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**The Multigrid Method Based on a
Wavelet Transformation and Schur
Complement**

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The Multigrid Method Based on a Wavelet Transformation and Schur Complement

Bjorn Engquist * Erding Luo †

Abstract

A new interpolation for the two-level method based on the wavelet transformation and the Schur complement is constructed. The new interpolation is useful for partial differential equations with highly oscillatory coefficients to which the homogenization theory is applicable. Convergence of the two-level method with the new interpolation is studied and numerical experiments are conducted.

1 Introduction

The multigrid method is widely applied in approximating solutions for differential equations. A distinct property of the method is that the convergence rate is fast and independent of the grid step. For differential equations with discontinuous coefficients or even oscillatory coefficients, the property is no longer possessed by the multigrid method. Various improvements to restore the fast convergence rate have been discussed in the literature. For more recent development, we refer the reader to [1, 4, 5, 6, 8].

For differential equations with oscillatory coefficients to which the homogenization theory is applicable, it is shown that directly applying homogenized coarse grid operator for the multigrid method is not as efficient as one would expect it to be (see [4, 5, 6, 8]). However, for the oscillatory differential equations, numerical experiments show that the use of the homogenized coarse grid operator can improve the performance of the multigrid method. In this paper, we take a different approach. We study the two-level method with

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a new interpolation operator. This operator arises from the wavelet transformation and the Schur complement. It is thus called here the wavelet interpolation operator. With this interpolation, we construct a Galerkin type coarse grid operator which we call the wavelet coarse grid operator. The wavelet coarse grid operator is an approximation for the corresponding homogenized operator. For oscillatory differential equations, the idea of using the wavelet coarse grid operator for the multigrid method follows suggestions in [4, 5, 6, 8]. An interesting property about these two operators is that they both can be explicitly generated as a combination of the original operator and the operators between the grid transformation. Therefore, applying these operators would enable us to apply the projection theory in analyzing convergence of the multigrid method. Furthermore, using the wavelet coarse grid operator instead of the homogenized operator directly can avoid a large number of computation, which might be needed for the generation of the homogenized operators [6]. We prove with these two operators that the spectral radius of the two-level method is less than that of the classic one under the Richardson smoothing iteration.

The wavelet operator and the Schur complement are introduced in Section 2. Section 3 discusses the two-level method and the wavelet operators for the multigrid method. Convergence of the two-level method with wavelet operators is analyzed in Section 4 and numerical illustrations are presented in Section 5.

2 Wavelet and Schur Complement

Consider an algebraic equation arising from the discretization of a partial differential equation on a grid level with equal step h ,

$$(2.1) \quad AU = f,$$

where A is a $N \times N$ symmetric positive definite matrix, U and f are vectors in \mathfrak{R}^N . As for one dimensional case with domain $[0, 1]$, $Nh = 1$. To solve equation (2.1) we first introduce wavelet transformation,

$$(2.2) \quad W = \begin{pmatrix} \hat{L} \\ \hat{H} \end{pmatrix}_{N \times N},$$

where

$$W^T W = W W^T = \hat{L}^T \hat{L} + \hat{H}^T \hat{H} = I,$$

and

$$\hat{L} \hat{L}^T = \hat{H} \hat{H}^T = I, \quad \hat{L} \hat{H}^T = \hat{H} \hat{L}^T = 0.$$

For the one dimensional case, \hat{L} and \hat{H} can be given by

$$(2.3) \quad \hat{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & \end{pmatrix}_{\frac{N}{2} \times N},$$

and

$$(2.4) \quad \hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & \end{pmatrix}_{\frac{N}{2} \times N}.$$

\hat{H} and \hat{L} play the roles of separating the high and low frequency components (see [9]). Applying W to equation (2.1), we get

$$\tilde{A}WU = WAW^TWU = Wf,$$

where $\tilde{A} = WAW^T$. The above equation can be decomposed as

$$(2.5) \quad \begin{pmatrix} \hat{L}A\hat{L}^T & \hat{L}A\hat{H}^T \\ \hat{H}A\hat{L}^T & \hat{H}A\hat{H}^T \end{pmatrix} \begin{pmatrix} \hat{L}U \\ \hat{H}U \end{pmatrix} = \begin{pmatrix} \hat{L}f \\ \hat{H}f \end{pmatrix}.$$

For brevity, set

$$\begin{aligned} T &= \hat{L}A\hat{L}^T, & B &= \hat{L}A\hat{H}^T, & D &= \hat{H}A\hat{H}^T, \\ U_{\hat{L}} &= \hat{L}U, & U_{\hat{H}} &= \hat{H}U, \\ f_{\hat{L}} &= \hat{L}f, & f_{\hat{H}} &= \hat{H}f. \end{aligned}$$

The Schur complement of \tilde{A} is

$$(2.6) \quad \begin{aligned} \tilde{A} &= \begin{pmatrix} \hat{L}A\hat{L}^T & \hat{L}A\hat{H}^T \\ \hat{H}A\hat{L}^T & \hat{H}A\hat{H}^T \end{pmatrix} = \begin{pmatrix} T & B \\ B^T & D \end{pmatrix} \\ &= \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} T - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix}. \end{aligned}$$

Thus, the inverse of \tilde{A} and its Schur complement are given by

$$(2.7) \quad \tilde{A}^{-1} = WA^{-1}W^T = \begin{pmatrix} \hat{L}A^{-1}\hat{L}^T & \hat{L}A^{-1}\hat{H}^T \\ \hat{H}A^{-1}\hat{L}^T & \hat{H}A^{-1}\hat{H}^T \end{pmatrix}$$

$$(2.8) \quad = \begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix} \begin{pmatrix} (T - BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}.$$

Rewrite (2.1) as

$$(2.9) \quad U = A^{-1}f,$$

and by applying wavelet transformation we have

$$(2.10) \quad WU = \begin{pmatrix} U_{\hat{L}} \\ U_{\hat{H}} \end{pmatrix} = WA^{-1}W^T Wf = \tilde{A}^{-1}Wf.$$

Now, $U_{\hat{L}}$ can be solved from (2.5) and (2.10), and by (2.7) $U_{\hat{L}}$, can be expressed as

$$(2.11) \quad U_{\hat{L}} = (\hat{L}A^{-1}\hat{L}^T)f_{\hat{L}} + \hat{L}A^{-1}\hat{H}^T f_{\hat{H}}.$$

By (2.8), $U_{\hat{L}}$ can be alternatively expressed as

$$(2.12) \quad U_{\hat{L}} = (T - BD^{-1}B^T)^{-1}(f_{\hat{L}} - BD^{-1}f_{\hat{H}}).$$

With (2.11) and (2.12), we establish

Lemma 2.1

$$(T - BD^{-1}B^T)^{-1} = \hat{L}A^{-1}\hat{L}^T.$$

Proof.

$$\begin{aligned} & (T - BD^{-1}B^T)(\hat{L}A^{-1}\hat{L}^T) \\ &= \hat{L}A\hat{L}^T\hat{L}A^{-1}\hat{L}^T - BD^{-1}B^T\hat{L}A^{-1}\hat{L}^T \\ &= \hat{L}A(I - \hat{H}^T\hat{H})A^{-1}\hat{L}^T - BD^{-1}B^T\hat{L}A^{-1}\hat{L}^T \\ &= I - \hat{L}A\hat{H}^T\hat{H}A^{-1}\hat{L}^T - \hat{L}A\hat{H}^T(\hat{H}A\hat{H}^T)^{-1}\hat{H}A\hat{L}^T\hat{L}A^{-1}\hat{L}^T \\ &= I - \hat{L}A\hat{H}^T\hat{H}A^{-1}\hat{L}^T - \hat{L}A\hat{H}^T(\hat{H}A\hat{H}^T)^{-1}\hat{H}A(I - \hat{H}^T\hat{H})A^{-1}\hat{L}^T \\ &= I. \end{aligned}$$

□

U can be solved from (2.9) and by (2.8), it can be written as

$$(2.13) \quad U = \begin{pmatrix} \hat{L} - BD^{-1}\hat{H} \\ \hat{H} \end{pmatrix}^T \begin{pmatrix} (T - BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} \hat{L} - BD^{-1}\hat{H} \\ \hat{H} \end{pmatrix} f.$$

Denote

$$(2.14) \quad I_{\hat{H}}^h = \sqrt{2}(\hat{L}^T - \hat{H}^T D^{-1}B^T) = \sqrt{2}(\hat{L}^T - \hat{L}_1^T),$$

where $\hat{L}_1 = BD^{-1}\hat{H}$ and

$$(2.15) \quad I_h^H = \frac{1}{2}(I_H^h)^T.$$

We can thus decompose U into two parts,

$$(2.16) \quad U = I_H^h(T - BD^{-1}B^T)^{-1}I_h^H f + \hat{H}^T D^{-1}\hat{H} f$$

or, equivalently

$$(2.17) \quad \begin{pmatrix} \hat{L} \\ D^{-1}\hat{H}A \end{pmatrix} U = \begin{pmatrix} (T - BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} \hat{L} - BD^{-1}\hat{H} \\ \hat{H} \end{pmatrix} f.$$

We end this section with the following lemma.

Lemma 2.2

$$(2.18) \quad A^{-1} = I_H^h(T - BD^{-1}B^T)^{-1}I_h^H + \hat{H}^T D^{-1}\hat{H}$$

$$(2.19) \quad = \hat{L}^T \hat{L} A^{-1} \hat{H}^T \hat{H} + \hat{H}^T \hat{H} A^{-1} \hat{H}^T \hat{H} + \hat{H}^T \hat{H} A^{-1} \hat{L}^T \hat{L} + \hat{L}^T \hat{L} A^{-1} \hat{L}^T \hat{L}.$$

Proof. The proof can be completed by noticing that $(T - BD^{-1}B^T)^{-1} = \hat{L}A^{-1}\hat{L}^T$. \square

3 Two-level Method

For equation (2.1), the approximation U_h^{n+1} at the $n + 1$ -th iteration with an initial value U_h^n is obtained from the two-level method. The associated algorithm typically consists of the following three steps.

1. Pre-smoothing step: apply γ_1 steps of a classical iteration S to (2.1) to obtain an approximation $U_h^{n+\frac{1}{2}}$. For convenience, we introduce the following notation:

$$U_h^{n+\frac{1}{2}} = S^{\gamma_1}(U_h^n, A, f).$$

2. Coarse grid correction step: introduce a coarse H -grid level on which a coarse grid operator A_H is defined, and then restrict the residual $r = f - AU_h^{n+\frac{1}{2}}$ to the H -grid level by a restriction I_h^H . Solve the correction equation $A_H e_H = I_h^H r$ and interpolate e_H by I_H^h back to h -grid level to update the approximation $\tilde{U} = U_h^{n+\frac{1}{2}} + I_H^h e_H$.

3. Post-smoothing step: repeat step 1 beginning with the approximation \tilde{U} ,

$$U_h^{n+1} = S^{\gamma_2}(\tilde{U}, A, f).$$

The iteration operator M of the two-level method is thus given by

$$(3.1) \quad M = S^{\gamma_2}(I - I_H^h A_H^{-1} I_h^H A)S^{\gamma_1}.$$

The correction equation in step 2 may be solved by recursively applying the two-level method, which coincides with the multigrid method.

Remark. We use subscript H in all the operators defined on the H-grid level, in order to distinguish from the operators defined on the h-grid level.

3.1 Smoothing Iteration

By (2.7), \tilde{A} can be decomposed as

$$(3.2) \quad \tilde{A} = \begin{pmatrix} T & 0 \\ 0 & D \end{pmatrix} - \begin{pmatrix} 0 & -B \\ -B^T & 0 \end{pmatrix}.$$

Based on this decomposition, the Richardson iterative method can be used as the smoothing iteration in the two-level method to obtain an approximation of (2.1). The procedure can be illustrated as follows. Since

$$\begin{pmatrix} T & 0 \\ 0 & D \end{pmatrix} WU_{new} \leftarrow \begin{pmatrix} T & 0 \\ 0 & D \end{pmatrix} WU_{old} - \omega \tilde{A} WU_{old} + \omega Wf,$$

we hence have

$$U_{new} \leftarrow SU_{old} + \omega W^T \begin{pmatrix} T^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} Wf,$$

where

$$(3.3) \quad S = I - \omega(\hat{L}^T T^{-1} \hat{L} + \hat{H}^T D^{-1} \hat{H})A,$$

and ω is a parameter.

3.2 Correction Equations

Consider (2.16) as an error correction equation by replacing f with the residual r and U with the error correction e . We thus have

$$(3.4) \quad e = I_H^h (T - BD^{-1}B^T)^{-1} I_h^H r + \hat{H}^T D^{-1} \hat{H} r.$$

The right hand side of (3.4) consists of two parts $r_{\hat{L}}$ and $r_{\hat{H}}$ where

$$r_{\hat{L}} = I_H^h (T - BD^{-1}B^T)^{-1} I_h^H r,$$

$$r_{\hat{H}} = \hat{H}^T D^{-1} \hat{H} r.$$

Assume that the term $r_{\hat{H}}$ is very small or, equivalently r is almost in $Range(\hat{L}^T)$, so that it can be negligible. Then, we obtain an approximate error correction equation of (3.4),

$$(3.5) \quad e = I_H^h (T - BD^{-1}B^T)^{-1} I_h^H r.$$

By the structure of $I_H^h (T - BD^{-1}B^T)^{-1} I_h^H$ and by introducing a coarse H-grid level, we construct the error correction equation on the level as

$$(3.6) \quad A_H e_H = I_h^H r,$$

where A_H is chosen to be

$$(3.7) \quad A_H = T - BD^{-1}B^T.$$

It follows that

$$A_H = I_h^H A I_H^h,$$

and thus A_H is a Galerkin type coarse grid operator with I_h^H as the interpolation and I_H^h as the prolongation. Furthermore, when the residual r is also in $ARange(\hat{L}^T)$, from (2.17) and (3.5),

$$e = I_H^h (T - BD^{-1}B^T)^{-1} I_h^H r = A^{-1} \hat{L}^T \hat{L} r,$$

which implies

$$\hat{L} A e = \hat{L} r = \hat{L} A \hat{L}^T r', \quad \text{for some } r'.$$

Based on the above equation, an error correction equation (3.6) on the coarse H-grid level can be simplified with a coarse grid operator defined as

$$(3.8) \quad A_H = \hat{L} A \hat{L}^T = T,$$

where A_H becomes the standard Galerkin type coarse grid operator when the operator $\frac{1}{\sqrt{2}} \hat{L}$ is as the interpolation and $2 \hat{L}^T$ is as the prolongation.

Classical iterative methods, such as Jacobi, Gauss-Seidel, usually derive the residual r with smoothing components in $Range(\hat{L}^T)$. When A is a smooth operator, such as laplacian, it is also true that the residual r is almost in $ARange(\hat{L}^T)$. When A is not smooth, this claim is no longer valid, and in this case, the construction of A_H as in (3.8) is not appropriate. Consequently, form (3.7) will be suggested.

Now consider the one dimensional case with \hat{L} and \hat{H} defined as in (2.3) and (2.4), respectively, and with A as the Laplacian operator. That is,

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{N \times N}.$$

By calculation, the corresponding \tilde{A} has the following components

$$T = \frac{1}{4h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}},$$

$$B = \frac{1}{4h^2} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}},$$

and

$$D = \frac{1}{4h^2} \begin{pmatrix} 6 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 6 \end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}} = \frac{2}{h^2} I - T.$$

The following properties can be established.

1. The eigenvalues of D are inside $(\frac{1}{h^2}, \frac{2}{h^2})$. Hence, $\|\hat{H}^T D^{-1} \hat{H}\|_2 = O(h^2)$ which is quite small. For general cases where the operator A is derived from elliptic operators with coefficients having lower and upper bounds, A is spectrally equivalent to the Laplacian operator. The eigenvalues of the corresponding D are inside $(\frac{c_1}{h^2}, \frac{c_2}{h^2})$, where c_1 and c_2 are positive constants independent of h .
2. D is a relatively sparse matrix. Hence, the computation of D is usually not too large.

4 Convergent Analysis

Consider projectors $P_{\hat{H}}$ and $P_{\hat{L}}$, such that

$$\begin{aligned} P_{\hat{H}} &= A^{1/2} \hat{H}^T D^{-1} \hat{H} A^{1/2}, \\ P_{\hat{L}} &= A^{1/2} \hat{L}^T T^{-1} \hat{L} A^{1/2}, \end{aligned}$$

which are A^{-1} orthogonal to each other. Define a new interpolation operator $I_h^H(\delta)$ with parameter δ by

$$(4.1) \quad I_h^H(\delta) = \frac{1}{\sqrt{2}}(\hat{L} - \delta B D^{-1} \hat{H}) = \frac{1}{\sqrt{2}}(\hat{L} - \delta \hat{L}_1) = \frac{1}{\sqrt{2}}(\hat{L} A^{1/2} (I - \delta P_{\hat{H}}) A^{-1/2}).$$

By taking its transpose, we define a new prolongation operator

$$(4.2) \quad I_H^h(\delta) = 2(I_h^H(\delta))^T.$$

Note that when $\delta = 1$, (4.1) and (4.2) are consistent with (2.15) and (2.14). We now construct a coarse grid operator $A_H(\delta)$ by

$$\begin{aligned} (4.3) \quad A_H(\delta) &= I_H^h(\delta) A I_H^h(\delta) \\ &= \hat{L} A \hat{L}^T - \delta (B D^{-1} \hat{H} A \hat{L}^T + \hat{L} A \hat{H}^T D^{-1} B^T) + \delta^2 B D^{-1} B^T \\ &= \hat{L} A \hat{L}^T - 2\delta \hat{L} A^{1/2} P_{\hat{H}} A^{1/2} \hat{L}^T + \delta^2 \hat{L} A^{1/2} P_{\hat{H}} A^{1/2} \hat{L}^T. \end{aligned}$$

In this section, we study the two-level method with the interpolation operator $I_h^H(\delta)$ in (4.1), the prolongation operator $I_H^h(\delta)$ in (4.2) and $A_H(\delta)$ in (4.3) as the coarse grid operator. Denote the smoothing iteration operator for the two-level method by S . Then, the operator for the two-level method with γ steps of pre- and post-smoothing iteration in the energy form is given by

$$(4.4) \quad M(\delta, \gamma) = G^2 - G A^{1/2} I_H^h(\delta) A_H^{-1}(\delta) I_h^H(\delta) A^{1/2} G,$$

where $G = A^{1/2} S^\gamma A^{-1/2}$. Two auxiliary operators $M_1(\delta, \gamma)$ and $M_2(\delta, \gamma)$ are needed to study properties of $M(\delta, \gamma)$. They are defined by

$$\begin{aligned} (4.5) \quad M_1(\delta, \gamma) &= G^2 - G A^{1/2} I_H^h(\delta) A_H^{-1}(0) I_h^H(\delta) A^{1/2} G, \\ &= G^2 - G P_{\hat{L}} G - \delta G (P_{\hat{H}} P_{\hat{L}} + P_{\hat{L}} P_{\hat{H}}) G \\ &\quad + \delta^2 G P_{\hat{H}} P_{\hat{L}} P_{\hat{H}} G, \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} M_2(\delta, \gamma) &= G^2 - GA^{1/2}I_h^h(0)A_H^{-1}(\delta)I_h^H(0)A^{1/2}G \\ &= G^2 - GA^{1/2}\hat{L}^T\hat{L}A^{-1}\hat{L}^T\hat{L}A^{1/2}G. \end{aligned}$$

Properties of $M(\delta, \gamma)$ can now be presented:

1. Since $A_H(\delta)$ is decreasing in δ over $[0, 1]$,

$$A_H^{-1}(1) > A_H^{-1}(0) = T^{-1}.$$

2. Assuming $\|G\|_2 < 1$, we have

$$\|M(\delta, \gamma)\|_2 \leq \|G\|_2.$$

3. For $0 \leq \delta \leq 1$,

$$0 < M(\delta, \gamma) \leq M_1(\delta, \gamma) \leq G^2.$$

4. When $\delta = 0$,

$$(4.7) \quad I_h^H(0) = \frac{1}{\sqrt{2}}\hat{L},$$

and

$$(4.8) \quad A_H(0) = \hat{L}A\hat{L}^T = T.$$

Hence, $M(0, \gamma)$ is an operator for the standard two-level method with an interpolation (4.7) and a Galerkin type coarse grid operator (4.8). When $\delta = 1$,

$$(4.9) \quad I_h^H(1) = \frac{1}{\sqrt{2}}(\hat{L} - BD^{-1}\hat{H}) = \frac{1}{\sqrt{2}}\hat{L}A^{1/2}(I - P_{\hat{H}})A^{-1/2},$$

and

$$(4.10) \quad A_H(1) = (\hat{L}A^{-1}\hat{L}^T)^{-1} = T - BD^{-1}B^T = \hat{L}A^{1/2}(I - P_{\hat{H}})A^{1/2}\hat{L}^T.$$

Hence, $M(1, \gamma)$ becomes an operator for our new two-level method with the wavelet interpolation (4.9) and the wavelet coarse grid operator (4.10). On the other hand, $M_1(1, \gamma)$ is an operator of the two-level method with the wavelet interpolation (4.9) and the coarse grid operator (4.8), and $M_2(1, \gamma)$ is an operator of the two-level method with the interpolation (4.7) and the wavelet coarse grid operator (4.10). From the constructions,

$$(4.11) \quad M(1, \gamma) = GP_{\hat{H}}G,$$

$$(4.12) \quad M_1(1, \gamma) = G(I - (I - P_{\hat{H}}) * P_{\hat{L}} * (I - P_{\hat{H}}))G,$$

$$(4.13) \quad M(0, \gamma) = G(I - P_{\hat{L}})G.$$

5. If $S = I - \hat{H}^T D^{-1} \hat{H} A = A^{-1/2} (I - P_{\hat{H}}) A^{1/2}$, then

$$(I - I_{\hat{H}}^h(1) A_{\hat{H}}^{-1}(1) I_{\hat{H}}^H(1) A) * S = (\hat{H}^T D^{-1} \hat{H} A) * S = 0,$$

which implies that the combination of the smoothing operator S and the coarse grid operator $I - I_{\hat{H}}^h(1) A_{\hat{H}}^{-1}(1) I_{\hat{H}}^H(1) A$ gives rise to an exact solver. On the other hand,

$$\begin{aligned} & (I - I_{\hat{H}}^h(0) A_{\hat{H}}^{-1}(0) I_{\hat{H}}^H(0) A) * S \\ &= A^{-1/2} (A^{1/2} \hat{L}^T T^{-1} \hat{L} A^{1/2} (I - A^{1/2} \hat{H}^T D^{-1} \hat{H} A^{1/2})) * A^{1/2} \\ &= A^{-1/2} P_{\hat{L}} (I - P_{\hat{H}}) A^{1/2}. \end{aligned}$$

6. If S is taken to be the Jacobi iterative method, i.e., $S = I - \alpha h^2 A$ with parameter α , then

$$\begin{aligned} A^{-1/2} M(1, \gamma) A^{1/2} &= S^\gamma A^{-1/2} P_{\hat{H}} A^{1/2} S^\gamma = S^\gamma \hat{H}^T D^{-1} \hat{H} A S^\gamma, \\ A^{-1/2} M(0, \gamma) A^{1/2} &= S^\gamma A^{-1/2} (I - P_{\hat{L}}) A^{1/2} S^\gamma = S^\gamma (A^{-1} - \hat{L}^T T^{-1} \hat{L}) A S^\gamma. \end{aligned}$$

Therefore, for the one dimensional case, since

$$\frac{h^2}{2} \hat{H}^T \hat{H} \leq \hat{H}^T D^{-1} \hat{H} \leq h^2 \hat{H}^T \hat{H},$$

we have

$$\|A^{-1/2} P_{\hat{H}} A^{-1/2}\|_2 = \|\hat{H}^T D^{-1} \hat{H}\|_2 \leq h^2.$$

With $\|AS^\gamma\|_2 = O(\frac{h^{-2}}{\gamma})$, we have

$$\|A^{-1/2} M(1, \gamma) A^{1/2}\|_2 = \|S^\gamma A^{-1/2} P_{\hat{H}} A^{1/2} S^\gamma\|_2 = O(\frac{1}{\gamma}).$$

When A is smooth, similar result for $M(0, \gamma)$

$$\|A^{-1/2} M(0, \gamma) A^{1/2}\|_2 = \|S^\gamma A^{-1/2} (I - P_{\hat{L}}) A^{-1/2} S^\gamma\|_2 = O(\frac{1}{\gamma}),$$

can be established by the fact that

$$(4.14) \quad \|A^{-1} - \hat{L}^T T^{-1} \hat{L}\|_2 = O(h^2).$$

However, when A is nonsmooth (see [4, 5, 6, 8]), the order of the approximation property [11] is $O(h)$ instead of (4.14). Hence, using $M(0, \gamma)$ as the operator for the multigrid method will not be as efficient as for the smooth case. Thus, for the nonsmooth case, $M(1, \gamma)$ is recommended to replace $M(0, \gamma)$ in order to restore all the properties that the standard multigrid method possesses, one of which is that the convergent rate is independent of the grid size.

Theorem 4.1 *Assume the Richardson smoothing iteration*

$$(4.15) \quad S = I - \frac{1}{2}(\hat{L}^T T^{-1} \hat{L} + \hat{H}^T D^{-1} \hat{H})A = \frac{1}{2}A^{-1/2}(2 - P_{\hat{L}} - P_{\hat{H}})A^{1/2}$$

for the two-level method is used. Then, we have

$$\rho(M(1, 1)) \leq \rho(M(0, 1)),$$

where $M(\delta, \gamma)$ is as in (4.4) and $\rho(\cdot)$ denotes the spectral radius.

Proof. We first show that the spectral radius of S defined as (4.15) is less than 1. For this, note that the two subspaces $\text{Range}(\hat{L}^T)$ and $\text{Range}(\hat{H}^T)$ are complement to each other in \mathfrak{R}^N . Thus, by $\hat{H}^T \hat{H} + \hat{L}^T \hat{L} = I$, a nontrivial vector can not be simultaneously in both spaces $\text{Range}(I - P_{\hat{H}})$ and $\text{Range}(I - P_{\hat{L}})$. Therefore,

$$-I < I - P_{\hat{H}} - P_{\hat{L}} < I.$$

This implies

$$\rho(S) < 1.$$

Set $G = A^{1/2}SA^{-1/2}$. Then,

$$G = I - \frac{1}{2}(P_{\hat{L}} + P_{\hat{H}}) = \frac{1}{2}(2 - P_{\hat{H}} - P_{\hat{L}}).$$

By calculation,

$$\begin{aligned} M(1, 1) &= \frac{1}{4}(I - P_{\hat{L}})P_{\hat{H}}(I - P_{\hat{L}}), \\ M(0, 1) &= \frac{1}{4}(2 - P_{\hat{H}})(I - P_{\hat{L}})(2 - P_{\hat{H}}). \end{aligned}$$

Since $I - P_{\hat{H}} \geq 0$ and $P_{\hat{L}} \geq 0$,

$$\begin{aligned} \rho(M(1, 1)) &= \frac{1}{4}\rho[(I - P_{\hat{L}})P_{\hat{H}}(I - P_{\hat{L}})] \leq \frac{1}{4}\rho[(I - P_{\hat{L}})(2 - P_{\hat{H}})^2(I - P_{\hat{L}})] \\ &= \frac{1}{4}\rho[(2 - P_{\hat{H}})(I - P_{\hat{L}})(2 - P_{\hat{H}})] = \rho(M(0, 1)). \end{aligned}$$

□

To compare $M_2(\delta, \gamma)$ with $M(0, \gamma)$, decompose \mathfrak{R}^N into two A-orthogonal subspaces S_1 and S_2 where

$$S_1 = A^{-1/2} \text{Range}(R), \quad S_2 = A^{-1/2} \text{Null}(R).$$

Here $R = \hat{L}^T \hat{L}$ which is idempotent and symmetric. For any $x \in \mathfrak{R}^N$,

$$Gx = x_1 + x_2$$

with $x_1 \in S_1$ and $x_2 \in S_2$, and thus, $M_2(1, \gamma)x_2 = x_2$, $M(0, \gamma)x_2 = x_2$, $x_1^T M_2(1, \gamma)x_1 = 0$, and $x_1^T M(0, \gamma)x_1 \geq 0$. Since $(\hat{L}A^{-1}\hat{L}^T)^{-1} \leq \hat{L}A\hat{L}^T$,

$$\begin{aligned} x^T M(0, \gamma)x &\geq x^T M_2(1, \gamma)x = 2x_1^T x_2 + x_2^T x_2 \geq 2(Gx)^T x_2 - x_2^T x_2 \\ &\geq -2\|x_2\|_2 \|Gx\|_2 - \|x_2\|_2^2, \end{aligned}$$

and

$$(4.16) \quad |x^T M_2(1, \gamma)x| \leq \max\{2\|x_2\|_2 \|Gx\|_2 + \|x_2\|_2^2, x^T M(0, \gamma)x\}.$$

Set

$$(4.17) \quad P = A^{-1/2}(I - R)A^{1/2}.$$

This is a A-orthogonal projector from \mathfrak{R}^N to S_2 along S_1 .

Theorem 4.2 *Let $M_2(1, \gamma)$, $M(0, \gamma)$ and P be defined as before. Then,*

$$\|M_2(1, \gamma)\|_2 \leq \max\{\|M(0, \gamma)\|_2, 2\|PG\|_2 + \|PG\|_2^2\}.$$

Proof. Note that

$$\begin{aligned} \|M_2(1, \gamma)\|_2 &= \max \frac{|x^T G(I - A^{1/2} \hat{L}^T (\hat{L}A\hat{L}^T)^{-1} \hat{L}A^{1/2} Gx)|}{x^T x} \\ &\leq \max\left\{ \frac{x^T G(I - A^{1/2} \hat{L}^T (\hat{L}A\hat{L}^T)^{-1} \hat{L}A^{1/2} Gx)}{x^T x}, 2 \frac{\|PGx\|_2}{\|x\|_2} + \frac{\|PGx\|_2^2}{\|x\|_2^2} \right\} \\ &= \max\{\|M(0, \gamma)\|_2, 2\|PG\|_2 + \|PG\|_2^2\}. \end{aligned}$$

This completes the proof. \square

The assumption that $\|PG\|_2$ is very small is similar to assumption (15) in Theorem 2 of [14], which can be fulfilled by taking a large iterative number γ (independent of h). From Theorem 4.2 is that $\|M_2(1, \gamma)\|_2 \leq \|M(0, \gamma)\|_2$ as γ becomes bigger. This means that as the smoothing iteration gets large, the multigrid method with only the wavelet coarse grid operator converges faster than that with the standard coarse grid operator. The next section provides numerical demonstrations of this result. For a small and practical smoothing iteration number, both the wavelet coarse grid operator and the wavelet interpolation are needed to improve the performance of the multigrid method.



5 Numerical Results

We consider the one dimensional elliptical differential operator

$$A(x) = -\frac{d}{dx}a\left(\frac{x}{\epsilon}\right)\frac{d}{dx}, \quad x \in (0, 1),$$

where $a(x/\epsilon) = 2.1 + 2 \sin(2\pi x/\epsilon)$. The corresponding homogenized operator is

$$A_\mu = -\mu \frac{d^2}{dx^2}, \quad \mu = \left(\int_0^1 \frac{1}{a(x)} dx\right)^{-1}.$$

Standard centered finite difference discretization gives

$$A_{\epsilon,h} = -D_+^j a_{j-1/2} D_-^j, \quad j = 1, 2, \dots, \frac{1}{h} - 1,$$

where $a_{j-1/2} = a((x_j - h/2)/\epsilon)$. Throughout the following numerical experiments, Har interpolation $\frac{1}{\sqrt{2}}\hat{L}$ in (2.3) is used. $\|\cdot\|_h$ denotes the discrete L_2 norm. We also calculate the energy norm for the multigrid method with operator $M_\mu(\gamma)$. This operator is constructed using the homogenized coarse grid operator. It has exactly the same form as $M_2(\delta, \gamma)$, except that the coarse grid operator $A_H(\delta)$ is replaced by the homogenized operator A_μ .

In Figure 1, ϵ is taken to be 1. This corresponds to the case where the operator A is smooth. As we can see in this figure, no operator is significantly superior than the others. For a small number of smoothing iteration, all operators perform equally well. As the smoothing iterative number increases, the performances of the operators $M(1, \gamma)$ and $M_1(1, \gamma)$ become gradually better than others.

Now, consider the case where A is not smooth by taking $\epsilon = \sqrt{2}h$. What we want to test are two things. First, we want to show numerically that the wavelet operator $A_H(1)$ as in (4.3) is an approximation of the corresponding homogenized operator A_μ . In Figure 2, the ratio of norms by comparing different operators is plotted. In this figure, it is clear that the wavelet operator is very close to the homogenized operator, while the Galerkin operator $A_H(0)$ is a bit far away from it as the grid size h decreases. Second, we compare the energy norms of the operators for the two-level method in Figures 3 and 4. Both figures are the same except the different iterative numbers for smoothing. We take small iterative steps in the graph on the left-hand side of figure 3 and the norms of both operators $M_2(1, \gamma)$ and $M_\mu(\gamma)$ are bigger than 1. $M_1(1, \gamma)$ is better than $M(0, \gamma)$. But, $M(1, \gamma)$ is the best among all. As we increase the smoothing iteration number as in the graph on the right-hand side, the wavelet operators $M(1, \gamma)$ becomes superior. This is also true for $M_2(1, \gamma)$.