An Elementary Proof of the Fundamental Property of a Perfect Gas

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ABSTRACT. A sharpened version of an important property of perfect gases proved by MONLÉON & PEDREGAL [1] is proved here by exploiting fully the fact that every perfect gas has an entropy function. In this manner, the more advanced machinery of weak convergence employed in the earlier version is avoided, and a more elementary and accessible proof emerges.

CONTENTS

1. INTRODUCTION

In the article [1], the equivalence of Kelvin's version of the second law of thermodynamics and the Clausius inequality was established. (We refer the reader to that article for an extensive discussion of the background in both classical and modern thermodynamics that supports this equivalence.) The main result on which the proof of equivalence in [1] rests is the "fundamental property of a perfect gas": *every right-continuous, non-decreasing function on* $(0, \infty)$ *that is initially zero and ultimately constant is the accumulation function of an absolutely continuous process of a pre-assigned perfect gas.* (See Section 2 for definitions of unfamiliar terms.) The proof of the fundamental property of a perfect gas in [1] employs measure theory and the machinery of weak convergence on Sobolev spaces. In the present paper, we present an elementary proof of the sharpened version of this property that is obtained by replacing *absolutely continuous process* by *Lipschitz continuous process.*

We are able to circumvent the use of more advanced tools in analysis by exploiting the fact that each perfect gas has a smooth entropy function. First of all, we use the fact that entropy and temperature, in place of volume and temperature, can be used to describe the states and processes of a pefect gas. In this way we are able to give a simpler description of the Carnot processes associated with accumulation functions that are also step functions. The second important property of entropy that we exploit is the fact that entropy is a potential for the vector field whose line integral describes heating divided by temperature integrated along processes. This permits us to replace the problem of proving that line integrals converge, which necessitates the use of weak convergence in [1], by the problem of proving that the potentials converge.

Our proof of the fundamental property of a perfect gas gives some insight into the question as to which special thermodynamical systems other than perfect gases might also have this fundamental property. In fact, our proof shows that a thermodynamical system will have the fundamental property if the system has a collection of processes on which the accumulation function can be expressed in the form given in relation (2) and on which the minimum and maximum temperatures can take on arbitrary positive values.

2. PERFECT GASES AND ACCUMULATION FUNCTIONS

For present purposes, a perfect gas can be prescribed by giving the *specific heat function* $c \in C^0(\mathbb{R}^{++}, \mathbb{R})$, with $\mathbb{R}^{++} := \{T \in \mathbb{R} \mid T > 0\}$, and the *latent heat* with respect to volume $\lambda \in \mathbb{R}^{++}$. The *state space* of a perfect gas is the set

$$\Sigma := \left\{(T, S) \in \mathbb{R}^2 \mid T \in \mathbb{R}^{++}\right\} \tag{1}$$

where T denotes the absolute temperature and S denotes the entropy. A *process* of a perfect gas is defined to be a Lipschitz continuous curve $(\hat{T}, \hat{S}) : [0,1] \to \Sigma$, and the heat gained by the gas in the process $(\hat{T}, \hat{S})$ is defined to be the line integral $\int\limits_{(\hat{T},\hat{S})} T\,dS = \int\limits_{(\hat{T},\hat{V})} c(T)\,dT + \frac{\lambda T}{V}\,dV$, where the function $\hat{V}$ describes the volume of the gas in the process and can be calculated from the following relation between entropy, volume and temperature:

$$S = \lambda \ln V + \int \frac{c(T)}{T} dT.$$

(This relation can be fully specified by requiring that the entropy vanish when the temperature and volume have preassigned values, but we do not need this specification here).

Not only the heat gained but also the distribution of heat gained with respect to temperature are essential for expressing the content of the second law of thermodynamics. Thus, following SERRIN [2], we define for each process $(\hat{T}, \hat{S})$ of a perfect gas and each temperature T the *net heat gained during* $(\hat{T}, \hat{S})$ *at or below the temperature* T:

$$C((\hat{T}, \hat{S}), T) = \int\limits_{\left\{t \in [0,1] \mid \hat{T}(t) \leq T\right\}} \hat{T}(t)\hat{S}^{\cdot}(t)dt\ . \tag{2}$$

The function $C((\hat{T}, \hat{S}), \cdot)$ is called the *accumulation function* for the given process $(\hat{T}, \hat{S})$. The accumulation function has the following properties:

Acc 1: $C((\hat{T},\hat{S}),\cdot):\mathbb{R}^{++}\to\mathbb{R}$ is right-continuous and satisfies

$$C((\hat{T},\hat{S}),T)=\begin{cases} 0 & if \quad 0<T<\min Rng\hat{T} \\ C((\hat{T},\hat{S}),\max Rng\hat{T}) & if \quad \max Rng\hat{T}\le T<\infty. \end{cases}$$

Acc 2: The accumulation function satisfies the *Clausius-Planck relation*:

$$\int_0^\infty \frac{C((\hat{T},\hat{S}),T)}{T^2}dT \;=\; \hat{S}(1)-\hat{S}(0). \tag{3}$$

Proofs of these properties of the accumulation function may be found in references cited by MONLÉON & PEDREGAL [1]. In our analysis of perfect gases, we shall use a third property of accumulation functions that rests on two types of "cutoff" operations. For each mapping $f:\mathbb{R}^{++}\to\mathbb{R}$ and $T_0>0$, we define the function $f_{T_0}:\mathbb{R}^{++}\to\mathbb{R}$ by the formula

$$f_{T_0}(T)=f(\min\{T,T_0\}). \tag{4}$$

A second kind of cutoff applies to a process $(\hat{T},\hat{S})$, with $\hat{T}$ non-decreasing, and a temperature $T_0>0$ and yields the mapping $(\hat{T},\hat{S})_{T_0}:[0,1]\to\mathbb{R}^{++}\times\mathbb{R}$ defined by

$$(\hat{T},\hat{S})_{T_0}=\begin{cases} t\longmapsto(\hat{T}(0),\hat{S}(0)) \\ t\mapsto\begin{cases} (\hat{T}(t),\hat{S}(t)) & if \quad 0\le t<\max\hat{T}^{-1}(\{T_0\}) \\ (T_0,\hat{S}(\max\hat{T}^{-1}(\{T_0\}))) & if \quad \max\hat{T}^{-1}(\{T_0\})\le t\le 1 \end{cases} \\ (\hat{T},\hat{S}) \end{cases}. \tag{5}$$

where the top formula in equation (5) applies when $0<T_0<\min Rng\hat{T}$, the second when $T_0\in Rng\hat{T}$,and the third when $\max Rng\hat{T}<T_0<\infty$. Thus, the cutoff $(\hat{T},\hat{S})_{T_0}$ follows the process $(\hat{T},\hat{S})$ until the last time when the temperature equals T_0 and thereafter freezes the temperature and entropy at the values corresponding to the cutoff time. It is an immediate consequence of the definition that the cutoff of a process is itself a process. We now can state the third property of accumulation functions needed in our analysis:

Acc 3: If $(\hat{T},\hat{S})$ is a process for which the temperature function $\hat{T}$ is non-decreasing, then for all $T_0\in\mathbb{R}^{++}$

$$C((\hat{T},\hat{S}),\cdot)_{T_0}=C((\hat{T},\hat{S})_{T_0},\cdot). \tag{6}$$

In other words, the cutoff of the accumulation function of a process with non-decreasing temperature is the accumulation function of the corresponding cutoff of the process.

Proof. First, consider the case where $0 < T_0 < \min Rng\hat{T}$. Relations (5), (4), (2), in which $(\hat{T},\hat{S})$ is replaced by $(\hat{T},\hat{S})_{T_0}$, along with Acc 1 tell us that both members of (6) are zero. Next, for the case where $\max Rng\hat{T} < T_0$, relation (5) implies that the right-hand side of (6) is $C((\hat{T},\hat{S}),\cdot)$. To examine the left-hand side of (6) when $\max Rng\hat{T} < T_0$, let $T \in (0,T_0]$ be given and note by (4) that $C((\hat{T},\hat{S}),\cdot)_{T_0}(T) = C((\hat{T},\hat{S}),T)$; for $T \in [T_0,\infty)$, $C((\hat{T},\hat{S}),\cdot)_{T_0}(T) = C((\hat{T},\hat{S}),T_0) = C((\hat{T},\hat{S}),T)$, because $C((\hat{T},\hat{S}),\cdot)$ is constant on $[\max Rng\hat{T},\infty)$. Hence, the left-hand side of (6) also is $C((\hat{T},\hat{S}),\cdot)$ when $\max Rng\hat{T} < T_0$. For the case $T_0 \in Rng\hat{T}$, we first let $T \in (0,T_0]$ be given and note by (2), (5), and the monotonicty of $\hat{T}$:

$$\begin{aligned}
C((\hat{T},\hat{S})_{T_0},T) &= \int\limits_{\{t\in[0,1]\,|\,\hat{T}_{T_0}(t)\le T\}} \hat{T}_{T_0}(t)\hat{S}^{\cdot}_{T_0}(t)dt \\
&= \int\limits_{\{t\in[0,\max \hat{T}^{-1}(\{T_0\})]\,|\,\hat{T}(t)\le T\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= \int\limits_{\{t\in[0,\max \hat{T}^{-1}(\{T\})]\,|\,\hat{T}(t)\le T\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= \int\limits_{\{t\in[0,1]\,|\,\hat{T}(t)\le T\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= C((\hat{T},\hat{S}),T) = C((\hat{T},\hat{S}),\cdot)_{T_0}(T). \qquad (7)
\end{aligned}$$

(In (7) we have written $\hat{T}_{T_0}$ for the first component of $(\hat{T},\hat{S})_{T_0}$ and $\hat{S}^{\cdot}_{T_0}$ for the derivative of the second component of $(\hat{T},\hat{S})_{T_0}$.) Finally, for $T \in (T_0,\infty)$, the first two relations in (7) and the monotonicity of $\hat{T}$ yield

$$\begin{aligned}
C((\hat{T},\hat{S})_{T_0},T) &= \int\limits_{\{t\in[0,\max \hat{T}^{-1}(\{T_0\})]\,|\,\hat{T}(t)\le T\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= \int\limits_{\{t\in[0,\max \hat{T}^{-1}(\{T_0\})]\,|\,\hat{T}(t)\le T_0\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= \int\limits_{\{t\in[0,1]\,|\,\hat{T}(t)\le T_0\}} \hat{T}(t)\hat{S}^{\cdot}(t)\,dt \\
&= C((\hat{T},\hat{S}),T_0) = C((\hat{T},\hat{S}),\cdot)_{T_0}(T),
\end{aligned}$$

and we may conclude that (6) holds also when $T_0 \in Rng\hat{T}$. ■

We close this section with another property of cutoffs that is essential to our elementary proof of the fundamental property of perfect gases.

Proposition 1. *Suppose that $f : \mathbb{R}^{++} \to \mathbb{R}$ is right-continuous, of bounded variation, and is "initially zero and unltimately constant", i.e., there exist $T_{low}(f)$ and $T_{up}(f)$ in $\mathbb{R}^{++}$ with $T_{low}(f) \leq T_{up}(f)$ such that for all $T \in \mathbb{R}^{++}$*

$$f(T) = \begin{cases} 0 & if \quad 0 < T < T_{low}(f) \\ f(T_{up}(f)) & if \quad T_{up}(f) \leq T < \infty. \end{cases} \tag{8}$$

Let D be a dense subset of $\mathbb{R}^{++}$ with the property that for every $T_0 \in D$,

$$\int_0^\infty \frac{f_{T_0}(T)}{T^2} dT = 0. \tag{9}$$

Then f is the zero function.

Proof. By (4) and (8), the cutoff f_{T_0} also is right-continuous and of bounded variation, and equation (9) may be written

$$\begin{aligned} 0 &= \int_0^{T_0} \frac{f_{T_0}(T)}{T^2} dT + \int_{T_0}^\infty \frac{f_{T_0}(T)}{T^2} dT \\ &= \int_0^{T_0} \frac{f(T)}{T^2} dT + \frac{f(T_0)}{T_0}, \end{aligned}$$

so that for every $T_0 \in D$,

$$f(T_0) = -T_0 \int_0^{T_0} \frac{f(T)}{T^2} dT. \tag{10}$$

Because f is integrable on bounded intervals, the right-hand side of (10) defines a continuous function

$$g(\tilde{T}) = -\tilde{T} \int_0^{\tilde{T}} \frac{f(T)}{T^2} dT$$

for all $\tilde{T} \in \mathbb{R}^{++}$. Thus, the right-continuous function f and the continuous function g agree on the dense set D, and it follows that $f = g$, so that f is continuous. Consequently, equation (10) holds for all $T_0 \in \mathbb{R}^{++}$ and, by the Fundamental Theorem of Calculus, f is continuously differentiable. Differentiation of (10) with respect to T_0 yields the relation $f'(T_0) = 0$ for all $T_0 \in \mathbb{R}^{++}$ and, by (8), $f = 0$. ■

3. THE FUNDAMENTAL PROPERTY OF A PERFECT GAS

We are now in a position to establish that *every right-continuous, non-decreasing function on $\mathbb{R}^{++}$ that is initially zero and is ultimately constant is the accumulation function of a process of a given perfect gas.* Our goal will be reached by proving the following theorem.

Theorem 2. *Let $f : \mathbb{R}^{++} \longrightarrow \mathbb{R}$ be right-continuous, non-decreasing, initially zero, and ultimately constant. There exists a sequence $n \longmapsto s_n$ of right-continuous, non-decreasing step functions, initially zero and ultimately constant, and a sequence $n \longmapsto (\hat{T}_n, \hat{S}_n)$ of processes of the perfect gas, with $\hat{T}_n$ and $\hat{S}_n$ both non-decreasing on $[0,1]$ for all $n \in \mathbb{N}\setminus\{0\}$, such that*

(1) $n \longmapsto s_n$ converges uniformly to f on $\mathbb{R}^{++}$; for all $T_0 \in \mathbb{R}^{++}$, $n \longmapsto (s_n)_{T_0}$ converges uniformly to f_{T_0} on $\mathbb{R}^{++}$; for all $n \in \mathbb{N}\setminus\{0\}$, $T_{low}(s_n) = T_{low}(f), T_{up}(s_n) = T_{up}(f)$;

(2) $n \longmapsto (\hat{T}_n, \hat{S}_n)$ converges uniformly on $[0,1]$ to a process $(\hat{T}, \hat{S})$; for every point of continuity T_0 of f, $n \longmapsto (\hat{T}_n, \hat{S}_n)_{T_0}$ converges uniformly on $[0,1]$ to the process $(\hat{T}, \hat{S})_{T_0}$;

(3) for all $n \in \mathbb{N}\setminus\{0\}$, $C((\hat{T}_n, \hat{S}_n), \cdot) = s_n$; $C((\hat{T}, \hat{S}), \cdot) = f$.

As will be apparent from the proof of Theorem 2, the processes $(\hat{T}_n, \hat{S}_n)$ are Carnot processes, i.e., consist only of adiabatic and isothermal segments. These Carnot processes trace out "staircase" curves in the state-space Σ consisting only of horizontal, adiabatic segments and vertical, isothermal segments, and, because both $\hat{T}_n$ and $\hat{S}_n$ are non-decreasing, the staircase curves move vertically upward and horizontally to the right with respect to the vertical S-axis and the horizontal T-axis. The desired process $(\hat{T}, \hat{S})$ in the fundamental property then may be viewed as a limit of such staircase curves. As was pointed out at length in [1], the idea of approximating a process by Carnot processes emerged early in the mathematical development of thermodynamics. The refinements of this idea that have evolved in recent years have dealt with the delicate issue of matching mathematical choices of collections of processes and collections of candidates for accumulation functions in order to achieve the relation $C((\hat{T}, \hat{S}), \cdot) = f$ in item (3) of Theorem 2. Because our goal has been to provide an elementary and essentially self-contained argument, the proof of Theorem 2 that follows is rather long. Nevertheless, the structure of the proof is easily discerned by reading items (1) and (2) in the statement of the theorem and relations (40) and (41) at the end of the proof.

Proof. Let f satisfy the hypothesis of Theorem 2, choose $T_{low}(f) \leq T_{up}(f)$ as in (8), and let $n \in \mathbb{N}\setminus\{0\}$ be given. First put

$$s_n(T) := \begin{cases} 0 & if \quad 0 < T < T_{low}(f) \\ f(T_{up}(f)) & if \quad T_{up}(f) \leq T < \infty. \end{cases} \tag{11}$$

Because f is bounded and non-decreasing, the set

$$\left\{T \in (T_{low}(f), T_{up}(f)) \mid f(T) - f(T-) \geq \frac{1}{n}\right\}$$

is finite, so we may list its elements $T_1 < \cdots < T_K$. Put $T_0 := T_{low}(f)$, $T_{K+1} := T_{up}(f)$, and note that, since f is right-continuous and has a left-hand limit at T_k, we may choose $\delta_n > 0$ such that for all $k \in \{0, 1, \cdots, K+1\}$

$$0 \leq f(T) - f(T_k) < \frac{1}{n} \qquad \text{for all } T \in [T_k, T_k + \delta_n) \tag{12}$$

and

$$-\frac{1}{n} < f(T) - f(T_k-) \leq 0 \qquad \text{for all } T \in (T_k - \delta_n, T_k). \tag{13}$$

Let $k \in \{0, 1, \cdots, K\}$ be given. To define s_n on $[T_k, T_{k+1})$, we note that f is right-continuous on $[T_k, T_{k+1})$ and, for each $T \in (T_k, T_{k+1})$, there holds $0 \leq f(T) - f(T-) < \frac{1}{n}$, so we may choose $\mu_n(T)$ such that

$$\left|f(\tilde{T}) - f(T)\right| < \frac{2}{n} \qquad \text{for all } \tilde{T} \in (T - \mu_n(T), T + \mu_n(T)). \tag{14}$$

The collection of intervals $\{(T - \mu_n(T), T + \mu_n(T)) \mid T \in (T_k, T_{k+1})\}$ together with $(T_k - \delta_n, T_k + \delta_n)$ and $(T_{k+1} - \delta_n, T_{k+1} + \delta_n)$ form an open cover of $[T_k, T_{k+1}]$, so we may choose $T_1^* < \cdots < T_{J(k)}^*$ in $[T_k, T_{k+1}]$ such that the pairwise-disjoint half-open intevals $[T_k, T_1^*), [T_{J(k)}^*, T_{k+1})$,and $\left\{[T_j^*, T_{j+1}^*) \mid j \in \{1, \cdots, J(k) - 1\}\right\}$ cover $[T_k, T_{k+1})$ and

$$\begin{gathered} 0 \leq T_1^* - T_k < \delta_n, \quad 0 \leq T_{k+1} - T_{J(k)}^* < \delta_n \\ 0 \leq T_{j+1}^* - T_j^* < \mu_n(T_j^*) \quad \text{for } j \in \{1, \cdots, J(k) - 1\}. \end{gathered} \tag{15}$$

We now put

$$s_n(T) := \begin{cases} f(T_k) & if \quad T_k \leq T < T_1^* \\ f(T_{k+1}-) & if \quad T_{J(k)}^* \leq T < T_{k+1} \\ f(T_j^*) & if \quad T_j^* \leq T < T_{j+1}^* \text{ and } j \in \{1, \cdots, J(k) - 1\}. \end{cases} \tag{16}$$

By (16) and (11), s_n is a right-continuous step function that is initially zero and ultimately constant and that agrees with f on $(0, T_{low}(f)) \cup (T_{up}(f), \infty)$. Because f is non-decreasing, so is s_n, and, by relations (11)-(16)

$$\|s_n - f\| := \sup_{T \in \mathbb{R}^{++}} |s_n(T) - f(T)| < \frac{2}{n}, \tag{17}$$

so that $n \mapsto s_n$ converges uniformly to f on $\mathbb{R}^{++}$. Moreover, for each $T_0 \in \mathbb{R}^{++}$, relation(4), applied to f and to s_n, and relation (17) yield for each $T \in \mathbb{R}^{++}$

$$|(s_n)_{T_0}(T) - f_{T_0}(T)| = |s_n(\min\{T_0, T\}) - f(\min\{T_0, T\})| < \frac{2}{n},$$

and it follows that $n \longmapsto (s_n)_{T_0}$ converges uniformly to f_{T_0} on $\mathbb{R}^{++}$. This completes the proof of item (1) of Theorem 1.

We establish items (2) and (3) of Theorem 1 first for the case when $T_{low}(f) < T_{up}(f)$ and f is not identically zero. Put:

$$\dot{T} := 2(T_{up}(f) - T_{low}(f) \in \mathbb{R}^{++} \tag{18}$$

and

$$\dot{S} = \frac{2f(T_{up}(f))}{T_{low}(f)} \in \mathbb{R}^{++}, \tag{19}$$

and let $n \longmapsto s_n$ be chosen as in item (1) of Theorem 1. Let $n \in \mathbb{N}\backslash\{0\}$ be given. Choose $T_0 := T_{low}(f) < T_1 < \cdots < T_{N(n)} := T_{up}(f)$ so that, for every $i \in \{0, 1, \cdots, N(n) - 1\}$, s_n is constant on each interval $[T_i, T_{i+1})$ and so that

$$\Delta Q_i := s_n(T_i) - s_n(T_i-) \in \mathbb{R}^{++}. \tag{20}$$

Moreover, we put

$$\begin{aligned} \Delta S_i \ &: = \frac{\Delta Q_i}{T_i} \quad \text{for } i \in \{0, 1, \cdots, N(n)\}, \\ \Delta T_i \ &: = T_i - T_{i-1} \quad \text{for } i \in \{1, \cdots, N(n)\}. \end{aligned} \tag{21}$$

Next, we define $2N(n) + 1$ intervals $\{[t_k, t_{k+1}] \mid k \in \{0, 1, \cdots, 2N(n)\}\}$ by putting $t_0 := 0$ and defining recursively

$$t_{2i+1} := t_{2i} + \frac{\Delta S_i}{\dot{S}} \quad \text{for } i \in \{0, 1, \cdots, N(n)\} \tag{22}$$

and

$$t_{2i+2} := t_{2i+1} + \frac{\Delta T_{i+1}}{\dot{T}} \quad \text{for } i \in \{0, 1, \cdots, N(n) - 1\}. \tag{23}$$

By (22), (21), (20), (16), and (11) there hold

$$\begin{aligned}
\sum_{i=0}^{N(n)}(t_{2i+1}-t_{2i}) &= \sum_{i=0}^{N(n)}\left(\frac{s_n(T_i)-s_n(T_i-)}{T_i}\Big/\frac{2f(T_{up}(f))}{T_{low}(f)}\right)\\
&= \sum_{i=1}^{N(n)}\left(\frac{s_n(T_i)-s_n(T_{i-1})}{T_i}\Big/\frac{2f(T_{up}(f))}{T_{low}(f)}\right)+\left(\frac{s_n(T_0)-s_n(T_0-)}{T_0}\Big/\frac{2f(T_{up}(f))}{T_{low}(f)}\right)\\
&\leq \frac{1}{2f(T_{up}(f))}\left(\sum_{i=1}^{N(n)}(s_n(T_i)-s_n(T_{i-1}))+s_n(T_{low}(f))-s_n(T_{low}(f)-)\right)\\
&= \frac{1}{2f(T_{up}(f))}\left(s_n(T_{N(n)})-s_n(T_{low}(f)-)\right)=\frac{1}{2f(T_{up}(f))}\left(f(T_{up}(f))-0\right)\\
&= \frac{1}{2},
\end{aligned} \tag{24}$$

and, by (23), (21), and (18),

$$\sum_{i=0}^{N(n)}(t_{2i+2}-t_{2i+1})=\frac{1}{\dot{T}}\sum_{i=1}^{N(n)}\Delta T_{i+1}=\frac{1}{2}. \tag{25}$$

Therefore, by (24) and (25),

$$t_{2N(n)+1}\leq 1, \tag{26}$$

and we may now define the process $(\hat{T}_n,\hat{S}_n):[0,1]\to\Sigma$ by:

$$\hat{T}_n(t):=\begin{cases} T_i & for \quad t\in[t_{2i},t_{2i+1}),\quad i\in\{0,1,\cdots,N(n)\}\\ T_i+\dot{T}(t-t_{2i+1}) & for \quad t\in[t_{2i+1},t_{2i+2}),\quad i\in\{0,1,\cdots,N(n)-1\}, \end{cases} \tag{27}$$

$$\hat{S}_n(t):=\begin{cases} \sum_{j=0}^{i-1}\Delta S_j+\dot{S}(t-t_{2i}) & for \quad t\in[t_{2i},t_{2i+1}),\quad i\in\{0,1,\cdots,N(n)\}\\ \sum_{j=0}^{i}\Delta S_j & for \quad t\in[t_{2i+1},t_{2i+2}),\quad i\in\{0,1,\cdots,N(n)-1\}, \end{cases} \tag{28}$$

and for $t\in[t_{2N(n)+1},1]$ by

$$\begin{aligned} \hat{T}_n(t) &:= T_{N(n)}=T_{up}(f)\\ \hat{S}_n(t) &:= \sum_{i=0}^{N(n)}\Delta S_i. \end{aligned} \tag{29}$$

(In (28) the sum $\sum_{j=0}^{i-1} \Delta S_j$ when $i = 0$ is defined to be zero.) Relations (22), (23),

(27)-(29) tell us that $\hat{T}_n$ and $\hat{S}_n$ are continuous, piecewise linear, non-decreasing, and the derivatives $\dot{\hat{T}}_n$ and $\dot{\hat{S}}_n$ satisfy for all but finitely many $t \in [0, 1]$

$$\begin{gathered} \dot{\hat{T}}_n(t) \in \left\{0, \dot{T}\right\}, \qquad \dot{\hat{S}}_n(t) \in \left\{0, \dot{S}\right\}, \\ \dot{\hat{T}}_n(t)\dot{\hat{S}}_n(t) = 0, \end{gathered} \tag{30}$$

and, for all $t \in [0, 1]$,

$$\begin{gathered} T_{low}(f) \leq \hat{T}_n(t) \leq T_{up}(f) \\ 0 \leq \ \hat{S}_n(t) \leq \ \sum_{i=0}^{N(n)} \Delta S_i \leq \frac{f(T_{up}(f))}{T_{low}(f)}. \end{gathered} \tag{31}$$

In fact, the definition of $\hat{T}_n$ tells us that $Rng\hat{T}_n = [T_{low}(f), T_{up}(f)]$ for all . Relations (30) and (31) mean that $n \longmapsto (\hat{T}_n, \hat{S}_n)$ is a uniformly bounded, equi(Lipschitz)continuous sequence of functions. By the Arzelà-Ascoli Theorem, we may choose a subsequence $k \longmapsto (\hat{T}_{n_k}, \hat{S}_{n_k})$ that converges uniformly on [0,1] to a function $(\hat{T}, \hat{S}) : [0, 1] \rightarrow \mathbb{R}^{++} \times \mathbb{R}$. By (30), (31), and the uniform convergence of $k \longmapsto (\hat{T}_{n_k}, \hat{S}_{n_k})$ to $(\hat{T}, \hat{S})$, $\hat{T}$ and $\hat{S}$ are non-decreasing, and there holds for all $t \in [0, 1]$

$$\begin{gathered} T_{low}(f) \leq \hat{T}(t) \leq T_{up}(f) \\ 0 \leq \hat{S}(t) \leq \tfrac{f(T_{up}(f))}{T_{low}(f)}, \end{gathered} \tag{32}$$

and, for all $t', t'' \in [0, 1]$

$$\begin{gathered} \left|\hat{T}(t') - \hat{T}(t'')\right| \leq \dot{T} \ \left|t' - t''\right| \\ \left|\hat{S}(t') - \hat{S}(t'')\right| \leq \dot{S} \ \left|t' - t''\right|. \end{gathered} \tag{33}$$

Since $Rng\hat{T}_n = [T_{low}(f), T_{up}(f)]$ for all $n \in \mathbb{N}\backslash\{0\}$, it follows that $Rng\hat{T} = [T_{low}(f), T_{up}(f)]$. Moreover, (32) and (33) imply that $(\hat{T}, \hat{S})$ is a process for the perfect gas.

In order to verify the uniform convergence of cutoffs asserted in item (2) of Theorem 1, we let $T_0 \in (0, \min Rng\hat{T})$ be given. By (5), $(\hat{T}, \hat{S})_{T_0}(t) = (\hat{T}(0), \hat{S}(0))$ for all $t \in [0, 1]$. By the uniform convergence of $k \longmapsto (\hat{T}_{n_k}, \hat{S}_{n_k})$ to $(\hat{T}, \hat{S})$, for k sufficiently large, $T_0 \in (0, \min Rng\hat{T}_{n_k})$, so that again by (5) $(\hat{T}_{n_k}, \hat{S}_{n_k})_{T_0}(t) = (\hat{T}_{n_k}(0), \hat{S}_{n_k}(0))$ for all $t \in [0, 1]$, and we conclude that $k \longmapsto (\hat{T}_{n_k}, \hat{S}_{n_k})_{T_0}$ converges uniformly to $(\hat{T}, \hat{S})_{T_0}$.

Next, let $T_0 \in (\max Rng\hat{T}, \infty)$ be given. Relation (5) tells us that $(\hat{T}, \hat{S})_{T_0} = (\hat{T}, \hat{S})$ and, for k sufficiently large, $(\hat{T}_{n_k}, \hat{S}_{n_k})_{T_0} = (\hat{T}_{n_k}, \hat{S}_{n_k})$, and the desired uniform convergence of $k \longmapsto (\hat{T}_{n_k}, \hat{S}_{n_k})_{T_0}$ to $(\hat{T}, \hat{S})_{T_0}$ is verified when $T_0 \in (\max Rng\hat{T}, \infty)$. Finally, let $T_0 \in Rng\hat{T}$ be given, and put

$$t^* := \max \hat{T}^{-1}(\{T_0\}) = \max \hat{T}^{-1}((0, T_0]). \tag{34}$$

Because $Rng\hat{T} = [T_{low}(f), T_{up}(f)] = Rng\hat{T}_n$ for all $n \in \mathbb{N}\backslash\{0\}$, it follows that $T_0 \in Rng\hat{T}_{n_k}$ for all $k \in \mathbb{N}\backslash\{0\}$, and we may put

$$t^*_{n_k} := \max \hat{T}^{-1}_{n_k}(\{T_0\}). \tag{35}$$

To complete the proof of item (2), it suffices to show that *if T_0 is a point of continuity of the given mapping f, then*

$$\liminf_{k\to\infty} t^*_{n_k} = t^*. \tag{36}$$

In fact, relation (36) implies that there is a subsequence $j \longmapsto n_{k_j}$ such that $t^* = \lim_{j\to\infty} t^*_{n_{k_j}}$, $j \longmapsto (\hat{T}_{n_{k_j}}, \hat{S}_{n_{k_j}})$ converges uniformly to $(\hat{T}, \hat{S})$ on [0,1], and, by (34) and (35),

$$T_0 = \hat{T}(t^*) = \lim_{j\to\infty} \hat{T}_{n_{k_j}}(t^*_{n_{k_j}}). \tag{37}$$

If we apply the definition (5) to $(\hat{T}_{n_{k_j}}, \hat{S}_{n_{k_j}})_{T_0}$ and to $(\hat{T}, \hat{S})_{T_0}$, then the cited properties of the subsequence $j \longmapsto n_{k_j}$ along with relation (37) imply that $j \longmapsto (\hat{T}_{n_{k_j}}, \hat{S}_{n_{k_j}})_{T_0}$ converges uniformly on [0,1] to $(\hat{T}, \hat{S})_{T_0}$. (In applying (5) one uses case by case the relations $Rng\hat{T} = [T_{low}(f), T_{up}(f)] = Rng\hat{T}_n$ for all $n \in \mathbb{N}\backslash\{0\}$.) Thus, we need only verify the italicized statement containing (36) or, equivalently, the statement: *if $\liminf_{k\to\infty} t^*_{n_k} \neq t^*$, then T_0 is not a point of continuity of f.* Suppose first that $t_\infty := \liminf_{k\to\infty} t^*_{n_k} > t^*$, so that we may choose a subsequence $j \longmapsto n_{k_j}$ such that $j \longmapsto t^*_{n_{k_j}}$ is convergent with limit $t_\infty > t^*$, and $j \longmapsto (\hat{T}_{n_{k_j}}, \hat{S}_{n_{k_j}})$ converges uniformly to $(\hat{T}, \hat{S})$ on [0,1]. Relations (35) and (34) imply

$$\hat{T}(t^*) = T_0 = \lim_{j\to\infty} \hat{T}_{n_{k_j}}(t^*_{n_{k_j}}) = \hat{T}(t_\infty), \tag{38}$$

so that $\hat{T}$ is constant on the non-trivial interval $[t^*, t_\infty]$. Because the monotone, continuous and piecewise linear functions $\hat{T}_{n_{k_j}}$ converge uniformly on $[t^*, t_\infty]$ to the constant function $\hat{T}\mid_{[t^*,t_\infty]}$and because $\dot{\hat{T}}_{n_{k_j}}(t) \in \{0, \dot{T}\}$ for all but finitely many $t \in [t^*, t_\infty]$ and for all $j \in \mathbb{N}\backslash\{0\}$, the total length of the intervals $\dot{\hat{T}}^{-1}_{n_{k_j}}(\{\dot{T}\})$ tends to zero as $j \to \infty$. Relations (27)-(29) then tell us that the total length of the intervals $\hat{S}_{n_{k_j}}^{\cdot -1}(\{\dot{S}\})$ tends to $t_\infty - t^* > 0$ as $j \to \infty$, and it follows that

$$\lim_{j\to\infty} (\hat{S}_{n_{k_j}}(t_\infty) - \hat{S}_{n_{k_j}}(t^*)) = \dot{S}(t_\infty - t^*) > 0. \tag{39}$$

Relations (20), (21), and (39) imply that for j sufficiently large, the oscillation in $s_{n_{k_j}}$ at T_0 is bounded below by the positive number $\dot{S}(t_\infty - t^*)T_{low}(f)/2$, and the uniform covergence of $j \longmapsto s_{n_{k_j}}$ to f tells us that f is not continuous at T_0. A similar argument applies when $t_\infty < t_*$, and this completes the proof of item (2).

The first statement in item (3) of Theorem 1 is an immediate consequence of relations (2), (20), (21), (27), and (28). This statement, Acc 2, and Acc 3 then tell us that for every $n \in \mathbb{N}\backslash\{0\}$and $T_0 > 0$ there holds

$$\begin{aligned}\int_0^\infty \frac{(s_n)_{T_0}(T)}{T^2}\,dT &= \int_0^\infty \frac{C((\hat{T}_n,\hat{S}_n),\cdot)_{T_0}(T)}{T^2}\,dT \\ &= \int_0^\infty \frac{C((\hat{T}_n,\hat{S}_n)_{T_0},T)}{T^2}\,dT \\ &= (\hat{S}_n)_{T_0}(1) - (\hat{S}_n)_{T_0}(0).\end{aligned} \tag{40}$$

Now, let T_0 be a point of continuity of f, and note that by items (1) and (2) established above, the first and last members of (40) converge. Upon letting n tend to infinity in (40), we find

$$\begin{aligned}\int_0^\infty \frac{f_{T_0}(T)}{T^2}\,dT &= \hat{S}_{T_0}(1) - \hat{S}_{T_0}(0) \\ &= \int_0^\infty \frac{C((\hat{T},\hat{S})_{T_0},T)}{T^2}\,dT \\ &= \int_0^\infty \frac{C((\hat{T},\hat{S}),\cdot)_{T_0}(T)}{T^2}\,dT,\end{aligned} \tag{41}$$

where Acc 2 and Acc 3 were used in the last two steps of the calculation. If we note that the set of points of continuity of f form a dense subset of $\mathbb{R}^{++}$ and that the function $T \longmapsto (f(T) - C((\hat{T},\hat{S}),(T)))/T^2$ is right-continuous, of bounded variation, and is initially zero and ultimately constant, then relation (41) and Proposition 1 tell us that $f = C((\hat{T},\hat{S}),\cdot)$. The proofs of items (2) and (3) are completed by noting that the assertions in these items also are valid in the trivial cases when $T_{low}(f) = T_{up}(f)$ or when f is the zero function. ∎

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