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On the asymptotic behavior of the equations describing the motion of an incompressible binary fluid

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Abstract

We determine the asymptotic behavior of the system of Cahn-Hillard/Euler's equations that control the dynamics inside the thin boundary layer separating two inviscid, incompressible, and nearly immiscible fluids. This model was proposed recently in order to replace the classical moving boundary model of two immiscible fluids, separated by the interface with the surface tension. We formally verify that these two problems are related. Using the method of matched asymptotic expansions, we show that when the width of the interface and the miscibility of the fluids converge to zero, then the system of Cahn-Hillard/Euler's equations converges asymptotically to the classical moving boundary problem. In addition, we analyze for the different timescales the behavior of the mixture inside the boundary layer.

1. Introduction

When studying the behavior of two immiscible fluids, separated by the interface with surface tension, it is normally assumed [So] that the condition

$$[P\mathbf{n}]_{\Gamma} = \sigma k \mathbf{n}, \tag{1}$$

is satisfied on the interface Γ. Then the equations governing fluid motion in each phase are solved subject to the condition (1) on the moving interface. Here P is the stress tensor, $\mathbf{n}$ is a unit normal vector to Γ, and k is the mean curvature of Γ. Also σ is a constant and $[\phi]_{\Gamma}$ denotes the jump in ϕ across Γ. The existence of a solution for this classical problem has only been shown for a small time or close to the exact solution ([St]). The possible complications can be seen in a particular case, when two phases are mixed by the action of the imposed flow. Then the amount of the interface area per unit volume increases with time, while the characteristic length of the microstructure created during the flow decreases. As the result, the flow becomes very complex, making both analytical and numerical analysis of the problem extremely difficult. In a view of this, the numerical simulations based on the classical formulation may provide misleading results.

In recent years the different approach to this type of problems has been suggested and used by several authors (Chella and Viñals in [CV], Gurtin, Polignone, and Viñals in [GPV], Starovoitov in [St], and Truskinovsky and Lowengrub in [LT]). Their basic idea is to replace the sharp interface between two phases by a thin boundary layer in which the relevant quantities vary continuously but may have large gradients. To track the interface one introduces and follows the dynamics of an order parameter that is assumed to have almost constant values within each phase. As a such parameter we could take, for example, a concentration of one of the fluids in the mixture. Then the classical problem can be replaced by a system of two equations, one describing the motion of the fluid and another describing the evolution of the order parameter. The thickness of the interface enters these equations as a small parameter.

This approach is based on the same principle as Ginzburg-Landau models of phase separation [GSS], phase-field models for solidification (see e.g. [Ca] or [CF]), and various models used to study the decay of fluctuations at the critical point [HH]. The particular model that we study here is referred to as “model H” in the literature on critical phenomena ([HH]) and was used by Siggia, Halperin, and Hohenberg [SHH] to study behavior at the critical points of single and binary fluids. We will, however, be interested in describing the behavior of the same model away from any critical points, when the phase separation is largely completed and the order parameter assumes almost constant values in each of the phases, separated by the interface of the small width ε over which the order parameter changes continuously.

The model under consideration was also obtained following the methods of continuum mechanics by Gurtin, Polignone, and Viñals in [PGV] and consists of the coupled Navier-Stokes/Cahn-Hilliard equations, that can be written for our choice of parameters as

$$\begin{cases} u_t^\varepsilon + \nabla u^\varepsilon \cdot v^\varepsilon = \varepsilon^{\frac{5}{2}} \Delta \mu^\varepsilon, \\ v_t^\varepsilon + \nabla v^\varepsilon v^\varepsilon = -\nabla p^\varepsilon - \varepsilon^2 \Delta u^\varepsilon \nabla u^\varepsilon, \end{cases} \tag{2}$$

for every $\varepsilon > 0$. Here v^ε denotes a velocity of the fluid, u^ε is the order parameter, and $\mu^\varepsilon = f(u^\varepsilon) - \varepsilon^2 \Delta u^\varepsilon$ is the chemical potential (the definition of f will be given in the next section). We will assume that the fluid is incompressible

$$div\ v^\varepsilon = 0, \tag{3}$$

and inviscid. Also, for simplicity, we will set the density of both phases identically equal to one.

Our goal will be to use the method of matched asymptotic expansions to determine the asymptotic limit of the system (2) as $\varepsilon \to 0$, as well as to study the behavior of the mixture inside the boundary layer. Observe that as $\varepsilon \to 0$, two fluids become immiscible. To simplify our analysis, we choose the mobility (the coefficient in front of chemical potential in the first equation) and capillarity coefficient in the Euler's equation in such a way that the relevant processes inside the boundary layer are separated in time or, in other words, occur on different timescales. Also we note that our analysis can be directly applied to the case of viscous fluid when the Reynolds number is large. For a low Reynolds number our technique does not seem to work.

Assuming that the initial velocity of the fluid $v(x, 0) \equiv O(\varepsilon)$, we show that

1. On the fast timescale $s = \frac{t}{\varepsilon}$ the interface Γ remains stationary and the order parameter remains equal to its initial value everywhere in the domain. The velocity v of the fluid also remains unchanged inside the regions away from Γ. However, near the interface, the tangential component of the velocity increases linearly in time, driven by the gradients in the initial distribution of the order parameter. Therefore, on the timescale t, the velocity of the fluid can be by the order of magnitude larger inside the boundary layer than away from the interface (in the bulk).

2. On the "regular" timescale t the high velocity inside the boundary layer forces the redistribution of the order parameter along the interface. Simultaneously the velocity inside the regions near Γ decreases down to the same order of magnitude as in the bulk. The resulting structure of the order parameter inside the boundary layer is such that at the leading order the capillarity term disappears from the Euler's equation in (2) by compensating the change in pressure across the interface. Furthermore, the interface itself remains stationary on the t timescale.

3. On the slow timescale $\sigma = \frac{t}{\sqrt{\varepsilon}}$ the order parameter has the structure, described in (b), for

every $\sigma > 0$. We use this to show that, near the interface, the equations in (2) can be decoupled by using the appropriate change of variables. In addition, we show that the equation for the order parameter u, written in the new variables, is the one-dimensional Cahn-Hilliard equation. It follows then that the asymptotic limit of (2) is the classical moving boundary problem with the condition (1) satisfied on the interface.

If the function u is interpreted as a concentration of one of the parameters in the mixture, then our result is in agreement with the conclusion in [LT], that the presence of concentration gradients in the boundary layer gives rise to an effective surface tension between the fluids.

For our analysis we use the method of matched asymptotic expansions in the same form as used, for example, by Pego in [Pe] and Rubinstein, Sternberg, and Keller in [RSK].

The convergence of (2) to the classical moving boundary problem was also formally shown by Starovoitov in [St] by employing a cruder version of the asymptotic analysis. In particular, Starovoitov did not analyze the dynamics of the mixture behavior near the interface. Moreover, although the expansions in [St] produce at the leading order the set of equations that is similar to ours, the presence of the viscosity term in [St] should cause problems at the higher order in the expansion for velocity.

2. Acknowledgments.

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3. Preliminaries.

As we have already discussed in the previous section, we study the asymptotic limit of the system

$$\begin{cases} u_t^{\varepsilon} + \nabla u^{\varepsilon} \cdot v^{\varepsilon} = \varepsilon^{\frac{5}{2}} \Delta \mu^{\varepsilon}, \\ v_t^{\varepsilon} + \nabla v^{\varepsilon} v^{\varepsilon} = -\nabla p^{\varepsilon} - \varepsilon^2 \Delta u^{\varepsilon} \nabla u^{\varepsilon}, \end{cases} \tag{4}$$

as $\varepsilon \to 0$. We suppose that the spatial variable $x \in \Omega \subset \mathbf{R}^n$, where Ω has a smooth boundary and that for every $\varepsilon > 0$

$$\left.\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}\right|_{\partial\Omega} = \left.\frac{\partial \mu^{\varepsilon}}{\partial \mathbf{n}}\right|_{\partial\Omega} = v^{\varepsilon} \cdot \mathbf{n}\big|_{\partial\Omega} = 0. \tag{5}$$

Also the chemical potential $\mu^{\varepsilon} = f\left(u^{\varepsilon}\right) - \varepsilon^2 \Delta u^{\varepsilon}$ and $div\ v^{\varepsilon} = 0$. The function $f(u) = W'(u)$ and $W(u)$ is a double-well potential. We will assume without loss of generality that $W(u) = \frac{\left(u^2 - 1\right)^2}{2}$. The energy functional corresponding to (2) is given by

$$E^{\varepsilon}\left[u^{\varepsilon}, v^{\varepsilon}\right] = \int_{\Omega} \left[\frac{\left|v^{\varepsilon}\right|^2}{2} + W\left(u^{\varepsilon}\right) + \varepsilon^2 \frac{\left|\nabla u^{\varepsilon}\right|^2}{2}\right] dx, \tag{6}$$

and is nonincreasing on the solutions of (2). We suppose that u^{ε} and v^{ε} satisfy the following initial condition:

$$E^{\varepsilon}\left[u_0^{\varepsilon}, v_0^{\varepsilon}\right] \le M\varepsilon, \tag{7}$$

for all $\varepsilon > 0$ and that, in addition, $\left|v_0^\varepsilon(x)\right| \le C\varepsilon$ for all $x \in \Omega$ and $\varepsilon > 0$. Here $u^\varepsilon(x, 0) = u_0^\varepsilon(x)$ and $v^\varepsilon(x, 0) = v_0^\varepsilon(x)$. Then, we have that

$$\text{(a) } sup_{t \ge 0} E^{\varepsilon}\left[u^\varepsilon, v^\varepsilon\right](t) \le M\varepsilon, \tag{8}$$

$$\text{(b) } sup_{t \ge 0} \int_\Omega \left(\left(u^\varepsilon\right)^2 - 1\right)^2 dx \le 2M\varepsilon. \tag{9}$$

Also we can adopt the following compactness result from [BK]:

Theorem 1 : *Assume that (7) is satisfied. Then for any sequence of* ε *'s tending to zero there exists a subsequence* ε_j *such that the limit* $\lim_{\varepsilon_j \to 0} u^{\varepsilon_j}(x, t) = u(x, t)$ *exists for a.e.* $(x, t) \in \Omega \times (0, \infty)$ *. The function* u *takes only the values* ± 1*, and there is a positive constant* C_1 *depending only on M such that:*

$$sup_{t \ge 0} \int_\Omega |\nabla u(x, t)| dx \le C_1.$$

Set

$$F^\varepsilon\left[u^\varepsilon\right] = \int_\Omega \left[\frac{W\left(u^\varepsilon\right)}{\varepsilon} + \varepsilon \frac{\left|\nabla u^\varepsilon\right|^2}{2}\right] dx$$

$$F^0[u] := \begin{cases} K \cdot Per_\Omega(\{u = 1\}), & \text{if } u(x) \in \{-1, 1\} \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise}, \end{cases}$$

where $K = \int_{-1}^{1} \sqrt{2W(s)}\, ds$ (in our case K=4/3) and $Per_\Omega(A)$ is a perimeter of A in Ω (for the definition of perimeter see e.g. [Gi]) . It turns out that F^0 is a $\Gamma\left(L^1(\Omega)\right)$-limit of F^ε (see e.g. [FT]). In other words, the following holds

Theorem 2 : *Let F^{ε} and F^0 be as above.*

(1) *If* $w^{\varepsilon} \to w^0$ *in* $L^1(\Omega)$ *then* $\liminf_{\varepsilon \to 0} F^{\varepsilon}[w^{\varepsilon}] \geq F^0[w^0]$.

(2) *For any* $w^0 \in L^1(\Omega)$ *there exists a family* (w^{ε}) *such that* $w^{\varepsilon} \to w^0$ *in* $L^1(\Omega)$ *and*

$$\lim_{\varepsilon \to 0} F^{\varepsilon}[w^{\varepsilon}] = F^0[w^0].$$

Using this theorem, we can assume that for every $t > 0$ there exists a front Γ_t separating two regions in Ω, where $u(x, t) \equiv 1$ and $u(x, t) \equiv -1$. As we will indicate later, some of our results will only be valid in $\boldsymbol{R}^2$. In this case the first part of the Theorem 2 shows that the total length of the front Γ_t is uniformly bounded in time.

In the remainder of this paper we will assume the following:

1. For every $t > 0$ and $\varepsilon > 0$ small, the functions $u^{\varepsilon}(x, t) = \pm 1 + O(\varepsilon)$ uniformly in Ω except in an ε - neighborhood of the front Γ_t. At distances greater than $O(\varepsilon)$ from Γ_t, the derivatives of u^{ε} will be presumed to be bounded independently of ε as $\varepsilon \to 0$.

2. For every $\varepsilon > 0$ the solutions of (2) are sufficiently smooth to justify our calculations.

3. Let $\Gamma_t^{\varepsilon} := \{u^{\varepsilon}(x, t) = 0\}$ for every $\varepsilon > 0$. We suppose that both Γ_t and Γ_t^{ε} are smooth, closed surfaces in the interior of the set Ω that evolve continuously in time and $\Gamma_t^{\varepsilon} \to \Gamma_t$ as $\varepsilon \to 0$ for all, but finitely many $t > 0$. Here the convergence is assumed in the following sense. Let A_t be the region of $\boldsymbol{R}^n$ enclosed by Γ_t and A_t^{ε} be the region enclosed by Γ_t^{ε}. We will say that Γ_t^{ε} converges to Γ_t as $\varepsilon \to 0$, if $meas\left(A_t^{\varepsilon} \Delta A_t\right) \to 0$ as $\varepsilon \to 0$, where $meas(B)$ denotes the Lebesgue measure of the set B while $A \Delta B$ is the symmetric difference between the sets A and B. Some additional restrictions will be imposed on the convergence of Γ_t^{ε} later on.

4. The width of the boundary layer is small in comparison to the distance separating layers and their radii of curvature.

4. Fast timescale

4.1. Outer Expansion

First, we develop the expansions for the solutions on the fast timescale $s = \frac{t}{\varepsilon}$. Then (2) takes the form

$$\begin{cases} \frac{1}{\varepsilon}\left(\tilde{u}_s^{\varepsilon} + \nabla\tilde{u}^{\varepsilon}\cdot\tilde{v}^{\varepsilon}\right) = \varepsilon^{\frac{5}{2}}\Delta\tilde{\mu}^{\varepsilon}, \\ \frac{1}{\varepsilon^2}\left(\tilde{v}_s^{\varepsilon} + \nabla\tilde{v}^{\varepsilon}\tilde{v}^{\varepsilon}\right) = -\nabla\tilde{p}^{\varepsilon} - \varepsilon^2\Delta\tilde{u}^{\varepsilon}\nabla\tilde{u}^{\varepsilon}. \end{cases} \tag{10}$$

where $\tilde{w}(x, s) = w(x, \varepsilon s)$ for any function w, while $\tilde{\mu}^{\varepsilon} = f\left(\tilde{u}^{\varepsilon}\right) - \varepsilon^2\Delta\tilde{u}^{\varepsilon}$ and $div\ \tilde{v}^{\varepsilon} = 0$. Consider now the outer expansion of the solutions of (10) away from the front Γ_s. We set

$$\begin{aligned} \tilde{u}^{\varepsilon}(x, s) &= \tilde{u}_0(x, s) + \tilde{u}_1(x, s)\varepsilon + \tilde{u}_2(x, s)\varepsilon^2 + \dots, \\ \tilde{v}^{\varepsilon}(x, s) &= \tilde{v}_0(x, s)\varepsilon^2 + \tilde{v}_1(x, s)\varepsilon^3 + \tilde{v}_2(x, s)\varepsilon^4 + \dots, \\ \tilde{p}^{\varepsilon}(x, s) &= \tilde{p}_0(x, s) + \tilde{p}_1(x, s)\varepsilon + \tilde{p}_2(x, s)\varepsilon^2 + \dots. \end{aligned} \tag{11}$$

Then, substituting these expressions into (10), we have for the lowest order terms

$$\begin{cases} \tilde{u}_{0s} = 0, \\ \tilde{v}_{0s} = -\nabla\tilde{p}_0, \\ div\ \tilde{v}_0 = 0. \end{cases} \tag{12}$$

It follows that there is no change in the leading order term for the order parameter u on the timescale s; the initial values are preserved. As we will see later, the same can be said about the leading term in the outer expansion for the velocity v.

4.2. Inner Expansion

To determine the behavior of the solution near the interface suppose, following [Pe], that $\varphi(x, s)$ is the signed distance to Γ_s. Suppose, in addition, that the location of Γ_s^ε can be approximated by the function

$$q^\varepsilon(x, s) := \varphi_1(x, s)\,\varepsilon + \varphi_2(x, s)\,\varepsilon^2 + \ldots, \tag{13}$$

where q^ε is first defined for every $x \in \Gamma_s$ through the relation $u^\varepsilon\left(x + q^\varepsilon(x, s)\,\boldsymbol{m}(x, s), s\right) = 0$ and then extended into the neighborhood of Γ_s by assuming that $q^\varepsilon(x + \alpha\boldsymbol{m}(x, s), s) = q^\varepsilon(x, s)$ for every small $\alpha > 0$. Here $\boldsymbol{m}(x, s) = \nabla\varphi(x, s)$ is the unit normal to the front Γ_s at $x \in \Gamma_s$, and Γ_s and Γ_s^ε are supposed to be such that the above definitions make sense.

We introduce a new variable $z = \dfrac{\varphi(x, s)}{\varepsilon}$ and define the functions $\bar{u}^\varepsilon(x, z, s)$ and $\bar{v}^\varepsilon(x, z, s)$ by

$$\bar{u}^\varepsilon\left(x, \frac{\varphi(x, s)}{\varepsilon}, s\right) = \tilde{u}^\varepsilon(x, s),$$

and

$$\bar{v}^\varepsilon\left(x, \frac{\varphi(x, s)}{\varepsilon}, s\right) = \tilde{v}^\varepsilon(x, s).$$

We define the functions $\bar{p}^\varepsilon(x, z, s)$ and $\bar{\mu}^\varepsilon(x, z, s)$ in a similar manner.

Assume for every $\bar{w}(x, z, s)$ that

(a) $\bar{w}$ does not change if z is fixed, while x changes in the direction normal to Γ_s, that is

$\bar{w}(x+\alpha \boldsymbol{m}(x,s), z, s) = \bar{w}(x,z,s)$ for every $\alpha > 0$.

(b) The limit $lim_{z \to \pm\infty} w_z(x,z,s) = 0$.

The assumption (a) implies that $\nabla_x \bar{w}(x,z,s) \cdot \boldsymbol{m}(x,s) = 0$ and one can show for every $\bar{w}\left(x, \frac{\varphi(x,s)}{\varepsilon}, s\right) = w(x,s)$ that the following hold

$$\nabla w = \nabla_x \bar{w} + \frac{1}{\varepsilon}\bar{w}_z \boldsymbol{m}, \tag{14}$$

$$w_s = \bar{w}_s + \frac{1}{\varepsilon}\bar{w}_z V, \tag{15}$$

$$\Delta w = \frac{1}{\varepsilon^2}\bar{w}_{zz} + \frac{1}{\varepsilon}k\bar{w}_z + \Delta_x \bar{w}. \tag{16}$$

Here $V(x,s) = \varphi_s(x,s)$ is the normal velocity of the front Γ_s and $k(x,s) = \Delta\varphi(x,s)$ is its mean curvature. In the new variables the equations (10) take the form

$$\begin{cases} \frac{1}{\varepsilon}\left(\bar{u}_s^\varepsilon + \frac{1}{\varepsilon}\bar{u}_z^\varepsilon V + \nabla_x \bar{u}^\varepsilon \cdot \bar{v}^\varepsilon + \frac{1}{\varepsilon}\bar{u}_z^\varepsilon\left(\boldsymbol{m}\cdot\bar{v}^\varepsilon\right)\right) = \varepsilon^{\frac{5}{2}}\left(\frac{1}{\varepsilon^2}\bar{\mu}_{zz}^\varepsilon + \frac{1}{\varepsilon}k\bar{\mu}_z^\varepsilon + \Delta_x\bar{\mu}\right), \\ \frac{1}{\varepsilon^2}\left(\bar{v}_s^\varepsilon + \frac{1}{\varepsilon}\bar{v}_z^\varepsilon V + \nabla_x \bar{v}^\varepsilon \bar{v}^\varepsilon + \frac{1}{\varepsilon}\bar{v}_z^\varepsilon\left(\boldsymbol{m}\cdot\bar{v}^\varepsilon\right)\right) \\ \quad = -\nabla_x \bar{p}^\varepsilon - \frac{1}{\varepsilon}\bar{p}_z^\varepsilon \boldsymbol{m} - \varepsilon^2\left(\nabla_x \bar{u}^\varepsilon + \frac{1}{\varepsilon}\bar{u}_z^\varepsilon \boldsymbol{m}\right)\left(\frac{1}{\varepsilon^2}\bar{u}_{zz}^\varepsilon + \frac{1}{\varepsilon}k\bar{u}_z^\varepsilon + \Delta_x \bar{u}\right). \end{cases} \tag{17}$$

Also we have that

$$div_x \bar{v}^\varepsilon + \frac{1}{\varepsilon}\left(\bar{v}_z^\varepsilon \cdot \boldsymbol{m}\right) = 0, \tag{18}$$

due to the incompressibility of the fluid and

$$\bar{\mu}^\varepsilon = f\left(\bar{u}^\varepsilon\right) - \bar{u}_{zz}^\varepsilon - \varepsilon k\bar{u}_z^\varepsilon - \varepsilon^2\Delta_x \bar{u}. \tag{19}$$

by the definition of the chemical potential. Moreover, by the definition of Γ_s^ε and (13),

$$\bar{u}^\varepsilon\left(x, \frac{q^\varepsilon}{\varepsilon}, s\right) \equiv 0 . \tag{20}$$

Note that q^ε is independent of z.

Next, suppose that

$$\begin{aligned} \bar{u}^\varepsilon(x, z, s) &= \bar{u}_0(x, z, s) + \bar{u}_1(x, z, s)\,\varepsilon + \bar{u}_2(x, z, s)\,\varepsilon^2 + \dots, \\ \bar{v}^\varepsilon(x, z, s) &= \bar{v}_0(x, z, s)\,\varepsilon^2 + \bar{v}_1(x, z, s)\,\varepsilon^3 + \bar{v}_2(x, z, s)\,\varepsilon^4 + \dots, \\ \bar{p}^\varepsilon(x, z, s) &= \bar{p}_0(x, z, s) + \bar{p}_1(x, z, s)\,\varepsilon + \bar{p}_2(x, z, s)\,\varepsilon^2 + \dots \end{aligned} \tag{21}$$

Substituting these expansions into (17) and collecting terms with the same powers of ε we obtain that

$$V(x, s) = 0, \tag{22}$$

$$\bar{u}_{0s}(x, z, s) = 0, \tag{23}$$

$$\left(\bar{p}_0 + \frac{\bar{u}_{0z}^2}{2}\right)_z = 0, \tag{24}$$

$$\bar{v}_{0s} = -\nabla_x \bar{p}_0 - \bar{u}_{0zz} \nabla_x \bar{u}_0 - \left((\bar{p}_1 + \bar{u}_{0z}\bar{u}_{1z})_z + k\bar{u}_{0z}^2 \right) \boldsymbol{m} . \tag{25}$$

In addition, due to (18),

$$\bar{v}_{0z} \cdot \boldsymbol{m} = 0 \tag{26}$$

and

$$div_x \bar{v}_0 + \bar{v}_{1z} \cdot \boldsymbol{m} = 0. \tag{27}$$

By taking a derivative of (25) in z, multiplying the result by $\boldsymbol{m}$ and using (26) we obtain that

$$\left((\bar{p}_1 + \bar{u}_{0z}\bar{u}_{1z})_z + k\bar{u}_{0z}^2 \right)_z = 0.$$

Since by our assumption the limit $lim_{z \to \pm\infty} w_z (x, z, s) = 0$ for every $w(x, z, s)$, then

$$(\bar{p}_1 + \bar{u}_{0z}\bar{u}_{1z})_z + k\bar{u}_{0z}^2 = 0,$$

and therefore

$$\bar{v}_{0s} = -\nabla_x \bar{p}_0 - \bar{u}_{0zz} \nabla_x \bar{u}_0. \tag{28}$$

By (24)

$$\bar{p}_0 (x, z, s) + \frac{\bar{u}_{0z}^2 (x, z)}{2} = \bar{p}_0 (x, \pm\infty, s), \tag{29}$$

and

$$[\tilde{p}_0]_{\Gamma_s} = 0.$$

Furthermore, by imposing the appropriate initial conditions on u^{ε}, we can use (24) to conclude that in fact,

$$\frac{\partial \tilde{p}_0}{\partial \boldsymbol{m}} = 0 \text{ on } \Gamma_s.$$

Then (12) implies that $\tilde{p}_0 = 0$ in the interior of the set enclosed by Γ_s (and $\tilde{v}_0$ remains unchanged in the outer expansion), hence by matching the inner and outer expansions, we obtain by (29) that

$$\bar{p}_0 (x, z) + \frac{\bar{u}_{0z}^2 (x, z)}{2} = 0.$$

Substituting $\bar{p}_0$ into (28), integrating in s, and using (23) we find that

$$\bar{v}_0 (x, z, s) = \bar{v}_0 (x, z, 0) + (\bar{u}_{0z} \nabla_x \bar{u}_{0z} - \bar{u}_{0zz} \nabla_x \bar{u}_0) s. \tag{30}$$

Notice that as $z \to \pm\infty$ the function $\bar{v}_0(x, z, s) \to \bar{v}_0(x, \pm\infty, 0)$ for every $s > 0$. This, along with the fact that the velocity remains unchanged in the outer expansion (due to the constant pressure), ensures that the matching condition for the velocity v holds in space variables for every $s > 0$.

By (30) the tangential velocity at some points in the boundary layer near Γ_s will increase linearly with the time s. Hence we will assume for the regular timescale t that the velocity v^{ε} is of the order one near the front Γ_t. This will provide for the correct matching of inner expansions of solutions of (2) for the fast timescale s and the timescale t. On the other hand, using (7), we can see that in the outer expansion the velocity v^{ε} cannot have the order of magnitude lower than $\sqrt{\varepsilon}$. We will begin by assuming that in the outer expansion, away from the front Γ_t, the function $v^{\varepsilon} = O(\varepsilon)$ for every $t > 0$.

5. Timescale t

By repeating for the t timescale our arguments from the previous section, we find that in the outer expansion the lower order terms should satisfy the following system of equations

$$u_{0t} = 0, \tag{31}$$

$$v_{0t} = -\nabla p_1, \tag{32}$$

$$\nabla p_0 = 0, \tag{33}$$

$$div\ v_0 = 0. \tag{34}$$

while we obtain from the expansion near Γ_t that

$$\bar{u}_{0t} + g\bar{u}_{0z} + \nabla_x \bar{u}_0 \cdot \bar{v}_0 = 0, \tag{35}$$

$$\bar{v}_{0t} + g\bar{v}_{0z} + \nabla_x \bar{v}_0 \bar{v}_0 = -\nabla_x \bar{p}_0 - \bar{u}_{0zz} \nabla_x \bar{u}_0 + hm, \tag{36}$$

$$g_z + div_x \bar{v}_0 = 0, \tag{37}$$

$$\boldsymbol{m} \cdot \bar{v}_{0z} = 0, \tag{38}$$

$$\left(\bar{p}_0 + \frac{\bar{u}_{0z}^2}{2}\right)_z = 0, \tag{39}$$

$$\varphi_{1t}(x, t) + \nabla_x \varphi_1(x, t) \cdot \bar{v}_0(x, \varphi_1(x, t), t) - g(x, \varphi_1(x, t), t) = 0, \tag{40}$$

$$\boldsymbol{m} \cdot \bar{v}_0 + V = 0 \tag{41}$$

Here $\varphi_1(x, t)$ is as defined in (13) and we have taken into account that

$$\bar{u}_0(x, \varphi_1(x, t), t) = 0. \tag{42}$$

In addition,

$$h = \bar{p}_{1z} + (\bar{u}_{0z}\bar{u}_{1z})_z + k\bar{u}_{0z}^2,$$

and

$$g = \boldsymbol{m} \cdot \bar{v}_1.$$

The following matching condition should also hold

$$lim_{z \to \pm\infty} \bar{v}_0(x, z, t) = 0. \tag{43}$$

Then using (39) we obtain that

$$[p_0]_{\Gamma_t} \equiv 0.$$

If we suppose that $p_0|_{\partial\Omega} \equiv 0$ then (33) implies that $p_0(x, t) \equiv 0$ in the outer expansion, away from the interface Γ_t. It follows then from (39) that

$$\bar{p}_0(x, z, t) + \frac{\bar{u}_{0z}^2(x, z, t)}{2} = 0. \tag{44}$$

By (38), (41), and (43) we have

$$V(x, t) = 0,$$

and

$$[v_0 \cdot \boldsymbol{m}]_{\Gamma_t} = 0, \tag{45}$$

$$\Gamma_t = \Gamma_0. \tag{46}$$

Hence, on the t timescale the front separating the fluids remains stationary.

Suppose for the remainder of this section that $\Omega \subset \boldsymbol{R}^2$. Since the normal component of $\bar{v}_0$ vanishes in the inner expansion (see (34)), we can show that v_0 would satisfy in the outer expansion

$$\begin{cases} v_{0t} = -\nabla p_1, \\ div\ v_0 = 0, \\ [p_1]_{\Gamma_0} = k \int\limits_{-\infty}^{\infty} \left(|\bar{v}_0|^2 - \bar{u}_{0z}^2 \right) dz, \end{cases} \tag{47}$$

with u_0 independent of t, while in the inner expansion

$$\begin{cases} \bar{u}_{0t} + g\bar{u}_{0z} + \nabla_x \bar{u}_0 \cdot \bar{v}_0 = 0, \\ \bar{v}_{0t} + g\bar{v}_{0z} + (\nabla_x (\bar{v}_0 \cdot \tau) \cdot \tau)\, \bar{v}_0 = \bar{u}_{0z} \nabla_x \bar{u}_{0z} - \bar{u}_{0zz} \nabla_x \bar{u}_0, \\ g_z + div_x \bar{v}_0 = 0, \end{cases} \tag{48}$$

where $\bar{v}_0$ is subject to the boundary conditions (43) and τ is a unit tangent vector to Γ_0. Parametrizing the curve Γ_0 with respect to its arclength s, we can rewrite the system (48) in terms of the variables z and s. Let $\xi(z, s, t) = \bar{u}_0(x(s), z, t)$ and $\eta(z, s, t) = \bar{v}_0(x(s), z, t) \cdot \tau(x(s))$ then

$$\begin{cases} \xi_t + g\xi_z + \eta\xi_s = 0, \\ \eta_t + g\eta_z + \eta_s \eta = \xi_z \xi_{zs} - \xi_{zz} \xi_s, \\ g_z + \eta_s + k\eta = 0. \end{cases} \tag{49}$$

Integrating the second equation in (49) in z by parts and using the third equation we obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\eta(z,s,t)\,dz + k(s)\int_{-\infty}^{\infty}\eta^2(z,s,t)\,dz = \frac{\partial}{\partial s}\int_{-\infty}^{\infty}\left(\xi_z^2(z,s,t) - \eta^2(z,s,t)\right)dz. \quad (50)$$

Suppose now that Γ_0 is a closed curve and that the region enclosed by Γ_0 is convex and $inf_{\Gamma_0} k(s) \geq \delta$ for some $\delta > 0$. Integrating (50) in s over Γ_0 we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{\Gamma_0}\eta(z,s,t)\,dsdz + \int_{-\infty}^{\infty}\int_{\Gamma_0}k(s)\eta^2(z,s,t)\,dsdz = 0. \quad (51)$$

We conjecture that (51) implies that $\int_{-\infty}^{\infty}\int_{\Gamma_0}\eta^2(z,s,t)\,dsdz \to 0$ and that $\frac{d}{ds}\int_{-\infty}^{\infty}\xi_z^2(z,s,t)\,dz \to 0$ as $t \to \infty$. Here the first identity implies that the lowest order term in the inner expansion of the tangential velocity converges to zero as $t \to \infty$. Indeed, by (7) we can assume for every $t > 0$ that

$$\int_{-\infty}^{\infty}\int_{\Gamma_0}\eta(z,s,t)\,dsdz < M^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}, \quad (52)$$

where $|\Omega|$ is the Lebesgue measure of the set Ω and M is defined in (7). Then , by (51),

$$lim_{t\to\infty}\int_{-\infty}^{\infty}\int_{\Gamma_0}\eta^2(z,s,t)\,dsdz = 0;$$

otherwise (52) will be violated. Using (50) we can formally conclude that

$$\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\xi_z^2(z,s,t)\,dz = 0. \qquad (53)$$

Furthermore, the latter result can be extended in the following way. Fix any $\Phi \in C^1(\mathbf{R})$ such that $\Phi(\xi) > 0$ for every $x \in \mathbf{R}$. Multiply the first equation in (49) by $\eta\Phi'(\xi)$ and add it to the second equation, multiplied by $\Phi(\xi)$. Then integrate the resulting equation over $\Gamma_0 \times \mathbf{R}$ by parts to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{\Gamma_0}\Phi(\xi)\,\eta(z,s,t)\,dsdz + \int_{-\infty}^{\infty}\int_{\Gamma_0}k(s)\,\Phi(\xi)\,\eta^2(z,s,t)\,dsdz = 0. \qquad (54)$$

Using the same reasoning as above we can conclude that

$$\frac{d}{ds}\int_{-\infty}^{\infty}\Phi(\xi)\,\xi_z^2(z,s,t)\,dz \to 0, \qquad (55)$$

as $t \to \infty$ or, differentiating in s and integrating by parts,

$$\int_{-\infty}^{\infty}\Phi(\xi)\,[\xi_z\xi_{sz} - \xi_s\xi_{zz}]\,dz \to 0, \qquad (56)$$

as $t \to \infty$. Assume that $\xi_z(s,z,\infty) \neq 0$ on $\Gamma_0 \times \mathbf{R}$ then, since the function Φ was chosen arbitrarily, we may deduce that

$$\xi_z(s,z,\infty)\,\xi_{sz}(s,z,\infty) - \xi_s(s,z,\infty)\,\xi_{zz}(s,z,\infty) = 0, \qquad (57)$$

for all $(s,z) \in \Gamma_0 \times \mathbf{R}$. The equation (57) can be rewritten as

$$\xi_z^2 \left(\frac{\xi_s}{\xi_z} \right)_z = 0 . \tag{58}$$

Since $\xi_z \neq 0$ on $\Gamma_0 \times \mathbf{R}$ by our assumption, we conclude that

$$\xi_s(s, z, \infty) = c(s)\, \xi_z(s, z, \infty) \tag{59}$$

for every $(s, z) \in \Gamma_0 \times \mathbf{R}$. On the other hand, by differentiating the equation (42), and letting $t \to \infty$, we have that

$$\xi_s(s, \varphi_1(s, \infty), \infty) + \xi_z(s, \varphi_1(s, \infty), \infty)\, \varphi_{1s}(s, \infty) = 0 , \tag{60}$$

and it follows that $\varphi_{1s}(s, \infty) = c(s)$. Therefore, by fixing some $s_0 \in \Gamma_0$ and setting $\psi(s) = \varphi_1(s_0, \infty) - \varphi_1(s, \infty)$, the function ξ can be written as

$$\xi(s, z, \infty) = \xi(s_0, z + \psi(s), \infty) . \tag{61}$$

We emphasize that the rigorous proof of these results can only be provided by studying the asymptotic behavior of the nonlinear system (49). Also, even formally, the different arguments should be used to show whether our results hold for Γ_0 that might contain affine parts or Γ_0 that is not closed, or not convex.

Remark 1. Notice that if we set $\eta = \xi_z$ and $g = -\xi_s$, when $k(s) \equiv 0$ on Γ_0 and $\xi \in C^2(\Gamma_0 \times \mathbf{R})$, then (ξ, η, g) is the stationary solution of the system (49).

Hence, we have established that on a fast timescale of order ε the gradients of the initial distribution of order parameter will cause the rapid acceleration of the fluid particles inside the boundary layer. On the timescale of order one, however, the transport effects associated with now moving particles will in turn "equilibrate" the order parameter along the boundary, leading to the solution structure, described by (61). At the same time the tangential component of the velocity of moving particles will decrease back to the same order of magnitude as in the outer expansion.

It follows from (47) that for $t >> 1$, the function v_0 in the outer expansion has to satisfy

$$\begin{cases} v_{0t} = -\nabla p_1, \\ div\ v_0 = 0, \\ [p_1]_{\Gamma_0} = -k \int_{-\infty}^{\infty} \bar{u}_{0z}^2 dz. \end{cases} \tag{62}$$

Assume that Γ_0 is connected. Then using (53) we conclude that the velocity v_0 will grow almost linearly in the outer expansion on the timescale t, provided that $k(s)$ is not identically constant on Γ_0.

6. Slow timescale.

Consider now the behavior of the system (2) on a slow timescale of the order $\frac{1}{\sqrt{\varepsilon}}$. Let $\sigma = \sqrt{\varepsilon} t$. Based on our results for the t timescale, we will assume that the velocity v on the σ timescale will be of order one everywhere in Ω, and that the leading order term $\underline{u}_0$ in the inner expansion of the order parameter u satisfies

$$\underline{u}_0 = \underline{u}_0(z + \psi(x, \sigma), \sigma), \tag{63}$$

for every $\sigma > 0$.

For every $w(x, t)$ suppose that $\hat{w}(x, \sigma) = w\left(x, \frac{\sigma}{\sqrt{\varepsilon}}\right)$ and for every $\overline{w}(x, z, t)$ suppose that $\underline{w}(x, z, \sigma) = \overline{w}\left(x, z, \frac{\sigma}{\sqrt{\varepsilon}}\right)$. Then (2) should be rewritten as

$$\begin{cases} \hat{u}_\sigma^\varepsilon + \nabla \hat{u}^\varepsilon \cdot \hat{v}^\varepsilon = \varepsilon^2 \Delta \hat{\mu}^\varepsilon, \\ \hat{v}_\sigma^\varepsilon + \nabla \hat{v}^\varepsilon \hat{v}^\varepsilon = -\frac{1}{\varepsilon} \nabla \hat{p}^\varepsilon - \varepsilon \Delta \hat{u}^\varepsilon \nabla \hat{u}^\varepsilon. \\ div\ \hat{v}^\varepsilon = 0, \end{cases} \tag{64}$$

where $\hat{\mu}^\varepsilon = f\left(\hat{u}^\varepsilon\right) - \varepsilon^2 \Delta \hat{u}^\varepsilon$, while the energy inequality (8) has to be replaced with

$$\hat{E}^\varepsilon\left[\hat{u}^\varepsilon, \hat{v}^\varepsilon\right] = \int_\Omega \left[\varepsilon \frac{\left|\hat{v}^\varepsilon\right|^2}{2} + W\left(\hat{u}^\varepsilon\right) + \varepsilon^2 \frac{\left|\nabla \hat{u}^\varepsilon\right|^2}{2}\right] dx \le M\varepsilon. \tag{65}$$

The system (64) leads to the following equations in the outer expansion

$$\hat{u}_{0\sigma} + \nabla \hat{u}_0 \cdot \hat{v}_0 = 0, \tag{66}$$

$$\nabla \hat{p}_0 = 0, \tag{67}$$

$$\hat{v}_{0\sigma} + \nabla \hat{v}_0 \hat{v}_0 = -\nabla \hat{p}_1, \tag{68}$$

$$div\ \hat{v}_0 = 0. \tag{69}$$

Near the front Γ_σ the system (64) can be rewritten to obtain

$$\begin{cases} \underline{u}_\sigma^\varepsilon + \frac{1}{\varepsilon} \underline{u}_z^\varepsilon V + \nabla_x \underline{u}^\varepsilon \cdot \underline{v}^\varepsilon + \frac{1}{\varepsilon} \underline{u}_z^\varepsilon \left(\boldsymbol{m} \cdot \underline{v}^\varepsilon\right) = \underline{\mu}_{zz}^\varepsilon + \varepsilon k \underline{\mu}_z^\varepsilon + \varepsilon^2 \Delta_x \underline{\mu}, \\ \underline{v}_\sigma^\varepsilon + \frac{1}{\varepsilon} \underline{v}_z^\varepsilon V + \nabla_x \underline{v}^\varepsilon \underline{v}^\varepsilon + \frac{1}{\varepsilon} \underline{v}_z^\varepsilon \left(\boldsymbol{m} \cdot \underline{v}^\varepsilon\right) \\ \quad = -\frac{1}{\varepsilon} \nabla_x \underline{p}^\varepsilon - \frac{1}{\varepsilon^2} \underline{p}_z^\varepsilon \boldsymbol{m} - \left(\nabla_x \underline{u} + \frac{1}{\varepsilon} \underline{u}_z^\varepsilon \boldsymbol{m}\right)\left(\frac{1}{\varepsilon} \underline{u}_{zz}^\varepsilon + k \underline{u}_z^\varepsilon + \varepsilon \Delta_x u\right), \end{cases} \tag{70}$$

and

$$div_x \underline{v}^\varepsilon + \frac{1}{\varepsilon}\left(\underline{v}_z^\varepsilon \cdot \boldsymbol{m}\right) = 0, \tag{71}$$

$$\underline{\mu}^\varepsilon = f\left(\underline{u}^\varepsilon\right) - \underline{u}_{zz}^\varepsilon - \varepsilon k \underline{u}_z^\varepsilon - \varepsilon^2 \Delta u_x, \tag{72}$$

where the variable z is as defined before. Hence in the inner expansion we will have

$$V + \boldsymbol{m} \cdot \underline{v}_0 = 0, \tag{73}$$

$$\underline{u}_{0\sigma} + \nabla_x \underline{u}_0 \cdot \underline{v}_0 + (\boldsymbol{m} \cdot \underline{v}_1)\, \underline{u}_{0z} = (f(\underline{u}_0) - \underline{u}_{0zz})_{zz}, \tag{74}$$

$$\underline{p}_{0z} + \underline{u}_{0z} \underline{u}_{0zz} = 0, \tag{75}$$

$$\nabla_x \underline{p}_0 + \underline{u}_{0zz} \nabla_x \underline{u}_0 = 0, \tag{76}$$

$$\underline{p}_{1z} + (\underline{u}_{0z} \underline{u}_{1z})_z + k \underline{u}_{0z}^2 = 0, \tag{77}$$

$$\begin{aligned} &\underline{v}_{0\sigma} + \nabla_x \underline{v}_0 \underline{v}_0 + (\boldsymbol{m} \cdot \underline{v}_1)\, \underline{v}_{0z} \\ &\quad = \left(-\nabla_x \underline{p}_1 - \underline{u}_{0zz} \nabla_x \underline{u}_1 - \underline{u}_{1zz} \nabla_x \underline{u}_0 - k \underline{u}_{0z} \nabla_x \underline{u}_0\right) - \alpha \boldsymbol{m}, \end{aligned} \tag{78}$$

$$\underline{v}_{0z} \cdot \boldsymbol{m} = 0, \tag{79}$$

$$div_x \underline{v}_0 + (\boldsymbol{m} \cdot \underline{v}_1)_z = 0, \tag{80}$$

$$\alpha = \left(\underline{p}_2 + \underline{u}_{0z} \underline{u}_{2z} + \frac{1}{2} \underline{u}_{1z}^2\right)_z + 2k \underline{u}_{0z} \underline{u}_{1z} + \underline{u}_{0z} \Delta_x \underline{u}_0 . \tag{81}$$

We are going to concentrate more closely on the equation (74) when $x \in \mathbf{R}^2$. We will use the same notation (with some obvious changes) as in the previous section. We already know from (61) that

$$\underline{u}_0(s, z, \sigma) = \underline{u}_0(s_0, z + \psi(s, \sigma), \sigma), \tag{82}$$

when $\sigma = 0$. Furthermore, this structure must be preserved during the evolution on the slow timescale, that is (82) must hold for every $\sigma > 0$. Set

$$y := z + \psi(s, \sigma) , \tag{83}$$

and

$$w(y, \sigma) := \underline{u}_0(s_0, y, \sigma) . \tag{84}$$

Substituting w into (74) and changing variables we obtain

$$w_\sigma + (\psi_\sigma + \psi_s(\underline{\nu}_0 \cdot \tau) + (\boldsymbol{m} \cdot \underline{\nu}_1))\, w_y = (f(w) - w_{yy})_{yy} . \tag{85}$$

In order to preserve the structure of (82) the factor

$$A(s, y, \sigma) := \psi_\sigma + \psi_s(\underline{\nu}_0 \cdot \tau) + (\boldsymbol{m} \cdot \underline{\nu}_1) ,$$

must be independent of s.

Taking the derivative of A with respect to y we obtain

$$\psi_s(\underline{\nu}_0 \cdot \tau)_y + (\boldsymbol{m} \cdot \underline{\nu}_1)_y = \alpha(y, \sigma) , \tag{86}$$

for some function α of y and σ. We can use this equation along with (80) (written in the new coordinates) to conclude that

$$\psi_s(\underline{\nu}_0 \cdot \tau)_y - \left[\psi_s(\underline{\nu}_0 \cdot \tau)_y + (\underline{\nu}_0 \cdot \tau)_s\right] = \alpha(y, \sigma) ,$$

and thus $(\underline{\nu}_0 \cdot \tau)_s = \alpha(y, \sigma) = 0$ since the function $(\underline{\nu}_0 \cdot \tau)$ is periodic in s. Hence, $(\underline{\nu}_0 \cdot \tau)$ is a function of (y, σ) and

$$(\boldsymbol{m} \cdot \underline{\nu}_1) = -\psi_s(\underline{\nu}_0 \cdot \tau) + \gamma(s, \sigma) ,$$

by (86), for some $\gamma : \mathbf{R} \times [0, \infty) \to \mathbf{R}$. Substituting this equation into (85) we have

$$w_\sigma + (\psi_\sigma + \gamma) w_y = (f(w) - w_{yy})_{yy}.$$

The coefficient in front of w_y should remain independent of s, thus, by denoting

$$\eta(\sigma) := \psi_\sigma(s, \sigma) + \gamma(s, \sigma),$$

and

$$W(y, \sigma) := w\left(y + \int_0^\sigma \eta(s)\, ds, \sigma\right),$$

we obtain the one-dimensional Cahn-Hilliard equation

$$W_\sigma = (f(W) - W_{yy})_{yy}. \tag{87}$$

Then, by using (87), we can conclude, following Pego ([Pe]), that for large σ the order parameter $\underline{u}_0(x, z, \sigma)$ should approach for every $x \in \Gamma_\sigma$ a translate $U(z - \lambda(x, \sigma))$ of U, where U is the solution of

$$\begin{cases} f(U) - \dfrac{d^2 U}{dz^2} = 0, \\ U(-\infty) = -1, \ U(\infty) = 1. \end{cases} \tag{88}$$

We will assume that the values of λ are uniformly bounded on its domain.

Hence, we may conclude for large $\sigma \gg 1$ that the asymptotic limit of (2) is

$$\begin{cases} \dot{\hat{v}}_0 = -\nabla \hat{p}, \\ div\ \hat{v}_0 = 0, \\ \dot{\hat{u}}_0 = 0, \end{cases} \tag{89}$$

in $\Omega \setminus \Gamma_\sigma$. Here $\dot{g}$, for any function g, denotes the advective time derivative $\dot{g} = g_t + v_0 \cdot \nabla g$. In addition we obtain using (77) and (88) that

$$[\hat{p}]_{\Gamma_\sigma} = -k\int_{-1}^{1}\sqrt{W(u)}\,du, \tag{90}$$

and using (73) that

$$[\hat{v}_{0m}]_{\Gamma_\sigma} = 0. \tag{91}$$

Comparing (90) with (1) we deduce that the asymptotic limit of (2) as $\varepsilon \to 0$ is the classical problem of motion of two immiscible, inviscid fluids. In addition, by Theorem 2, the total length of the front Γ_t is uniformly bounded in time, hence the interface stretching (the amount of interface length per unit volume) is bounded during the motion on the slow timescale as well .

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