On the ubiquity of fracture in nonlinear elasticity

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The purpose of the present paper is to point out a subtle and little known consequence of assuming a common ordering of principal stretches and principal stresses for an isotropic hyperelastic material [i.e., of assuming the Baker-Ericksen inequalities] . Now, as was shown by Ball [B1][B2], the condition of **polyconvexity** on a stored-energy function, together with appropriate growth hypotheses,ensures the existence of equilibrium solutions for the partial differential equations associated with appropriate boundary value problems in nonlinear elasticity. An incentive for adopting the polyconvexity requirement is the observation [Mo1],[B1] that the notion of **quasiconvexity** , which is essentially equivalent to the existence of smooth weak solutions, is very hard to verify explicitly; polyconvexity is not subject to this difficulty. What is, however, also special about polyconvexity and quasiconvexity for an isotropic material is that these properties [even the Baker-Ericksen inequalities alone] necessarily privilege those affine deformations which are **dilatations**: the stored-energy associated with a dilatation is smaller than the stored energy associated with any other affine deformation possessing the same determinant. We will demonstrate this fact in §3 and then point out in §4 certain fracture related consequences of this property which arise in those cases where the stored-energy function assigns to the dilatation of determinant $\delta > 0$ a value $A(\delta)$ which is not an everywhere convex function of $\delta \in (0,\infty)$.

In §2 below we introduce appropriate notation and terminology with which we can describe the equilibrium behavior of isotropic nonlinearly hyperelastic materials. Then in §3 we formulate as Theorem A the result informally described above. Finally, in §4 we point out [Theorem B] that in general the Baker-Ericksen inequalities lead, for certain ranges of the average density of the deformed material, to the existence of equilibria which are idealized **fractured** deformations. Moreover, we find that for fluids (and some other materials) such fractured minimizers are energy limits of minimizing sequences of continuous deformations in a manner analogous to deformations encountered in the study of twinned elastic crystals such as martensite (cf. e.g., [B&J], [J&K]).

It is worth pointing out that the idealized fracture deformations discussed here are not related to the deformations of cavitation type discussed by Ball [B3]. The presence of the latter depends on the *asymptotic* behavior of the stored-energy function for large deformation gradients, while the presence of our deformations depends only on the existence of intervals on which A is nonconvex.

§2. We shall consider a nonlinear elastic material which is isotropic in a reference configuration Ω consisting of a domain in $\mathbb{R}^n$. Putting x for the displacement vector to a point in the reference configuration and $u = u(x)$ for the displacement vector to the corresponding point in the deformed configuration, we know that in the absence of body forces the total stored energy associated with an orientation preserving and locally invertible displacement field $u \in W^{1,1}{}_{loc}(\Omega;\mathbb{R}^n)$ is given by

$$E[u] = \int_\Omega W(x, \text{grad } u(x))dx, \tag{1}$$

where $W\colon \Omega \times \text{lin}^+ \to \mathbb{R}$ is the *stored energy function* and $\text{lin}^+ = \text{lin}^+(\mathbb{R}^n)$ denotes the set of tensors with positive determinant. Moreover, because of isotropy and objectivity it is known ([T&N], [B1],[A],[C]) that $W(x,F) = \psi(x;v_1,\ldots,v_n)$, $\forall F \in \text{lin}^+$,

where for each $x \in \Omega$ $\psi(x;\cdot,\ldots,\cdot)$ is a function symmetric in its n arguments and $v_1,\ldots,v_n$ are the singular values associated with F, namely, the eigenvalues of $\sqrt{(F^T F)}$. Hereafter we shall take n=3 and refer to v_1, v_2, v_3 as the principal stretches associated with the tensor F, and we shall eventually suppose for convenience that the material is homogeneous, so that W is independent of location and the total stored energy associated with u is given by

$$E[u] = \int_\Omega \psi(v_1(x), v_2(x), v_3(x))dx, \tag{2}$$

with $v_i(x)$ the principal stretches associated with $F=(\text{grad}u)(x), x \in \Omega$. The eigenvalues of the Cauchy stress tensor associated with F, i.e. $T=F(W_{,F})^T$, are called the **principal stresses**.

In a fundamental 1977 article [B1] (cf. also [B2]) John Ball succeeded in combining advances in existence theory for nonlinear partial differential equations due to Morrey [Mo1],[Mo2] with the requirements of nonlinear elasticity to provide an important existence theory for the latter discipline. Morrey had shown that lower semicontinuity of a variational functional such as (1) is essentially equivalent to the requirement that the integrand have the property he termed **quasiconvexity** [Mo1],[Mo2],[B1],[B2]. This is the requirement that for each bounded open set $D \subset \mathbb{R}^3$, for each $F \in \text{lin}+$ and for each test field $\varsigma \in C_0^\infty(D;\mathbb{R}^3)$ the following inequality hold:

$$\int_D W(y, F + \text{grad}\,\varsigma(x))dx \geq \int_D W(y, F)dx, \quad \forall y \in \Omega. \tag{3}$$

Roughly speaking "each affine displacement $u = Fx + z$, $x \in D$, is a minimizer for total energy E, relative to its own boundary data on D".

In view of the difficulty involved in directly verifying (3), Morrey [Mo1],[Mo2] and Ball [B1],[B3] presented two other, more easily verifiable, properties which could ensure some features of quasiconvexity.

The weaker of these is **rank 1 convexity** [E],[B1],[B3]. This is the requirement that W be convex on all closed line segments in lin^+ whose endpoints differ by a tensor of rank 1 :

$$W(x,F + (1-\lambda)a\otimes b) \leq \lambda\, W(x,F) + (1-\lambda)W(x,F+a\otimes b), \quad \forall \lambda \in (0,1), \tag{4}$$

for all $F \in \text{lin}^+$ and $a,b \in \mathbb{R}^3$ for which $F+a \otimes b \in \text{lin}^+$. *Strict* rank 1 convexity is the requirement that (4) hold with strict inequality. Ball demonstrated [B1],[B3] that rank 1 convexity is a *consequence* of quasiconvexity while strict rank 1 convexity is equivalent to the requirement that every $W^{1,1}_{loc}(\Omega;\mathbb{R}^3)$ weak solution of the elasticity [Euler-Lagrange] equations for $E[u]= \int W(x,\text{grad}u(x))dx$ fail to exhibit jump discontinuities in gradient across any smooth 2-surface. The stronger of these conditions is **polyconvexity** [Mo1], [Mo2],[B1],[B2]. This is the requirement that for all $x \in \Omega$

$W(x,\cdot) = G^x \circ Z$, where $Z:\text{lin}^+ \rightarrow \text{lin}^+ \times \text{lin}^+ \times (0,\infty)$ is defined by $Z(F) = (F,\text{adj}F,\det F)$, $\forall F \in \text{lin}+$,and G^x: conv(range Z) $\rightarrow \mathbb{R}$ is convex, so that:

$$G^x(\lambda F+(1-\lambda)H,\ \lambda \mathrm{adj}F+(1-\lambda)\mathrm{adj}H, \lambda \det F+(1-\lambda)\det H) \qquad (5)$$
$$\leq \lambda G^x(F,\mathrm{adj}F,\det F) + (1-\lambda)G^x(H,\mathrm{adj}H,\det H),\ \forall \lambda \in [0,1],$$

for each F,H $\in$ lin+. Here adjF denotes the *adjugate* of F, which is the tensor associated with the transposed matrix of minors of the standard matrix of F. Morrey proved [Mo1],[Mo2] that when W is polyconvex it is necessarily quasiconvex, as well. Consequently Ball's existence theorem for nonlinear elasticity applies to cases in which W is polyconvex and satisfies appropriate growth conditions. We note that objectivity and isotropy of W ensure that polyconvexity of W is equivalent to the following constraint on the function ψ [B1]

$$\psi(x;v_1,v_2,v_3) = \varphi^x(v_1,v_2,v_3;v_1v_2,\ v_2v_3,\ v_3v_1;v_1v_2v_3) \qquad (6)$$

where the function φ^x is convex and $\varphi^x(w_1,w_2,w_3;y_1,y_2,y_3;z)$ is symmetric in the w's and in the y's and is nondecreasing in each w_i and each y_j.

§3. We are now in a position to present the major result of this paper. In what follows we will utilize the following norm on $\mathrm{lin}(\mathbb{R}^3)$ [hereafter denoted by lin]: $|F|^2 = \mathrm{tr}(F^TF)$, and we will continue to use lin^+ to denote the set of tensors with strictly positive determinant.

THEOREM A Suppose that the homogeneous isotropic stored-energy function $W \in C^1(\mathrm{lin}^+)$ always provides the same ordering for the principal stresses as for the principal stretches [$v_i > v_j \Rightarrow t_i \geq t_j$, where $\{t_k\}$ are eigenvalues of the Cauchy stress T] and satisfies the following growth condition

(*) $W(F) \to \infty$ as $|F| \to \infty$ or $\det F \to 0+$,

while the function ψ appearing in (2) is continuous on $(0,\infty)^3$. Then for each $\delta \in (0,\infty)$ the dilatation u_δ with $\det(\mathrm{grad}\ u_\delta) = \delta$ has the least energy of all affine displacements with gradient having determinant δ:

$$W(\delta^{1/3}I) \leq W(F),\ \ \forall F \in \mathrm{lin}^+ \text{ s.t. } \det F= \delta,\ \ \forall\ \delta \in (0,\infty). \qquad (7)$$

That is, the function $\psi \in C((0,\infty)^3)$ appearing in (2) satisfies for each $\delta \in (0,\infty)$:

$$\psi(\delta^{1/3}, \delta^{1/3}, \delta^{1/3}) \leq \psi(v_1,v_2,v_3),\ \forall\ v_i \in (0,\infty) \text{ s.t. } v_1v_2v_3=\delta. \qquad (8)$$

Proof: For fixed $\delta > 0$, the condition (*) ensures that the continuous function ψ attains its infimum on the smooth closed surface $C_\delta = \{(a,b,c) \in (0,\infty)^3 \mid abc = \delta\}$ at some triple $(w_1,w_2,w_3) \in C_\delta$. Initially we shall restrict attention to the case in which ψ is C^1 and the assumed ordering inequalities hold with $\geq$ replaced by $>$. Since $\psi \in C^1((0,\infty)^3)$ and C_δ is nowhere degenerate it follows that there is a scalar ["Lagrange multiplier"] $\mu \in \mathbb{R}$ such that

$$\mathrm{grad}\psi(w_1,w_2,w_3) = \mu \,\mathrm{grad} p(w_1,w_2,w_3),$$

where $p(a,b,c)=abc$, $\forall (a,b,c) \in (0,\infty)^3$. That is,

$$\psi_{,1}(w_1,w_2,w_3)=\mu w_2 w_3, \quad \psi_{,2}(w_1,w_2,w_3)=\mu w_3 w_1, \quad \psi_{,3}(w_1,w_2,w_3)=\mu w_1 w_2. \tag{9}$$

It follows that

$$w_1 \psi_{,1}(w_1,w_2,w_3) = w_2 \psi_{,2}(w_1,w_2,w_3) = w_3 \psi_{,3}(w_1,w_2,w_3) \tag{9'}$$

Now it was proved by Baker and Ericksen [Ba&E] that our hypothesis on W, with $\geq$ strengthened to $>$, implies the following inequalities:

$$[v_i \psi_{,i}(v_1,v_2,v_3) - v_j \psi_{,j}(v_1,v_2,v_3)]/(v_i - v_j) > 0 \text{ whenever } v_i \neq v_j \tag{10}$$

Taken together (9') and (10) imply $w_1=w_2=w_3=\delta^{1/3}$, as asserted. Note that this argument ensures the uniqueness of the minimizer (w_1,w_2,w_3).

To deal with the case in which W only satisfies the weakened Baker-Ericksen inequalities where > 0 is replaced by ≥ 0, consider for each $\varepsilon > 0$ the function W_ε defined by

$$W_\varepsilon(F) = W(F) + \varepsilon|F|, \quad \forall F \in \mathrm{lin}^+. \tag{11}$$

It is well known that the norm term in (11) is a strictly convex function on $\mathrm{lin}(\mathbb{R}^3)$. Consequently by the argument in the preceding paragraph the infimum of the function $\psi_\varepsilon = \psi + \varepsilon(v_1^2 + v_2^2 + v_3^2)^{1/2}$ on C_δ is attained at the (unique) minimizer $(\delta^{1/3},\delta^{1/3},\delta^{1/3})$ for each $\varepsilon > 0$. In view of (*) it follows that the infimum of ψ (i.e. W) is attained there, as well.

Finally, we shall remove the condition that $\psi \in C^1$. Let ρ be a C^∞ nonnegative function on lin vanishing outside the unit ball $B_1(0) = \{F \in \mathrm{lin} \mid |F| < 1\}$ and satisfying (+)$\int \rho \, dm = 1$, where m denotes

Lebesgue measure, i.e., the translation invariant Haar measure on lin, normalized so that the set of those tensors F for which the standard matrix consists solely of entries $a_{ij} \in [0,1]$ has measure unity. For example, $\rho(F) = c\exp 1/(1-|F|^2)$, $|F|<1$, $\rho(F)=0$ otherwise, with c chosen so that (+) holds, is of this type. Now it follows from (*) that for some $\eta=\eta(\delta) > 0$ sufficiently small the infimum of W on C_δ is attained only at points (v_1,v_2,v_3) with all $v_i \in [\eta, 1/\eta]$. For each h>0 sufficiently small let the *regularized* function W^h be defined by

$$W^h(F) = \psi^h(v_1,v_2,v_3)= h^{-9}\int\rho((F-G)/h)W(G)\ dm$$

for all $F\in \text{lin}^+$ satisfying $|F| \in [\eta,1/\eta]$. Then each ψ^h is clearly a C^∞ rank one function and ψ^h converges to ψ uniformly on C_δ as $h\to 0$. By our previous reasoning, each ψ^h attains its minimum at the point $(\delta^{1/3},\delta^{1/3},\delta^{1/3})$ whence the same is true for ψ, as well. This completes the proof.

Remark: I am indebted to John Ball for supplying the above much improved replacement for an argument I had initially developed in the case of polyconvexity.

COROLLARY Suppose that the isotropic stored-energy function $W\in C^1(\text{lin}^+)$ satisfies the following growth condition

(*) $W(F) \to\infty$ as $|F|\to\infty$ or det $F\to 0+$,

and is polyconvex. Then the dilatation u_δ satisfying $\det(\text{grad } u_\delta) = \delta$ has the least energy of all affine displacements with gradient having determinant δ:

$$W(\delta^{1/3}I) \le W(F),\quad \forall F\in \text{lin}^+ \text{ s.t. } \det F= \delta,\quad \forall\ \delta\in(0,\infty).$$

That is, the function $\psi\in C((0,\infty)^3)$ appearing in (2) satisfies for each $\delta\in(0,\infty)$:

$$\psi(\delta^{1/3},\delta^{1/3},\delta^{1/3}) \le \psi(v_1,v_2,v_3),\ \forall\ v_i\in(0,\infty) \text{ s.t. } v_1v_2v_3=\delta. \quad (8)$$

This result follows from the fact that a polyconvex function is necessarily rank 1 convex and hence satisfies the weakened Baker-Ericksen inequalities [B3].

§4. It will be shown next that, in general, property (8) leads to fracture-like features for certain equilibria of a hyperelastic material, as was mentioned in the Introduction. In view of (8) it will be convenient to introduce the notation

$A(\delta) := W(\delta^{1/3}I) = \psi(\delta^{1/3}, \delta^{1/3}, \delta^{1/3}), \quad \forall \delta \in (0,\infty)$. In general the function A need not be everywhere convex; we will describe below even polyconvex materials for which convexity of A fails. However by (*) A does tend to ∞ as $\delta \to 0$ or $\delta \to \infty$. Thus there are in general one or more intervals $[\delta_0,\delta_1]$ on which the graph of A lies above the chord joining $(\delta_0,A(\delta_0))$ and $(\delta_1,A(\delta_1))$. Given a point $\delta^* \in (\delta_0,\delta_1)$ for one such (maximal) interval assume for definiteness that $A(\delta_1) \geq A(\delta_0)$ [otherwise interchange δ_0 and δ_1]. We seek an equilibrium solution for the following problem: Take as reference configuration Ω the closed unit ball of $\mathbb{R}^3$ with the requirement that Ω consist of the homogeneous and isotropic material under consideration, and require that the deformed configuration $u(\Omega)$ be such that its volume is δ^* times the volume of the unit ball, i.e. the average density in the deformed configuration is $1/\delta^*$ times the density in the reference configuration. Now there is a unique $\vartheta \in (0,1)$ such that $\delta^* = \vartheta\delta_0 + (1-\vartheta)\delta_1$. This leads to an idealized "equilibrium solution" which possesses a crack (cf. [N]) in the form of a spherical shell in the deformed configuration. Namely,

$$u(x) = \delta_0^{1/3}x, \quad |x| \in [0,\vartheta^{1/3}), \qquad (13)$$
$$= \delta_1^{1/3}x, \quad |x| \in (\vartheta^{1/3},1],$$

so that the deformed configuration consists of the ball of radius $r_0 = (\delta_0\vartheta)^{1/3}$ together with the spherical shell of inner radius $r_1 = (\delta_1\vartheta)^{1/3}$ and outer radius $r_2 = (\delta_1)^{1/3}$. It is easily seen by an elementary comparison argument that this deformation u does indeed supply the smallest stored energy consistent with the conditions of the problem. On the other hand, when the material is a fluid so that W is a function of det F alone, one can obtain approximate equilibrium solutions <u>without</u> fracture [for which the deformed configuration $u_h(\Omega)$ is not a ball] as follows. Given a positive integer h, consider first the (discontinuous) deformation v_h of Ω defined by $v_h(-x) = -v_h(x)$, with v_h defined on the hemisphere $\Omega^+ = \{x \in \Omega | \ x_1 \geq 0\}$ by

$$v_h(x) = \delta_o^{1/3}(x-y_{i-1}) + v_h(y_{i-1}),\ \ x_1 \in [r_{i-1}, s_{i-1})$$
$$= \delta_1^{1/3}(x-z_{i-1}) + v_h(z_{i-1}),\ \ x_1 \in [s_{i-1}, r_i)$$

where $r_j = j/h, y_j = (r_j,0,0)$ and $z_j = (s_j,0,0)$ with $s_j \in (r_j, r_{j+1})$ determined by the condition that $\mathrm{vol}(\Omega \cap \{x|x_1 \in (r_j,s_j)\}) = \vartheta\,\mathrm{vol}(\Omega \cap \{x|x_1 \in (r_j,r_{j+1})\})$. Note that $v_h(\Omega)$ is a connected set. Setting $C'_j := \Omega \cap \{x|x_1 \in (r_j,s_j)\}, C''_j := \Omega \cap \{x|x_1 \in (s_j,r_{j+1})\}$ it is readily seen that $\mathrm{grad}v_h$ is a dilatation with determinant δ_o in C'_j and with determinant δ_1 in C''_j, $\forall\ j \in \{0,n-1\}$. Next we construct a continuous deformation $u_{h,\eta}$ which approximates v_h appropriately, and we show that $E[u_{h,\eta}] \to E[v_h] = \vartheta W(\delta_o^{1/3}I) + (1-\vartheta)W(\delta_1^{1/3}I) = E[u]$ as $\eta \to 0$. Let $\eta = \eta_h << 1/h$ denote a number to be specified later and consider the function u_h defined as follows:

$$u_{h,\eta}(x) = v_h(x)\ , \qquad x \in B_j := \{x \in \Omega |\ x_1 \in [r_j+\eta, s_j-\eta] \text{ or } x_1 \in [s_j+\eta, r_{j+1}-\eta]\}$$
$$= \lambda_{h,j}(x_1)(x-z'_j) + u_h(z'_j),\quad x \in D_j := \{x \in \Omega |\ x_1 \in (r_j-\eta, r_j+\eta)\}$$
$$= \lambda^*_{h,j}(x_1)(x-z^*_j) + u_h(z^*_j),\ x \in D^*_j := \{x \in \Omega |\ x_1 \in (s_j-\eta, s_j+\eta)\}, 1 \le j \le h,$$

where $z'_j = (r_j-\eta,\ 0,0)$, $z^*_j = (s_j-\eta,\ 0,0)$. Here $\lambda_{h,j}$, and $\lambda^*_{h,j}$ denote smooth strictly monotone functions satisfying

$$\lambda_{h,j}(r_j+\eta) = \lambda^*_{h,j}(s_j-\eta) = \delta_o^{1/3},\ \ \lambda_{h,j}(r_j-\eta) = \lambda^*_{h,j}(s_j+\eta) = \delta_1^{1/3} \qquad (14a)$$

as well as the constraints

$$\mathrm{vol}u_{h,\eta}(D_j) = \int_{D_j}[(\lambda_{h,j})^3(x_1) + (x_1-r_j+\eta)(\lambda_{h,j})^2(x_1)(\lambda_{h,j})'(x_1)]$$
$$= \mathrm{vol}v_h(D_j),$$

(14b)

$$\mathrm{vol}u_{h,\eta}(D^*_j) = \int_{D_j}[(\lambda^*_{h,j})^3(x_1) + (x_1-r_j+\eta)(\lambda^*_{h,j})^2(x_1)(\lambda^*_{h,j})'(x_1)]$$
$$= \mathrm{vol}v_h(D^*_j).$$

The above integrands occur because for a deformation of the form $Y(x) = \lambda(x_1)(x-Z) + Z^*$, $\mathrm{grad}Y(x) = \lambda(x_1)I + (x-Z) \otimes \lambda'(x_1)\mathbf{e}_1$ whence

$$\det[\mathrm{grad}Y](x) = (\lambda(x_1))^3 + (x_1-Z_1)(\lambda(x_1))^2\lambda'(x_1).$$

It is readily seen by integration by parts in x_1 that the integrals in (14b) do tend to zero with rate 0(1) as $\eta/h \to 0$. Consequently by our assumption $W[u_{h,\eta}] \to W[v_h]=\vartheta W(\delta_0{}^{1/3}I)+(1-\vartheta)W(\delta_1{}^{1/3}I)$ whenever $\eta_h/h \to 0$.

To summarize, the isoperimetric problem [$⨍$ denotes the integral mean]:

$$\int_\Omega W(\text{grad}u)dx \to \inf \qquad \text{s.t. } ⨍_\Omega \det(\text{grad}u)dx = \delta^* \tag{15}$$

has infimum $\vartheta W(\delta_0{}^{1/3}I)+(1-\vartheta)W(\delta_1{}^{1/3}I)$, but solutions corresponding to this value consist of discontinuous [BV] deformations. On the other hand, if W is a function of det F alone or is separately sublinear at ∞ [$\limsup_{v_i \to +\infty} \psi/v_i < \infty$] there are *continuous* deformations with energy arbitrarily close to the infimum. We state matters more precisely as:

THEOREM B Consider the isoperimetric problem (14) with $\delta^* = \vartheta\delta_0+(1-\vartheta)\delta_1$, $\vartheta \in (0,1)$, where $[\delta_0,\delta_1]$ is a maximal interval in which the graph of A lies above its chord. Then there is no minimizer for (15) in the class of continuous fields $v \in W^{1,1}{}_{loc}(\Omega;\mathbb{R}^3)$. However if the material is a fluid or is separately sublinear at ∞, the infimum in (15) is that attained by the idealized fractured deformation (13): $E_{inf} = \vartheta A(\delta_0)+(1-\vartheta)A(\delta_1)$.

<u>Proof:</u> We observe that for any field v satisfying the given isoperimetric constraint in (15) one has by use of (8):

$$E[v]=\int_\Omega W(\text{grad}v(x))dx \geq \int_\Omega A(\delta(x))dx = \int_{(0,\infty)} A(\delta)d\omega(\delta), \tag{16a}$$

where $\delta(x) = \det(\text{grad}v(x))$ and ω denotes the distribution function for $\delta(\cdot)= \delta_v$. Correspondingly, the constraint can be written as

$$\int_{(0,\infty)} \delta d\omega(\delta) = \delta^*. \tag{16b}$$

It is easily verified that any minimizer ω for the last integral in (16a) subject to (16b) must be supported on $[\delta_0,\delta_1]$, so that the set of x for which $\delta(x) \in [\delta_0,\delta_1]$ must be of full measure. Furthermore by an elementary application of Jensen's inequality,

$$\inf \int_{(0,\infty)} A(\delta)d\omega(\delta) = \vartheta A(\delta_0)+(1-\vartheta)A(\delta_1) \tag{17}$$

and is attained only for measures ω supported by points $\delta \in [\delta_0,\delta_1]$ at which the graph of A coincides with the graph of its convex hull A^{**}, while the latter includes the line segment joining $(\delta_0,A(\delta_0))$ to $(\delta_1,A(\delta_1))$. This completes the argument.

REMARK C Although the deformation (13) involving a single fracture is the *simplest* deformation giving the minimal energy E_{inf}, it is readily seen that for every partition of Ω into regular subregions $\{\Omega_j\}$ there is a deformation of energy E_{inf}, with image in the ball of radius $\delta_1{}^{1/3}$, consisting of regions $u(\Omega_j)$ each of which involves a fractured portion of the dilation of Ω_j by $\delta_1{}^{1/3}I$ analogous to that produced by (13). Thus there are minimizers involving fractures throughout the material, giving an alternative meaning to "ubiquity".

REMARK D Replacing the isoperimetric problem (15) by the Dirichlet type problem

$$\int_\Omega W(\mathrm{grad}u)dx \to \inf \quad \text{s.t. } u(x) = \delta_1{}^{1/3} x, \ \forall x \in \partial\Omega, \tag{18}$$

for continuous deformations $u \in W^{1,p}{}_{loc}(\Omega;\mathbb{R}^3)$, p>3, leads to the following observation. In contrast to (15b) all such deformations satisfy the isoperimetric constraint

$$⨍_\Omega \det(\mathrm{grad}u)dx = \delta_1, \tag{19}$$

(cf.[M&M]). The Jensen inequality again shows that the deformation in (13) gives a stored energy E[u] strictly dominated by the infimum value in (17), but the "fractured" deformation in question belongs to the larger space $BV(\Omega;\mathbb{R}^3)$ rather than $W^{1,p}{}_{loc}(\Omega;\mathbb{R}^3)$, so that we have here a *gap* somewhat analogous to those encountered in the context of the **Lavrentiev phenomenon** [B&M],[H&M],[Bu&M1],[Bu&M2].

We now provide an example of a polyconvex stored energy function for which the hypotheses of Theorem A apply. Namely, consider the function $W(F) = (\det F)^{-a} + (\det F)^b + k\,\mathrm{tr}(\mathrm{adj}F)$, $a\geq 1, b\geq 1$ so that $A(\delta) = \psi(\delta^{1/3},\delta^{1/3},\delta^{1/3}) = \delta^{-a} + \delta^b + 3k\delta^{2/3}$. It is easily verified that if k>0 is sufficiently large then the above stored energy function is of the desired type. For b=1, Theorem B's 2nd result holds.
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