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Convergence of a Multigrid Method for Elliptic Equations with Highly Oscillatory Coefficients

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Abstract

Standard multigrid methods are not so effective for equations with highly oscillatory coefficients. New coarse grid operators based on homogenized operators are introduced to restore the fast convergence rate of multigrid methods. Finite difference approximations are used for the discretization of the equations. Convergence analysis is based on the homogenization theory. Proofs are given for a two-level multigrid method with the homogenized coarse grid operator for two classes of two dimensional elliptic equations with Dirichlet boundary conditions.

Key Words. elliptic equation, oscillation, finite difference, multigrid method, homogenization theory, convergence

AMS subject classifications. 65N06, 65N12, 65N55

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1 Introduction

Consider the multigrid method arising from the finite difference approximations to elliptic equations with highly oscillatory coefficients of the following type

$$L_\epsilon u_\epsilon(x) = -\sum_{i,j} \frac{\partial}{\partial x_i} a^\epsilon_{ij}(x) \frac{\partial}{\partial x_j} u_\epsilon(x) = f(x), \tag{1.1}$$

where $a^\epsilon_{ij}(x) = a_{ij}(x, \frac{x}{\epsilon}), a_{ij}(x,y) = a_{ji}(x,y)$, strictly positive, continuous and 1-periodic in each component of y. Also, the operator L_ϵ is uniformly elliptic. That is, there exist two positive constants q and Q independent of ϵ such that

$$0 < q\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq Q\sum_{i=1}^{n} \xi_i^2, \tag{1.2}$$

for any non-zero $\xi = (\xi_i) \in \Re^n$. Here, ϵ represents the length of the oscillations, and is assumed to be very small. Equations of the above type have important practical applications, examples including the study of elasticity and heat conduction for composite materials. One major mathematical technique to deal with these equations is the homogenization theory. The theory associates the original equation with its microstructure to some macrostructure effective equation that does not have oscillatory coefficients [3]. By homogenization, as ϵ approaches zero, the solution $u_\epsilon(x)$ of (1.1) converges to the solution $u(x)$ of the following homogenized equation,

$$L_\mu u(x) = \sum_{i,j} A_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x), \tag{1.3}$$

where A_{ij} is a constant given by

$$A_{ij} = \sum_k \int_0^1 (a_{ij} - a_{ik}\frac{\partial \chi^j(y)}{\partial y_k})dy. \tag{1.4}$$

Here $\chi^j(y)$ is 1-periodic in y such that

$$-\sum_{i,j} \frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j})\chi^k = -\sum_i \frac{\partial}{\partial y_i} a_{ik}(y), \tag{1.5}$$

and the homogenized operator L_μ retains the ellipticity property of the operator L_ϵ. We are interested in cases where the microstructure is important in itself. This means that the oscillatory equation (1.1) must be solved and can not be replaced by its homogenized equation (1.3), which only provides an average quantity. It could also be necessary to compute the ϵ-scale explicitly if the microstructure is not periodic and there is only an approximative homogenized form. The regular elliptic problem (1.1) can be discretized in many ways and typically the discretized solution u_ϵ^h converges as the stepsize $h \to 0$ for all $\epsilon > 0$. Standard estimate alone for the five point discretization gives

$$|u_\epsilon^h - u_\epsilon| \le Ch^2 \|u_\epsilon\|_{W^{4,\infty}},$$

which together with

$$\|u_\epsilon\|_{W^{4,\infty}} = O(\epsilon^{-4})$$

from [2] implies

$$|u_\epsilon^h - u_\epsilon| \le Ch^2\epsilon^{-4}$$

and converges if $h\epsilon^{-2} \to 0$ as $h \to 0$, $\epsilon \to 0$. For various extensions of the estimate, see [13]. Moreover the convergence theory in [1] guarantees the convergence for our problem (1.1) with centered difference approximation under the condition that h/ϵ is fixed to be irrational and $h \to 0$, $\epsilon \to 0$.

Multigrid methods are usually not so effective when applied to equation (1.1). Standard construction of coarse grid operators may generate operators with different properties from those of the fine grid operators [2, 4, 10]. To restore the high efficiency of the multigrid method, a new operator for the coarser grid operator is developed in [6, 7]. This new operator is called a homogenized coarse grid operator and is based on the homogenized form of the equation. For full multigrid or multigrid with more general coefficients, the homogenized operator can be numerically calculated from the finer grids based on the local solution of the so called cell problem [6]. For numerical examples on model problems and on the approximation of heat conduction in composite materials, we refer the reader to [7].

One difficulty for these problems (1.1) is that the smaller eigenvalues do not correspond to very smooth eigenfunctions. It is thus not easy to represent these eigenfunctions on the coarser grids. Nevertheless, after classical smoothing iterations on the fine grid, we know that the high frequency eigenmodes of the errors can be reduced and only the low frequency eigenmodes are significant. Thus, following [11], one may realize that the low frequency eigenmodes can be approximated by the corresponding homogenized eigenmodes. This is the reason why effective or homogenized operators are useful when defining the coarse grid operator.

In this paper, using the newly developed homogenized coarse grid operators, we analyze the convergence of the two-level multigrid method, applied to two classes of two dimensional problems as (1.1) with Dirichlet boundary conditions. In section 2, we consider equations with coefficients oscillatory along one coordinate direction only. In section 3, we consider equations with coefficients oscillatory diagonally. We show that as both ϵ and h go to zero, our two-level multigrid method converges when the number of smoothing iteration γ as a function of h is large enough and the ratio h/ϵ is not in a small resonance set. More precisely the convergence is proved under the following conditions:

- For the first case in section 2,

$$\gamma \geq C h^{-4/3} \ln h;$$

- For the second case in section 3,

$$\gamma \geq C h^{-5/3} \ln h,$$

if h belongs to the set $S(\epsilon, h_0)$ of Diophantine numbers,

$$\begin{aligned} S(\epsilon, h_0) &= \{0 \leq h \leq h_0 \mid |\frac{kh}{\epsilon} - i| \geq |\frac{\tau}{|k|^{3/2}}|, \\ & \quad for \quad i = 1, 2, \cdots, \quad [\frac{kh_0}{\epsilon}] + 1, 0 \neq k \in Z\}, \end{aligned} \tag{1.6}$$

where $S(\epsilon, h_0) \subseteq [0, h_0](h_0 > 0)$ with measure $|S(\epsilon, h_0)| \geq (1 - 3\tau)h_0$. Convergence under this condition is termed *the convergence essentially independent of* ϵ in [5]. Our analysis provides a theoretical explanation for the computational results presented in [6, 7]. The bounds on γ given above are overly pessimistic compared to the numerical experiments, but the dependence of γ on h exists in the computations in [6, 7]. The effect of not requiring $h \in S$ is also reflected in the numerical tests in [6, 12].

The above convergence on γ is quite modest but better than that for Jacobi and Gauss-Seidel iterative methods, even for the constant coefficient model problem. The optimal SOR iterative method gives an estimate $\gamma = O(h^{-1})$ for the model problem. For our oscillatory problem as presented in this paper, however the convergence rate under SOR is substantially slower than that under the multigrid method (see Table 1 in [7]). The SOR technique is efficient as a smoother for the oscillatory problem (see Figure 12 in [7]). If the coarse grid operator is defined by a direct arithmetic averaging, the eigenmode analysis considered in sections 2 and 3 generates an estimate $\gamma = O(h^{-2})$ for the multigrid method. The difference on the convergence between the correctly homogenized coarse grid operators and other operators is qualitatively consistent with the computational results in

[6, 7]. The $O(h^{-2})$ estimate means that there is no multigrid effect and the convergence is only produced by the smoothing iterations. As shown in [3], the l_2 difference between the inverse of the analytic operator L_ϵ and that of the corresponding homogenized operator L_μ is of the order $O(\epsilon)$. What this implies is that the eigenmode analysis considered in this paper cannot give estimates better than $\gamma = O(h^{-1})$, which is close to the estimate for an one dimensional problem $\gamma \geq Ch^{-6/5}\ln h$ in [12]. In special cases, it is possible to design prolongation, restriction and coarse grid operators under which the resulting method corresponds to a direct solver [8]. This type of algorithm and methods based on special discretizations with built-in a priori knowledge of the oscillatory behavior is outside the scope of this paper.

Since in the sequel of the paper the following lemma is often applied, we introduce it here.

Lemma 1.1 *[5] Suppose $g(x,y) \in C^3([0,1]\times[0,1])$ and is 1-periodic in y. Let $x_k = kh, k = 1,\cdots,N$ and $Nh = 1$. If $h \in S(\epsilon, h_0)$, we have*

$$(1.7) \qquad |\sum_{k=1}^{j} g(x_k, x_k/\epsilon)h - \int_0^{x_j}\int_0^1 g(x,y)dydx| \leq C\frac{h}{\tau}, \quad 0 \leq j \leq N.$$

Throughout the paper, we denote the domain $(0,1)\times(0,1)$ by Ω, and $\bar{\Omega}/\Omega$ by $\partial\Omega$. We discretize the domain by the same number of grid points N with equal step size $h = \frac{1}{N}$ both in x- and y-directions. The step size h is chosen to belong to $S(\epsilon, h_0)$. And, the ratio of h to the wavelength ϵ is fixed to be a strictly irrational number. $\bar{\Omega}_h$ denotes the set of grid points $(ih, jh) \in \bar{\Omega}$, Ω_h for $(ih, jh) \in \Omega$, and $\partial\Omega_h$ for $(ih, jh) \in \partial\Omega$. C and c represent some constants that are independent of ϵ and h. D_+^i and D_-^i are standard forward and backward finite difference operators in the x-direction; D_+^j and D_-^j are similarly defined for the y-direction. $\|\cdot\|_h$ denotes the discrete L_2−norm, indexed by $1,\cdots,N-1$.

2 Oscillation Along a Coordinate Direction

2.1 Model Equation

Consider as a special case of (1.1) a two-dimensional elliptic problem with coefficients oscillatory in x-direction only

$$(2.1) \qquad \begin{aligned} -\frac{\partial}{\partial x}a_\epsilon(x)\frac{\partial\phi_\epsilon}{\partial x} - \frac{\partial}{\partial y}a_\epsilon(x)\frac{\partial\phi_\epsilon}{\partial y} &= f(x,y), \quad (x,y) \in \Omega, \\ \phi_\epsilon(x,y) &= 0, \quad (x,y) \in \partial\Omega, \end{aligned}$$

where $a_\epsilon(x)$ is a strictly positive continuous function, $a_\epsilon(x) = a(x/\epsilon) = a(x/\epsilon + 1)$, and the operator on the left hand side of (2.1) satisfies the property of (1.2). From (1.3), the corresponding homogenized equation of (2.1) is:

$$\begin{aligned} -\mu\frac{\partial^2\phi}{\partial x^2} - \bar{a}\frac{\partial^2\phi}{\partial y^2} &= f(x,y), \quad (x,y) \in \Omega, \\ \phi(x,y) &= 0, \quad (x,y) \in \partial\Omega, \end{aligned} \tag{2.2}$$

where $\mu = (\int_0^1 1/a(x)dx)^{-1}$ and $\bar{a} = \int_0^1 a(x)dx$ are the harmonic and the arithmetical averages of the coefficient $a(x)$, respectively. As ϵ goes to zero, the solution ϕ_ϵ of (2.1) converges to the solution ϕ of (2.2).

Consider now a corresponding discretized equation of (2.1),

$$L_{\epsilon,h}u_{ij}^h = -D_+^i a_i D_-^i u_{ij}^h - D_+^j b_i D_-^j u_{ij}^h = f_{ij}^h, \quad (i,j) \in \Omega_h \tag{2.3}$$

where $a_i = a_\epsilon(x_i - \frac{h}{2})$, $b_i = a_\epsilon(x_i)$, $\quad i = 1, \cdots, N$. Denote the discretization of the homogenized operator $-\mu\frac{\partial^2}{\partial x^2} - \bar{a}\frac{\partial^2}{\partial y^2}$ in (2.2) by

$$L_{\mu,h} = -\mu_h D_+^i D_-^i - b_h D_+^j D_-^j, \tag{2.4}$$

where $\mu_h = (h\sum_{i=1}^N \frac{1}{a_i})^{-1}, b_h = \sum_{i=1}^N b_i h$. The operator of the two-level method with the homogenized coarse grid operator [6, 7] can be expressed as

$$M = (I - I_H^h L_H^{-1} I_h^H L_{\epsilon,h})S^\gamma. \tag{2.5}$$

For simplicity, in the sequel of the paper, I_h^H and I_H^h always denote the weighting restriction and bilinear interpolation, respectively. The coarse grid operator L_H is taken to be the corresponding homogenized operator $L_{\mu,H}$, and the smoothing operator

$$S = I - \alpha L_{\epsilon,h},$$

where α is the inverse of the largest eigenvalue of the fine grid operator $L_{\epsilon,h}$, has order of h^{-2}.

2.2 Convergence Analysis

Consider a simplified operator M_1 defined by

$$M_1 = (I - L_{\mu,h}^{-1}L_{\epsilon,h})(I - \alpha L_{\epsilon,h})^\gamma = (I - L_{\mu,h}^{-1}L_{\epsilon,h})S^\gamma. \tag{2.6}$$

Theorem 2.1 *If the ratio of h to ϵ is fixed and $h \in S(\epsilon, h_0)$, then there exist two constants C and ρ_0 such that*

$$\|M_1\|_h \leq \rho_0 < 1, \tag{2.7}$$

whenever $\gamma \geq Ch^{-1-1/3}ln(h)$.

The proof of Theorem 2.1 uses the following lemmas.

Lemma 2.1 *Assume*

$$Z_i = \frac{h}{\epsilon}(i - \mu_h \sum_{k=1}^{i} \frac{1}{a_k}), \quad i = 0, \cdots, N.$$

Then, Z_i is bounded and

$$L_{\epsilon,h} Z_i = -\frac{1}{\epsilon} D_+^i a_i, \quad i = 1, \cdots, N-1.$$

Proof. By Lemma 1.1, taking $j = N$ we have

$$|\mu_h^{-1} - \int_0^1 \frac{1}{a(y)} dy| \leq Ch.$$

Hence,

$$\frac{h}{2\epsilon}(i - \mu_h \sum_{k=1}^{i} \frac{1}{a_k}) = \frac{1}{2\epsilon}(\frac{ih}{\mu_h} - ih \int_0^1 \frac{1}{a(y)} dy - \sum_{k=1}^{i} \frac{1}{a_k} h + ih \int_0^1 \frac{1}{a(y)} dy) = O(1).$$

Thus Z_i is bounded for $i = 0, \cdots, N$ and

$$a_i(1 - \epsilon D_-^i Z_i) = \mu_h, \quad i = 1, \cdots, N-1.$$

□

Lemma 2.2 *Assume η_i satisfies*

$$\begin{aligned} -\epsilon D_-^i \eta_i &= b_i - b_h, \quad i = 1, \cdots, N, \\ \eta_0 &= 0. \end{aligned}$$

Then, η_i is bounded.

Proof. The proof follows directly from Lemma 1.1. □

Lemma 2.3 *Assume U_{ij} satisfies*

$$\begin{aligned} L_{\mu,h}U_{ij} &= \lambda_\epsilon \phi^h_{ij}, \quad (i,j)\in\Omega_h, \\ U_{ij} &= 0, \quad (i,j)\in\partial\Omega_h, \end{aligned} \tag{2.8}$$

where $(\lambda_\epsilon, \phi^h_{ij})$ is a normalized eigenpair for $L_{\epsilon,h}$. That is,

$$\begin{aligned} L_{\epsilon,h}\phi^h_{ij} &= \lambda_\epsilon \phi^h_{ij}, \quad (i,j)\in\Omega_h, \\ \phi^h_{ij} &= 0, \quad (i,j)\in\partial\Omega_h. \end{aligned}$$

Then,

$$\|U\|_1^2 = \sum_{i,j=1}^{N} (D^i_- U_{ij} h)^2 = O(\lambda_\epsilon); \tag{2.9}$$

$$\sum_{i,j=1}^{N-1} ((D^i_+ D^i_- U_{ij})^2 + (D^i_- D^j_- U_{ij})^2 + (D^j_+ D^j_- U_{ij})^2)h^2 = O(\lambda_\epsilon^2). \tag{2.10}$$

Proof. Multiplying by U_{ij} on both sides of

$$L_{\mu,h}U_{ij} = L_{\epsilon,h}\phi^h_{ij},$$

and then summing by parts implies (2.9).

For (2.10), note first for any grid function U_{ij} vanishing on $\partial\Omega_h$ we have

$$\begin{aligned} \sum_{i,j=1}^{N-1} (D^i_+ D^i_- U_{ij})(D^j_+ D^j_- U_{ij}) &= \sum_{i,j=1}^{N} (D^i_+ D^i_- U_{ij})(D^j_+ D^j_- U_{ij}) \\ &= \sum_{i,j=1}^{N} (D^i_- D^j_- U_{ij})^2 \geq 0. \end{aligned} \tag{2.11}$$

Multiply on both sides of (2.8) by $D^i_+ D^i_- U_{ij}$ and then sum over i, j,

$$\begin{aligned} &\sum_{i,j=1}^{N-1} \mu_h (D^i_+ D^i_- U_{ij})^2 \\ \leq\ &\sum_{i,j=1}^{N-1} \mu_h (D^i_+ D^i_- U_{ij})^2 + \sum_{i,j=1}^{N-1} b_h (D^i_+ D^i_- U_{ij})(D^j_+ D^j_- U_{ij}) \\ =\ &\sum_{i,j=1}^{N-1} \lambda_\epsilon \phi_{ij} (D^i_+ D^i_- U_{ij}). \end{aligned}$$

Thus

$$\sum_{i,j=1}^{N-1} \mu_h (D_+^i D_-^i U_{ij})^2 \le C\lambda_\epsilon^2, \tag{2.12}$$

for some constant C. Similarly

$$\sum_{i,j=1}^{N-1} \mu_h (D_+^j D_-^j U_{ij})^2 \le C\lambda_\epsilon^2. \tag{2.13}$$

The rest of the proof follows easily from (2.11), (2.12) and (2.13). □

Lemma 2.4 *Assume ϕ_{ij}^h, U_{ij} satisfies conditions in Lemma 2.3. Then,*

$$\|\phi^h - U\|_h = O(\epsilon\lambda_\epsilon). \tag{2.14}$$

Proof. Introduce the following discrete function

$$G_{ij} = U_{ij} - \epsilon Z_i D_-^i U_{ij} - \phi_{ij}^h, \quad (i,j) \in \bar{\Omega}_h,$$

where Z_i is defined as in Lemma 2.1. Such G_{ij} vanishes at boundary, i.e.,

$$G_{ij} = 0, \quad (i,j) \in \partial\Omega_h.$$

Simple calculation then implies

$$\begin{aligned} & L_{\epsilon,h} G_{ij} \\ = \ & -D_-^i(\epsilon a_{i+1} Z_i D_+^i D_-^i U_{ij}) - \epsilon b_i Z_i D_+^j D_-^j D_-^i U_{ij} + (b_h - b_i) D_+^j D_-^j U_{ij} \\ = \ & -D_-^i(\epsilon a_{i+1} Z_i D_+^i D_-^i U_{ij}) - \epsilon b_i Z_i D_+^j D_-^j D_-^i U_{ij} + \epsilon D_-^i \eta_i D_+^j D_-^j U_{ij}, \end{aligned}$$

where $(i,j) \in \Omega_h$ and η_i is as in Lemma 2.2. Multiply G_{ij} on both sides and then sum over i, j,

$$\begin{aligned} & \sum_{i,j=1}^{N} L_{\epsilon,h} G_{ij} G_{ij} h^2 \\ = \ & \epsilon\Big(\sum_{i,j=1}^{N-1} a_{i+1} Z_i D_+^i D_-^i U_{ij} D_+^i G_{ij} + \sum_{i=1}^{N-1}\sum_{j=0}^{N-1} b_i Z_i D_+^j D_-^i U_{ij} D_+^j G_{ij} \\ & - \sum_{i,j=1}^{N-1} \eta_i D_+^j D_-^j U_{ij} D_+^i G_{ij} - \sum_{i,j=1}^{N} \eta_{i-1} D_-^i D_-^j D_+^j U_{ij} G_{ij}\Big) h^2 \end{aligned}$$

$$
\begin{aligned}
&= \epsilon(\sum_{i,j=1}^{N-1} a_{i+1} Z_i D_+^i D_-^i U_{ij} D_+^i G_{ij} + \sum_{i=1}^{N-1}\sum_{j=0}^{N-1} b_i Z_i D_+^j D_-^i U_{ij} D_+^j G_{ij} \\
&\quad - \sum_{i,j=1}^{N-1} \eta_i D_+^j D_-^j U_{ij} D_+^i G_{ij} + \sum_{i=1}^{N-1}\sum_{j=0}^{N-1} \eta_i D_+^i D_+^j U_{ij} D_+^j G_{i+1j})h^2 \\
&\leq C\epsilon(\sqrt{\sum_{i,j=1}^{N-1} (D_+^i D_-^i U_{ij} h)^2 + \sum_{i,j=1}^{N-1} (D_+^j D_-^j U_{ij} h)^2 + \sum_{i,j=1}^{N} (D_-^j D_-^i U_{ij} h)^2)} \\
&\quad \times \sqrt{\sum_{i,j=1}^{N} (D_-^i G_{ij} h)^2 + (D_-^j G_{ij} h)^2}.
\end{aligned}
$$

By Lemma 2.3,

$$
\sum_{i,j=1}^{N} L_{\epsilon,h} G_{ij} G_{ij} h^2 \leq C\epsilon\lambda_\epsilon \sqrt{\sum_{i,j=1}^{N} (D_-^i G_{ij} h)^2 + (D_-^j G_{ij} h)^2}. \tag{2.15}
$$

Hence,

$$
\sqrt{\sum_{i,j=1}^{N} (D_-^i G_{ij} h)^2 + (D_-^j G_{ij} h)^2} \leq O(\epsilon\lambda_\epsilon).
$$

By Poincare inequality,

$$
\|G\|_h = O(\epsilon\lambda_\epsilon),
$$

and by Lemma 2.1 and Lemma 2.3,

$$
\|\epsilon Z D_-^i U\|_h \leq \epsilon \max_{0\leq i\leq N} |Z_i| \|U\|_1 \leq C\epsilon\sqrt{\lambda_\epsilon}.
$$

Hence,

$$
\|\phi^h - U\|_h \leq \|G\|_h + \|\epsilon Z D_-^i U\|_h \leq C\epsilon\lambda_\epsilon,
$$

which implies

$$
\|(I - L_{\mu,h}^{-1} L_{\epsilon,h})\phi^h\|_h \leq C\epsilon\lambda_\epsilon. \tag{2.16}
$$

The proof of lemma is completed. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Denote the eigenvalues of $L_{\epsilon,h}$ and Δ_h (Laplacian operator) by λ_{ij}^ϵ and λ_{ij}, respectively, for $i, j = 1, \cdots, N-1$. Then, by dividing the set of eigenvalues into two subsets, say $\{\lambda_{ij}^{\epsilon}, \quad i^2+j^2 \le k_0^2\}$ and $\{\lambda_{ij}^{\epsilon}, \quad (N-1)^2 \ge i^2+j^2 \ge k_0^2\}$ for some k_0, we can split the eigenspace of $L_{\epsilon,h}$ into two orthogonal subspaces. Namely, the space of low frequency expanded by the eigenfunctions whose corresponding eigenvalues belong to the first set, and that of high frequency expanded by the eigenfunctions whose corresponding eigenvalues belong to the second set. By minimax principle of eigenvalues, it follows easily

$$c\lambda_{ij} \le \lambda_{ij}^{\epsilon} \le C\lambda_{ij},$$

for some constants c and C.

For any normalized vector ξ such that

$$\xi = \Sigma_{i,j=1}^{N-1}\beta_{ij}\phi_{ij}^{\epsilon},$$

where $\Sigma_{i,j=1}^{N-1}\beta_{ij}^2 = 1$, we have

$$\begin{aligned} M_1\xi &= \Sigma_{i,j=1}^{N-1}\beta_{ij}M_1\phi_{ij}^{\epsilon} \\ &= \Sigma_{i^2+j^2\le k_0^2}\beta_{ij}M_1\phi_{ij}^{\epsilon} + \Sigma_{i^2+j^2>k_0^2}\beta_{ij}M_1\phi_{ij}^{\epsilon}. \end{aligned}$$

Thus,

$$\|M_1\xi\|_h \le \|\Sigma_{i^2+j^2\le k_0^2}\beta_{ij}M_1\phi_{ij}^{\epsilon}\|_h + \|\Sigma_{i^2+j^2>k_0^2}\beta_{ij}M_1\phi_{ij}^{\epsilon}\|_h. \tag{2.17}$$

For the rest of the proof, we want to show $\|M_1\xi\|_h < 1$. To do this, we show that the two terms on the right hand side of (2.17) are both less than $1/2$ in two separate steps.

Step 1: The low frequency subspace.

$$\begin{aligned} I_1 &= \|\Sigma_{i^2+j^2\le k_0^2}\beta_{ij}M_1\phi_{ij}^{\epsilon}\|_h \\ &= \|\Sigma_{i^2+j^2\le k_0^2}\beta_{ij}(I-L_{\mu,h}^{-1}L_{\epsilon,h})(I-\alpha L_{\epsilon,h})^{\gamma}\phi_{ij}^{\epsilon}\|_h \\ &\le \Sigma_{i^2+j^2\le k_0^2}|\beta_{ij}|\|(I-L_{\mu,h}^{-1}L_{\epsilon,h})\phi_{ij}^{\epsilon}\|_h. \end{aligned} \tag{2.18}$$

By Lemma 2.4,

$$\|(I-L_{\mu,h}^{-1}L_{\epsilon,h})\phi_{ij}^{\epsilon}\|_h \le C\epsilon\lambda_{ij}^{\epsilon} \le C\epsilon\lambda_{ij}. \tag{2.19}$$

The corresponding eigenvalues of the Laplacian operator Δ_h can be explicitly expressed as

$$\lambda_{ij} = \frac{4}{h^2}(sin^2(\frac{\pi}{2}ih) + sin^2(\frac{\pi}{2}jh)), \quad i,j = 1,\cdots,N-1.$$

It follows from Taylor expansion,

$$\lambda_{ij} = (i^2+j^2)(C + O((ih)^2 + (jh)^2)).$$

Hence,

$$I_1 \le C\epsilon \Sigma_{i^2+j^2\le k_0^2}|\beta_{ij}|(i^2+j^2).$$

By the constraint $\Sigma_{i^2+j^2\le k_0^2}\beta_{ij}^2 \le 1$,

$$I_1 \le C\epsilon k_0^3.$$

Since the ratio of h to ϵ is fixed to be an irrational number, h has the same order as ϵ. We thus have

$$I_1 \le Chk_0^3. \tag{2.20}$$

Therefore, in order to make $I_1 < \frac{1}{2}$, it's sufficient to have

$$k_0 \le Ch^{-1/3}. \tag{2.21}$$

Step 2: The high frequency subspace.

$$\begin{aligned} I_2 &= \|\Sigma_{i^2+j^2>k_0^2}\beta_{ij}M_1\phi_{ij}^\epsilon\|_h \\ &\le \|L_{\epsilon,h}^{-\frac{1}{2}}(I - L_{\epsilon,h}^{\frac{1}{2}}L_{\mu,h}^{-1}L_{\epsilon,h}^{\frac{1}{2}})\|_h\|L_{\epsilon,h}^{\frac{1}{2}}(1-\alpha L_{\epsilon,h})^\gamma(\Sigma_{i^2+j^2>k_0^2}\beta_{ij}\phi_{ij}^\epsilon)\|_h. \end{aligned}$$

Since $\|L_{\epsilon,h}^{-\frac{1}{2}}\|_h \le C$ and $\|I - L_{\epsilon,h}^{\frac{1}{2}}L_{\mu,h}^{-1}L_{\epsilon,h}^{\frac{1}{2}}\|_h \le C$,

$$\begin{aligned} I_2 &\le C\sqrt{\Sigma_{i^2+j^2>k_0^2}\beta_{ij}^2\|L_{\epsilon,h}^{\frac{1}{2}}(1-\alpha L_{\epsilon,h})^\gamma\phi_{ij}^\epsilon\|_h^2} \\ &\le C \max_{k_0^2\le\lambda\le\lambda_{N-1N-1}^\epsilon}|\lambda^{\frac{1}{2}}(1-\alpha\lambda)^\gamma|. \end{aligned} \tag{2.22}$$

Assume $1+2\gamma > \frac{1}{\alpha k_0^2} = \frac{\lambda_{N-1N-1}^\epsilon}{k_0^2}(\ge C\frac{h^{-2}}{k_0^2})$. Then

$$I_2 \le Ck_0(1-\alpha k_0^2)^\gamma. \tag{2.23}$$

For $I_2 \le \frac{1}{2}$, it is sufficient to have

$$\gamma \ge C\frac{h^{-2}}{k_0^2}lnk_0(\ge \frac{1}{\alpha k_0^2}). \tag{2.24}$$

Combining (2.21) and (2.24), we have

$$\gamma \ge Ch^{-1-\frac{1}{3}}\ln h.$$

The proof is completed. $\square$

We now present our main result.

Theorem 2.2 *There exists a constant C such that the operator M defined by (2.5) satisfies*

$$\rho(M) \leq \rho_0 < 1,$$

whenever h belongs to $S(\epsilon, h_0)$ and

$$\gamma \geq C h^{-1-1/3} ln h.$$

Before we carry out the proof, we need to establish the following lemma. Let

$$M_2 = (L_{\mu,h}^{-1} - I_H^h L_H^{-1} I_h^H) L_{\epsilon,h} S^\gamma.$$

Lemma 2.5 *For some constant C,*

$$\|M_2\|_h \leq \frac{C}{\gamma}.$$

Proof. Since $L_{\mu,h}$ and $L_H = L_{\mu,H}$ are the homogenized operators defined respectively on fine and coarse grid with constant coefficients, they are well behaved and satisfy the following approximate property [9, 10],

$$\|L_{\mu,h}^{-1} - I_H^h L_H^{-1} I_h^H\|_h \leq C h^2. \tag{2.25}$$

for some constant C. Therefore,

$$\begin{aligned} \|M_2\|_h &\leq \|L_{\mu,h}^{-1} - I_H^h L_H^{-1} I_h^H\|_h \|L_{\epsilon,h} S^\gamma\|_h \\ &\leq C h^2 h^{-2}/\gamma \\ &\leq \frac{C}{\gamma}. \end{aligned} \tag{2.26}$$

□

Proof of Theorem 2.2. Note that

$$M = M_1 + M_2.$$

Therefore,

$$\|M\|_h \leq \|M_1\|_h + \|M_2\|_h.$$

Since $\rho(M) \leq \|M\|_h$, the rest of the proof follows easily from Theorem 2.1 and Lemma 2.5.

□

3 Oscillation Along the Diagonal Direction

3.1 Model Equation

Consider as another special case of (1.1) a two-dimensional elliptic problem with coefficients oscillatory diagonally

$$
(3.1) \qquad \begin{aligned} -\frac{\partial}{\partial x} a_\epsilon(x-y)\frac{\partial \phi_\epsilon}{\partial x} - \frac{\partial}{\partial y} a_\epsilon(x-y)\frac{\partial \phi_\epsilon}{\partial y} &= f(x,y), \quad (x,y)\in\Omega, \\ \phi_\epsilon(x,y) &= 0, \quad (x,y)\in\partial\Omega, \end{aligned}
$$

where $a_\epsilon(x)$ is a strictly positive continuous function, $a_\epsilon(x) = a(x/\epsilon) = a(x/\epsilon+1)$, and the operator on the left hand side of (3.1) satisfies the property of (1.2). It follows from (1.3) that the corresponding homogenized equation of (3.1) is

$$
(3.2) \qquad \begin{aligned} -\frac{\bar{a}+\mu}{2}\frac{\partial^2\phi}{\partial x^2} + (\bar{a}-\mu)\frac{\partial^2\phi}{\partial x\partial y} - \frac{\bar{a}+\mu}{2}\frac{\partial^2\phi}{\partial y^2} &= f(x,y), \quad (x,y)\in\Omega, \\ \phi(x,y) &= 0, \quad (x,y)\in\partial\Omega, \end{aligned}
$$

where $\bar{a} = \int_0^1 a(x)dx$, and $\mu = (\int_0^1 1/a(x)dx)^{-1}$. As ϵ goes to zero, the solution ϕ_ϵ of (3.1) converges to the solution ϕ of (3.2).

Now, consider a corresponding discretized equation of (3.1),

$$
(3.3) \qquad L_{\epsilon,h}u_{ij}^h = -D_+^i a_{ij} D_-^i u_{ij}^h - D_+^j b_{ij} D_-^j u_{ij}^h = f_{ij}^h, \quad (i,j)\in\Omega_h
$$

where

$$
a_{ij} = a(\frac{x_i - h/2 - y_j}{\epsilon}), \quad b_{ij} = a(\frac{x_i - y_j + h/2}{\epsilon}), \quad (i,j)\in\bar{\Omega}_h.
$$

Here, we assume the discretized coefficients have the following property,

$$
a_{j0} = b_{0N-j+1}, \quad j = 0,\cdots,N.
$$

Denote the discretization of the homogenized operator

$$
-\frac{\bar{a}+\mu}{2}\frac{\partial^2}{\partial x^2} + (\bar{a}-\mu)\frac{\partial^2}{\partial x\partial y} - \frac{\bar{a}+\mu}{2}\frac{\partial^2}{\partial y^2}
$$

in (3.2) by

$$
(3.4) \qquad L_{\mu,h} = -\frac{\mu_h + a_h}{2}(D_+^i D_-^i + D_+^j D_-^j) + \frac{\mu_h - a_h}{2}(D_+^i D_+^j + D_-^i D_-^j),
$$

where $\mu_h = (h\sum_{k=1}^{N}\frac{1}{a_{k0}})^{-1}$ and $a_h = h\sum_{k=1}^{N} a_{k0}$. Note also that the operator of the two-level method can be expressed as

$$M = (I - I_H^h L_H^{-1} I_h^H L_{\epsilon,h})S^\gamma = (I - I_H^h L_H^{-1} I_h^H L_{\epsilon,h})(I - \alpha L_{\epsilon,h})^\gamma, \tag{3.5}$$

where α is the inverse of the largest eigenvalue of $L_{\epsilon,h}$, has order of h^{-2} and $L_H = L_{\mu,H}$.

3.2 Convergence Analysis

For the simplified operator M_1 defined in (2.6) we prove:

Theorem 3.1 *If the ratio of h to ϵ is strictly irrational and h is in $S(\epsilon, h_0)$, then there exist two constants C and ρ_0 such that*

$$\|M_1\|_h \le \rho_0 < 1 \tag{3.6}$$

whenever $\gamma \ge Ch^{-1-2/3}ln(h)$.

Before proving Theorem 3.1, we establish some lemmas and a theorem.

Lemma 3.1 *Let two discrete functions on $\bar{\Omega}_h$ be defined by*

$$Z_{ij}^1 = \frac{h}{2\epsilon}(i - j - \mu_h\sum_{k=1}^{i}\frac{1}{a_{kj}} + \mu_h\sum_{k=1}^{j}\frac{1}{b_{0k}}), \quad (i,j) \in \bar{\Omega}_h, \tag{3.7}$$

$$Z_{ij}^2 = \frac{h}{2\epsilon}(j - i + \mu_h\sum_{k=1}^{i}\frac{1}{a_{k0}} - \mu_h\sum_{k=1}^{j}\frac{1}{b_{ik}}), \quad (i,j) \in \bar{\Omega}_h. \tag{3.8}$$

Then, Z_{ij}^1, Z_{ij}^2 are bounded and

$$L_{\epsilon,h}Z_{ij}^1 = -\frac{1}{\epsilon}D_+^i a_{ij}, \quad L_{\epsilon,h}Z_{ij}^2 = -\frac{1}{\epsilon}D_+^j b_{ij}, \tag{3.9}$$

for $(i,j) \in \Omega_h$.

Proof. Notice that by the assumption of the coefficients,

$$\sum_{k=1}^{N}\frac{h}{a_{kj}} = \sum_{k=1}^{N}\frac{h}{b_{ik}} = \mu_h^{-1}, \quad (i,j) \in \bar{\Omega}_h.$$

applying the operator $L_{\epsilon,h}$ to Z^1_{ij} implies the first part of (3.9). Rewrite Z^1_{ij} as follows

$$Z^1_{ij} = \frac{h}{2\epsilon}(i - \mu_h \sum_{k=1}^{i} \frac{1}{a_{kj}}) - \frac{h}{2\epsilon}(j - \mu_h \sum_{k=1}^{j} \frac{1}{b_{0k}}).$$

By Lemma 1.1,

$$\frac{h}{2\epsilon}(i - \mu_h \sum_{k=1}^{i} \frac{1}{a_{kj}}) = O(1), \quad \frac{h}{2\epsilon}(j - \mu_h \sum_{k=1}^{j} \frac{1}{b_{0k}}) = O(1).$$

Hence, Z^1_{ij} is bounded for $(i,j) \in \bar{\Omega}_h$ and

$$a_{ij}(1 - \epsilon D^i_- Z^1_{ij}) = \frac{a_{ij} + \mu_h}{2}, \quad -\epsilon b_{ij} D^j_- Z^1_{ij} = \frac{b_{ij} - \mu_h}{2}.$$

The result can be deduced similarly for Z^2_{ij}. We can also show that $Z^2_{ij} = -Z^1_{ij}$ and

$$b_{ij}(1 - \epsilon D^j_- Z^2_{ij}) = \frac{b_{ij} + \mu_h}{2}, \quad -\epsilon a_{ij} D^i_- Z^2_{ij} = \frac{a_{ij} - \mu_h}{2}.$$

This proves the lemma. □

Remark. The explicit forms of Z^1_{ij} and Z^2_{ij} depend on a_ϵ being a function of $x - y$. For the general angular dependences, these forms would not be possible.

Lemma 3.2 *Let a discrete function η_{ij} be defined by*

$$\eta_{ij} = \frac{h}{\epsilon}(\sum_{k=1}^{i} a_{kj} - i a_h) + \frac{h}{\epsilon}(j a_h - \sum_{k=1}^{j} b_{0k}), \quad (i,j) \in \bar{\Omega}_h.$$

Then, η_{ij} is bounded and

$$\epsilon D^i_- \eta_{ij} = a_{ij} - a_h, \quad \epsilon D^j_- \eta_{ij} = a_h - b_{ij}, \quad (i,j) \in \Omega_h.$$

Proof. Using the symmetry properties of the coefficients, proof of Lemma 3.2 is similar to the proof of Lemma 3.1. □

Lemma 3.3 *Assume U_{ij} satisfies*

$$\begin{aligned} L_{\mu,h} U_{ij} &= \lambda_\epsilon \phi^h_{ij}, \quad (i,j) \in \Omega_h, \\ U_{ij} &= 0, \quad (i,j) \in \partial\Omega_h, \end{aligned} \tag{3.10}$$

where $(\lambda_\epsilon, \phi^h_{ij})$ *is a normalized eigenpair of* $L_{\epsilon,h}$*. Then,*

$$\|U\|_1^2 = \sum_{i,j=0}^{N-1} [(D^i_+ U_{ij} h)^2 + (D^j_+ U_{ij} h)^2] = O(\lambda_\epsilon); \tag{3.11}$$

and

$$\begin{aligned}&\sum_{i,j=1}^{N-1} ((D^i_+ D^i_- U_{ij})^2 + (D^i_- D^j_- U_{ij})^2 + (D^i_+ D^j_+ U_{ij})^2 + (D^j_+ D^j_- U_{ij})^2) h^2 \\ &= O(\lambda_\epsilon^2).\end{aligned} \tag{3.12}$$

Proof. First, we observe that

$$\sum_{i,j=0}^{N-1} [(D^i_+ \phi^h_{ij} h)^2 + (D^j_+ \phi^h_{ij} h)^2] = O(\lambda_\epsilon).$$

Multiplying both sides of (3.10) by U_{ij} and then summing over $i, j = 1, \cdots, N-1$

$$\sum_{i,j=1}^{N-1} L_{\mu,h} U_{ij} U_{ij} = \sum_{i,j=1}^{N-1} L_{\epsilon,h} \phi_{ij} U_{ij},$$

which implies (3.11). Since U_{ij} vanishes at the boundary,

$$\begin{aligned}&\frac{1}{2} \sum_{i,j=1}^{N-1} [(D^i_+ D^i_- U_{ij})^2 + (D^j_+ D^j_- U_{ij})^2] \geq \sum_{i,j=1}^{N-1} (D^i_+ D^i_- U_{ij})(D^j_+ D^j_- U_{ij}) \\ &= \sum_{i,j=1}^{N} (D^i_- D^j_- U_{ij})^2 = \sum_{i,j=0}^{N-1} (D^i_+ D^j_+ U_{ij})^2.\end{aligned} \tag{3.13}$$

Multiplying (3.10) by $D^i_+ D^i_- U_{ij}$,

$$\begin{aligned}&(D^i_+ D^i_- U_{ij})^2 + 2\frac{a_h - \mu_h}{a_h + \mu_h} (D^i_+ D^j_+ U_{ij})(D^i_+ D^i_- U_{ij}) + (D^i_+ D^j_+ U_{ij})^2 + \\ &(D^i_+ D^i_- U_{ij})^2 + 2\frac{a_h - \mu_h}{a_h + \mu_h} (D^i_- D^j_- U_{ij})(D^i_+ D^i_- U_{ij}) + (D^i_- D^j_- U_{ij})^2 + \\ &((D^j_+ D^j_- U_{ij})(D^i_+ D^i_- U_{ij}) - (D^i_+ D^j_+ U_{ij})^2) + \\ &((D^j_+ D^j_- U_{ij})(D^i_+ D^i_- U_{ij}) - (D^i_- D^j_- U_{ij})^2) \\ = \;& \frac{2}{a_h + \mu_h} \lambda_\epsilon \phi^h_{ij} D^i_+ D^i_- U_{ij}, \quad (i,j) \in \Omega_h.\end{aligned}$$

By the uniform ellipticity property of the homogenized operator (3.4),

$$\sum_{i,j=1}^{N-1}((D_+^iD_-^iU_{ij})^2+(D_+^iD_+^jU_{ij})^2+(D_-^iD_-^jU_{ij})^2)h^2=O(\lambda_\epsilon^2). \tag{3.14}$$

A similar argument shows

$$\sum_{i,j=1}^{N-1}((D_+^jD_-^jU_{ij})^2+(D_+^iD_+^jU_{ij})^2+(D_-^iD_-^jU_{ij})^2)h^2=O(\lambda_\epsilon^2). \tag{3.15}$$

The rest of the proof follows from (3.13), (3.14) and (3.15). □

Lemma 3.4 *Let U_{ij} be as in Lemma 3.3 and assume it satisfies the following boundary condition,*

$$D_+^iU_{ij}+D_-^iU_{ij}=D_+^jU_{ij}+D_-^jU_{ij}=0,\quad (i,j)\in\partial\Omega_h. \tag{3.16}$$

Then,

$$\begin{aligned}\|U\|_2^2 &= \sum_{i,j=0}^{N}((D_+^iD_-^iU_{ij})^2+(D_+^jD_-^jU_{ij})^2)h^2\\ &+ \sum_{i,j=0}^{N-1}(D_+^iD_+^jU_{ij}h)^2+\sum_{i,j=1}^{N}(D_-^iD_-^jU_{ij}h)^2=O(\lambda_\epsilon^2/h).\end{aligned} \tag{3.17}$$

Proof. Since

$$D_-^iU_{Nj}=D_-^iU_{ij}+h\sum_{k=i}^{N-1}D_+^iD_-^iU_{kj},$$

we have

$$(D_-^iU_{Nj})^2\le(D_-^iU_{ij})^2+\sum_{i=1}^{N-1}(D_+^iD_-^iU_{ij})^2h.$$

Thus,

$$\sum_{j=1}^{N}(D_-^iU_{Nj})^2h\le\sum_{i,j=1}^{N}(D_-^iU_{ij}h)^2+\sum_{i,j=1}^{N}(D_+^iD_-^iU_{ij}h)^2=O(\lambda_\epsilon^2).$$

The rest of the proof follows from combining (3.12) with the following relation

$$D_+^iD_-^iU_{Nj}=-\frac{2}{h}D_+^iU_{N-1j}.$$

□

Theorem 3.2 *Suppose ϕ^h_{ij} and U_{ij} satisfy assumptions in Lemma 3.3 and Lemma 3.4. Then,*

$$\|\phi^h - U\|_h = O(\sqrt{\epsilon}\lambda_\epsilon). \tag{3.18}$$

Proof. Consider first the following discrete functions

$$\begin{aligned}
G_{ij} &= G^1_{ij} + G^2_{ij},\\
G^1_{ij} &= \frac{1}{2}U_{ij} - \frac{\epsilon}{2}(Z^1_{ij}D^i_+U_{ij} + Z^2_{ij}D^j_+U_{ij}) - \frac{1}{2}\phi^h_{ij},\\
G^2_{ij} &= \frac{1}{2}U_{ij} - \frac{\epsilon}{2}(Z^1_{ij}D^i_-U_{ij} + Z^2_{ij}D^j_-U_{ij}) - \frac{1}{2}\phi^h_{ij},
\end{aligned}$$

for $(i,j) \in \bar{\Omega}_h$. By the assumption (3.16), G_{ij} vanishes at boundary. That is,

$$G_{ij} = 0, \quad (i,j) \in \partial\Omega_h.$$

For G^1_{ij}, we have

$$\begin{aligned}
2D^i_-G^1_{ij} &= D^i_-U_{ij} - \epsilon(D^i_-(Z^1_{ij}D^i_+U_{ij}) + D^i_-(Z^2_{ij}D^j_+U_{ij})) - D^i_-\phi^h_{ij}\\
&= (D^i_-U_{ij} - \epsilon D^i_-Z^1_{ij}D^i_+U_{i-1j}) - \epsilon Z^1_{ij}D^i_-D^i_+U_{ij}\\
&\quad -\epsilon D^i_-Z^2_{ij}D^j_+U_{ij} - \epsilon Z^2_{i-1j}D^i_-D^j_+U_{ij} - D^i_-\phi^h_{ij},
\end{aligned}$$

$$\begin{aligned}
2D^j_-G^1_{ij} &= D^j_-U_{ij} - \epsilon(D^j_-(Z^1_{ij}D^i_+U_{ij}) + D^j_-(Z^2_{ij}D^j_+U_{ij})) - D^j_-\phi^h_{ij}\\
&= (D^j_-U_{ij} - \epsilon D^j_-Z^2_{ij}D^j_+U_{ij-1}) - \epsilon Z^2_{ij}D^j_-D^j_+U_{ij}\\
&\quad -\epsilon D^j_-Z^1_{ij}D^i_+U_{ij} - \epsilon Z^1_{ij-1}D^j_-D^i_+U_{ij} - D^j_-\phi^h_{ij}.
\end{aligned}$$

Thus

$$\begin{aligned}
2L_{\epsilon,h}G^1_{ij} &= D^i_+\frac{a_{ij}+\mu_h}{2}D^i_-U_{ij} + D^i_+\frac{a_{ij}-\mu_h}{2}D^j_+U_{ij}\\
&\quad -\epsilon D^i_+(a_{ij}Z^1_{ij}D^i_+D^i_-U_{ij} + a_{ij}Z^2_{i-1j}D^i_-D^j_+U_{ij})\\
&\quad +D^i_+\frac{b_{ij}-\mu_h}{2}D^j_+U_{ij} + D^j_+\frac{b_{ij}+\mu_h}{2}D^j_-U_{ij}\\
&\quad -\epsilon D^j_+(b_{ij}Z^2_{ij}D^j_-D^j_+U_{ij} + b_{ij}Z^1_{ij-1}D^j_-D^i_+U_{ij})\\
&\quad +L_\epsilon\phi^h_{ij}.
\end{aligned} \tag{3.19}$$

Similarly

$$
\begin{aligned}
2D_+^i G_{ij}^2 &= D_+^i U_{ij} - \epsilon(D_+^i(Z_{ij}^1 D_-^i U_{ij}) + D_+^i(Z_{ij}^2 D_-^j U_{ij})) - D_+^i \phi_{ij}^h \\
&= (D_+^i U_{ij} - \epsilon D_+^i Z_{ij}^1 D_-^i U_{i+1j}) - \epsilon Z_{ij}^1 D_-^i D_+^i U_{ij} \\
&\quad -\epsilon D_+^i Z_{ij}^2 D_-^j U_{ij} - \epsilon Z_{i+1j}^2 D_+^i D_-^j U_{ij} - D_+^i \phi_{ij}^h,
\end{aligned}
$$

$$
\begin{aligned}
2D_+^j G_{ij}^2 &= D_+^j U_{ij} - \epsilon(D_+^j(Z_{ij}^1 D_-^i U_{ij}) + D_+^j(Z_{ij}^2 D_-^j U_{ij})) - D_+^j \phi_{ij}^h \\
&= (D_+^j U_{ij} - \epsilon D_+^j Z_{ij}^2 D_-^j U_{ij+1}) - \epsilon Z_{ij}^2 D_+^j D_-^j U_{ij} \\
&\quad -\epsilon D_+^j Z_{ij}^1 D_-^i U_{ij} - \epsilon Z_{ij+1}^1 D_+^j D_-^i U_{ij} - D_+^j \phi_{ij}^h.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
2L_{\epsilon,h} G_{ij}^2 &= D_-^i \frac{a_{i+1j} + \mu_h}{2} D_+^i U_{ij} + D_-^i \frac{a_{i+1j} - \mu_h}{2} D_+^j U_{ij} \\
&\quad + D_-^j \frac{b_{ij+1} - \mu_h}{2} D_-^i U_{ij} + D_-^j \frac{b_{ij+1} + \mu_h}{2} D_+^j U_{ij} \\
&\quad -\epsilon D_-^i (a_{i+1j} Z_{ij}^1 D_-^i D_+^i U_{ij} + a_{i+1j} Z_{i+1j}^2 D_+^i D_-^j U_{ij}) \\
&\quad -\epsilon D_-^j (b_{ij+1} Z_{ij}^2 D_-^j D_+^j U_{ij} + b_{ij+1} Z_{ij+1}^1 D_+^j D_-^i U_{ij}) \\
&\quad + L_\epsilon \phi_{ij}^h. \qquad (3.20)
\end{aligned}
$$

Define operators L_1 and L by

$$
L_1 = \frac{a_{ij} + b_{ij} - 2a_h}{2}(D_+^i D_-^i + D_+^i D_+^j + D_-^i D_-^j + D_+^j D_-^j),
$$

and

$$
\begin{aligned}
L = \frac{1}{2}(\\
& D_+^i \frac{a_{ij} + \mu_h}{2} D_-^i + D_-^i \frac{a_{i+1j} - \mu_h}{2} D_-^j + D_-^j \frac{b_{ij+1} - \mu_h}{2} D_-^i + D_+^j \frac{b_{ij} + \mu_h}{2} D_-^j \\
+ \ & D_-^i \frac{a_{i+1j} + \mu_h}{2} D_+^i + D_+^i \frac{a_{ij} - \mu_h}{2} D_+^j + D_+^j \frac{b_{ij} - \mu_h}{2} D_+^i + D_-^j \frac{b_{ij+1} + \mu_h}{2} D_+^j).
\end{aligned}
$$

Observe that

$$
L_{\epsilon,h} \phi_{ij}^h = L_{\mu,h} U_{ij} = L U_{ij} - L_1 U_{ij}, \quad (i,j) \in \Omega_h, \qquad (3.21)
$$

which together with (3.19) and (3.20) implies

$$
\begin{aligned}
& 2L_{\epsilon,h}G_{ij} = 2L_{\epsilon,h}G^1_{ij} + 2L_{\epsilon,h}G^2_{ij} = -2L_1U_{ij} \\
- \quad & \epsilon D^i_+(a_{ij}Z^1_{ij}D^i_+D^i_-U_{ij} + a_{ij}Z^2_{i-1j}D^i_-D^j_+U_{ij}) \\
- \quad & \epsilon D^j_+(b_{ij}Z^2_{ij}D^j_-D^j_+U_{ij} + b_{ij}Z^1_{ij-1}D^j_-D^i_+U_{ij}) \\
- \quad & \epsilon D^i_-(a_{i+1j}Z^1_{ij}D^i_-D^i_+U_{ij} + a_{i+1j}Z^2_{i+1j}D^i_+D^j_-U_{ij}) \\
- \quad & \epsilon D^j_-(b_{ij+1}Z^2_{ij}D^j_-D^j_+U_{ij} + b_{ij+1}Z^1_{ij+1}D^j_+D^i_-U_{ij}).
\end{aligned} \tag{3.22}
$$

By Lemma 3.2 and summing by parts,

$$
\begin{aligned}
& \sum_{i,j=1}^{N-1}(a_{ij}-a_h)(D^j_+D^j_-U_{ij})G_{ij} = \epsilon\sum_{i,j=1}^{N-1} D^i_-\eta_{ij}D^j_+D^j_-U_{ij}G_{ij} \\
= \quad & -\epsilon\sum_{i,j=1}^{N} \eta_{i-1j}D^i_-(D^j_+D^j_-U_{ij}G_{ij}) \\
= \quad & -\epsilon\sum_{i,j=1}^{N} \eta_{i-1j}D^i_-D^j_+D^j_-U_{ij}G_{i-1j} - \epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij} \\
= \quad & -\epsilon\sum_{i,j=1}^{N} \eta_{ij}D^i_+D^j_-D^j_+U_{ij}G_{ij} - \epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij} \\
= \quad & \epsilon\sum_{i,j=1}^{N} D^j_-(\eta_{ij}G_{ij})D^i_+D^j_+U_{ij-1} - \epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij} \\
= \quad & \epsilon\sum_{i,j=0}^{N-1} D^j_+(\eta_{ij}G_{ij})D^i_+D^j_+U_{ij} - \epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij} \\
= \quad & \epsilon\sum_{i,j=0}^{N-1} D^j_+\eta_{ij}D^i_+D^j_+U_{ij}G_{ij} + \epsilon\sum_{i,j=0}^{N-1}\eta_{ij+1}D^j_+G_{ij}D^i_+D^j_+U_{ij} \\
& -\epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij} \\
= \quad & \sum_{i,j=0}^{N-1}(a_h-b_{ij+1})D^i_+D^j_+U_{ij}G_{ij} + \epsilon\sum_{i,j=0}^{N-1}\eta_{ij+1}D^j_+G_{ij}D^i_+D^j_+U_{ij} \\
& -\epsilon\sum_{i,j=1}^{N}\eta_{i-1j}D^j_+D^j_-U_{ij}D^i_-G_{ij}.
\end{aligned}
$$

By the symmetry property of the coefficients, $a_{ij} = b_{ij+1}$, we get

$$\sum_{i,j=1}^{N-1} (a_{ij} - a_h)(D_+^j D_-^j + D_+^i D_+^j) U_{ij} G_{ij}$$

$$= \epsilon \sum_{i,j=0}^{N-1} \eta_{ij+1} D_+^j G_{ij} D_+^i D_+^j U_{ij} - \epsilon \sum_{i,j=1}^{N} \eta_{i-1j} D_+^j D_-^j U_{ij} D_-^i G_{ij}. \tag{3.23}$$

Proceeding in the same way as before, we obtain

$$\sum_{i,j=1}^{N-1} (b_{ij} - a_h)(D_+^j D_-^j + D_-^i D_-^j) U_{ij} G_{ij}$$

$$= -\epsilon \sum_{i,j=1}^{N} \eta_{ij} D_-^i G_{ij} D_+^j D_-^j U_{i-1j} + \epsilon \sum_{i,j=0}^{N-1} \eta_{ij} D_-^i D_-^j U_{ij+1} D_+^j G_{ij}. \tag{3.24}$$

$$\sum_{i,j=1}^{N-1} (a_{ij} - a_h)(D_+^i D_-^i + D_-^i D_-^j) U_{ij} G_{ij}$$

$$= \epsilon \sum_{i,j=0}^{N-1} \eta_{ij+1} D_+^j G_{ij} D_+^i D_-^i U_{ij} - \epsilon \sum_{i,j=0}^{N-1} \eta_{ij} D_-^i D_-^j U_{i+1j} D_+^i G_{ij}. \tag{3.25}$$

$$\sum_{i,j=1}^{N-1} (b_{ij} - a_h)(D_+^i D_-^i + D_+^i D_+^j) U_{ij} G_{ij}$$

$$= \epsilon \sum_{i,j=1}^{N} \eta_{ij-1} D_-^j G_{ij} D_+^i D_-^i U_{ij} - \epsilon \sum_{i,j=1}^{N} \eta_{ij} D_+^i D_+^j U_{i-1j} D_-^i G_{ij}. \tag{3.26}$$

Together with (3.23), (3.24), (3.25) and (3.26) imply

$$\sum_{i,j=1}^{N} \frac{a_{ij} + b_{ij} - 2a_h}{2} (D_+^j D_-^j + D_-^i D_-^j + D_+^i D_-^i + D_+^i D_+^j) U_{ij} G_{ij} h^2$$

$$\leq C\epsilon \|U\|_2 \sqrt{\sum_{i,j=0}^{N-1} [(D_+^i G_{ij} h)^2 + (D_+^j G_{ij} h)^2]}$$

$$\leq C\sqrt{\epsilon}\lambda_\epsilon \sqrt{\sum_{i,j=0}^{N-1} [(D_+^i G_{ij} h)^2 + (D_+^j G_{ij} h)^2]}. \tag{3.27}$$

Furthermore,

$$\epsilon \sum_{i,j=0}^{N-1} [D_+^i(a_{ij} Z_{ij}^1 D_+^i D_-^i U_{ij} + a_{ij} Z_{i-1j}^2 D_-^i D_+^j U_{ij})] G_{ij} h^2$$

$$= \epsilon \sum_{i,j=0}^{N-1} [a_{i+1j} Z_{i+1j}^1 D_+^i D_-^i U_{i+1j} + a_{i+1j} Z_{ij}^2 D_-^i D_+^j U_{i+1j}] D_+^i G_{ij} h^2$$

$$\leq C\epsilon \left(\sqrt{\sum_{i,j=0}^{N-1} (D_+^i D_-^i U_{i+1j})^2 h^2} + \sqrt{\sum_{i,j=0}^{N-1} (D_+^i D_+^j U_{ij})^2 h^2} \right) \sqrt{\sum_{i,j=0}^{N-1} (D_+^i G_{ij})^2 h^2}$$

$$= C\sqrt{\epsilon}\lambda_\epsilon \sqrt{\sum_{i,j=0}^{N-1} (D_+^i G_{ij})^2 h^2}. \tag{3.28}$$

The exactly same order can be similarly established for the last three terms in (3.22). Consequently, from (3.27) and (3.28) it follows

$$\sqrt{\sum_{i,j=1}^{N} (D_-^i G_{ij} h)^2 + (D_-^j G_{ij} h)^2} = O(\sqrt{\epsilon}\lambda_\epsilon).$$

By Poincare inequality,

$$\|G\|_h = O(\sqrt{\epsilon}\lambda_\epsilon).$$

By Lemma 3.1 and Lemma 3.3,

$$\|\epsilon Z^1(D_0^i U - D_0^j U)\|_h \leq \epsilon \max_{i,j \in \bar{\Omega}_h} |Z_{ij}^1| \|U\|_1 = O(\epsilon\lambda_\epsilon),$$

which implies,

$$\|(I - L_{\mu,h}^{-1} L_{\epsilon,h})\phi^h\|_h \leq C\sqrt{\epsilon}\lambda_\epsilon. \tag{3.29}$$

The proof is therefore completed. □

Remark. Theorem 3.2 is consistent with the result for the continuous case established in [11].

Proof of Theorem 3.1. Replacing inequalities (2.20) and (2.21) by (3.30) and (3.31) below, the proof follows exactly the same procedure as in the proof of Theorem 2.1.

By Theorem 3.2, we have

$$I_1 \leq C\sqrt{h} k_0^3. \tag{3.30}$$

Therefore, $I_1 < \frac{1}{2}$ whenever

$$k_0 \leq Ch^{-1/6}. \tag{3.31}$$

For $I_2 \leq \frac{1}{2}$, it is sufficient to have

$$\gamma \geq C\frac{h^{-2}}{k_0^2} lnk_0 (\geq \frac{1}{\alpha k_0^2}). \tag{3.32}$$

Combining (3.31) and (3.32), we have

$$\gamma \geq Ch^{-1-\frac{2}{3}} \ln h.$$

□

We conclude this section with the following theorem.

Theorem 3.3 *There exists a constant C such that the operator M defined by (2.5) satisfies*

$$\rho(M) \leq \rho_0 < 1,$$

whenever the step size h belongs to $S(\epsilon, h_0)$ and

$$\gamma \geq Ch^{-1-2/3} lnh.$$

4 Conclusion

The results of this paper strongly indicate the role which homogenization plays in the convergence analysis. If, for example, the coarse grid operator is replaced by the averaged operator in an one dimensional problem [6], the direct estimate for the multigrid convergence rate is not asymptotically better than just using the damped Jacobi smoothing operator. This follows from the effect of the oscillations on the low eigenmodes. The homogenized coarse grid operator reduces the number of smoothing operation from $O(h^{-2})$ to $O(h^{-6/5} \ln h)$, when the step size h belongs to the set $S(\epsilon, h_0)$ of *Diophantine numbers.* In [12], it has also been shown that the number of smoothing iteration needed for the convergence of the multigrid method with the averaged coarse grid operator dominates the one with the homogenized coarse grid operator.

There are some inequalities in the implementation of the proof, which potentially could be improved so that a sharper convergence rate is possible. One such improvement is to enlarge the space of low eigenmodes, which can be approximated by the corresponding homogenized eigenmodes. It might be possible to improve (3.18) to (2.14), which we think is the sharpest inequality one can establish. We established the same inequality (2.14) for the space of low eigenmodes both for an one dimensional problem and for a two dimensional problem with coefficients oscillatory along a coordinate direction [12]. However, the portion of the eigenmodes that can be approximated by the homogenized ones in the latter case is relatively much smaller than in the former. That's why we need $O(h^{-4/3}\ln h)$ for the number of smoothing iterations for the two dimensional case instead of $O(h^{-6/5}\ln h)$ for the one dimensional case.

Nevertheless, from the homogenization analysis, we understand that there always exists a boundary layer [3, 14], which makes it hard to get the first lower order correction of the eigenfunctions. The case discussed in section 2 of this paper, which is equivalent to an one dimensional problem, doesn't have such a boundary layer. We hence get an estimate as in (2.14). For the case discussed in section 3, all we can establish is (3.18), which consists of the result established in [11] for the continuous case. It hence defines a smaller low eigenspace. Numerical examples also tell us that there are some differences between these two cases. A complete understanding of the first lower order correction for the eigenfunctions is required in order to further improve the estimates.

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