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- A New Model -

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1. Introduction

In this note we consider a new model for one-dimensional phase transitions in elastic solids. The governing equations are

$$u_t - v_x = 0, \tag{1.1}$$

$$v_t - \sigma_x = 0, \tag{1.2}$$

$$\sigma = c^2 (u - D), \tag{1.3}$$

and

$$\epsilon^2 \left(D_{tt} - \lambda^2 D_{xx}\right) + \epsilon D_t = \frac{\beta\left(\sigma - \mu c^2 D\left(D^2 - 1\right)\right)}{c^2} \tag{1.4}$$

where

$$0 < c, \lambda, \beta, \text{ and } \epsilon \text{ and } 0 < \mu \leq 1. \tag{1.5}$$

Here, u may be thought of as the strain, v the velocity, and σ the stress at a material point x at time t and D is a dimensionless order parameter which describes the phase of the solid at x at time t.[1] Equation (1.1) is merely the statement that u and v are derivable from a motion, ϕ, via

$$v = \phi_t \text{ and } u = \phi_x$$

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[1] u like D, is dimensionless; v has dimensions (length)/(time) as do the parameters c and λ; σ has dimensions $(\text{length})^2/(\text{time})^2$; β and μ are dimensionless; and finally ϵ has dimensions (time).

and (1.2) is the balance of momentum.

In the formal limit where $\epsilon \to 0^+$ (and the left hand side of (1.4) vanishes) the order parameter D completely characterizes u and σ through the limiting constitutive equations:

$$u = (1-\mu)\,D + \mu D^3 \quad \text{and} \quad \sigma = \mu c^2 D\left(D^2 - 1\right).^2 \tag{1.6}$$

The constraint $0 < \mu \leq 1$ guarantees that $(1.6)_1$ is monotone increasing and thus in this limit D and u are related in a 1-1 fashion. The separate phases of the material are the regions where $(1.6)_2$ is monotone increasing, here $D < -\frac{1}{\sqrt{3}}$ and $D > \frac{1}{\sqrt{3}}$, and the spinodal region is the interval $-\frac{1}{\sqrt{3}} < D < \frac{1}{\sqrt{3}}$ where $(1.6)_2$ is decreasing.

The model proposed is similar to the better studied Landau-Ginzburg model where (1.1) – (1.3) are maintained but (1.4) is replaced by the diffusion equation

$$\epsilon\, D_t - \lambda^2\epsilon^2 D_{xx} = \frac{\beta\left(\sigma - \mu c^2 D\left(D^2-1\right)\right)}{c^2}\ {}^3 \tag{1.7}$$

A principle difference between the Landau-Ginzburg model and ours is that the former propagates information about D at infinite speed whereas our model transmits such information at speeds $\pm\lambda$. In fact our model can be written as a simple first order transport process with all fields propagating information at finite speeds. To see this we first note that (1.4) may be written as the first order system

$$\left.\begin{aligned} D_t - Q_x &= \frac{\beta C}{\epsilon}, \\ Q_t - \lambda^2 D_x &= -\frac{Q}{\epsilon}, \\ \text{and} \qquad C_t &= \frac{\left(\sigma - \mu c^2 D\left(D^2-1\right) - c^2 C\right)}{\epsilon c^2} \end{aligned}\right\}. \tag{1.8}$$

[2] We shall see later that in the $\epsilon = 0^+$ limit the left hand side of (1.4) may give a non zero contribution to σ ; in fact, it is exactly in these situations that we see microstructure.

[3] For details see Caginalp[1-3].

Of course u, v and σ satisfy (1.1) – (1.3). If we now introduce $m_1 - m_4$ via

$$\left.\begin{aligned} u &= m_1 + m_2 + \frac{c^2}{c^2-\lambda^2}\left(m_3+m_4\right), \\ v &= c\left(m_1-m_2\right) + \frac{\lambda c^2}{c^2-\lambda^2}\left(m_3-m_4\right), \\ D &= m_3+m_4, \\ \text{and} \quad Q &= \lambda\left(m_3-m_4\right) \end{aligned}\right\} \tag{1.9}$$

or equivalently

$$\left.\begin{aligned} m_1 &= \frac{1}{2}\left(u + \frac{v}{c} - \frac{c^2}{c^2-\lambda^2}D - \frac{cQ}{c^2-\lambda^2}\right), \\ m_2 &= \frac{1}{2}\left(u - \frac{v}{c} - \frac{c^2 D}{c^2-\lambda^2} + \frac{cQ}{c^2-\lambda^2}\right), \\ m_3 &= \frac{1}{2}\left(D + \frac{Q}{\lambda}\right), \\ \text{and} \quad m_4 &= \frac{1}{2}\left(D - \frac{Q}{\lambda}\right) \end{aligned}\right\} \tag{1.10}$$

we find that if $u, v, \sigma, D, Q, \mathcal{C}$ satisfy (1.1) – (1.3) and (1.8), then $m_1 - m_4$ and $\mathcal{C}$ satisfy the transport equations

$$\left.\begin{aligned} m_{1t} - c m_{1x} &= \frac{\beta c^2 \mathcal{C}}{2\epsilon\left(c^2-\lambda^2\right)} + \frac{\lambda c\left(m_3-m_4\right)}{2\epsilon\left(c^2-\lambda^2\right)}, \\ m_{2t} + c m_{2x} &= -\frac{\beta c^2 \mathcal{C}}{2\epsilon\left(c^2-\lambda^2\right)} - \frac{\lambda c\left(m_3-m_4\right)}{2\epsilon\left(c^2-\lambda^2\right)}, \\ m_{3t} - \lambda m_{3x} &= \frac{\beta \mathcal{C}}{2\epsilon} - \frac{\left(m_3-m_4\right)}{2\epsilon}, \\ m_{4t} + \lambda m_{4x} &= \frac{\beta \mathcal{C}}{2\epsilon} + \frac{m_3-m_4}{2\epsilon}, \\ \text{and} \quad \mathcal{C}_t &= \frac{\sigma - \mu c^2 D\left(D^2-1\right) - c^2 \mathcal{C}}{\epsilon c^2} = \frac{\left(u - (1-\mu)D - \mu D^3 - \mathcal{C}\right)}{\epsilon} \end{aligned}\right\} \tag{1.11}$$

where u and D are given by $(1.9)_1$, and $(1.9)_3$ and $\sigma = c^2 (u - D)$. We will actually use (1.11) when carrying out numerical simulations, for details see section 4.

Another difference between our model and the Landau-Ginzburg formulation is we find that for $0 < c \leq \lambda$ our system is capable of supporting both dynamic phase transitions and strong shocks in either phase of arbitrary strength whereas if $0 < \lambda < c$ not all single phase strong shocks are possible. This distinction is lost in the Landau-Ginzburg formulation. For details see section 2.

The model also has some features in common with the viscosity-capillarity models proposed and studied by Slemrod, [4-6]. These models focus on the system (1.1) and (1.2) when the stress σ is given by the constitutive equation

$$\sigma = \sigma_E (u) + \epsilon v_x - \epsilon^2 \lambda^2 u_{xx}$$

and $\sigma_E (u)$ is the composite function defined by the relations (1.6). Such a model could be derived formally from ours via a Chapman-Enskog expansion in ϵ. This model, like the Landau-Ginzburg, is not hyperbolic, and it also has the unpleasant feature that when considering boundary inputs one is forced to prescribe not only state variables like u, σ, v, D, or Q, but their derivatives.

The organization of the paper is as follows. In section 2 we examine the possible equilibrium solutions of (1.1) – (1.4). Such solutions include ones exhibiting phase transitions and microstructure; the latter being oscillatory solutions to the equilibrium equations. In this section we also explore travelling wave solutions to (1.1) – (1.4). These come in two flavors. The first represent dynamic phase transitions. As $\epsilon \to 0^+$ these converge to piecewise constant functions which satisfy the reduced system (1.1) and (1.2) with u and σ given by (1.6) and they have the additional property that the state behind the wave, typically labeled with a minus subscript, is in one phase (say $D_- < -\frac{1}{\sqrt{3}}$) and the state ahead of the wave, typically labeled with a plus subscript, is in the other phase (say $D_+ > \frac{1}{\sqrt{3}}$). These limit solutions are undercompressive shocks of the reduced system (1.1), (1.2), and (1.6). Specifically, when s is positive, the positive wave speeds, $p_\pm$, of the reduced system based on the states ahead and behind the shock

$$p_\pm := \frac{c\mu^{1/2}\sqrt{3D_\pm^2 - 1}}{\sqrt{1 - \mu + 3\mu D_\pm^2}} \tag{1.12}$$

satisfy $p_+ > s > 0$ and $p_- > s > 0$, whereas when s is negative, the negative wave speeds, $n_\pm$, of the reduced system based on the states ahead and behind the shock

$$n_\pm := \frac{-c\mu^{1/2}\sqrt{3D_\pm^2 - 1}}{\sqrt{1 - \mu + 3\mu D_\pm^2}} \tag{1.13}$$

satisfy $n_+ < s < 0$ and $n_- < s < 0$. There are also travelling wave solutions which represent single phase shocks. These profiles are not necessarily monotone, this latter property depends on the size

of β relative to the strength of the shock. These solutions exist so long as $|s| < \lambda$ and this condition is guaranteed when $c \leq \lambda$. In the limit $\epsilon \to 0^+$, these travelling waves converge piecewise constant solutions of the reduced system (1.1), (1.2), and (1.6) and they have the additional property that the states on either side of the shock lie in a single phase; that is either D_- and D_+ are both less than $-\frac{1}{\sqrt{3}}$ or both greater than $\frac{1}{\sqrt{3}}$. These limit shocks also satisfy the Lax entropy or shock conditions, that is:
when $s > 0$

$$p_- > s > p_+, \tag{1.14}$$

and when $s < 0$

$$n_+ < s < n_-. \tag{1.15}$$

Here $p_\pm$ and $n_\pm$ are the positive and negative wave speeds defined in (1.12) and (1.13).

In section 3 we develop a set of "energy" identities for (1.1) – (1.4). These are insensitive to the relative size of c and λ. We use these estimates to obtain long time information about solutions when $\epsilon > 0$ is fixed and either conservative or mildly dissipative boundary conditions are imposed on the boundaries of a fixed domain which we take to be the interval $0 < x < 1$. These identities are not sufficient to yield convergence Theorems when ϵ tends to zero for general initial and boundary data.

In section 4 we present some numerical simulations. We run these with $\beta = \epsilon = \mu = \lambda = 1$ when $c = \frac{1}{2}$, and we work with the transport formulation of the problem, namely equations (1.9) – (1.11). Specific quarter plane problems with piecewise constant data are run and snapshots of the solution are displayed at times $t = 10, 100$, and 1000. All fields are plotted against the self similar variable $\frac{x}{t}$. Since the solutions are constant in the region $\frac{x}{t} > 1$, we confine our displays to the interval $0 \leq \frac{x}{t} \leq 1$. These solutions exhibit both single phase and phase transitional shocks and also rarefaction waves which lie in a single phase. As t tends to infinity these particular solutions to (1.9) – (1.11), when regarded as functions of $\frac{x}{t}$ and t, converge to solutions of the reduced system(1.1) – (1.3) and (1.6) which are functions of $\frac{x}{t}$ only, this point is driven home by looking at the snapshots at time $t = 1000$.

2. Equilibrium Solutions and Travelling Waves.

We first turn to the equilibrium solutions of the system (1.1) – (1.4). Equations (1.1) and (1.2) imply that the velocity v and stress σ must be spatially constant; i.e.,

$$v \equiv v_0 \text{ and } \sigma \equiv \sigma_0 := \mu c^2 \alpha_0 \tag{2.1}$$

and (1.4) reduces to

$$D_{xx} + \Omega^2 \left(\alpha_0 + D\left(1 - D^2\right)\right) = 0 \tag{2.2}$$

where

$$\Omega^2 := \frac{\beta\mu}{\lambda^2\epsilon^2} > 0. \tag{2.3}$$

We concentrate first on the case $\alpha_0 = 0$. Here there are three constant solutions, namely $D = -1, 0$, and 1. There are also nonconstant solutions and these satisfy the energy identity

$$D_x^2 = \frac{\Omega^2}{2}\left(\left(D^2-1\right)^2 - E\right) \tag{2.4}$$

for constants E satisfying $0 \le E < 1$. When $E = 0$, the solutions of (2.4) represent equilibrium phase transitions which monotonically connect -1 to 1 or 1 to -1 as x ranges over $(-\infty, \infty)$. When $0 < E < 1$, the right hand side of (2.4) may be written as $\frac{\Omega^2}{2}\left(D_2^2 - D^2\right)\left(D_1^2 - D^2\right)$ where $D_1^2 = 1 - E^{1/2}$, $D_2^2 = 1 + E^{1/2}$, and $D_2^2 = 2 - D_1^2$. In this case the nonconstant equilibrium solutions are periodic with half period, $L_{1/2}$, given by

$$L_{1/2} = \frac{2^{1/2}}{\Omega}\int_{-D_1}^{D_1}\frac{dD}{\sqrt{2-D_1^2-D^2}\sqrt{D_1^2-D^2}} = \frac{2^{1/2}}{\Omega}\int_{-1}^{1}\frac{ds}{\sqrt{2-D_1^2\left(1+s^2\right)}\sqrt{1-s^2}}\,. \tag{2.5}$$

We note that $L_{1/2}$ is monotone increasing in D_1 on $(0,1)$ and that

$$L_{1/2}\left(0^+\right) = \frac{\pi}{\Omega} \text{ and } \lim_{D_1 \to 1-} L_{1/2} = \infty. \tag{2.6}$$

The limit relation (2.6) implies that on a fixed interval, say $0 \le x \le 1$, there are at most $2 \times \left[\frac{\Omega}{\pi}\right]$ nontrivial equilibrium solutions satisfying $D_x(0) = D_x(1) = 0$. [4,5] Here $[\gamma]$ is the greatest integer $\le \gamma$. These oscillatory solutions are referred to as having equilibrium microstructure at zero stress.

We now look at the situation when $\alpha_0 \ne 0$. If $|\alpha_0| > \frac{2}{3^{3/2}}$, the cubic $\alpha_0 + D\left(1-D^2\right) = 0$ has only one real root and this root is the unique bounded equilibrium solutions of (2.2). If $\frac{-2}{3^{3/2}} < \alpha_0 < \frac{2}{3^{3/2}}$ (and $\alpha_0 \ne 0$), there are three real roots of the cubic $\alpha_0 + D\left(1-D^2\right) = 0$ and each is a solution of (2.2). In this situation there are no equilibrium phase transitions connecting the left (respectively right) most root of the cubic to the right (respectively left) most root as x goes from minus to plus infinity. For α_0 in this parameter range there are nonconstant equilibrium solutions to (2.2) and these satisfy the energy identity

$$\frac{D_x^2}{2} + \Omega^2\left(\alpha_0 D - \frac{\left(1-D^2\right)^2}{4}\right) = -\Omega^2 E \tag{2.7}$$

[4]Note that this boundary condition is really a statement about the flux Q which evolves according to $(1.8)_2$ and at equilibrium satisfies $Q = \lambda^2\epsilon D_x$.

[5]The trivial equilibrium solutions are of course $D \equiv 1$ and $D \equiv -1$.

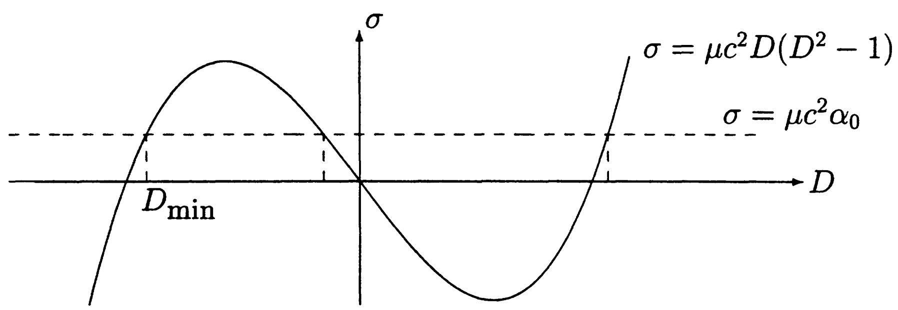
for any $E \in [E_1(\alpha_0), E_2(\alpha_0))$. Here, $E_1(\alpha_0)$ is the value of the unique positive local minimum of $\frac{(1-D^2)^2}{4} - \alpha_0 D$ and $D_{\min}(\alpha_0)$ is the location of this minima. $E_2(\alpha_0)$ is the value of the unique local maximum of $\frac{(1-D^2)^2}{4} - \alpha_0 D$. We note that if $E_1(\alpha_0) < E < E_2(\alpha_0,)$, the quadratic $\frac{(1-D^2)^2}{4} - \alpha_0 D - E$ may be written as

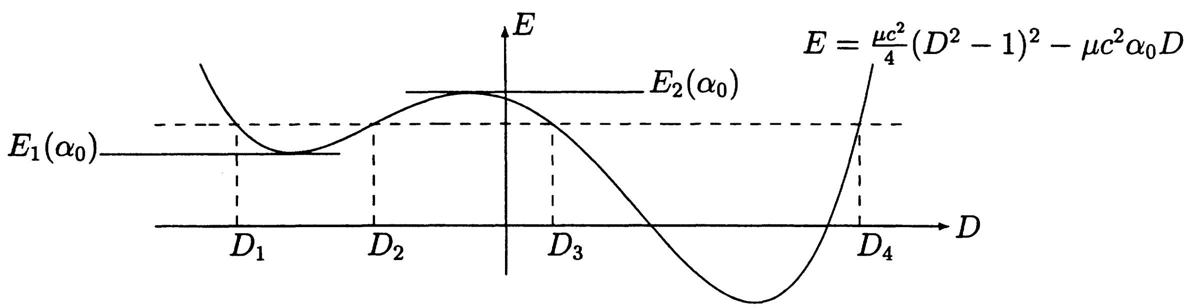
$$\frac{(1-D^2)^2}{4} - \alpha_0 D - E = \frac{1}{4}(D - D_1)(D - D_2)(D_3 - D)(D_4 - D) \tag{2.8}$$

where $D_1 < D_2 < D_3 < D_4$ are the four real roots of the equation $\frac{(1-D^2)^2}{4} - \alpha_0 D - E = 0$. For $\frac{-2}{3^{3/2}} < \alpha_0 < 0$, the roots satisfy $\lim_{E \searrow E_1(\alpha_0)} D_{3,4}(E) = D_{\min}(\alpha_0)$ whereas for $0 < \alpha_0 < \frac{2}{3^{3/2}}$, the roots satisfy $\lim_{E \searrow E_1(\alpha_0)} D_{1,2}(E) = D_{\min}(\alpha_0)$. These last two observations imply that if $\alpha_0 \in \left(\frac{-2}{3^{3/2}}, \frac{2}{3^{3/2}}\right)$ is not equal to zero and if $E = E_1(\alpha_0)$, then the nonconstant equilibrium solutions are standing solitary waves satisfying

$$\lim_{|x| \to \infty} D(x) = D_{\min}(\alpha_0) \quad \text{and} \quad D(0) = \begin{cases} D_2(E_1(\alpha_0)), & \alpha_0 < 0 \\ D_3(E_1(\alpha_0)), & \alpha_0 > 0 \end{cases}$$

or translates thereof. For $E_1(\alpha_0) < E < E_2(\alpha_0)$, we get periodic equilibria which span the roots D_2 and D_3.





We now turn our attention to travelling wave solutions of (1.1) – (1.4). These are functions of $\xi = \frac{x-st}{\epsilon}$. For definiteness we focus on the case where $s > 0$ though analogous results obtain when $s < 0$. In what follows we adopt the convention that a minus subscript refers to the state at $\xi = -\infty$ and a plus the state at $\xi = +\infty$. The states at plus and minus infinity will be related via the equilibrium relations (1.6); that is

$$u_{\mp} = (1-\mu)\, D_{\mp} + \mu D_{\mp}^{3} \quad \text{and} \quad \sigma_{\mp} = \mu c^2 D_{\mp} \left(D_{\mp}^2 - 1\right) \tag{2.9}$$

where $0 < \mu \leq 1$. It is easily checked that if (1.1) – (1.4) has solutions of the desired type, then for every $\xi \in (-\infty, \infty)$

$$s\,(u\,(\xi) - u_{-}) = -\,(v\,(\xi) - v_{-}) \tag{2.10}$$

$$s\,(v\,(\xi) - v_{-}) = -\,(\sigma\,(\xi) - \sigma_{-}) \tag{2.11}$$

$$\sigma\,(\xi) = c^2\,(u\,(\xi) - D\,(\xi)) \tag{2.12}$$

and

$$\left(\lambda^2 - s^2\right) D_{\xi\xi} + s D_{\xi} = \frac{\beta\,(\mu c^2 D\,(D^2 - 1) - \sigma)}{c^2}. \tag{2.13}$$

Equations (2.10) and (2.11) are first integrals of the travelling wave version of (1.1) and (1.2) and they imply that the states at $\xi = \pm\infty$ obey the Rankine-Hugoniot equations

$$s\,(u_{+} - u_{-}) = -\,(v_{+} - v_{-})\,, \quad s\,(v_{+} - v_{-}) = -\,(\sigma_{+} - \sigma_{-})\,, \quad \text{and } s^2 = \frac{\sigma_{+} - \sigma_{-}}{u_{+} - u_{-}} \tag{2.14}$$

where, of course, $u_{\mp}, \sigma_{\mp}$, and $D_{\mp}$ satisfy (2.9). Equations (2.10) – (2.13) may be combined to yield

$$\sigma\,(\xi) = \sigma_{-} + \frac{s^2 c^2}{c^2 - s^2}\,(D\,(\xi) - D_{-})\,. \tag{2.15}$$

where $D\,(\cdot)$ satisfies

$$\left(\lambda^2 - s^2\right) D_{\xi\xi} + s D_{\xi} = \beta\mu\,(D - D_{-}) \left(D^2 + D_{-}D + D_{-}^2 - 1 - \frac{s^2}{\mu\,(c^2 - s^2)}\right) \tag{2.16}$$

for all $\xi \in (-\infty, \infty)$. In what follows we shall restrict our attention to the situation where

$$-\frac{2}{3^{1/2}} \left(1 + \frac{s^2}{\mu\,(c^2 - s^2)}\right)^{1/2} < D_{-} \leq -1. \tag{2.17}$$

Exactly the same arguments apply to the situation where

$$1 < D_{-} < \frac{2}{3^{1/2}} \left(1 + \frac{s^2}{\mu\,(c^2 - s^2)}\right)^{1/2}. \tag{2.18}$$

We shall also assume that $\lambda \geq c$. Since s satisfies

$$s^2 = \frac{\sigma_+ - \sigma_-}{u_+ - u_-} < c^2, \tag{2.19}$$

this condition on λ guarantees that $\lambda^2 - s^2 > 0$.

We note that if (2.17) holds, then the quadratic $D^2 + D_- D + D_-^2 - 1 - \frac{s^2}{\mu(c^2-s^2)}$ has two real roots

$$D_I := -\frac{D_-}{2} - \sqrt{1 + \frac{s^2}{\mu\left(c^2 - s^2\right)} - \frac{3D_-^2}{4}} < D_+ := -\frac{D_-}{2} + \sqrt{1 + \frac{s^2}{\mu\left(c^2 - s^2\right)} - \frac{3D_-^2}{4}} \tag{2.20}$$

and in this parameter range (2.16) reduces to

$$\left(\lambda^2 - s^2\right) D_{\xi\xi} + sD_\xi = \beta\mu\left(D - D_-\right)\left(D - D_I\right)\left(D - D_+\right). \tag{2.21}$$

The desired phase transitions are solutions of (2.21) satisfying

$$\lim_{\xi \to -\infty} D(\xi) = D_- \ \text{ when } \ \lim_{\xi \to \infty} D(\xi) = D_+ > \frac{1}{\sqrt{3}}. \tag{2.22}$$

A variant of the theorem stated below may be found in Greenberg [7]. Once stated, the result is a simple exercise to prove.

Theorem 2.1. *For $0 < s$ there are travelling wave solutions of (2.21) of the desired type. These solutions satisfy the reduced equation*

$$D_\xi = K\left(D - D_-\right)\left(D_+ - D\right) \tag{2.23}$$

provided

$$K = \left(\frac{\beta\mu}{2\left(\lambda^2 - s^2\right)}\right)^{1/2} \tag{2.24}$$

and D_- is given parametrically in terms of s by

$$D_- = -\frac{s}{3\left(2\beta\mu\left(\lambda^2 - s^2\right)\right)^{1/2}} - \sqrt{1 + \frac{s^2}{\mu\left(c^2 - s^2\right)} - \frac{s^2}{6\beta\mu\left(\lambda^2 - s^2\right)}}. \tag{2.25}$$

The other roots D_I and D_+ are computed by inserting (2.25) into (2.20). It is easily verified that at $s = 0^+$

$$D_- = -1, D_I = 0, D_+ = 1, \ \frac{dD_-}{ds} = -\frac{1}{3\lambda\left(2\beta\mu\right)^{1/2}},$$

$$\frac{dD_I}{ds} = \frac{5}{12\lambda\left(2\beta\mu\right)^{1/2}}, \ \text{and} \ \frac{dD_+}{ds} = -\frac{1}{12\lambda\left(2\beta\mu\right)^{1/2}}$$

and thus for s positive and small enough the roots D_I and D_+ satisfy $0 < D_I$ and $D_+ < 1$ ■

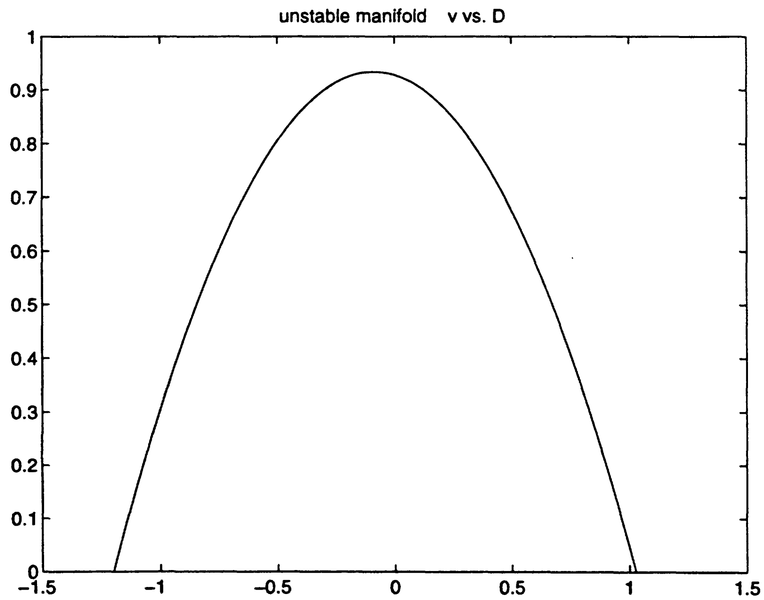
The solutions characterized by the preceding theorem are only part of the story on elementary phase change solutions when $D_- < -1$ and $s > 0$. Coexisting with these solutions is an additional one parameter family of solutions which satisfy $D_- < -1$ and $D_+ = \mathcal{D}(D_-) > 1$. For these solutions s is again given by

$$0 < s = \frac{c\sqrt{\mu\left(D_+^2 + D_+D_- - 1\right)}}{\sqrt{1 + \mu\left(D_+^2 + D_+D_- + D_-^2 - 1\right)}}. \tag{2.26}$$

To establish the existence of these solutions one writes (2.21) as the first order system

$$D_\xi = v \text{ and } v_\xi = \frac{-sv}{\lambda^2 - s^2} - \frac{\beta\mu}{\lambda^2 - s^2}\left(D - D_-\right)\left(D - D_I\right)\left(D_+ - D\right) \tag{2.27}$$

where $D_I = -\left(D_- + D_+\right)$ and s is given by (2.26). For a fixed value of D_- satisfying $-1 - \delta < D_- < -1$ with $0 < \delta$ small enough we note that when $D_+ = -D_-$ the unstable manifold of (2.27) through $(D_-, 0)$ with $D > D_-$ connects to $(D_I, 0) = (0, 0)$, whereas when $D_+ = 1$, the unstable manifold through $(D_-, 0)$, with $D > D_-$ satisfies $v(\cdot) > 0$ and $\lim_{\xi\to\infty} D(\xi) = +\infty$. These two facts, together with an elementary continuity or shooting argument, guarantee the existence of a $D_+ \in (1, -D_-)$ such that the unstable manifolds through $(D_-, 0)$ with $D > D_-$ satisfies $\lim_{\xi\to\infty} (D(\xi), v(\xi)) = (D_+, 0)$. A picture of this manifold is shown below for the system (2.27) when $\beta = \mu = \lambda = 1,\ c = \frac{1}{2}$ and $D_- = -1.2$. In this case $D_+ = 1.02775\ldots$ and $s = .2282\ldots$.



When $D_- > 1$ and $s > 0$ we obtain two one parameter families of solutions representing change of phase waves. One family satisfies $-1 < D_+ < -\frac{1}{\sqrt{3}}$ and the other $-D_- < D_+ < -1$. We also obtain analogous phase transitions when $s < 0$.

That all of these solutions represent undercompressive shocks of the type described in Section 1 follows from (1.12)and (2.26).

We now turn our attention to the existence of strong shocks which lie in a single phase. For definiteness we take the phase to be the region $D > \frac{1}{\sqrt{3}}$ though analogous results are true for the region $D < -\frac{1}{\sqrt{3}}$. We let D_+ and D_- be two numbers satisfying

$$\frac{1}{\sqrt{3}} < D_+ < D_-, \tag{2.28}$$

$u_\pm$ and $\sigma_\pm$ be as defined in (2.9), and let s be given by (2.26).

We again assume $c \leq \lambda$. This guarantees $\lambda^2 - s^2 > 0$. Finally we let $D_* < -D_-$ be the third root of the cubic equation

$$\sigma_- + \frac{c^2 s^2}{c^2 - s^2}(D_* - D_-) = \mu c^2 D_* \left(D_*^2 - 1\right). \tag{2.29}$$

We next note that (2.13) and (2.15) may be written as the first order system

$$D_\xi = v \quad \text{and} \quad v_\xi = -\frac{sv}{(\lambda^2 - s^2)} + \frac{\beta\mu}{(\lambda^2 - s^2)}(D - D_-)(D - D_+)(D - D_*) \tag{2.30}$$

where again $D_* < -D_-$ and $\frac{1}{\sqrt{3}} < D_+ < D_-$.

The critical points of (2.29) are given by

$$(D, v) = (D_-, 0)\ ,\ (D_+, 0),\ \text{and}\ \ (D_*, 0).$$

The first and third critical points are saddle points while the second is either a stable node or a stable spiral point. The former situation is true if $\frac{s^2}{4(\lambda^2 - s^2)} > \beta\mu(D_- - D_+)(D_+ - D_*)$ and the latter if $\frac{s^2}{4(\lambda^2 - s^2)} \leq \beta\mu(D_- - D_+)(D_+ - D_*)$. In either situation we shall now show there is a connecting orbit from $(D_-, 0)$ to $(D_+, 0)$. We let $v = v_u(D)$, $D < D_-$ be the unstable manifold through $(D_-, 0)$. It satisfies

$$v_u(D) \sim \gamma_+(D - D_-),\ \ D_+ << D < D_- \tag{2.31}$$

where

$$\gamma_+ = \frac{s}{2(\lambda^2 - s^2)}\left(\sqrt{1 + \frac{4\beta\mu(D_- - D_+)(D_- - D_*)(\lambda^2 - s^2)}{s^2}} - 1\right) \tag{2.32}$$

and

$$\frac{d^2 v_u}{dD^2} < 0 \tag{2.33}$$

so long as

$$\frac{\beta\mu}{s}(D - D_-)(D - D_+)(D - D_*) < v_u(D) < 0, \; D < D_-. \tag{2.34}$$

The energy

$$E(\xi) := \frac{\beta\mu}{(\lambda^2 - s^2)} \int_{D_+}^{D(\xi)} (D_- - s)(s - D_+)(s - D_*)\, ds + \frac{1}{2} v_u^2(D(\xi)) \tag{2.35}$$

associated with any solution on this manifold satisfies

$$\lim_{\xi \to -\infty} E(\xi) = \frac{\beta\mu}{(\lambda^2 - s^2)} \int_{D_+}^{D_-} (D_- - s)(s - D_+)(s - D_*)\, ds \tag{2.36}$$

and

$$\frac{dE}{d\xi} = -\frac{s}{(\lambda^2 - s^2)} v_u^2(D(\xi)) \tag{2.37}$$

and the right hand side of (2.37) is negative so long as (2.34) holds. Since both D_- and $D_* < -D_-$ are local maxima of the potential energy

$$P(D) = \frac{\beta\mu}{(\lambda^2 - s^2)} \int_{D_+}^{D} (D_- - s)(s - D_+)(s - D_*)\, ds \tag{2.38}$$

satisfying

$$P(D_*) > P(D_-) \tag{2.39}$$

the identities (2.36) and (2.37) imply that solutions on the unstable manifold $v = v_u(D)$ with $D < D_-$ are in the basin of attraction of $(D_+, 0)$ and this guarantees the existence of the connecting orbit. These orbits are the strong shock solutions when $\frac{1}{\sqrt{3}} < D_+ < D_-$. The same argument yields shocks with s given by (2.27) and $D_- < D_+ < \frac{-1}{\sqrt{3}}$ as well as shocks in either phase when $s < 0$.

That the Lax conditions (1.14) and (1.15) hold for these strong shocks follows from (2.26) and the fact that the function $\frac{c\mu^{\frac{1}{2}}\sqrt{3D^2-1}}{\sqrt{1-\mu+3\mu D^2}}$ is increasing on $D > \frac{1}{\sqrt{3}}$. This concludes Section 2.

3. Large Time Behavior of Solutions to (1.1) - (1.4)

In this section we consider the large time behavior of solutions to (1.1) – (1.4). We assume the existence of sufficiently regular solutions to these equations defined on the domain

$\{(x,t) \mid 0 \le x \le 1 \text{ and } 0 \le t\}$. Initial conditions for D, D_t, v, and σ are prescribed at $t = 0$ and at the boundaries we assume

$$D_x\left(0^+,t\right) = D_x\left(1^-,t\right) = 0^6 \tag{3.1}$$

and

$$\sigma\left(0^+,t\right) - K_0 v\left(0^+,t\right) = \sigma\left(1^-,t\right) + K_1 v\left(1^-,t\right) = 0. \tag{3.2}$$

The constants K_i, $i = 0$ or 1, in (3.2) are non-negative. When $0 < K_i < \infty$ these boundary conditions are dissipative and they guarantee that $\sigma(0^+,t)$, $v(0^+,t)$, $\sigma(1^-,t)$, and $v(1^-,t)$ are in $L_2[0,\infty)$. The limiting cases where $K_i = 0$ or ∞ are referred to as energy conserving conditions and when either holds they guarantee that the mechanical energy flux, σv, vanishes at the point in questions.

Our main result is if (3.1) and (3.2) hold, then as t tends to infinity σ and v converge to zero and D converges to a function $D_\infty(\cdot)$ which satisfies

$$\lambda^2\epsilon^2 D_{\infty_{xx}} + \beta\mu D_\infty\left(1 - D_\infty^2\right) = 0, \quad 0 < x < 1 \tag{3.3}$$

and the boundary conditions (3.1).

We shall first establish this result when the constants K_i satisfy $0 < K_i < \infty$. The same result obtains when σ vanishes at one of the endpoints and v vanishes at the other. When v vanishes at both endpoints we obtain instead that $\sigma \equiv \sigma_\infty$, a constant, and D_∞ satisfies

$$\beta\sigma_\infty = c^2\beta\mu D_\infty\left(D_\infty^2 - 1\right) - c^2\lambda^2\epsilon^2 D_{\infty_{xx}}, \quad 0 \le x \le 1$$

and (3.1).

If we insert (1.3) into (1.1) we see that the original system may be rewritten as

$$\frac{\sigma_t}{c^2} - v_x = -D_t, \tag{3.4}$$

$$v_t - \sigma_x = 0, \tag{3.5}$$

and

$$c^2\epsilon^2\left(D_{tt} - \lambda^2 D_{xx}\right) + c^2\epsilon D_t + c^2\beta\mu D\left(D^2 - 1\right) = \beta\sigma. \tag{3.6}$$

We also note that the time derivatives σ_t, v_t, and D_t satisfy the differentiated version of (3.4) – (3.6), namely the system

$$\frac{\sigma_{tt}}{c^2} - v_{tx} = -D_{tt}, \tag{3.7}$$

$$v_{tt} - \sigma_{tx} = 0, \tag{3.8}$$

[6]Once again this boundary condition is equivalent to the condition that Q vanishes at $x = 0^+$ and $x = 1^-$.

and

$$c^2\epsilon^2\left(D_{ttt}-\lambda^2 D_{txx}\right)+c^2\epsilon D_{tt}=\beta\sigma_t+c^2\beta\mu\left(1-3D^2\right)D_t.^7 \tag{3.9}$$

If we now multiply (3.4) by $\beta\sigma$, (3.5) by βv, and (3.6) by D_t and add the resulting expressions we obtain the identity

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{\beta}{2}\left(\frac{\sigma^2}{c^2}+v^2\right)+\frac{c^2\epsilon^2}{2}\left(D_t^2+\lambda^2 D_x^2\right)+\frac{c^2\beta\mu}{4}\left(D^2-1\right)^2\right\}\\ -&\frac{\partial}{\partial x}\left\{\beta\sigma v+c^2\lambda^2\epsilon^2 D_t D_x\right\}+c^2\epsilon D_t^2=0.\end{aligned} \tag{3.10}$$

Similarly, multiplying (3.7) by $\beta\sigma_t$, (3.8) by βv_t, and (3.9) by D_{tt} and adding yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{\beta}{2}\left(\frac{\sigma_t^2}{c^2}+v_t^2\right)+\frac{c^2\epsilon^2}{2}\left(D_{tt}^2+\lambda^2 D_{tx}^2\right)\right\}\\ -&\frac{\partial}{\partial x}\left\{\beta\sigma_t v_t+c^2\lambda^2\epsilon^2 D_{tt}D_{tx}\right\}+c^2\epsilon D_{tt}^2=c^2\beta\mu\left(1-3D^2\right)D_t D_{tt}.\end{aligned} \tag{3.11}$$

The identity (3.10), the boundary condition (3.1), and the fact that when $0<K_i<\infty$ equation (3.2) implies that

$$\sigma v\left(1^-,t\right)=-\frac{1}{2}\left(K_1 v^2\left(1^-,t\right)+\frac{\sigma^2\left(1^-,t\right)}{K_1}\right), \tag{3.12}$$

and

$$\sigma v\left(0^+,t\right)=\frac{1}{2}\left(K_0 v^2\left(0^+,t\right)+\frac{\sigma^2\left(0^+,t\right)}{K_0}\right), \tag{3.13}$$

all combine to yield

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left\{\frac{\beta}{2}\left(\frac{\sigma^2}{c^2}+v^2\right)+\frac{c^2\epsilon^2}{2}\left(D_t^2+\lambda^2 D_x^2\right)+\frac{c^2\beta\mu}{4}\left(D^2-1\right)^2\right\}(x,t)\,dx\\ +&\frac{\beta}{2}\left[K_0 v^2\left(0^+,t\right)+\frac{\sigma^2\left(0^+,t\right)}{K_0}\right]+\frac{\beta}{2}\left[K_1 v^2\left(1^-,t\right)+\frac{\sigma^2\left(1^-,t\right)}{K_1}\right]\\ +&c^2\epsilon\int_0^1 D_t^2(x,t)\,dx=0\end{aligned} \tag{3.14}$$

[7] At $t=0$ we prescribe σ, v, D and D_t on $0\le x\le 1$ and we assume these data are compatible with the boundary conditions (3.1) and (3.2). The data for the time differentiated system (3.7) – (3.9) is computed from the previously assigned data and the equation (3.4) – (3.6). We also assume that the initial distributions of σ, v, D, and D_t are compatible with the time differentiated version of the boundary conditions (3.1) and (3.2).

and

$$\int_0^1 \left\{ \frac{\beta}{2}\left(\frac{\sigma^2}{c^2}+v^2\right)+\frac{c^2\epsilon^2}{2}\left(D_t^2+\lambda^2 D_x^2\right)+\frac{c^2\beta\mu}{4}\left(D^2-1\right)^2\right\}(x,t)\,dx$$

$$+\ \frac{\beta}{2}\int_0^t \left\{\left[K_0 v^2\left(0^+,s\right)+\frac{\sigma^2\left(0^+,s\right)}{K_0}\right]+\left[K_1 v^2\left(1^-,s\right)+\frac{\sigma^2\left(1^-,s\right)}{K_1}\right]\right\} ds$$

$$+\ c^2\epsilon\int_0^t \left(\int_0^1 D_s^2\left(x,s\right)dx\right)ds = \tag{3.15}$$

$$\int_0^1 \left\{ \frac{\beta}{2}\left(\frac{\sigma^2}{c^2}+v^2\right)+\frac{c^2\epsilon^2}{2}\left(D_t^2+\lambda^2 D_x^2\right)+\frac{c^2\beta\mu}{4}\left(D^2-1\right)^2\right\}\left(x,0^+\right)dx.$$

Assuming smooth enough initial data so that the right hand side of (3.15) is finite we see that for all $t \geq 0$, $D(\cdot,t)$ is in $L_4[0,1]$ with derivative $D_x(\cdot,t)$ in $L_2[0,1]$ and that these quantities are bounded independently of t. This observation in turn implies that for each $t \geq 0$, $D(\cdot,t)$ is in $L_\infty[0,1]$ with a bound which is independent of t.

We now turn our attention to the identity (3.11). We focus first on the right hand side of the identity. We note that

$$c^2\beta\mu\left(1-3D^2\right)D_t D_{tt} \leq \frac{c^2\epsilon}{2}D_{tt}^2+\frac{c^2\beta^2\mu^2}{2\epsilon}\left|1-3D^2\right|_\infty^2 D_t^2 \tag{3.16}$$

where

$$\left|1-3D^2\right|_\infty = \sup_{\substack{0\leq x\leq 1\\ 0\leq t}} \left|1-3D^2\right|(x,t) \tag{3.17}$$

is, by our remark in the last paragraph, finite with a bound depending only on the right hand side of (3.15) and the parameters $0<c$, $0<\lambda$, $0<\beta$, $0<\epsilon$, and $0\leq\mu\leq 1$. If we now integrate (3.11) over $[0,1]$, make use of the fact that (3.1) and (3.2) (with $0<K_i<\infty$) imply that

$$D_{xt}\left(0^+,t\right)=D_{xt}\left(1^-,t\right)=0, \tag{3.18}$$

$$\left(\sigma_t v_t\right)\left(1^-,t\right)=-\frac{1}{2}\left(K_1 v_t^2\left(1^-,t\right)+\frac{\sigma_t^2\left(1^-,t\right)}{K_1}\right). \tag{3.19}$$

and

$$\left(\sigma_t v_t\right)\left(0^+,t\right)=\frac{1}{2}\left(K_0 v_t^2\left(0^+,t\right)+\frac{\sigma_t^2\left(0^+,t\right)}{K_0}\right), \tag{3.20}$$

and exploit (3.16) we obtain

$$\frac{d}{dt}\int_0^1 \left\{\frac{\beta}{2}\left(\frac{\sigma_t^2}{c^2}+v_t^2\right)+\frac{c^2\epsilon^2}{2}\left(D_{tt}^2+\lambda^2 D_{tx}^2\right)\right\}(x,t)\,dx$$

$$+\quad \frac{\beta}{2}\left[K_0 v_t^2\left(0^+,t\right)+\frac{\sigma_t^2\left(0^+,t\right)}{K_0}\right]+\frac{\beta}{2}\left[K_1 v_t^2\left(1^-,t\right)+\frac{\sigma_t^2}{K_1}\left(1^-,t\right)\right] \tag{3.21}$$

$$+\quad \frac{c^2\epsilon}{2}\int_0^1 D_{tt}^2(x,t)\,dx \le \frac{c^2\beta^2\mu^2\left|1-3D^2\right|_\infty^2}{2\epsilon}\int_0^1 D_t^2(x,t)\,dx.$$

Integration of (3.21) also yields

$$\frac{1}{2}\int_0^1 \left\{\beta\left(\frac{\sigma_t^2}{c^2}+v_t^2\right)+\frac{c^2\epsilon^2}{2}\left(D_{tt}^2+\lambda^2 D_{tx}^2\right)\right\}(x,t)\,dx$$

$$+\quad \frac{\beta}{2}\left\{\left[K_0 v_s^2\left(0^+,s\right)+\frac{\sigma_s^2\left(0^+,s\right)}{K_0}\right]+\left[K_1 v_s^2\left(1^-,s\right)+\frac{\sigma_s^2\left(1^-,s\right)}{K_1}\right]\right\}ds$$

$$+\quad \frac{c^2\epsilon}{2}\int_0^t\left(\int_0^1 D_{ss}^2(x,s)\,dx\right)ds \le \tag{3.22}$$

$$\frac{1}{2}\int_0^1 \left\{\beta\left(\frac{\sigma_t^2}{c^2}+v_t^2\right)+\frac{c^2\epsilon^2}{2}\left(D_{tt}^2+\lambda^2 D_{tx}^2\right)\right\}\left(x,0^+\right)dx$$

$$+\quad \frac{c^2\beta^2\mu^2\left|1-3D^2\right|_\infty^2}{2\epsilon}\int_0^t\left(\int_0^1 D_s^2(x,s)\,dx\right)ds.$$

We assume that the data at $t=0^+$ are such that the first integral on the right hand side of (3.22) is finite and this assumption, together with the observation that (3.15) implies that $D_t \in L_2([0,1]\times[0,\infty))$ guarantees the boundedness of the left hand side of (3.22) independently of $t \geq 0$. Additionally, (3.15) and (3.22) along with (3.4) – (3.6) imply that for each $t \geq 0$ $\sigma, v, \sigma_t, v_t, \sigma_x, v_x, D, D_t, D_x, D_{tt}, D_{tx}$, and D_{xx} are bounded in $L_2[0,1]$ independent of t.

For each $n = 1, 2, \ldots$ we define the functions

$$(\sigma_n, v_n, D_n)(x,t) := (\sigma, v, D)(x, n+t), \quad 0 \le x \le 1 \text{ and } t \ge 0, \tag{3.23}$$

and note that they satisfy (3.4) – (3.6) while their time derivatives satisfy (3.7) – (3.9). These functions satisfy the same $L_2[0,1]$ and $L_\infty[0,1]$ bounds that obtain for $\sigma, v,$ and D. Moreover, (3.15) and (3.22) imply that as $n \to \infty$ $D_{n,t}$ and $D_{n,tt}$ converge strongly to zero in $L_2([0,1] \times [0,\infty))$ while the traces of $\sigma_n, \sigma_{n,t}, v_n$, and $v_{n,t}$ evaluated at $x = 0^+$ and $x = 1^-$ converge strongly to zero in $L_2[0,\infty)$. We also have that the traces of σ_n and v_n at $x = 0^+$ and $x = 1^-$ converge uniformly to zero; i.e., $\lim_{n\to\infty} \sup_{t\geq 0} [|v_n| + |\sigma_n|](a,t) = 0$ where $a = 0^+$ or 1^-. These observations imply that we may, without loss in generality, assume that for any $T > 1$ the functions (σ_n, v_n) converge strongly in $L_2([0,1] \times [0,T])$ to functions $(\sigma_\infty, v_\infty)$ which satisfy

$$\sigma_{\infty t} - v_{\infty x} = 0 \quad \text{and} \quad v_{\infty t} - \sigma_{\infty x} = 0, \;\; 0 \leq x \leq 1 \text{ and } 0 \leq t \leq T \tag{3.24}$$

and the overdetermined boundary conditions

$$\sigma_\infty\left(0^+,t\right) = v_\infty\left(0^+,t\right) = \sigma_\infty\left(1^-,t\right) = v_\infty\left(1^-,t\right) = 0, \; 0 \leq t \leq T. \tag{3.25}$$

The boundary conditions along with $T > 1$ and the fact that (3.24) is equivalent to the characteristic system

$$(\sigma_\infty + v_\infty)_t - (\sigma_\infty + v_\infty)_x = 0 \text{ and } (\sigma_\infty - v_\infty)_t + (\sigma_\infty - v_\infty)_x = 0$$

imply that σ_∞ and v_∞ must be zero at $t = 0^+$ and this in turn guarantees that $(\sigma_\infty, v_\infty) \equiv (0,0)$ on $L_2([0,1] \times [0,T])$ where again $T > 1$ is arbitrary.

In like manner we may assume, without loss in generality, that for any $T > 1$ the functions D_n converge strongly in $L_2([0,1] \times [0,T])$ to a function D_∞ which satisfies the same bounds as our base solution D and the additional relations:

$$D_{\infty t} = D_{\infty tt} = 0, \;\; 0 \leq x \leq 1 \text{ and } 0 \leq t \leq T \tag{3.26}$$

$$\lambda^2\epsilon^2 D_{\infty xx} + \beta\mu D_\infty\left(1 - D_\infty^2\right) = 0, \;\; 0 \leq x \leq 1 \tag{3.27}$$

and

$$D_{\infty x}\left(0^+\right) = D_{\infty x}\left(1^-\right) = 0. \tag{3.28}$$

As noted previously, the conclusions about the large time behavior are valid if we replace the dissipative boundary conditions by the limit cases where $K_i = 0$ or ∞ and σ vanishes at least one boundary and v at the other. For definiteness, we shall illustrate this when

$$\sigma\left(0^+,t\right) = v\left(1^-,t\right) = 0. \tag{3.29}$$

Again we take $T > 1$ to be arbitrary. The a priori information on the solution guarantees the existence of limit functions σ_∞, v_∞, and D_∞ which satisfy

$$D_{\infty,t} = 0, \tag{3.30}$$

$$\beta\sigma_\infty = c^2\beta\mu D_\infty\left(D_\infty^2 - 1\right) - c^2\lambda^2\epsilon^2 D_{\infty xx}, \tag{3.31}$$

and

$$\sigma_{\infty t} - v_{\infty x} = 0 \text{ and } v_{\infty t} - \sigma_{\infty x} = 0 \tag{3.32}$$

for $0 \le x \le 1$ and $0 \le t \le T$ and boundary conditions

$$\sigma_\infty\left(0^+, t\right) = v_\infty\left(1^-, t\right) = 0,\ 0 \le t \le T \tag{3.33}$$

and

$$D_{\infty x}\left(0^+, t\right) = D_{\infty x}\left(1^-, t\right) = 0,\ \ 0 \le t \le T.$$

Equations (3.30) and (3.31) imply that $\dfrac{\partial\sigma_\infty}{\partial t} = 0$, $0 \le x \le 1$ and $0 \le t \le T$ and this, when combined with $(3.32)_1$, guarantees that

$$\sigma_\infty = \Sigma_\infty(x),\ \ 0 \le x \le 1 \tag{3.34}$$

and

$$v_\infty = V_\infty(t),\ \ 0 \le t \le T. \tag{3.35}$$

Additionally, $(3.32)_2$, (3.34), and (3.35) imply that

$$\Sigma_\infty = A_\infty + B_\infty x \text{ and } V_\infty = C_\infty + B_\infty t \tag{3.36}$$

where $A_\infty, B_\infty, C_\infty$ are constants. Finally, (3.33) and (3.36) imply that $A_\infty = B_\infty = C_\infty = 0$ which is the desired result. If instead of $(3.33)_1$ we impose $v = 0$ at $x = 0^+$ we obtain $C_\infty = B_\infty = 0$ and that $\sigma_\infty \equiv A_\infty$, an arbitrary constant.

Finally, we note that (3.14) implies that the energy

$$\mathcal{E}(t) := \int_0^1 \left\{\frac{\beta}{2}\left(\frac{\sigma^2}{c^2} + v^2\right) + \frac{c^2\epsilon^2}{2}\left(D_t^2 + \lambda^2 D_x^2\right) + \frac{c^2\beta\mu}{4}\left(D^2 - 1\right)^2\right\}(x,t)\,dx \tag{3.37}$$

is nonnegative and monotone decreasing and in the cases where σ_∞ and v_∞ are zero the energy converges as $t \to \infty$ to

$$\mathcal{E}_\infty := \frac{c^2\epsilon^2\lambda^2}{2}\int_0^1 \left(D_{\infty x}^2 + \frac{\Omega^2}{2}\left(D_\infty^2 - 1\right)^2\right)(x)\,dx \tag{3.38}$$

where $\Omega^2 = \frac{\beta\mu}{\lambda^2\epsilon^2}$ and D_∞ is a solution of (3.27) and (3.28). The issue before us is the limiting value of the integral

$$E_\infty = \int_0^1 \left(D_{\infty x}^2 + \frac{\Omega^2}{2}\left(D_\infty^2 - 1\right)^2\right)(x)\,dx. \tag{3.39}$$

We note that if $\left[\frac{\Omega}{\pi}\right] = 0$, then E_∞ must be zero and D_∞ must be identically equal to either plus or minus one, whereas if $\left[\frac{\Omega}{\pi}\right]$ is a positive integer, there are exactly $\left[\frac{\Omega}{\pi}\right] + 1$ possible values for the limit E_∞, namely the numbers

$$E_{\infty,k} = \begin{cases} 0, \text{ if } k = 0 \\ \\ 2^{1/2}\Omega k D_{1,k} \int_{-1}^{1} \sqrt{2 - D_{1,k}^2 (1+s^2)}\sqrt{1-s^2}ds + \dfrac{\Omega^2 D_{1,k}^2 \left(2 - D_{1,k}^2\right)}{2}, \ 1 \le k \le \left[\dfrac{\Omega}{\pi}\right] \end{cases} \tag{3.40}$$

where $D_{1,k} \in (0,1)$ is the unique positive solution of

$$\frac{2^{1/2}k}{\Omega} \int_{-1}^{1} \frac{ds}{\sqrt{2 - D_{1,k}^2 (1+s^2)}\sqrt{1-s^2}} = 1. \tag{3.41}$$

The above identities follow from the observation that solutions of (3.27) and (3.28) have the first integral

$$D_{\infty x}^2 = \Omega^2 \left(2 - D_{1,k}^2 - D^2\right)\left(D_{1,k}^2 - D^2\right) \tag{3.42}$$

where $D_{1,k} \in (0,1)$. Solutions associated with (3.42) are periodic with half period, $L_{1/2}$, given by

$$L_{1/2} = \frac{2^{1/2}}{\Omega} \int_{-D_{1,k}}^{D_{1,k}} \frac{dD}{\sqrt{2 - D_{1,k}^2 - D^2}\sqrt{D_{1,k}^2 - D^2}} = \frac{2^{1/2}}{\Omega} \int_{-1}^{1} \frac{ds}{\sqrt{2 - D_{1,k}^2 (1+s^2)}\sqrt{1-s^2}} \tag{3.43}$$

and the identity (3.41) guarantees that $D_{\infty x}(1) = 0$. Corresponding to the solution of (3.41), there are exactly two equilibrium solutions of (3.27) and (3.28), one satisfying $D_\infty(0) = D_{1,k}$ and the other satisfying $D_\infty(0) = -D_{1,k}$, and each have exactly k interior zeros where $k = 1, 2, \ldots \left[\frac{\Omega}{\pi}\right]$. The estimates available to us give no information on the basin of attraction of each of these possible equilibria. This concludes section 3.

4. Numerical Simulations

In this section we present some numerical simulations for (1.1) – (1.3) and (1.8). All simulations were run with the following parameter values:

$$\beta = \mu = \epsilon = \lambda = 1 \text{ and } c = \frac{1}{2}. \tag{4.1}$$

With this choice, (1.6) becomes

$$u = D^3 \text{ and } \sigma = .25D\left(D^2 - 1\right). \tag{4.2}$$

The algorithm we chose to integrate the equations was a split advection-reaction scheme. During the first halfstep of the update we passively advect $m_1 - m_4$ and $\mathcal{C}$ according to

$$m_{1t} - \frac{1}{2} m_{1x} = 0 \text{ and } m_{2t} + \frac{1}{2} m_{2x} = 0, \tag{4.3}$$

$$m_{3t} - m_{3x} = 0 \text{ and } m_{4t} + m_{4x} = 0 \tag{4.4}$$

and

$$\mathcal{C}_t = 0. \tag{4.5}$$

From the updated fields $m_1 - m_4$ and $\mathcal{C}$ we compute the intermediate updates for u, v, D, and Q from (1.9). These serve as initial conditions for the second halfstep where we solve

$$u_t = v_t = 0 \tag{4.6}$$

and

$$D_t = \mathcal{C}, \; Q_t = -Q, \text{ and } \mathcal{C}_t = u - D^3 - \mathcal{C} \tag{4.7}$$

From this update we compute $m_1 - m_4$ using (1.10) and repeat the algorithm. We used second order updates for both the advection and reaction stages and chose $\Delta t = \Delta x = 0.1$ throughout. Both of the simulations we shall show involved solving the system in the quarter plane $x > 0$ and $t > 0$ and both were run with the constant boundary conditions:

$$D\left(0^+, t\right) = -1.2 \text{ and } u\left(0^+, t\right) = (-1.2)^3. \tag{4.8}$$

These boundary conditions were imposed during the advection stage through the use of $(1.9)_1$ and $(1.9)_3$ which enables us to compute boundary updates for m_2 and m_4 from those of m_1 and m_3.

The first simulation was run with the initial conditions

$$D\left(x, 0^+\right) = .6, \; u\left(x, 0^+\right) = (.6)^3, \text{ and } (V, Q, \mathcal{C})\left(x, 0^+\right) = (0,0,0), \; 0 < x \tag{4.9}$$

and the second was run with

$$D\left(x, 0^+\right) = (2)^{\frac{1}{3}}, \; u\left(x, 0^+\right) = 2, \text{ and } (V, Q, \mathcal{C}) = (0,0,0), \; 0 < x. \tag{4.10}$$

Both simulations exhibit the same phase change under compressive shock spanning the states $D_- = -1.2$ and $D_+ = 1.0609\ldots$ and propagating with speed $s_1 = .2378\ldots$. In the first simulation, the state $D = 1.0609\ldots$ is connected to $D = .6$ by a single phase shock which propagates at speed $s_2 = .3636\ldots$

It is worth noting that though the state $D = .6$ lies in the phase $D > \frac{1}{\sqrt{3}}$, the shock connecting $D = 1.0609\ldots$ to $D = .6$ is not monotone decreasing. This lack of monotonicity is because the point $(D, V) = (.6, 0)$ is a spiral attractor of the system (2.30).

In the second simulation the state $D = 1.0609\ldots$ connects to $D = 2^{\frac{1}{3}}$ via a rarefaction wave. The local wave speeds activated by this wave comprise the interval $[.4195, .4444]$. All of the features described above are easily seen in the snapshots at time = 1000.

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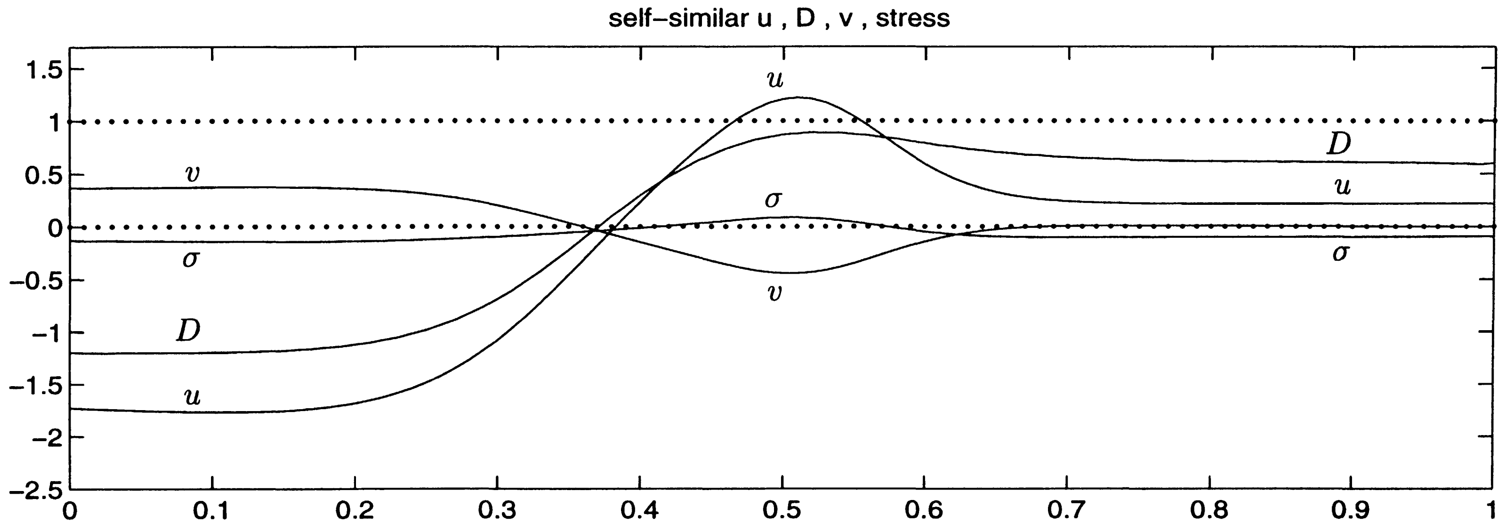
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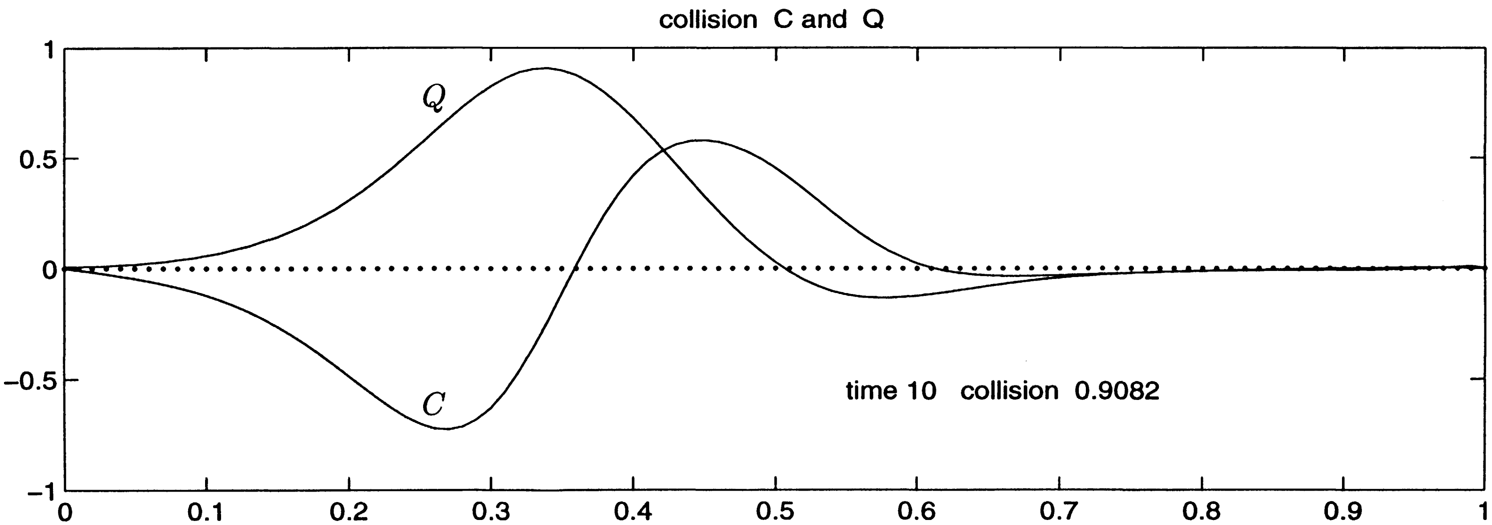
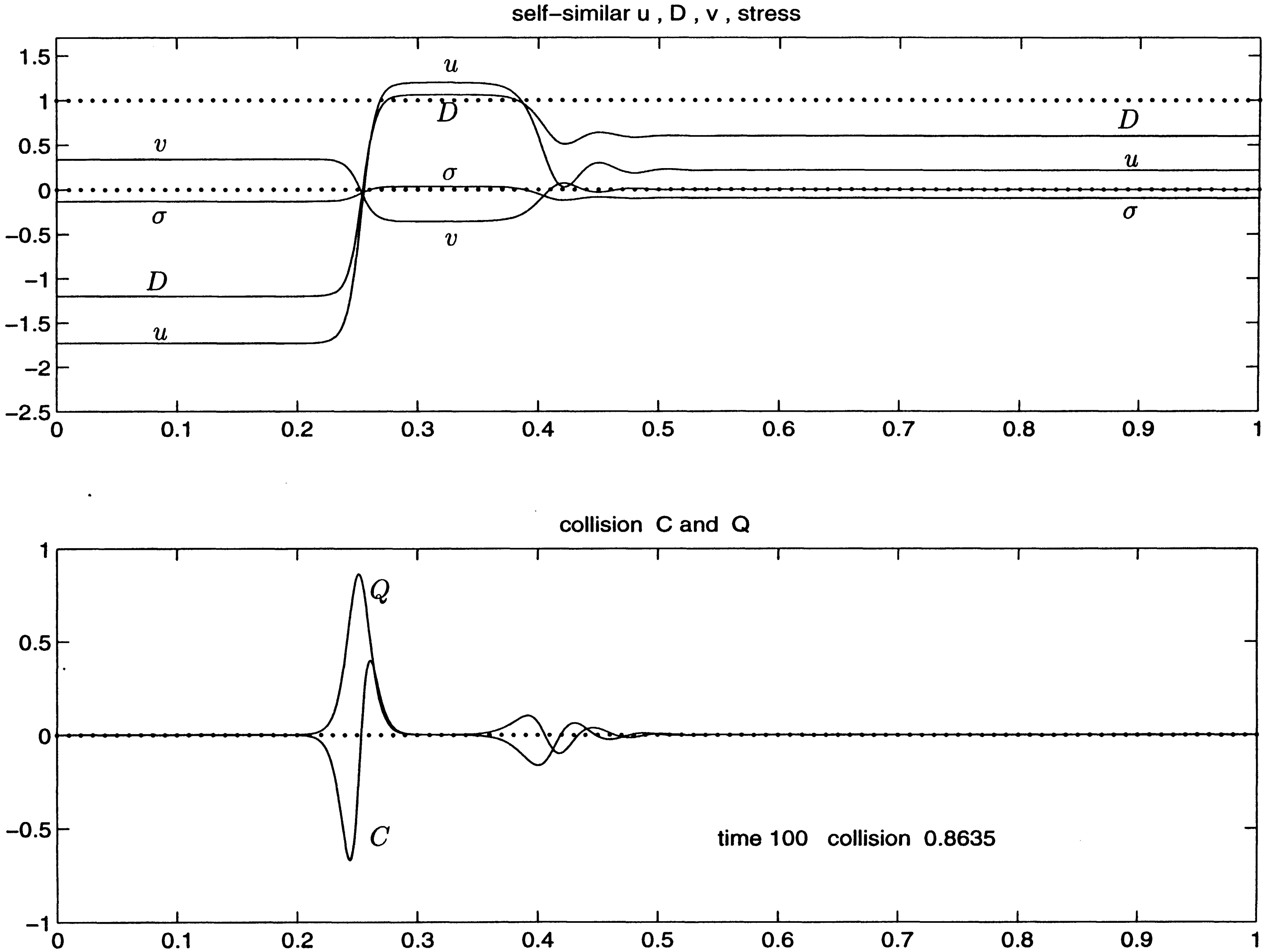
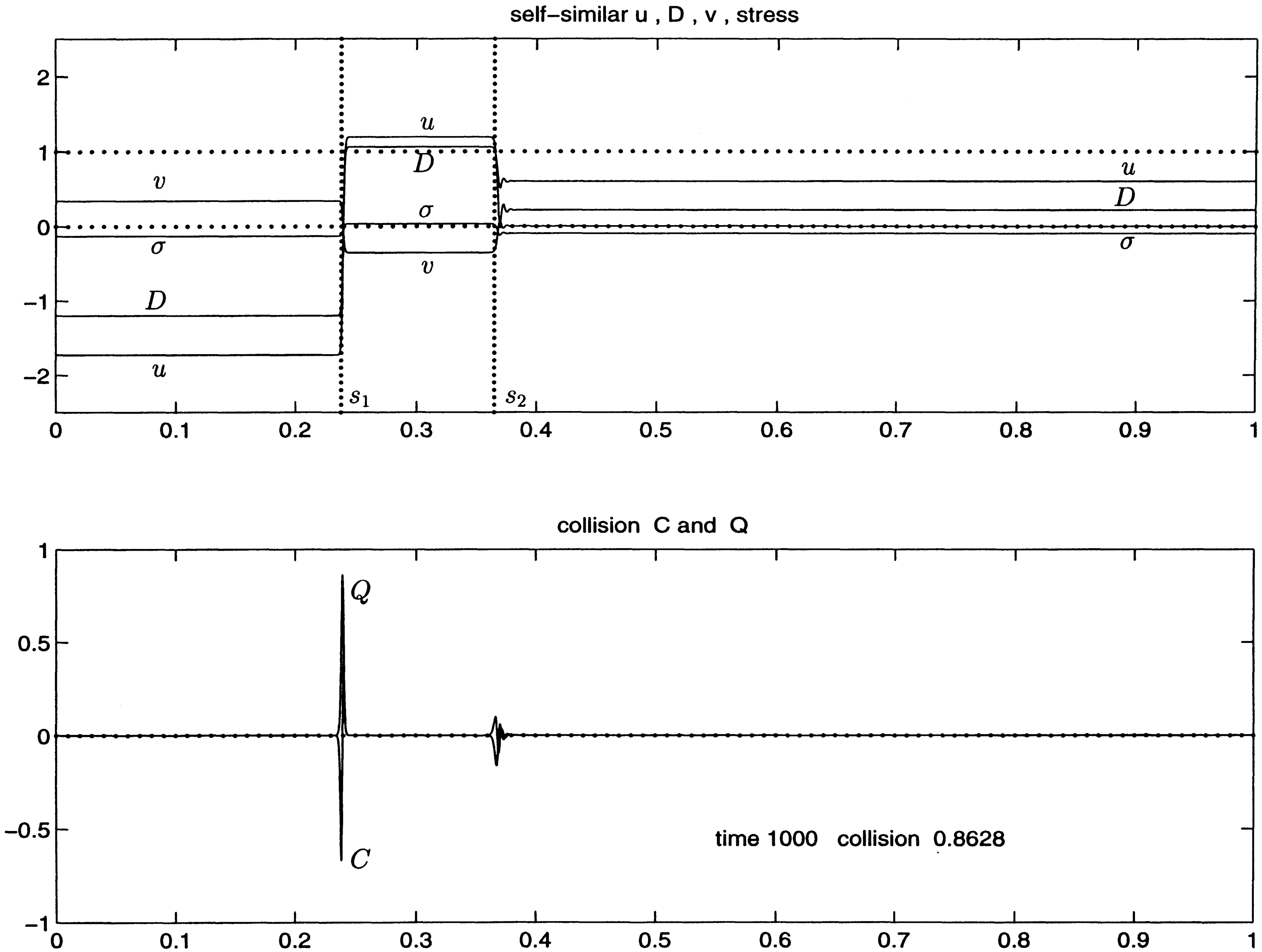
u
D
v
σ
time 10 collision 0.9082


collision C and Q
Q
C
time 10 collision 0.9082



self-similar u , D , v , stress
u
D
v
σ
D
u
σ
v
σ
D
u
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
collision C and Q
Q
C
time 100 collision 0.8635
1
0.5
0
−0.5
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1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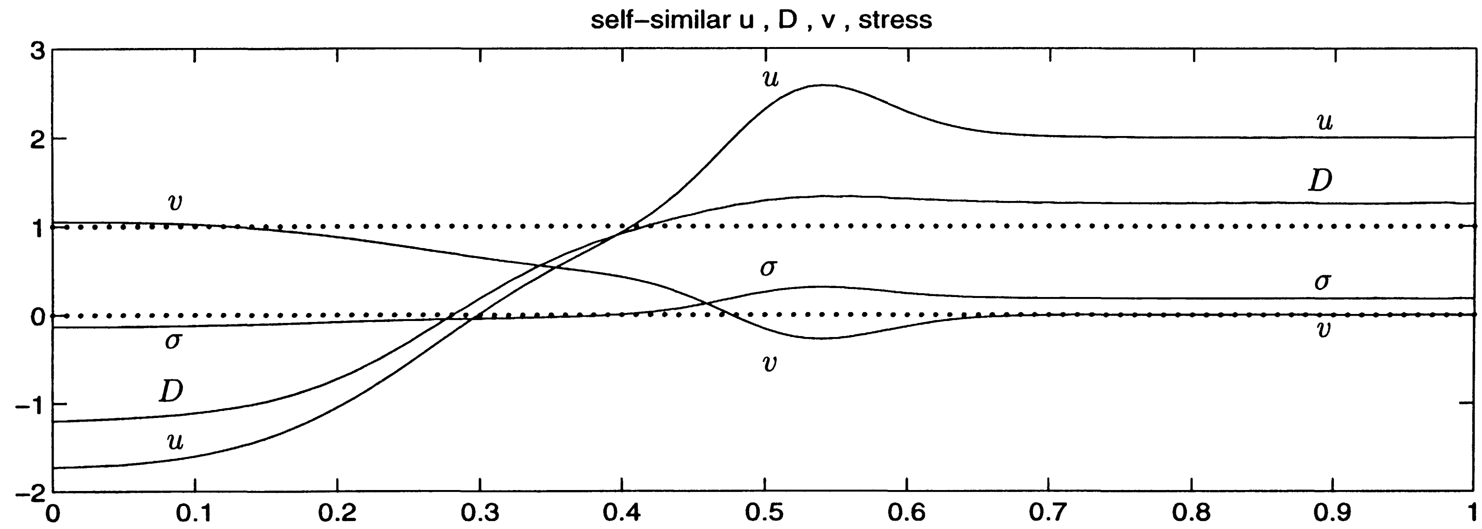
u
D
v
σ
s_1
s_2
collision C and Q
Q
C
time 1000 collision 0.8628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u
u
D
v
σ
σ
σ
v
v
D
u
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2
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0
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0
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0.2
0.3
0.4
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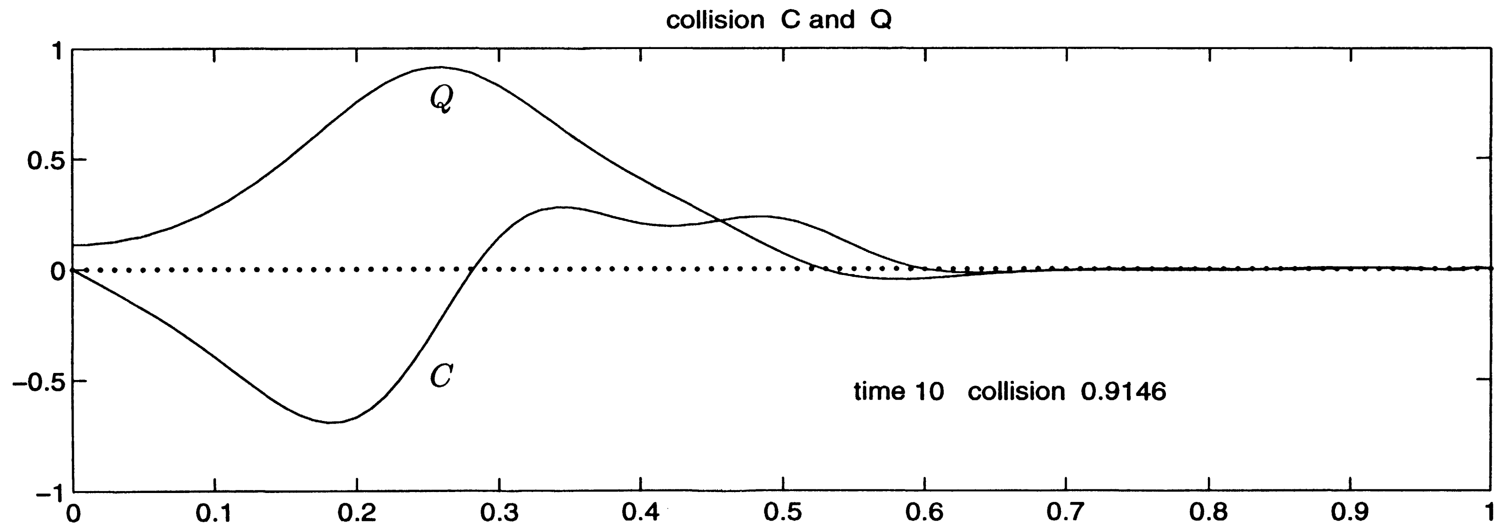
0.6
0.7
0.8
0.9
1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Q
C
time 10 collision 0.9146
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0
−0.5
−1
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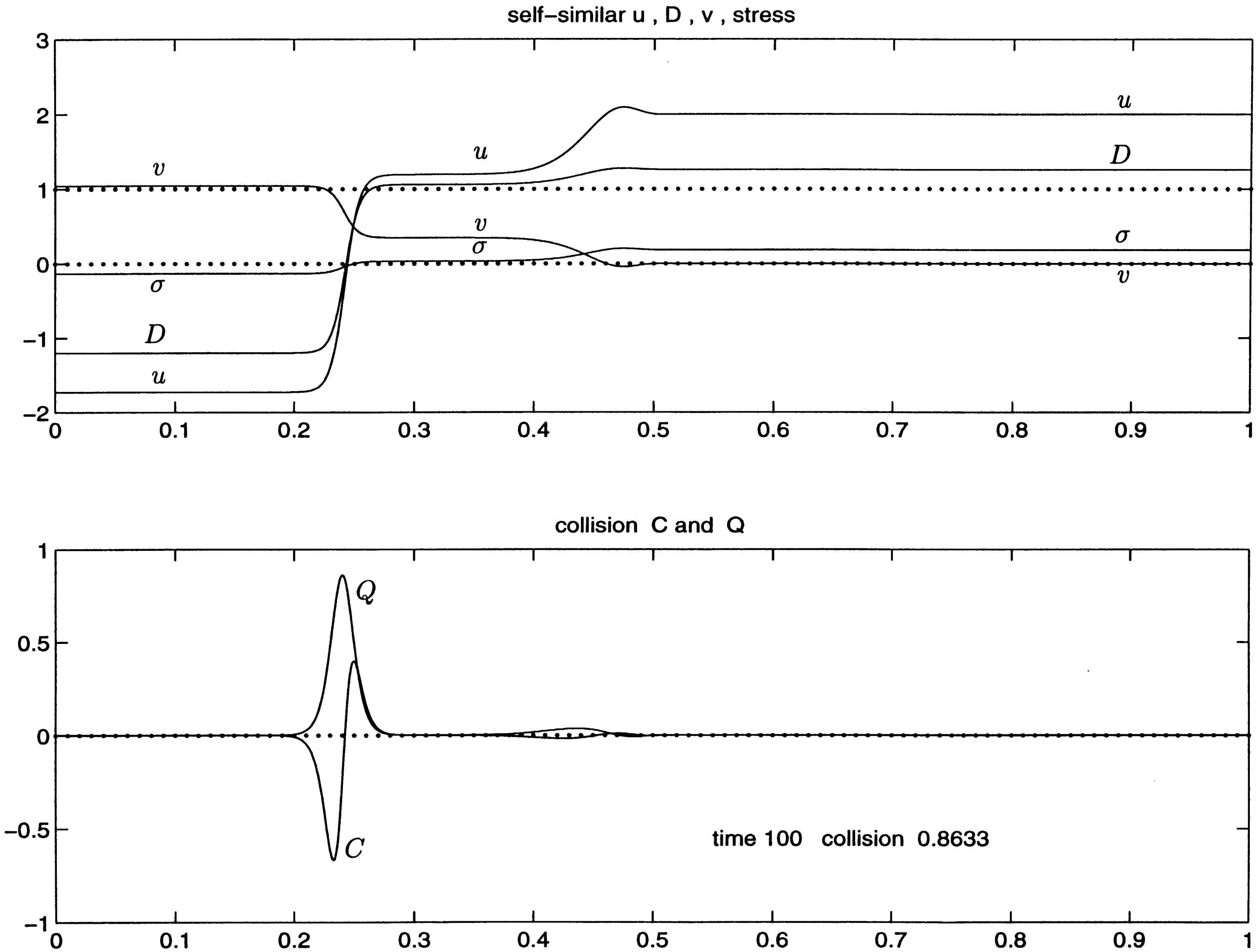
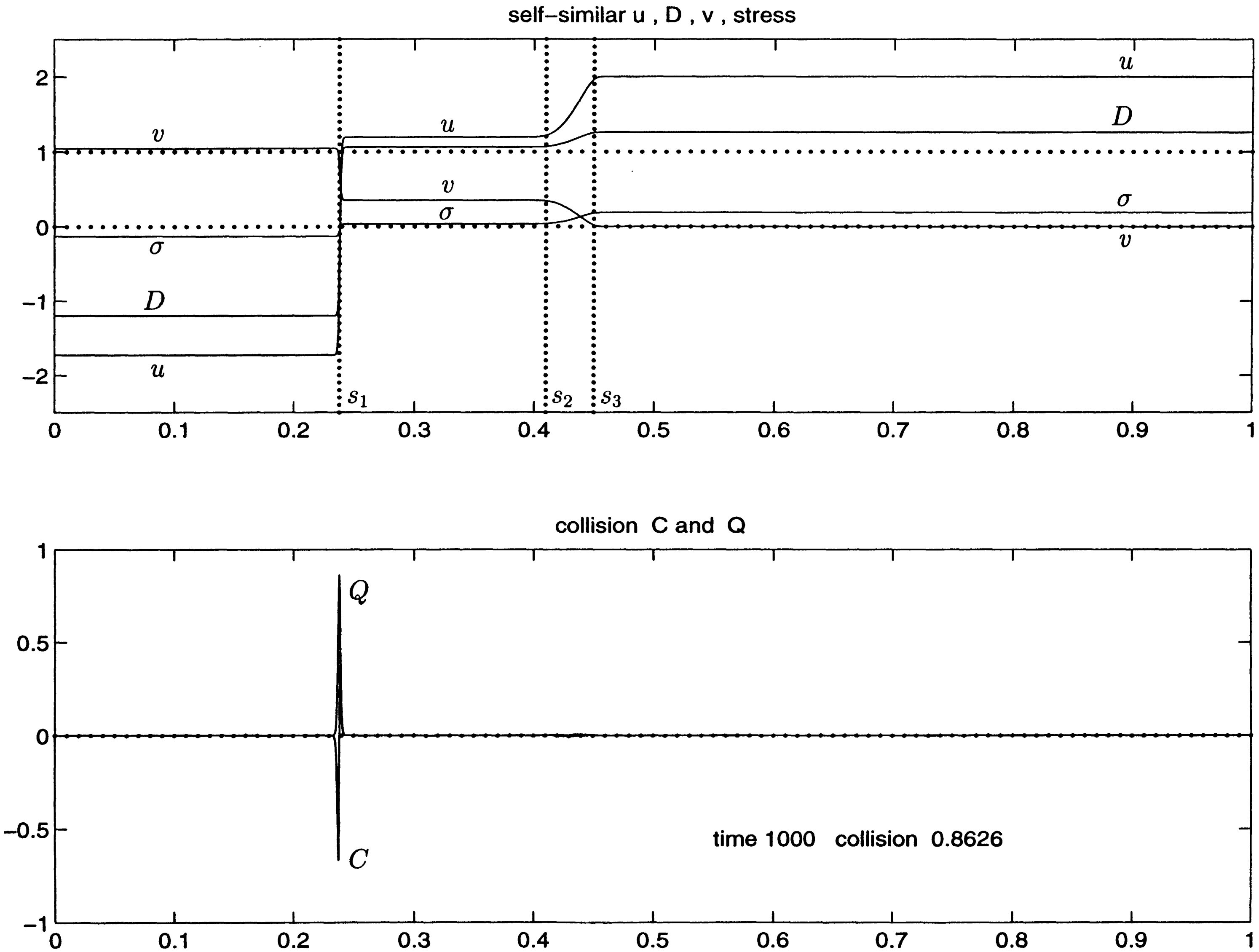
0.5
0.6
0.7
0.8
0.9
1



self-similar u , D , v , stress
collision C and Q
time 100 collision 0.8633



self-similar u , D , v , stress
u
D
v
u
v
σ
σ
σ
v
D
u
s_1
s_2
s_3
2
1
0
−1
−2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
collision C and Q
Q
C
time 1000 collision 0.8626
1
0.5
0
−0.5
−1