbf{m}(u, B(x_0,R)) - \varepsilon &< \mathbf{m}(u, B(x_0,r)) \le \liminf_{k\to\infty} \mathbf{m}(u_{n_k}, B(x_0,r)) \\ &\le \liminf_{k\to\infty} F(u_{n_k}, B(x_0,r)) \le \lim_{n\to\infty} F(u_n, B(x_0,R)) < \mathcal{F}_{\mathrm{loc}}(u, B(x_0,R)) + \varepsilon.\end{aligned}$$

This proves the assertion. □

3.7. Lemma. *Let $R \in \mathcal{E}(u,x_0), B := B(x_0,R)$, and let $v \in W^{1,q}(B;\mathbb{R}^d)$ be such that $\mathbf{T}u = \mathbf{T}v$ on ∂B. Then the function w defined by*

$$w(x) := \begin{cases} u(x) & \text{if } x \notin B \\ v(x) & \text{if } x \in B \end{cases}$$

satisfies

$$\mathcal{R}(w, \partial B) = 0.$$

Proof. Let $\varepsilon > 0$. Since $\mathcal{R}(u,\partial B) = 0$ and $F(v,\cdot)$ is absolutely continuous with respect to $\mathcal{L}^N$, we may find $\delta > 0$ such that

$$\mathcal{F}_{\mathrm{loc}}(u, S_\delta) + F(v, B \cap S_\delta) < \varepsilon \tag{3.6}$$

where $S_\delta := A(x_0, R-\delta, R+\delta)$. Let $u_n \in W^{1,p}(S_\delta;\mathbb{R}^d)\cap W^{1,q}_{\mathrm{loc}}(S_\delta;\mathbb{R}^d)$ be such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(S_\delta;\mathbb{R}^d)$$

and

$$\sup_n F(u_n, S_\delta) < \varepsilon. \tag{3.7}$$

By virtue of Rellich-Kondrachov compactness theorem we may suppose that

$$u_n \to u \quad \text{strongly in } L^p(S_\delta;\mathbb{R}^d). \tag{3.8}$$

As in the proof of Lemma 3.6, we may find a set $\mathcal{E}^\circ \subset (R,\ R+\delta)\cap\mathcal{E}(u,x_0)$ such that $\mathcal{L}^1\big((R,\ R+\delta)\setminus\mathcal{E}^\circ\big) = 0$, and for each $r \in \mathcal{E}^\circ$ there is a subsequence u_{n_j} (depending on r) such that

$$\mathbf{T}u_{n_j} \rightharpoonup \mathbf{T}u \quad \text{weakly in } W^{1,p}(\partial B(x_0,r);\mathbb{R}^d), \tag{3.9}$$

where

$$\mathbf{T} = \mathbf{T}[\partial B(x_0,r)].$$

Choose $r_k \in \mathcal{E}^\circ$, $r_k \searrow R$ such that $M(\phi)(r_k) \le M(\phi)(R)+\varepsilon$ and, using (2.4),

$$\begin{aligned} &\|\mathbf{P}[x_0,R,r_k]u\|_{W^{1,q}(A(x_0,R,r_k);\mathbb{R}^d)} < 1/k,\\ &F(\mathbf{P}[x_0,R,r_k]u,\ A(x_0,R,r_k)) < 1/k. \end{aligned} \tag{3.10}$$

For each r_k we relabel the subsequence $\{u_{n_j}\}$ satisfying (3.9) as $\{u_n^{(k)}\}$. We write

$$\mathbf{P}_k := \mathbf{P}_k[x_0,R,r_k], \quad \mathbf{E}_k := \mathbf{E}_k[x_0,R,r_k],$$

and set

$$\theta_n^{(k)} := \begin{cases} 0 & \text{on } \partial B(x_0,R),\\ \mathbf{T}_k(u_n^{(k)}-u) & \text{on } \partial B(x_0,r_k), \end{cases}$$

$$w_n^{(k)} := \begin{cases} v & \text{in } B(x_0,R),\\ \mathbf{P}_k u + \mathbf{E}_k\theta_n^{(k)} & \text{on } B(x_0,r_k)\setminus B(x_0,R),\\ u_n^{(k)} & \text{in } S_\delta\setminus B(x_0,r_k). \end{cases}$$

Since $\theta_n^{(k)} \rightharpoonup 0$ weakly in $W^{1,p}(\partial A(x_0,R,r_k);\mathbb{R}^d)$ as $n\to\infty$, we have

$$w_n^{(k)} \to \mathbf{P}_k u \quad \text{strongly in } W^{1,q}(A(x_0,R,r_k);\mathbb{R}^d)$$

and thus

$$F(w_n^{(k)}, A(x_0,R,r_k)) \to F(\mathbf{P}_k u, A(x_0,R,r_k)).$$

Let

$$z_k := w_{n_k}^{(k)},$$

where, according to (3.8) and (3.10), the increasing sequence $\{n_k\}_{k=1}^{\infty}$ is selected in such a way that

$$\|u_{n_k}^{(k)} - u\|_{L^p(S_\delta)} < 1/k,$$
$$\|w_{n_k}^{(k)} - \mathbf{P}_k u\|_{W^{1,q}(A(x_0,R,r_k);\mathbb{R}^d)} < 2/k,$$

and

$$F(w_{n_k}^{(k)}, A(x_0, R, r_k)) < 2/k. \tag{3.11}$$

Then $\{z_k\}$ is bounded in $W^{1,p}(S_\delta;\mathbb{R}^d)$, $z_k \in W^{1,p}_{\mathrm{loc}}(S_\delta;\mathbb{R}^d)$, and $z_k \to w$ in $L^p(S_\delta;\mathbb{R}^d)$. It follows that $z_k \rightharpoonup w$ weakly in $W^{1,p}(S_\delta;\mathbb{R}^d)$, and using (3.6), (3.7) and (3.11) we have

$$\begin{aligned} F(z_k, S_\delta) \le\ & F(v, A(x_0, R-\delta, R)) \\ & + F(w_{n_k}^{(k)}, A(x_0, R, r_k)) + F(u_{n_k}^{(k)}, A(x_0, r_k, R+\delta)) \\ \le\ & \varepsilon + 2/k + \varepsilon. \end{aligned}$$

and thus

$$\mathcal{F}_{\mathrm{loc}}(w, S_\delta) \le \liminf_{k\to\infty} F(z_k, S_\delta) < 2\varepsilon.$$

Hence $\mathcal{R}(w, \partial B) < 2\varepsilon$ and the conclusion follows by letting ε tend to 0. □

3.8. Definition. We define

$$\mathbf{m}^*(u, U) := \lim_{\delta\to 0^+} \mathbf{m}^\delta(u, U),$$

where

$$\mathbf{m}^\delta(u, U) := \inf\left\{ \sum_{i=1}^{\infty} \mathbf{m}(u, B_i) \colon B_i = B(x_i, r_i) \subset\subset U,\ r_i \in (0,\delta) \cap \mathcal{E}(u, x_i), \right.$$
$$\left. \overline{B}_i \text{ are disjoint}, \mathcal{R}\Big(u, U \setminus \bigcup_{i=1}^{\infty} B_i\Big) = 0 \right\}.$$

We remark that by Besicovitch Covering Theorem we may always find countable families $\{B_i\}$ of balls under the conditions of Definition 3.8.

3.9. Lemma. $\mathbf{m}^*(u, U) \le \mathcal{F}_{\mathrm{loc}}(u, U)$ *for every open set* $U \subset \Omega$.

Proof. Let $\delta > 0$ and let B_i be such that $B_i = B(x_i, r_i) \subset\subset U$, $r_i \in (0,\delta) \cap \mathcal{E}(u, x_i)$, $\overline{B}_i$ are disjoint, and $\mathcal{R}(u, U \setminus \bigcup_{i=1}^{\infty} B_i) = 0$. Then, using measure properties of $\mathcal{F}_{\mathrm{loc}}(u, \cdot)$ and Lemma 3.6 we obtain

$$\begin{aligned} \mathcal{F}_{\mathrm{loc}}(u, U) &\ge \mathcal{F}_{\mathrm{loc}}\left(u, \bigcup_{i=1}^{\infty} B_i\right) \\ &\ge \sum_{i=1}^{\infty} \mathbf{m}(u, B_i) \ge \mathbf{m}^\delta(u, U). \end{aligned}$$

It follows that

$$\mathcal{F}_{\mathrm{loc}}(u, U) \ge \lim_{\delta\to 0^+} \mathbf{m}^\delta(u, U) = \mathbf{m}^*(u, U).$$

□

3.10. Lemma. *If $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ is a continuous function verifying*

$$\frac{1}{C}|\xi|^p \le f(\xi) \le C(1+|\xi|^q)$$

for some constant $C > 0$, then $\mathcal{F}_{\text{loc}}(u,U) \le \mathbf{m}^*(u,U)$ for every open set $U \subset \Omega$.

Proof. Fix $\varepsilon > 0$, $\delta = 1/k, k \in \mathbb{N}$, and choose B_i such that $B_i := B(x_i, R_i) \subset\subset U$, $R_i \in (0,\delta) \cap \mathcal{E}(u,x_i)$, $\overline{B}_i$ are disjoint, $\mathcal{R}(u, U \setminus \bigcup_{i=1}^{\infty} B_i) = 0$, and

$$\sum_{i=1}^{\infty} \mathbf{m}(u,B_i) < \mathbf{m}^\delta(u,U) + \varepsilon. \tag{3.12}$$

Since $\mathcal{R}(u,\cdot)$ is a finite, Radon measure, we may choose m large enough so that

$$\mathcal{F}_{\text{loc}}(u,V) < \varepsilon, \quad \text{where } V := U \setminus \bigcup_{i=1}^{m} \overline{B}_i.$$

Let $v_i \in W^{1,q}(B_i;\mathbb{R}^d)$ be such that $\mathbf{T}v_i = \mathbf{T}u$ and

$$F(v_i,B_i) \le \mathbf{m}(u,B_i) + \frac{\varepsilon}{m}. \tag{3.13}$$

Setting

$$u_k := \begin{cases} u & \text{in } V, \\ v_i & \text{in } B_i, \end{cases}$$

by Lemma 3.7

$$\mathcal{R}(u_k, \partial B_i) = 0 \quad \text{for all } i \in \{1,\dots,m\},$$

and by (3.12), (3.13), we have

$$\begin{aligned} \mathcal{F}_{\text{loc}}(u_k,U) &\le \mathcal{F}_{\text{loc}}(u,V) + \sum_{i=1}^{m} \mathcal{F}_{\text{loc}}(v_i,B_i) \\ &\le \varepsilon + \sum_{i=1}^{m} F(v_i,B_i) \\ &\le \varepsilon + \sum_{i=1}^{m} \left(\mathbf{m}(u,B_i) + \frac{\varepsilon}{m}\right) \\ &\le \mathbf{m}^\delta(u,U) + 3\varepsilon. \end{aligned} \tag{3.14}$$

Next we prove that

$$\lim_{k\to\infty} ||u_k - u||_p = 0. \tag{3.15}$$

Indeed, using Poincaré's inequality and the coercivity hypothesis we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \int_U |u_k - u|^p \, dx &= \limsup_{k\to\infty} \sum_{i=1}^m \int_{B_i} |v_i - u|^p \, dx \\
&\le \limsup_{k\to\infty} \sum_{i=1}^m \int_{B_i} r_i^p \, |\nabla v_i - \nabla u|^p \, dx \\
&\le \limsup_{k\to\infty} \sum_{i=1}^m 2^{p-1} r_i^p \left(\int_{B_i} |\nabla u|^p \, dx + \int_{B_i} |\nabla v_i|^p \, dx \right) \\
&\le \limsup_{k\to\infty} \sum_{i=1}^m C\, 2^{p-1} r_i^p \left(\mathcal{F}_{\rm loc}(u, B_i) + F(v_i, B_i) \right) \\
&\le \limsup_{k\to\infty} C\, 2^{p-1} \, k^{-p} \left(\mathcal{F}_{\rm loc}(u, U) + \sum_{i=1}^m F(v_i, B_i) \right),
\end{aligned}$$

where we have used the fact that $R_i < 1/k$. By Lemma 3.9, and by (3.12) and (3.13)

$$\sum_{i=1}^m F(v_i, B_i) \le \mathbf{m}^\delta(u, U) + 2\varepsilon \le \mathbf{m}^*(u, U) + 2\varepsilon \le \mathcal{F}_{\rm loc}(u, U) + 2\varepsilon < +\infty,$$

and we conclude that (3.15) holds. This, together with (3.14) and the coercivity hypothesis, implies that $u_k \rightharpoonup u$ in $W^{1,p}(U; \mathbb{R}^d)$. Finally, once again due to the coercivity assumption,

$$w \mapsto \mathcal{F}_{\rm loc}(w, U) \quad \text{is sequentially weakly lower semicontinuous in } W^{1,p}(U; \mathbb{R}^d),$$

and so, by (3.14),

$$\mathcal{F}_{\rm loc}(u, U) \le \limsup_{k\to\infty} \mathcal{F}_{\rm loc}(u_k, U) \le \lim_{k\to\infty} \mathbf{m}^{1/k}(u, U) + 3\,\varepsilon.$$

The result now follows by letting $\varepsilon \to 0$. □

Using the above lemmas, we proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.6 we have

$$\liminf_{\substack{r\to 0 \\ r\in\mathcal{E}(u,x_0)}} \frac{\mathcal{F}_{\rm loc}(u, B(x_0, r))}{\mathbf{m}(u, B(x_0, r))} \ge 1.$$

Let $t > 1$ and set

$$E_t := \left\{ x \in \Omega \colon \limsup_{\substack{r\to 0 \\ r\in\mathcal{E}(u,x)}} \frac{\mathcal{F}_{\rm loc}(u, B(x, r))}{\mathbf{m}(u, B(x, r))} \ge t \right\}.$$

We claim that

$$\mathcal{R}(u, E_t) = 0. \tag{3.16}$$

Fix $\tau, \varepsilon > 0$ with $t > \tau > 1$, and define for any $\delta > 0$

$$\begin{aligned}\mathcal{X}_\delta &:= \{B(x,r) : x \in E_t, 0 < r < \delta, r \in \mathcal{E}(u,x), B(x,r) \subset \Omega, \\ &\qquad \mathcal{F}_{\text{loc}}(u, B(x,r)) > \tau\, \mathbf{m}(u, B(x,r))\}, \\ U_\delta &:= \bigcup \{B(x,r) \colon B(x,r) \in \mathcal{X}_\delta\}.\end{aligned}$$

Set

$$U_0 := \bigcap_{\delta > 0} U_\delta.$$

Clearly $E_t \subset U_0$, and since $U_\delta \searrow U_0$ we may find $\rho = \rho(\varepsilon)$ such that $\mathcal{R}(u, U_\rho \backslash U_0) < \varepsilon$. Choose a compact set $K \subset U_0$ such that $\mathcal{F}_{\text{loc}}(u, U_0 \setminus K) < \varepsilon$, and define

$$\mathcal{Y}_\delta := \{B(x,r) \colon x \in U_\rho \setminus K,\ 0 < r < \delta,\ r \in \mathcal{E}(u,x),\ B(x,r) \subset U_\rho \setminus K\}.$$

Let $0 < \delta < \rho$. Since $\mathcal{X}_\delta \cup \mathcal{Y}_\delta$ is a fine covering of U_ρ, by Besicovitch's Covering Theorem we may find a countable, disjoint, subcovering such that

$$U_\delta = \bigcup_{i=1}^{\infty} B_i \cup \bigcup_{j=1}^{\infty} \hat{B}_j \cup \mathcal{N}$$

where $B_i := B(x_i, r_i) \in \mathcal{X}_\delta, \hat{B}_j := B(y_j, r_j) \in \mathcal{Y}_\delta$ and $\mathcal{R}(u, \mathcal{N}) = 0$. By Lemma 3.6 we have

$$\begin{aligned}\mathcal{F}_{\text{loc}}(u, U_\rho) &= \sum_{i=1}^{\infty} \mathcal{F}_{\text{loc}}(u, B_i) + \sum_{j=1}^{\infty} \mathcal{F}_{\text{loc}}(u, \hat{B}_j) \\ &\geq \sum_{i=1}^{\infty} \tau\, \mathbf{m}(u, B_i) + \sum_{j=1}^{\infty} \mathbf{m}(u, \hat{B}_j) \\ &= \tau \left[\sum_{i=1}^{\infty} \mathbf{m}(u, B_i) + \sum_{j=1}^{\infty} \mathbf{m}(u, \hat{B}_j)\right] + (1-\tau) \sum_{j=1}^{\infty} \mathbf{m}(u, \hat{B}_j) \\ &\geq \tau\, m^\delta(u, U_\rho) - (\tau - 1) \sum_{j=1}^{\infty} \mathcal{F}_{\text{loc}}(u, \hat{B}_j) \\ &\geq \tau\, m^\delta(u, U_\rho) - (\tau - 1) \mathcal{F}_{\text{loc}}(u, U_\rho \setminus K).\end{aligned}$$

Letting $\delta \to 0^+$ and using Lemma 3.10, we have

$$\begin{aligned}\mathcal{F}_{\text{loc}}(u, U_\rho) &\geq \tau\, m^*(u, U_\rho) - (\tau - 1)\, \mathcal{F}_{\text{loc}}(u, U_\rho \setminus K) \\ &\geq \tau\, \mathcal{F}_{\text{loc}}(u, U_\rho) - (\tau - 1)\, \mathcal{F}_{\text{loc}}(u, U_\rho \setminus K),\end{aligned}$$

so that, as $E_t \subset U_0 \subset U_\rho$,

$$(\tau - 1)\,\mathcal{R}(u, E_t) \le (\tau - 1)\,\mathcal{F}_{\rm loc}(u, U_\rho) \le (\tau - 1)\,\mathcal{F}_{\rm loc}(u, U_\rho \setminus K) \le \varepsilon\,(\tau - 1)$$

Since $\tau > 1$, letting $\varepsilon \to 0$ we conclude (3.16). □

Proof of Theorem 3.1.

STEP 1. Assume first that the coercivity hypothesis holds, i.e. $C^{-1}|\xi|^p \le f(\xi) \le C(1 + |\xi|^q)$ for some constant $C > 0$.

Let $\eta := \frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}\,\mathcal{L}^N$ be the absolutely continuous part of $\mathcal{R}(u, \cdot)$. By Theorem 3.2 for η a. e. $x_0 \in \Omega$

$$\text{(3.17)}\qquad \begin{aligned} \frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\substack{r\to 0\\ r\in\mathcal{E}(u,x_0)}} \frac{\mathcal{F}_{\rm loc}(u, B(x_0,r))}{\mathbf{m}(u, B(x_0,r))}\,\frac{\mathbf{m}(u, B(x_0,r))}{\mathcal{L}^N(B(x_0,r))} \\ &= \lim_{\substack{r\to 0\\ r\in\mathcal{E}(u,x_0)}} \frac{\mathbf{m}(u, B(x_0,r))}{\mathcal{L}^N(B(x_0,r))}. \end{aligned}$$

In addition, we know that $\mathcal{L}^N$ a. e. we may choose $x_0 \in \Omega$ so that x_0 is a Lebesgue point for $\nabla u(x_0)$ and

$$u_\varepsilon(y) := \frac{u(x_0 + \varepsilon y) - u(x_0)}{\varepsilon} \rightharpoonup \nabla u(x_0) y \quad \text{weakly in } W^{1,p}(B(0,1); \mathbb{R}^d).$$

Write

$$\begin{aligned} v(y) &:= \nabla u(x_0) y, \\ v_n &:= u_{1/n}. \end{aligned}$$

Assume that v_n are represented in such a way that $v_n = \mathbf{T}[0,r]v_n$ for all $r \in (0,1)$. By Rellich's compact imbedding theorem and Fatou's lemma, as in the proof of Lemma 3.6 we find a set $\mathcal{N} \subset (0,1)$ with $\mathcal{L}^1(\mathcal{N}) = 0$, and a subsequence (not relabelled for convenience), such that

$$\begin{aligned} &\int_{\partial B(0,r)} |v_n - v|^p \, dH^{N-1}(y) \to 0, \\ &\liminf_{n\to\infty} \int_{\partial B(0,r)} |\nabla v_n(x)|^p \, dH^{N-1}(x) < +\infty \end{aligned}$$

for all $r \in (0,1) \setminus \mathcal{N}$. Let

$$E := ((0,1) \setminus \mathcal{N}) \cap \bigcap_{n=1}^{\infty} n\mathcal{E}(u, x_0).$$

It is clear that $\mathcal{L}^1((0,1) \setminus E) = 0$. Fix $\sigma \in E$. Then there is a subsequence v_{n_j} such that

$$\frac{\sigma}{n_j} \in \mathcal{E}(u, x_0)$$

and

$$v_{n_j} \rightharpoonup v \quad \text{weakly in } W^{1,p}(\partial B(0,\sigma);\mathbb{R}^d). \tag{3.18}$$

In view of (3.17) we have

$$\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{n_j\to\infty} \frac{\mathbf{m}(u, B(x_0,\sigma/n_j))}{\mathcal{L}^N(B(x_0,\sigma/n_j))}$$

and by an obvious rescaling we obtain

$$\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{n_j\to\infty} \frac{\mathbf{m}(v_{n_j}, B(0,\sigma))}{\mathcal{L}^N(B(0,\sigma))}.$$

Using Lemma 3.5 and (3.18) we conclude that

$$\begin{aligned}\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \frac{\mathbf{m}(\nabla u(x_0)y, B(0,\sigma))}{\mathcal{L}^N(B(0,\sigma))}\\ &= Qf(\nabla u(x_0)).\end{aligned}$$

STEP 2. Finally, we remove the coercivity hypothesis. By (1.1) we know already that

$$\mathcal{R}(u,\cdot) = \bar{f}\,\mathcal{L}^N + \mu$$

where μ is a Radon measure, singular with respect to $\mathcal{L}^N$, and $\bar{f} \geq Qf(\nabla u)$. It remains to prove that

$$\bar{f} \leq Qf(\nabla u) \quad \mathcal{L}^N\text{a. e.} \tag{3.19}$$

Consider the perturbations $f^\varepsilon(\xi) := f(\xi) + \varepsilon|\xi|^p$, $\varepsilon > 0$, with corresponding relaxed energy $\mathcal{F}^\varepsilon_{\text{loc}}(u,\cdot)$ and associated Radon measure $\mathcal{R}^\varepsilon(u,\cdot)$. Using the result obtained on Step 1,

$$\frac{d\mathcal{R}^\varepsilon(u,\cdot)}{d\mathcal{L}^N}(x_0) = Qf^\varepsilon(\nabla u(x_0)).$$

Clearly $\mathcal{R}(u,\cdot) \leq \mathcal{R}^\varepsilon(u,\cdot)$, hence

$$\bar{f}(x) \leq \liminf_{\varepsilon\to 0^+} Qf^\varepsilon(\nabla u(x)) \quad \mathcal{L}^N\text{a. e. } x \in \Omega.$$

It is easy to show that

$$\lim_{\varepsilon\to 0^+} Qf^\varepsilon(\xi) = Qf(\xi)$$

for all $\xi \in \mathbb{M}^{d\times N}$, and we conclude the proof of (3.19). □

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