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We construct an example confirming that the interfacial energy density of the Γ-limit of a family of nonconvex functionals, cannot be computed, in general, by assuming that the local behavior of a sequence of vector-valued minimizers near the interface is unidirectional.

1. Introduction.

Consider the nonconvex energy functional

$$E[u] := \int_\Omega W(u(x))\,dx\,, \tag{1}$$

where $\Omega \in \mathbf{R}^N$ is an open, bounded, and strongly Lipschitz domain, $u : \Omega \to \mathbf{R}^p$, and W supports two phases. Depending on the constraints placed on u, in general there are more than one solution of the minimization problem for (1). In order to identify a selection criterion for resolving this non-uniqueness, one can study the properties of the limits of minimizers for the family of perturbed and rescaled energies

$$F_\epsilon[u] \;:= \frac{1}{\epsilon}\int_\Omega W(u)\,dx + \epsilon\int_\Omega h^2(x,\nabla u)\,dx\,. \tag{2}$$

The relevant type of convergence in this context is the $\Gamma -$ convergence, as introduced by De Giorgi [6] (see also [1] or [5]).

The characterization of the Γ-limit of the sequence F_ϵ was studied under the assumption that $h = ||\cdot||$ in the scalar-valued case by Modica ([10]), and in the vector-valued case by Baldo ([2]), Fonseca and Tartar ([8]), Kohn and Sternberg ([9]), and Sternberg ([12]). In the former case it was found that $\{F_\epsilon\}$ $\Gamma-$ converges to the functional given by

$$F_0[u] := \begin{cases} \overline{K}\, Per_\Omega\{u=a\}\,, & \textit{if } u \in \{a,b\} \textit{ a.e.},\\ +\infty\,, & \textit{otherwise}, \end{cases} \tag{3}$$

where

$$\overline{K} := 2\inf\left\{\int_{-1}^{1}\sqrt{W(g(s))}|g'(s)|\,ds \;:\; g \text{ is} \right.$$
$$\left. \text{piecewise } C^1,\; g(-1)=a,\; g(1)=b\right\}. \tag{4}$$

Here $Per_\Omega\{u=a\}$ denotes the perimeter of A in Ω (see, for example, [7] for the definition). Notice that the interfacial energy density $\overline{K}$ is constant and is defined as an infimum of the integral in (4) over the curves connecting the points a and b.

A more general choice of h was considered in the vector-valued case by Barroso and Fonseca in [3]. They found that the $\Gamma(L^1(\Omega))$-limit of the family of functionals in (2) is given by

$$F_0[u] := \begin{cases} \int_{\Omega\cap\partial^*\{u=a\}} K(x,a,b,\nu)\,dH^{N-1}(x)\,, & \textit{if } u \in V_{a,b}\,,\\ +\infty\,, & \textit{otherwise}. \end{cases} \tag{5}$$

Here W satisfies a certain growth condition and attains its minimum value of zero at exactly two points a and b, while h grows at most linearly in the last argument and satisfies some technical continuity conditions. The symbol H^{N-1} denotes the $N-1-$ dimensional Hausdorff measure, the set $V_{a,b}$ is defined by

$$V_{a,b} := \{f \in BV(\Omega)\,|\, f(x) \in \{a,b\} \textit{ a.e. in } \Omega\}\,,$$

the vector $\nu(x)$ is normal to the interface $\Omega \cap \partial^*\{u = a\}$ at the point x, where $\partial^*\{u = a\}$ is the reduced boundary of the set $\{u = a\}$ (see, for example, [7] for the definition). In addition

$$K(x, a, b, \nu(x)) := \inf_{\substack{\xi \in A(a,b,\nu(x)) \\ L>0}} \Phi_\nu[\xi, L], \tag{6}$$

$$\Phi_\nu[\xi, L] := \int_{Q_\nu} [LW(\xi(y)) + \tfrac{1}{L}(h^\infty(x, \nabla\xi(y))^2] \, dy, \tag{7}$$

$$\begin{aligned} A(a, b, \nu) := \{\xi \in H^1(Q_\nu; \mathbf{R}^p) \ : \ \xi(y) = a \text{ if } y \cdot \nu = -\tfrac{1}{2}, \\ \xi(y) = b \text{ if } y \cdot \nu = \tfrac{1}{2}, \text{ and } \xi \text{ is periodic} \\ \text{with period one in the directions of } \nu_1, \ldots, \nu_{N-1}\}. \end{aligned} \tag{8}$$

Here, the vectors $\nu_1, \ldots, \nu_{N-1}, \nu$ form an orthonormal basis of $\mathbf{R}^N$, the open unit cube Q_ν is centered at the origin with two of its faces normal to ν and the *recession function* h^∞ is given by

$$h^\infty(x, B) := \limsup_{t\to\infty} \frac{h(x, tB)}{t}. \tag{9}$$

Remark: Following the discussion at the beginning of this section, we note that the requirement of minimality of

$$\int_{\Omega\cap\partial^*\{u=a\}} K(x, a, b, \nu(x)) \, dH^{N-1}(x) \tag{10}$$

represents the selection criterion for resolving the possible non-uniqueness of the minimizers of (1).

We conclude this introduction by mentioning that the scalar versions of the problem considered in [3] were studied by Bouchitté ([4]) and Owen and Sternberg ([11]).

From the definition of the interfacial energy density K one can see that it is determined by the local structure of the sequence of minimizers of (2) near the interface $\Omega \cap \partial^*\{u = a\}$.

In particular, in the scalar-valued case the methods of convex analysis allow one to conclude that for a small ϵ the minimizers of (2) are essentially locally constant along the boundary of the set $\Omega \cap \partial^* \{u = a\}$ (see, for example, [4]). This implies that the functions ξ in (8) can be taken as depending on a single variable in the direction of the vector ν. However, the same property might not hold, in general, in a vector-valued case, as the local behavior of minimizers near the interface can be more complex.

This conclusion is confirmed in this note as we provide an example of functions W and h in (2) for which the functions ξ in (8) cannot be restricted to changing only in the direction normal to the boundary. The example is based on studying the behavior of the minimizers for the "blown-up" problem (6).

2. Main Results.

From now on, we will suppose for simplicity that $(x_1, x_2) \in \Omega \subset \mathbf{R}^2$ and that $\mathbf{u} : \Omega \to \mathbf{R}^2$. In this section we will use boldface letters to represent both vectors in $\mathbf{R}^2$ and $\mathbf{R}^2-$ valued functions. Assume that the function h in (2) is independent of $\mathbf{x}$, positively homogeneous of degree one, convex, and satisfies the coercivity condition

$$\frac{h(p)}{||p||} \geq C, \tag{11}$$

where $C > 0$ is constant. Also suppose that the function W in (2) has a superlinear growth, while its minimum is equal to 0 and is achieved at exactly two vectors, $\mathbf{a}$ and $\mathbf{b}$. Observe that since h is positively homogeneous of degree one, it is equal to its own recession function, that is

$$h^\infty(p) = h(p),$$

for every $p \in M^{2\times2}$.

Let

$$h(\nabla \mathbf{u}) = |div\, \mathbf{u}| \ , \tag{12}$$

and

$$W(\mathbf{u}) = (1 - u_2^2)^2. \tag{13}$$

It is trivial to verify that both W and h satisfy only some of the restrictions imposed in the previous paragraph. In particular, W attains its minimum value on the set $\{(u_1, u_2) \in \mathbf{R}^2 \,|\, u_2 = \pm 1\}$ and h, while being convex and positively homogeneous of degree one, does not satisfy (11). Later we will, however, use the small perturbations of W and h in order to extend our results to the required classes of functions.

Suppose that $\mathbf{i}$ is the unit vector in the direction of x_1-axis and $\mathbf{j}$ is the unit vector in the direction of x_2-axis. Let

$$Q_{\mathbf{j}} =: \left\{(x_1, x_2) \in \mathbf{R}^2 \,|\, -\frac{1}{2} \le x_1, x_2 \le \frac{1}{2}\right\} ,$$

while

$$I := \left\{x_1 \in \mathbf{R} \,|\, |x_1| \le \frac{1}{2}\right\} ,$$

and

$$J := \left\{x_2 \in \mathbf{R} \,|\, |x_2| \le \frac{1}{2}\right\} .$$

Define

$$\begin{aligned} \overline{A}(\mathbf{a}, \mathbf{b}, \nu) \ := \{ \zeta \ : \ Q_{\mathbf{j}} \to \mathbf{R}^2 \,|\, \exists \xi \in A(\mathbf{a}, \mathbf{b}, \nu) \ \textit{such that} \\ \zeta(x_1, x_2) = \xi(x_1, x_2) \ \textit{a.e. in}\ \Omega \} \ , \end{aligned} \tag{14}$$

to be the set of restrictions to $Q_{\mathbf{j}}$ of functions from $A(\mathbf{a}, \mathbf{b}, \nu)$ (see (8)).

Fix a function $u_2 : Q_{\mathbf{j}} \to \mathbf{R}$ such that $u_2(x_1, \cdot) \in C^2(J)$ for every $x_1 \in I$, while $<0, u_2> \in \overline{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$, and

$$\frac{\partial u_2}{\partial x_2}\left(x_1, \pm\frac{1}{2}\right) \equiv 0 \, . \tag{15}$$

Now consider any $u_1 : Q_{\mathbf{j}} \to \mathbf{R}$ that satisfies

$$\frac{\partial u_1}{\partial x_1}(x_1, x_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_2}{\partial x_2}(s, x_2)\, ds - \frac{\partial u_2}{\partial x_2}(x_1, x_2), \tag{16}$$

$$u_1\left(x_1, \pm\frac{1}{2}\right) \equiv 0. \tag{17}$$

It is easy to verify that u_1 is a local minimizer of the functional

$$\Pi_{u_2}[w] := \int_{Q_{\mathbf{j}}} \left(\frac{\partial w}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 dx\,,$$

where $< w, u_2(0,\cdot) > \in C^1(Q_{\mathbf{j}}; \mathbf{R}^2) \cap \overline{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$. By evaluating Π_{u_2} at u_1 , we obtain that

$$\Pi_{u_2}[u_1] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_2}{\partial x_2}\, dx_1\right)^2 dx_2\,.$$

Choose $\phi \in C^2(\mathbf{R})$ to be monotone increasing and such that $\phi(x) \cdot sgn(x) = 1$, whenever $|x| \geq \frac{1}{2}$. For every $x \in Q_{\mathbf{j}}$ and every small $\epsilon > 0$ set

$$\overline{u}_2(x_1, x_2) = \phi\left(\frac{2x_2 - |x_1|}{\epsilon}\right), \tag{18}$$

$$\overline{u}_1(x_1, x_2) = x_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \overline{u}_2}{\partial x_2}(s, x_2)\, ds - \int_0^{x_1} \frac{\partial \overline{u}_2}{\partial x_2}(s, x_2) ds\,. \tag{19}$$

Then the function $\overline{\mathbf{u}} =< \overline{u}_1, \overline{u}_2 >$ satisfies the conditions (15-17) and $\overline{\mathbf{u}} \in \overline{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$. Hence we can extend $\overline{\mathbf{u}}$ periodically with period one in the direction of the x_1- axis to obtain $\overline{\mathbf{U}} \in A(-\mathbf{j}, \mathbf{j}, \mathbf{j})$.

One can calculate both $\overline{u}_1$ and $div\ \overline{\mathbf{u}}$ explicitly in terms of ϕ. In particular, for every $x \in Q_{\mathbf{j}}$ we have

$$\begin{aligned}\overline{u}_1(x_1, x_2) = 4\, x_1 &\left(\phi\left(\frac{2x_2}{\epsilon}\right) - \phi\left(\frac{2x_2}{\epsilon} - \frac{1}{2\epsilon}\right)\right) \\ - 2\, sgn(x_1) &\left(\phi\left(\frac{2x_2}{\epsilon}\right) - \phi\left(\frac{2x_2 - |x_1|}{\epsilon}\right)\right),\end{aligned}$$

$$div\ \overline{\mathbf{u}} = 4\left(\phi\left(\frac{2x_2}{\epsilon}\right) - \phi\left(\frac{2x_2}{\epsilon} - \frac{1}{2\epsilon}\right)\right).$$

Then, by definition of ϕ,

$$|div\ \overline{\mathbf{u}}| \leq 8\,. \tag{20}$$

Next, define

$$\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) = \inf_{\substack{\xi\in\mathcal{A}(-\mathbf{j},\mathbf{j},\mathbf{j})\\ L>0}} \Phi_{\mathbf{j}}[\xi, L]\,, \tag{21}$$

where $\Phi_{\mathbf{j}}$ is as defined in (7) and

$$\mathcal{A}(\mathbf{a},\mathbf{b},\nu) := \{\xi \in A(\mathbf{a},\mathbf{b},\nu) \mid \nabla\xi \times \nu = \mathbf{0}\ a.e.\ in\ Q_{\mathbf{j}}\}\,,$$

consists of those functions in $A(\mathbf{a},\mathbf{b},\nu)$ that remain constant in any direction, perpendicular to the vector ν. Then, following [8] one can find that

$$\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) = 2\int_{-1}^{1}\sqrt{W(s)}\,ds\,,$$

which implies that $\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) = \frac{8}{3}$ for our choice of W.

Proposition 1: *The inequality*

$$\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) > K(-\mathbf{j},\mathbf{j},\mathbf{j}) \tag{22}$$

holds for the functions h and W defined in (12) and (13).

Proof: In order to prove (22) we will use the following lower bound in L on $\Phi_{\mathbf{j}}\left[\overline{\mathbf{U}}, L\right]$:

$$\begin{aligned} K(-\mathbf{j},\mathbf{j},\mathbf{j}) &\leq \inf_{L>0}\Phi_{\mathbf{j}}[\overline{\mathbf{U}}, L] \\ &= \Lambda[\overline{\mathbf{u}}] := 2\left(\int_{Q_{\mathbf{j}}} W(\overline{\mathbf{u}})\,dx\right)^{1/2}\left(\int_{Q_{\mathbf{j}}} h^2(\nabla\overline{\mathbf{u}})\,dx\right)^{1/2}, \end{aligned} \tag{23}$$

and verify that $\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) > \Lambda[\overline{\mathbf{u}}]$ for a particular value of the parameter ϵ in (18). Indeed, by (20)

$$\begin{aligned}\Lambda[\overline{\mathbf{u}}] &= 2\left(\int_{Q_\mathbf{j}} W(\overline{u}_2)\,dx\right)^{1/2}\left(\int_{Q_\mathbf{j}} (div\ \overline{\mathbf{u}})^2\,dx\right)^{1/2} \\ &\leq 16\left(\int_{Q_\mathbf{j}} W(\overline{u}_2)\,dx\right)^{1/2}. \end{aligned} \tag{24}$$

Then, by the definition of W and ϕ, the right-hand side of (24) can be made arbitrarily small by chosing $\epsilon > 0$ small enough in the definition (18) of $\overline{u}_2$. Hence there exists a $\overline{\mathbf{u}}$ such that

$$K(-\mathbf{j},\mathbf{j},\mathbf{j}) \leq \Lambda[\overline{\mathbf{u}}] < \frac{8}{3} = \kappa(-\mathbf{j},\mathbf{j},\mathbf{j})\,.$$

■

This example can be extended to a more general choice of the functions W and h. In particular, by considering

$$\begin{aligned} W(\mathbf{u}) &:= \alpha u_1^4 + (1-u_2^2)^2, \\ h(\nabla\mathbf{u}) &:= |div\,\mathbf{u}| + \beta|\nabla\mathbf{u}|, \end{aligned}$$

for small $\alpha > 0$ and $\beta > 0$, we obtain W and h that satisfy the requirements imposed at the beginning of this section. That is, W attains its minimum at exactly two vectors $\mathbf{a}$ and $\mathbf{b}$ and has a superlinear growth, while h is convex, positively homogeneous of degree one, and coercive. The inequality (22) can be proven for this new set of functions W and h by using Proposition 1, the continuity of $\Lambda[\overline{\mathbf{u}}]$ with respect to α and β, the boundedness of the function $\overline{\mathbf{u}}$, and the easily verifiable fact that $\kappa(-\mathbf{j},\mathbf{j},\mathbf{j}) = \frac{8}{3}(1+\beta)$. Here Λ, $\overline{\mathbf{u}}$, and κ are as defined in (23), (18-19), and (21).

Therefore, when the family of functionals in (2) is defined over a set of the vector-valued functions, the interfacial energy density K cannot be computed, in general, by restricting the set A in (8) to its proper subset $\mathcal{A}$ of the functions that are independent of x_1-variable. In other words, unlike in the scalar-valued case, it is not, in general, optimal for an element of a sequence of the vector-valued minimizers of (2) to be locally constant along the interface $\Omega \cap \partial^*\{\mathbf{u} = \mathbf{a}\}$ (The isotropic case (3-4) can be considered here as one of the exceptions). Hence, we conclude that the local behavior of such vector-valued minimizers near the interface can be significantly more complex when compared to the behavior of the minimizers that are scalar-valued.

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References.

1. Attouch, H.: Variational convergence for functions and operators. Boston, Mass.-London: Pitman 1984
2. Baldo, S.: Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. Ann. Inst. H. Poincaré - Analyse Non Linéaire. **7**, 37-65 (1990)
3. Barroso, A. and Fonseca, I.: Anisotropic singular perturbations - the vectorial case. Proc. Royal Soc. Edin. **124**, 527-571 (1994)
4. Bouchitté, G.: Singular perturbations of variational problems arising from a two-phase transition model. Appl. Math. Optim. **21**, 289-314 (1990)
5. Dal Maso, G.: An introduction to Γ-Convergence. Prog – ress in nonlinear differential equations and their applications. Boston Basel Stuttgart: Birkhäuser 1993

6. De Giorgi, E.: Sulla convergenza di alcune successioni d'integrali del tipo dell'area. Rend. Mat. **8**, 277-294 (1975)

7. Evans, L.C. and Gariepy, R.F.: Lecture notes on measure theory and fine properties of functions. Boca Raton, FL: CRC Press 1992

8. Fonseca, I. and Tartar, L.: The gradient theory of phase transitions for systems with two potential wells. Proc. Royal Soc. Edin. **111A**, 89-102 (1989)

9. Kohn, R. and Sternberg, P.: Local minimizers and singular perturbations. Proc. Royal Soc. Edin. **111A**, 69-84 (1989)

10. Modica, L.: Gradient theory of phase transitions and minimal interface criterion. Arch. Rat. Mech. Anal. **98**, 123-142 (1987)

11. Owen, N.C. and Sternberg, P.: Nonconvex variational problems with anisotropic perturbations. Nonlinear Anal. **16**, 705-719 (1991)

12. Sternberg, P.: The effect of a singular perturbation on nonconvex variational problems. Arch. Rat. Mech. Anal. **101**, 209-260 (1988)