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Summary

A general theorem characterizing the interaction of concentrations and oscillations effects associated to sequences of gradients bounded in L^p, $p>1$, is proved. The oscillations are recorded in the Young measure while the concentrations are encoded in the varifold.

Contents

1. Introduction

Oscillatory phenomena and the characterization of limits of nonlinear quantities of oscillating sequences have been sucessfully analyzed by means of Young measures. These measures where first introduced by Young [37] to study non-convex problems in optimal control theory and to provide the appropriate framework for the description of generalized minimizers in the calculus of variations. Recently Young measures have become an important tool in the study of nonlinear partial differential equations ([10], [12], [13], [30], [32], [33], [34], [36]) and the analysis of oscillatory behavior in non-convex variational principles that arise in models of solid-solid phase transitions ([7], [8]). Characterizations of Young measures associated to minimizing sequences of such functionals as well as to general sequences of gradients bounded in $L^p(\Omega)$ have been found in [20] and [21]. See also [28].

One of the main drawbacks of Young measures is that they miss completely concentration effects. Indeed, sequences may share the same Young measure and yet one may exhibit concentrations while the other does not. Several ways of understanding and manipulating concentrations have been proposed. We refer the reader to [14], [15], [18], [19], [23], [24], [29], [35] and [36] for some of these methods. Another possibility is to use varifolds or indicator measures following the works [3], [4], [17], [27]. This is the point of view that we will take here, and focus on sequences that are constrained to be gradients. A similar approach has been employed in [2] for unconstrained sequences that are bounded in L^1.

The notion of a varifold has been used to describe certain nonlinear limits of oscillating measures, and it plays a role complementary to that of the Young measure. In fact, the Young measure associated to a sequence $\{u_n\}$, which is bounded in $L^p(\Omega)$, describes the effect of oscillations on the limits of $\{f(u_n)\}$ whenever the nonlinearity f has growth of order strictly less than p, while the varifold describes the effect of concentrations on the limits of $\{g(u_n)\}$ when g grows asymptotically as the pth power. We will be more precise in Section 3.

Our goal is to understand the relation between the varifold and the Young measure that are generated by a sequence of gradients which is bounded in $L^p(\Omega)$, $p > 1$. We hope to address the case $p = 1$ in a future work. A detailed description of Young measures generated by sequences of gradients bounded in $L^p(\Omega)$ was obtained in [21] (see Theorem 2.3 below).

To describe our main result we consider an open, bounded set $\Omega \subset \mathbf{R}^N$ and a sequence $\{f_j\}$ of functions from Ω to $\mathbf{R}^d$ which is bounded in $L^p(\Omega)$ for some $p > 1$. There exists a subsequence, still denoted $\{f_j\}$, and a family $\nu = \{\nu_x\}_{x\in\Omega}$ (called the *Young measure*) of probability measures ν_x on $\mathbf{R}^d$, as well as a Radon, nonnegative measure Λ on $\Omega \times S^{d-1}$ (called the *varifold*) with the following properties (see [6], [17] and Section 3). For all continuous functions θ that vanish on $\partial\Omega$, $\theta \in C_0(\Omega)$, all continuous functions φ on $\mathbf{R}^d$ with growth of order strictly less than p, i.e.

$$|\varphi(\xi)| \leq C(1 + |\xi|^r), \quad 1 \leq r < p,$$

and for all continuous functions ψ on $\mathbf{R}^d$ that are homogeneous of degree p, we have

$$\int_\Omega \theta(x)\varphi(f_j(x))\,dx \to \int_\Omega \theta(x)\int_{\mathbf{R}^d}\varphi(\xi)\,d\nu_x(\xi)\,dx,$$

$$\begin{aligned}\int_\Omega \theta(x)\psi(f_j(x))\,dx &\to \int_{\Omega\times S^{d-1}} \theta(x)\psi(\xi)\,d\Lambda(x,\xi)\\ &= \int_\Omega \theta(x)\int_{S^{d-1}}\psi(\xi)\,d\lambda_x(\xi)\,d\pi(x).\end{aligned}$$

Here π is the projection of Λ onto Ω, λ_x are probability measures (for π-a.e. $x \in \Omega$), $\Lambda = \lambda \otimes \pi$, $\lambda = \{\lambda_x\}_{x\in\Omega}$, is the slicing decomposition of Λ ([15]) and $\mathcal{S} = \mathcal{S}^{d-1}$ is the unit sphere in $\mathbf{R}^d$. In what follows, we will refer to (ν, Λ) as the *Young measure-varifold pair*.

If $\{u_j\}$ is a bounded sequence in $W^{1,p}(\Omega)$, if $f_j = \nabla u_j$ and if the target space $\mathbf{R}^d$ is identified with the space $\mathbf{M} = \mathbf{M}^{m\times N}$ of $m \times N$ matrices, we say that (ν, Λ) is a $W^{1,p}(\Omega)$-*Young measure-varifold pair*, and we abbreviate saying that (ν, Λ) is a YM-V pair. The Young measures that arise in such pairs (the so-called $W^{1,p}(\Omega)$ *Young measures*) were characterized in [21]. The following example shows that there are restrictions involving both ν and Λ. Let $p = m = N$ and consider the N-homogeneous function $\psi(A) = \det A$. Then by the above

$$\int_\Omega \theta(x) \det \nabla u_j \, dx \to \int_\Omega \theta(x) \int_S \det A \, d\lambda_x(A) \, d\pi(x).$$

On the other hand, we know that ([5], [27]) $\det \nabla u_j \overset{*}{\rightharpoonup} \det \nabla u$ in the sense of measures, where u is the weak limit of u_j in $W^{1,p}(\Omega)$. We conclude that

$$(\det \nabla u) \, d\mathcal{L}^N = \left(\int_S \det A \, d\lambda_x(A) \right) d\pi,$$

where $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbf{R}^N$.

The main result of this paper is the following characterization theorem for YM-V pairs.

Theorem 1.1 *Let $p > 1$. (ν, Λ) is a YM-V pair, where $\nu = \{\nu_x\}_{x\in\Omega} \otimes \mathcal{L}^N$, $\Lambda = \{\lambda_x\}_{x\in\Omega} \otimes \pi$, if and only if*

1.
$$\nabla u(x) = \int_{\mathbf{M}} A \, d\nu_x(A), \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

for some $u \in W^{1,p}(\Omega)$;

2.
$$\varphi(\nabla u(x)) \le \int_{\mathbf{M}} \varphi(A) \, d\nu_x(A), \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

for every quasiconvex φ for which the limit

$$\lim_{|A|\to\infty} \frac{\varphi(A)}{1 + |A|^p}$$

exists;

3.
$$\int_{\mathbf{M}} \psi(A) \, d\nu_x(A) \le \frac{d\pi}{d\mathcal{L}^N}(x) \int_S \psi(A) \, d\lambda_x(A), \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

for every p-homogeneous, continuous function ψ such that $Q\psi(0) = 0$, where $Q\psi$ denotes the quasiconvexification of ψ;

4.
$$\int_S \psi(A) \, d\lambda_x(A) \ge 0, \quad \pi_s \text{ a.e. } x \in \Omega,$$

for every p-homogeneous, continuous function ψ such that $Q\psi(0) = 0$, where π_s is the singular part of π with respect to $\mathcal{L}^N$.

We remind the reader that (see [5], [11]) a function φ, defined on $\mathbf{M}$, is said to be *quasiconvex* if

$$\varphi(F) \leq \frac{1}{|\Omega|}\int_\Omega \varphi(F+\nabla u(x))\,d\mathcal{L}^N(x),$$

for all matrices F and all test functions $u \in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$. If φ is not quasiconvex then its *quasiconvexification*, $Q\varphi$, is defined to be

$$Q\varphi(F) = \inf_u \frac{1}{|\Omega|}\int_\Omega \varphi(F+\nabla u(x))\,dx,$$

for all matrices F. The infimum is taken again over the set of functions $u \in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$. In addition, if

$$|\varphi(\xi)| \leq C\,(1+|\xi|^p),$$

then

$$Q\varphi(F) = \inf_{u\in W_0^{1,p}(\Omega)} \frac{1}{|\Omega|}\int_\Omega \varphi(F+\nabla u(x))\,dx.$$

Equivalently, $Q\varphi$ can be characterized as the largest quasiconvex function below φ ([1], [11]).

Parts 1 and 2 of Theorem 1.1 correspond to the characterization of the underlying Young measure, and were proved in [20] and [21]. Part 3 provides the interaction between the Young measure and the absolutely continuous part of the varifold. Part 4 represents the restriction on the varifold in the set where the singular part π_s is concentrated. An interesting consequence of this result is that there are no restrictions on the singular measure π_s.

A key tool in the proof of the above theorem is the following decomposition result for sequences of gradients that are bounded in $L^p(\Omega)$, for some $p > 1$. It states, in particular, that every such sequence can be written as a sum of a sequence $\{\nabla z_j\}$ (of gradients!) whose p-th power is equiintegrable, and a remainder that converges to zero in measure (and hence almost uniformly). Using the terminology introduced earlier, we may say that $\{\nabla z_j\}$ carries the oscillations, while the remainder accounts for the concentration effects.

Lemma 1.2 *(Decomposition Lemma) Let $\Omega \subset \mathbf{R}^N$ be an open, bounded set and let $\{w_n\}$ be a bounded sequence in $W^{1,p}(\Omega)$ taking values in $\mathbf{R}^m$. There exists a subsequence, $\{w_j\}$, and a sequence $\{z_j\} \subset W^{1,p}(\Omega)$ such that*

$$\mathcal{L}^N\left(\{z_j \neq w_j \text{ or } \nabla z_j \neq \nabla w_j\}\right) \to 0, \tag{1.1}$$

as $j \to \infty$, and $\{|\nabla z_j|^p\}$ is equiintegrable. If Ω is Lipschitz (or, more generally, an extension domain), then each z_j may be chosen to be a Lipschitz function.

Note that (1.1) implies that both sequences $\{\nabla z_j\}$ and $\{\nabla w_j\}$ generate the same Young measure.

Some remarks are in order. A similar result was derived independently by Kristensen [22]. The lemma above is in fact a consequence of the characterization of $W^{1,p}$-Young measures obtained in [21] (see Theorem 2.3). The point is that one can give a short, direct proof of Lemma 1.2, while the approach via [21] is rather indirect and implicitly relies on the lower semicontinuity results of Acerbi and Fusco [1]. In fact, once the lemma is proved one can considerably shorten the arguments in [1] and [21] (see [26] for this point of view). Our proof of Lemma 1.2 (see Section 4) still relies on essentially the same tools as [1], namely L^p estimates for maximal functions and Lipschitz extensions of $W^{1,p}$ functions off small sets, but we think that an approach that uses the decomposition result as a starting point might be more intuitive. Kristensen's proof ([22]), on the other hand, uses Iwaniec's estimates for perturbed Hodge decompositions. These estimates, however, in turn rely on L^p estimates involving the sharp maximal function. Finally, the result may be viewed as an L^p counterpart of a theorem by Kewei Zhang [38] which states that if $\{\nabla w_j\}$ is bounded in L^q for some $q > 1$ and generates a Young measure with support contained in a ball $B = B(0, R) \subset \mathbb{M}$ (i.e. $\operatorname{supp}(\nu_x) \subset B$ for a.e. $x \in \Omega$) then there exists a sequence z_j with

$$|\nabla z_j| \le C(N)R \quad \text{and} \quad \mathcal{L}^N(\{z_j \ne w_j \text{ or } \nabla z_j \ne \nabla w_j\}) \to 0.$$

2. Preliminaries

Let $\mathbb{M} = \mathbb{M}^{m\times N}$ be the set of $m \times N$ matrices, and for any number $p > 0$ consider the class

$$\mathcal{H}_p = \{f \in C(\mathbb{M}) : f \text{ is positively homogeneous of degree } p\},$$

where $C(\mathbb{M})$ is the set of continuous functions on $\mathbb{M}$. If $f \in \mathcal{H}_p$ then $f(tA) = t^p f(A)$ for all $A \in \mathbb{M}$ and $t > 0$. It is easy to show that homogeneity entails $Q\psi(0) = 0$ whenever $\psi \in \mathcal{H}_p$ and $Q\psi(0)$ is finite. Let X_p denote the set of continuous functions in $\mathbb{M}$ with growth of order at most p, i.e.

$$X_p = \{\varphi \in C(\mathbb{M}) : |\varphi(A)| \le C(1 + |A|^p)\}.$$

X_p is a Banach space under the natural norm

$$\|\varphi\| = \left\| \frac{\varphi(\cdot)}{1 + |\cdot|^p} \right\|_{L^\infty(\mathbb{M})}.$$

Finally, we consider the class

$$\mathcal{E}_p = \left\{\varphi \in C(\mathbb{M}) : \text{ there exists } f \in \mathcal{H}_p, \lim_{|A|\to\infty} \frac{\varphi(A) - f(A)}{|A|^p} = 0\right\}.$$

The properties of $\mathcal{E}_p$ are listed in the proposition below.

Proposition 2.1

1. *For every $\varphi \in \mathcal{E}_p$ there is a unique $f \in \mathcal{H}_p$ such that*

$$\lim_{|A|\to\infty} \frac{\varphi(A) - f(A)}{|A|^p} = 0.$$

The function f is the recession function of φ of degree p, φ_p^∞, defined by

$$\varphi_p^\infty(A) = \lim_{t\to\infty} \frac{\varphi(tA)}{t^p}.$$

2. *$\mathcal{E}_p$ is a closed, separable subspace of X_p and $\mathcal{H}_p$ is a closed subspace of $\mathcal{E}_p$.*

3. *If $f \in \mathcal{H}_p$ then*

$$\|f\| = \|f\|_{L^\infty(S)},$$

where S is the unit sphere in $\mathbf{M}$.

The proof of this proposition is elementary. The only fact that requires some comment is the separability of $\mathcal{E}_p$. Indeed, using the map

$$C^\infty\left(\overline{B}(0,1)\right) \to \mathcal{E}_p,$$
$$\theta \to \left(A \mapsto \theta\left(\frac{A}{1+|A|}\right)|A|^p\right),$$

one can easily verify that $\mathcal{E}_p$ is isomorphic to the space of continuous functions on the unit ball of $\mathbf{M}$, equipped with the sup norm. This space is separable due to the compactness of the unit ball. If we compare the space $\mathcal{E}_p$ with the space considered in [21],

$$E_p = \left\{\varphi \in C(\mathbf{M}) : \lim_{|A|\to\infty} \frac{\varphi(A)}{1+|A|^p} \text{ exists}\right\},$$

we see that $\mathcal{E}_p$ corresponds to the compactification of $\mathbf{M}$ by a sphere at ∞ while E_p corresponds to the one-point compactification. More general compactifications have been considered in [14], [28], [29] and [30].

We will use the following lemma, whose proof is elementary and left to the reader.

Proposition 2.2 *If ψ is Lipschitz continuous on the unit sphere S and homogeneous of degree p, $p \geq 1$, then there is a constant $C > 0$ (depending on ψ) such that*

$$|\psi(A) - \psi(B)| \leq C\left(|A|^{p-1} + |B|^{p-1}\right)|A - B|,$$

for any pair of matrices A, B.

A remark that will be used often in Sections 5 and 6 is the following. Given a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ and a sequence of functions $\{f_j\}$ taking values in $\mathbf{R}^d$ with $\{f_j\}$ bounded in $L^p(\Omega)$, it can be shown that if

$$\lim_{j\to\infty} \int_\Omega \theta(x)\varphi(f_j(x))\,dx = \int_\Omega \theta(x) \int_{\mathbf{R}^d} \varphi(\xi)\,d\nu_x(\xi)\,dx \tag{2.1}$$

for all $\theta \in C_0(\Omega)$, $\varphi \in C_0^\infty(\mathbf{R}^d)$, then (2.1) still holds for all $\varphi \in C(\mathbf{R}^d)$ such that $\{\varphi(f_j)\}$ is equiintegrable, and in particular, for all φ on $\mathbf{R}^d$ which grow slower than $1 + |\xi|^p$. Therefore $\nu = \{\nu_x\}_{x\in\Omega}$ is *the* Young measure associated to $\{f_j\}$. We conclude that in order to identify the Young measure generated by $\{f_j\}$, it suffices to study the limits (2.1) for $\theta \in C_0(\Omega)$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$. Also, it can be shown that

$$\int_\Omega \int_{\mathbf{R}^d} |\xi|^p\,d\nu_x(\xi)\,dx < \infty. \tag{2.2}$$

The main result in [21] is a characterization of $W^{1,p}$-Young measures in terms of Jensen's inequality for quasiconvex functions.

Theorem 2.3 *Let $p > 1$. Then $\nu = \{\nu_x\}_{x\in\Omega}$ is a $W^{1,p}$-Young measure if and only if*

1.

$$\nabla u(x) = \int_{\mathbf{M}} A\,d\nu_x(A), \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

for some $u \in W^{1,p}(\Omega)$;

2.

$$\varphi(\nabla u(x)) \leq \int_{\mathbf{M}} \varphi(A)\, d\nu_x(A), \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

for every quasiconvex φ for which the limit

$$\lim_{|A|\to\infty} \frac{\varphi(A)}{1+|A|^p}$$

exists;

3.

$$\int_\Omega \int_{\mathbf{M}} |A|^p \, d\nu_x(A)\, dx < \infty.$$

3. The Representation Formula

We introduced the space $\mathcal{E}_p$ in order to recover weak limits associated to sequences $\{\varphi(\nabla u_j)\}$ for $\varphi \in \mathcal{E}_p$ and any sequence $\{u_j\}$ that is bounded in $W^{1,p}(\Omega)$. The representation of weak limits for such functions in terms of Young measures is only valid if one can rule out concentration effects (see [9]). To account for possible development of concentrations, we associate to $\{\nabla u_j\}$ a measure Λ on $\Omega \times \mathcal{S}$, called the *varifold associated to* $\{\nabla u_j\}$. We first recall that for an $\mathbf{M}$-valued Radon measure μ on an open set Ω the polar decomposition (see [17]) is given by $d\mu = \alpha\, d\lambda$, where λ is the total variation of μ and $\alpha : \Omega \longrightarrow \mathcal{S}$ is the density of μ, i.e. the Radon-Nikodym derivative of μ with respect to its total variation λ (see [16]). To define the varifold we use the following general representation result for weak * limits of $\mathbf{M}$-valued Radon measures (see [17]).

Theorem 3.1 *Let $\{\mu_j\}$ be a sequence of $\mathbf{M}$-valued measures on Ω with polar decomposition $\alpha_j d\lambda_j$. Assume that $\mu_j \overset{*}{\rightharpoonup} \mu$ in the sense of measures. There exists a subsequence, still denoted $\{\mu_j\}$, and a non-negative, finite, Radon measure $\Lambda = \lambda_x \otimes \pi$ on $\Omega \times \mathcal{S}$ such that for every $f \in \mathcal{C}_0(\Omega \times \mathbf{R}^d)$*

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega f(x, \alpha_j(x))\, d\lambda_j(x) &= \int_{\Omega\times\mathcal{S}} f(x,y)\, d\Lambda(x,y) \\ &= \int_\Omega \int_{\mathcal{S}} f(x,y)\, d\lambda_x(y)\, d\pi(x).\end{aligned}$$

The decomposition $\Lambda = \lambda_x \otimes \pi$ is obtained via the slicing measures technique (see [15]), and by means of the Radon-Nikodym Theorem we write $\pi = \pi_a \mathcal{L}^N + \pi_s$ where $\pi_a = \frac{d\pi}{d\mathcal{L}^N}$ and π_s is singular with respect to $\mathcal{L}^N$.

Given a sequence $\{u_n\}$, bounded in $W^{1,p}(\Omega)$, we may regard the sequence $\Big\{|\nabla u_n|^{p-1}$ $\nabla u_n \mathcal{L}^N\Big\}$ as a bounded sequence of $\mathbf{M}$-valued Radon measures. Hence, according to Theorem 3.1, associated to a subsequence there exists a varifold, and this suggests the following definition.

Definition 3.2 *A finite, Radon measure Λ supported on $\Omega \times \mathcal{S}$ is a $W^{1,p}$-varifold if there exists a bounded sequence in $W^{1,p}(\Omega)$, $\{u_n\}$, such that for every $f \in \mathcal{C}_0(\Omega \times \mathbf{M})$*

$$\lim_{n\to\infty} \int_\Omega f\left(x, \frac{\nabla u_n}{|\nabla u_n|}\right) |\nabla u_n|^p \, dx = \int_{\Omega\times\mathcal{S}} f(x,A)\, d\Lambda(x,A).$$

In particular, if ψ is homogeneous of degree p and $\theta \in C_0(\Omega)$ then

$$\lim_{n\to\infty}\int_\Omega \theta(x)\psi(\nabla u_n)\,dx = \int_{\Omega\times S}\theta(x)\psi(A)\,d\Lambda(x,A)$$
$$= \int_\Omega \theta(x)\int_S \psi(A)\,d\lambda_x(A)\,d\pi(x).$$

In order to see how the YM-V pair determines the limits of $\{\varphi(\nabla u_n)\}$ for sequences $\{u_n\}$ bounded in $W^{1,p}(\Omega)$ and having oscillatory and concentrating features, consider $\varphi \in \mathcal{E}_p$. By definition,

$$\lim_{|A|\to\infty}\frac{\varphi(A)-\varphi_p^\infty(A)}{|A|^p} = 0,$$

which implies that $\{\varphi(\nabla u_n) - \varphi_p^\infty(\nabla u_n)\}$ is weakly relatively compact in $L^1(\Omega)$. For this sequence, the representation in terms of the Young measure is valid. On the other hand $\varphi_p^\infty \in \mathcal{H}_p$, and the limit for $\{\varphi_p^\infty(\nabla u_n)\}$ is therefore given by the varifold. Hence, we have the representation formula

$$\lim_{n\to\infty}\int_\Omega \theta(x)\varphi(\nabla u_n(x))\,dx = \int_\Omega \theta(x)\int_{\mathbf{M}} \left(\varphi(A)-\varphi_p^\infty(A)\right)\,d\nu_x(A)\,dx$$
$$+\int_\Omega \theta(x)\int_S \varphi_p^\infty(A)\,d\lambda_x(A)\,d\pi(x).$$

It is this formula that motivated our study of YM-V pairs.

4. Proof of the decomposition lemma

In this section we will prove Lemma 1.2. As mentioned in the introduction, our argument uses maximal functions and their properties, and we recall some well-known facts (see [31]).

Given a Borel measurable function $u : \mathbf{R}^N \to \mathbf{R}^d$, the *maximal function* of u is defined by

$$M(u)(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)|\,dy.$$

If $u \in W^{1,p}(\Omega)$ then we set

$$M^*(u)(x) = M(u)(x) + M(\nabla u)(x),$$

and if $p > 1$, then

$$\|M^*(u)\|_{L^p(\Omega)} \le C(N,p)\,\|u\|_{W^{1,p}(\Omega)}. \tag{4.1}$$

Lemma 4.1 *Let $p > 1$ and let $w \in W^{1,p}(\mathbf{R}^N;\mathbf{R}^m)$. Given $\lambda > 0$ there exists a Lipschitz function z in $\mathbf{R}^N$ such that $w = z$ on $\{M(\nabla w) < \lambda\}$ and the Lipschitz constant for z is bounded by $C(N)\lambda$, where $C(N)$ is a constant depending only upon dimension.*

For the proof see, e.g., [16].

The proof of Lemma 1.2 will be divided into two steps. In the first step we consider an *extension domain* Ω, i.e., an open, bounded set Ω for which there exists an extension operator $T : W^{1,p}(\Omega) \to W^{1,p}(\mathbf{R}^N)$ such that

$$Tu(x) = u(x), \quad x\in\Omega, \qquad \|Tu\|_{W^{1,p}(\mathbf{R}^N)} \le C\,\|u\|_{W^{1,p}(\Omega)}.$$

In the second step we remove this restriction on Ω, generalizing the result for arbitrary open sets.

Proof of Lemma 1.2.

Step 1. Assume that Ω is an extension domain. Let $\{w_n\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbf{R}^m)$. In the sequel, we identify w_n with its extension $Tw_n \in W^{1,p}(\mathbf{R}^N;\mathbf{R}^m)$.

By (4.1) the sequence $\{M(\nabla w_n)\}$ is bounded in $L^p(\Omega)$ and so (see [6] and (2.2)), there exists a subsequence (not relabeled) and a parametrized measure $\mu = \{\mu_x\}_{x\in\Omega}$ such that

$$\int_\Omega \int_{\mathbf{R}} |s|^p \, d\mu_x(s)\, dx < \infty, \tag{4.2}$$

and whenever $\{f(M(\nabla w_n))\}$ converges weakly in $L^1(\Omega)$, its weak limit is given by

$$\overline{f}(x) = \langle \mu_x, f\rangle, \quad \mathcal{L}^N\text{a.e. } x \in \Omega.$$

Let $k \in \mathbf{N}$ and consider the truncation map $T_k : \mathbf{R} \to \mathbf{R}$ given by

$$T_k(x) = \begin{cases} x, & |x| \le k, \\ k\frac{x}{|x|}, & |x| > k. \end{cases}$$

Clearly $\{T_k(M(\nabla w_n))\}$ is a bounded sequence in $L^\infty(\Omega)$, therefore equiintegrable, and so given $a \in L^\infty(\Omega)$ we have

$$\begin{aligned} \lim_{k\to\infty}\lim_{n\to\infty} \int_\Omega a(x)\, |T_k(M(\nabla w_n))(x)|^p \, dx &= \lim_{k\to\infty} \int_\Omega a(x) \int_{\mathbf{R}} |T_k(s)|^p \, d\mu_x(s)\, dx \\ &= \int_\Omega \int_{\mathbf{R}} a(x)\, |s|^p \, d\mu_x(s)\, dx, \end{aligned} \tag{4.3}$$

where we have used (4.2) and the Dominated Convergence Theorem. For every $k \in \mathbf{N}$, choose $n(k)$ with $n(k) > n(k-1)$, such that

$$\left| \lim_{n\to\infty} \int_\Omega |T_k(M(\nabla w_n))(x)|^p \, dx - \int_\Omega |T_k(M(\nabla w_m))(x)|^p \, dx \right| < \frac{1}{k}$$

whenever $m \ge n(k)$. Setting $a \equiv 1$, (4.3) reduces to

$$\lim_{k\to\infty} \int_\Omega \left|T_k(M(\nabla w_{n(k)}))(x)\right|^p dx = \int_\Omega \int_{\mathbf{R}} |s|^p \, d\mu_x(s)\, dx. \tag{4.4}$$

We claim that

$$\left|T_k(M(\nabla w_{n(k)}))\right|^p \rightharpoonup \overline{f} \text{ in } L^1(\Omega), \tag{4.5}$$

where

$$\overline{f}(x) = \int_{\mathbf{R}} |s|^p \, d\mu_x(s).$$

Indeed, fix $b \in L^\infty(\Omega)$, $l \in \mathbf{N}$ and let $k > l$. Clearly

$$\begin{aligned} \int_\Omega b(x)\left|T_k(M(\nabla w_{n(k)}))(x)\right|^p dx \le{}& \|b\|_{L^\infty(\Omega)} \int_\Omega \left|T_k(M(\nabla w_{n(k)}))(x)\right|^p dx \\ &- \int_\Omega \left(\|b\|_{L^\infty(\Omega)} - b(x)\right) \left|T_l(M(\nabla w_{n(k)}))(x)\right|^p dx, \end{aligned}$$

and so, taking first the limit as $k \to \infty$, followed by the limit as $l \to \infty$, and by virtue of (4.3) and (4.4), we conclude that

$$\limsup_{k\to\infty} \int_\Omega b(x) \left|T_k(M(\nabla w_{n(k)}))(x)\right|^p dx \le \int_\Omega \int_{\mathbf{R}} b(x)\,|s|^p \, d\mu_x(s)\, dx. \tag{4.6}$$

Similarly, (4.6) holds for $-b$ in place of b; hence

$$\lim_{k\to\infty} \int_\Omega b(x) \left|T_k(M(\nabla w_{n(k)}))(x)\right|^p dx = \int_\Omega b(x)\overline{f}(x)\, dx,$$

proving (4.5). Set

$$R_k = \left\{x \in \mathbf{R}^N : M(\nabla w_{n(k)})(x) \ge k\right\}.$$

By Lemma 4.1 there exist Lipschitz functions z_k such that

$$z_k = w_{n(k)} \quad \text{a.e. on } \mathbf{R}^N \setminus R_k, \qquad |\nabla z_k(x)| \le C(N)k, \quad \text{a.e. } x \in \mathbf{R}^N.$$

Therefore, by (4.1) and because Ω is bounded

$$\begin{aligned}\mathcal{L}^N\left(\Omega \cap \left\{z_k \ne w_{n(k)} \text{ or } \nabla z_k \ne \nabla w_{n(k)}\right\}\right) &\le \mathcal{L}^N\left(R_k \cap \Omega\right)\\ &\le \frac{1}{k^p}\int_\Omega \left|M(\nabla w_{n(k)})\right|^p dx,\end{aligned}$$

and this term tends to zero as $k \to \infty$. In addition, for $\mathcal{L}^N$ a.e. $x \in \Omega \setminus R_k$ we have

$$|\nabla z_k(x)| = \left|\nabla w_{n(k)}(x)\right| \le \left|M(\nabla w_{n(k)})(x)\right| = \left|T_k(M(\nabla w_{n(k)})(x))\right|,$$

while if $x \in R_k$ then

$$|\nabla z_k(x)| \le C(N)k = C(N)\left|T_k(M(\nabla w_{n(k)})(x))\right|.$$

We conclude that

$$|\nabla z_k(x)|^p \le C\left|T_k(M(\nabla w_{n(k)})(x))\right|^p \quad \text{a.e. } x \in \Omega,$$

which, together with (4.5), yields equiintegrability of $\{|\nabla z_k|^p\}$.

Step 2. Let Ω be an open, bounded domain of $\mathbf{R}^N$, and let $\{w_j\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbf{R}^m)$. Without loss of generality we may assume that there exists $w_0 \in W^{1,p}(\Omega;\mathbf{R}^m)$ such that

$$w_j \rightharpoonup w_0 \quad \text{in } W^{1,p}(\Omega;\mathbf{R}^m), w_j \to w_0 \quad \text{in } L^p(\Omega;\mathbf{R}^m),$$

i.e. if $w_j = w_0 + \tilde{w}_j$,

$$\tilde{w}_j \rightharpoonup 0 \quad \text{in } W^{1,p}(\Omega;\mathbf{R}^m), \tilde{w}_j \to 0 \quad \text{in } L^p(\Omega;\mathbf{R}^m).$$

Let $\{\Omega_n\}$ be an increasing sequence of compactly contained subdomains of Ω, with $\mathcal{L}^N(\Omega\setminus\Omega_n) \to 0$, and choose cut-off functions $\eta_n \in C_0^\infty(\Omega;[0,1])$ such that $\eta_n = 1$ if $x \in \Omega_n$. We have

$$\limsup_{n\to\infty}\limsup_{j\to\infty} \|\eta_n \tilde{w}_j\|_{L^p(\Omega)} = 0$$

and

$$\limsup_{n\to\infty}\limsup_{j\to\infty}\|\nabla(\eta_n\tilde{w}_j)\|_{L^p(\Omega)} = \limsup_{n\to\infty}\limsup_{j\to\infty}\|\tilde{w}_j\otimes\nabla\eta_n + \eta_n\nabla\tilde{w}_j\|_{L^p(\Omega)}$$
$$\leq \limsup_{j\to\infty}\|\nabla\tilde{w}_j\|_{L^p(\Omega)} < \infty.$$

A standard diagonalization procedure yields a bounded subsequence in $W^{1,p}(\Omega;\mathbf{R}^m)$, $\{\eta_n\tilde{w}_{j(n)}\}$, which we extend by zero to $\mathbf{R}^N$. Now the argument used in Step 1 applies to this sequence, so that we obtain a sequence $\{z_k\}$ of Lipschitz functions such that

$$\alpha_k = \mathcal{L}^N\left(\Omega\cap\left\{z_k \neq \eta_{n(k)}\tilde{w}_{j(n(k))} \text{ or } \nabla z_k \neq \nabla\left(\eta_{n(k)}\tilde{w}_{j(n(k))}\right)\right\}\right) \to 0,$$

as $k\to\infty$, and $\{|\nabla z_k|^p\}$ is equiintegrable. We conclude that $\{|\nabla(w_0+z_k)|^p\}$ is equiintegrable and

$$\mathcal{L}^N\left(\Omega\cap\left\{w_{j(n(k))}\neq w_0+z_k \text{ or } \nabla w_{j(n(k))}\neq\nabla(w_0+z_k)\right\}\right)\leq \alpha_k + \mathcal{L}^N\left(\Omega\setminus\Omega_{n(k)}\right)$$

and this term converges to zero as $k\to\infty$. ∎

5. Characterization of YM-V: necessary conditions

We devote this section to the proof of the necessity part of Theorem 1.1. We may assume that Ω is smooth as otherwise we can first consider smooth subsets of Ω and then exhaust Ω by such sets. Conditions 1 and 2 were established in [20] and [21]. To prove 3 and 4 we split Λ into a part, $P\nu$, that is determined by the Young measure and a remainder, $\tilde{\Lambda}$, that is related to pure concentrations effects.

For $\psi\in C(\mathcal{S})$ (with p-homogeneous extension $\tilde{\psi}$) and $\theta\in C_0(\Omega)$, let

$$\langle P\nu_x,\psi\rangle = \langle \nu_x,\tilde{\psi}\rangle = \int_{\mathbf{M}}\tilde{\psi}(A)\,d\nu_x(A)\,dx,$$

and $P\nu = \{P\nu_x\}_{x\in\Omega}\otimes\mathcal{L}^N$, i.e.,

$$\langle P\nu,\theta\otimes\psi\rangle = \int_\Omega\theta(x)\int_{\mathbf{M}}\tilde{\psi}(A)\,d\nu_x(A)\,dx.$$

Let $\tilde{\Lambda} = \Lambda - P\nu$. Suppose that $\{\nabla u_j\}$ generates the YM-V pair (ν,Λ). In Steps 2 and 3 below we will show that u_j can be decomposed as $u_j = z_j + v_j$ where $\{|\nabla z_j|^p\}$ is equiintegrable, $\{\nabla z_j\}$ generates the YM-V pair $(\nu, P\nu)$, and $\{\nabla v_j\}$ generates the YM-V pair $(\delta_0\otimes\mathcal{L}^N,\tilde{\Lambda})$.

Step 1. Reformulation of conditions 3 and 4.

We claim that 3 and 4 are equivalent to requiring that

i) $\tilde{\Lambda}$ is a nonnegative, finite, Radon measure on $\Omega\times\mathcal{S}$;

ii) if $\tilde{\Lambda} = \left\{\tilde{\lambda}_x\right\}_{x\in\Omega}\otimes\tilde{\pi}$ is the slicing decomposition of $\tilde{\Lambda}$, where $\tilde{\lambda}_x$ are probability measures on $\mathbf{M}$, then for $\tilde{\pi}$ a.e. $x\in\Omega$

$$\langle\tilde{\lambda}_x,\psi\rangle\geq 0 \tag{5.1}$$

for all $\psi \in \mathcal{H}_p$ such that $Q\psi(0) = 0$.

Assume first that 3 and 4 hold. Since Λ and $P\nu$ are finite, nonnegative Radon measures, it follows that $\tilde{\Lambda}$ is a finite Radon measure. In addition, if $\theta \in C_0(\Omega)$, $\theta \geq 0$, and if $\psi \in C(S)$, $\psi \geq 0$, then $Q\tilde{\psi}(0) = 0$ and we have by 3 and 4

$$\begin{aligned}\langle \tilde{\Lambda}, \theta \otimes \psi \rangle =& \int_\Omega \theta(x) \int_S \psi(A)\, d\lambda_x(A)\, d\pi_s(x) \\ &+ \int_\Omega \theta(x) \left[\frac{d\pi}{d\mathcal{L}^N}(x) \int_S \psi(A)\, d\lambda_x(A) - \int_{\mathrm{M}} \tilde{\psi}(A)\, d\nu_x(A) \right] dx \\ \geq& 0.\end{aligned} \tag{5.2}$$

Hence $\tilde{\Lambda} \geq 0$, proving i).

In order to prove ii), fix $\theta \in C_0(\Omega)$, $\theta \geq 0$, $\psi \in \mathcal{H}_p$, $Q\psi(0) = 0$, and using the slicing decomposition of $\tilde{\Lambda}$ and (5.2) we deduce that

$$\int_\Omega \theta(x) \langle \tilde{\lambda}_x, \psi \rangle \, d\tilde{\pi}(x) = \langle \tilde{\Lambda}, \theta \otimes \psi \rangle \geq 0. \tag{5.3}$$

The arbitrariness of θ yields the existence of a $\tilde{\pi}$-null set E_ψ such that if $x \in \Omega \setminus E_\psi$ then

$$\langle \tilde{\lambda}_x, \psi \rangle \geq 0.$$

Let $\{\psi_k\}$ be a countable, dense set in $\mathcal{H}_p$, and define

$$E = \bigcup_k \bigcup_{\{n : Q(\psi_k + (1/n)|A|^p)(0) = 0\}} E_{\psi_k + (1/n)|A|^p}. \tag{5.4}$$

It is clear that $\tilde{\pi}(E) = 0$. Fix $x \in \Omega \setminus E$, $\psi \in \mathcal{H}_p$, $Q\psi(0) = 0$, and choose a subsequence $\{\psi_{k_i}\}$ such that

$$\psi_{k_i} \to \psi \quad \text{in } L^\infty(S), \qquad \|\psi_{k_i} - \psi\|_{L^\infty(S)} < \frac{1}{n_i},$$

where $n_i \to \infty$. Then

$$\begin{aligned}\psi_{k_i}(A) + \frac{1}{n_i} |A|^p &\geq \psi_{k_i}(A) + |A|^p \, \|\psi_{k_i} - \psi\|_{L^\infty(S)} \\ &\geq \psi_{k_i}(A) + |\psi_{k_i}(A) - \psi(A)| \\ &\geq \psi(A),\end{aligned}$$

and so

$$Q(\psi_{k_i} + \frac{1}{n_i} |\cdot|^p)(0) \geq Q\psi(0) = 0.$$

By homogeneity

$$Q(\psi_{k_i} + \frac{1}{n_i} |\cdot|^p)(0) = 0.$$

Finally, using the definition of E, $x \notin E_{\psi_{k_i} + (1/n_i)|\cdot|^p}$, therefore

$$0 \leq \lim_{i \to \infty} \langle \tilde{\lambda}_x, \psi_{k_i} + \frac{1}{n_i} |\cdot|^p \rangle = \langle \tilde{\lambda}_x, \psi \rangle, \tag{5.5}$$

concluding the proof of ii).

Conversely, if i) and ii) hold, using (5.2), (5.3) and $\theta = \chi_{B(a,\rho)}$, $a \in \Omega$, $\rho > 0$, we have

$$\int_{B(a,\rho)} \langle \lambda_x, \psi \rangle \, d\pi_s + \int_{B(a,\rho)} \left(\frac{d\pi}{d\mathcal{L}^N}(x) \langle \lambda_x, \psi \rangle - \langle \nu_x, \psi \rangle \right) dx \geq 0$$

for $\psi \in \mathcal{H}_p$, $Q\psi(0) = 0$. Conditions 3 and 4 follow by virtue of the Radon-Nikodym Theorem. Note that a priori the exceptional sets could depend on ψ, but the argument outlined for the definition of E above would entail the existence of π_s- and $\mathcal{L}^N$-negligible sets for which 3 and 4 hold for all $\psi \in \mathcal{H}_p$, such that $Q\psi(0) = 0$.

In light of Step 1, the rest of this section will be dedicated to proving (5.1).

Step 2. Construction of $\{z_j\}$.

By the decomposition lemma (Lemma 1.2) there exists a sequence of Lipschitz functions $\{z_j\}$ such that $\{|\nabla z_j|^p\}$ is equiintegrable in Ω and the set

$$R_j = \{x \in \Omega : z_j(x) \neq u_j(x), \nabla z_j(x) \neq \nabla u_j(x)\}$$

satisfies

$$\mathcal{L}^N(R_j) \to 0. \tag{5.6}$$

In particular, $\{\nabla z_j\}$ generates the YM-V pair $(\nu, P\nu)$.

Step 3. Construction of $\{v_j\}$.

Let $v_j = u_j - z_j$. We claim that $\{\nabla v_j\}$ generates the YM-V pair $(\delta_0 \otimes \mathcal{L}^N, \tilde{\Lambda})$. In particular, $\tilde{\Lambda} \geq 0$. The assertion regarding the Young measure follows from (5.6). To study the varifold generated by $\{\nabla v_j\}$ consider $\theta \in \mathcal{C}_0(\Omega)$ and $\psi \in \mathcal{H}_p$ such that $\psi|_S$ is Lipschitz. In view of Proposition 2.2 and Hölder's inequality we have

$$\begin{aligned}
&\left| \int_\Omega \theta(x)\psi(\nabla v_j)\, dx - \int_\Omega \theta(x)\left(\psi(\nabla u_j) - \psi(\nabla z_j)\right) dx \right| \\
&= \left| \int_{R_j} \theta(x)\left(\psi(\nabla u_j - \nabla z_j) - \psi(\nabla u_j) + \psi(\nabla z_j)\right) dx \right| \\
&\leq C\,\|\theta\|_\infty \int_{R_j} \left[\left(|\nabla u_j - \nabla z_j|^{p-1} + |\nabla u_j|^{p-1} \right) |\nabla z_j| + |\nabla z_j|^p \right] dx \\
&\leq C\,\|\theta\|_\infty \left[\left(\int_{R_j} |\nabla z_j|^p \, dx \right)^{1/p} + \int_{R_j} |\nabla z_j|^p \, dx \right].
\end{aligned}$$

Since $\mathcal{L}^N(R_j) \to 0$ as $j \to \infty$ and $\{|\nabla z_j|^p\}$ is equiintegrable, the last term goes to zero as $j \to \infty$ and thus, using Step 2, we conclude that

$$\int_\Omega \theta(x)\psi(\nabla v_j)\, dx \to \langle \Lambda - P\nu, \theta \otimes \psi \rangle = \langle \tilde{\Lambda}, \theta \otimes \psi \rangle.$$

By density, the result extends to all $\psi \in \mathcal{H}_p$ and the claim is proved.

Step 4. We prove that for $\tilde{\pi}$ a.e. $x \in \Omega$

$$\langle \tilde{\lambda}_x, \psi \rangle \geq 0 \tag{5.7}$$

for all $\psi \in \mathcal{H}_p$ with $Q\psi(0) = 0$.

We first make the additional assumption that ψ is Lipschitz on $\mathcal{S}$. Let $\theta \in C_0^\infty(B(a,\rho))$, $0 \le \theta \le 1$. By the definition of $Q\psi$, Proposition 2.2 and Hölder's inequality, we have

$$\begin{aligned} 0 &\le \int_{B(a,\rho)} \psi\left(\nabla(\theta v_j)\right) \\ &= \int_{B(a,\rho)} \psi\left(\theta \nabla v_j + v_j \otimes \nabla\theta\right) dx \\ &\le \int_{B(a,\rho)} \theta^p \psi(\nabla v_j)\, dx + C \int_{B(a,\rho)} \left(|\theta \nabla v_j|^{p-1} + |v_j \otimes \nabla\theta|^{p-1}\right) |v_j \otimes \nabla\theta|\, dx \\ &\le \int_{B(a,\rho)} \theta^p \psi(\nabla v_j)\, dx + C(\theta) \left[\left(\int_{B(a,\rho)} |v_j|^p\, dx\right)^{1/p} + \int_{B(a,\rho)} |v_j|^p\, dx\right]. \end{aligned}$$

Now $v_j \rightharpoonup 0$ in $W^{1,p}(B(a,\rho))$ as $j \to \infty$, and thus $v_j \to 0$ in $L^p(B(a,\rho))$. By Step 3, the sequence $\{\nabla v_j\}$ generates the varifold $\tilde\Lambda$. Therefore taking the limit as $j \to \infty$ in the above inequality, we obtain

$$0 \le \langle \tilde\Lambda, \theta^p \otimes \psi \rangle.$$

The assertion follows (for $\psi \in \text{Lip}(\mathcal{S})$) by taking an increasing sequence $\theta_i \to \chi_{B(a,\rho)}$ and applying the dominated convergence theorem. Hence

$$\int_{B(a,\rho)} \langle \tilde\lambda_x, \psi \rangle\, d\bar\pi(x) \ge 0,$$

and the Radon-Nikodym Theorem yields the existence of a set $E_\psi \subset \Omega$, $\bar\pi(E_\psi) = 0$, such that (5.7) holds if $x \notin E_\psi$. Defining E as in (5.4) and following the argument (5.4)-(5.5), we finally remove the restriction that ψ be Lipschitz on $\mathcal{S}$ to conclude that (5.7) holds for $\psi \in \mathcal{H}_p$, $Q\psi(0) = 0$, proving ii).

6. Characterization of YM-V: sufficient conditions

Suppose that the pair (ν, Λ) satisfies the conditions of Theorem 1.1. We have to construct a sequence $\{u_j\}$, bounded in $W^{1,p}(\Omega)$, such that (ν, Λ) is the YM-V pair generated by $\{\nabla u_j\}$.

As in the beginning of Section 5, we write $\Lambda = P\nu + \tilde\Lambda$ where

$$\langle P\nu, \theta \otimes \psi \rangle = \int_\Omega \theta(x) \int_{\mathbf{M}} \tilde\psi(A)\, d\nu_x(A)\, dx$$

for $\theta \in C_0(\Omega)$, $\psi \in C(\mathcal{S})$ and with $\tilde\psi$ the p-homogeneous extension of ψ. From Section 5, Step 1, we know that $\tilde\Lambda$ is a nonnegative, finite, Radon measure and

$$\langle \tilde\lambda_x, \psi \rangle \ge 0 \tag{6.1}$$

for all $\psi \in \mathcal{H}_p$ with $Q\psi(0) = 0$, where $\left\{\tilde\lambda_x\right\}_{x\in\Omega} \otimes \bar\pi$ denotes the slicing decomposition of $\tilde\Lambda$.

Step 1. We claim that it suffices to find $\{z_j\}$, $\{v_j\}$ bounded sequences in $W^{1,p}(\Omega)$ such that

$$\{|\nabla z_j|^p\} \text{ is equiintegrable,} \quad \{\nabla z_j\} \text{ generates the YM-V pair } (\nu, P\nu), \tag{6.2}$$

and

$$\{\nabla v_j\} \text{ generates the YM-V pair } (\delta_0 \otimes \mathcal{L}^N, \tilde{\Lambda}), \tag{6.3}$$

setting, as before, $u_j = z_j + v_j$. Indeed, note that since $p > 1$ then $\{|\nabla v_j|\}$ is equiintegrable, and so given $\lambda > 0$ and in view of (2.1)

$$\mathcal{L}^N(\{|\nabla v_j| > \lambda\}) \leq \frac{1}{\lambda}\int_\Omega |\nabla v_j|\, dx \to \frac{1}{\lambda}\langle \delta_0 \otimes \mathcal{L}^N, |\cdot|\rangle = 0. \tag{6.4}$$

Thus given $\theta \in C_0(\Omega)$ and $\varphi \in C_0^\infty(\mathbf{M})$ we have

$$\left|\int_\Omega \theta(x)\,[\varphi(\nabla u_j) - \varphi(\nabla z_j)]\, dx\right| \leq \|\theta\|_\infty C \int_\Omega |\nabla v_j|\, dx \to 0$$

as $j \to \infty$ and this implies that the Young measure associated to $\{\nabla u_j\}$ is also ν.

Similarly, if $\theta \in C_0(\Omega)$, $\psi \in \mathcal{H}_p$, $\psi|_S$ Lipschitz, by Proposition 2.2 for fixed $\lambda > 0$

$$\begin{aligned}
&\left|\int_\Omega \theta(x)\,(\psi(\nabla u_j) - \psi(\nabla z_j) - \psi(\nabla v_j))\, dx\right| \\
&\leq C\int_{\{|\nabla v_j| \leq \lambda\}} (|\psi(\nabla u_j) - \psi(\nabla z_j)| + |\psi(\nabla v_j)|)\, dx \\
&+ C\int_{\{|\nabla v_j| \geq \lambda\}} (|\psi(\nabla u_j) - \psi(\nabla v_j)| + |\psi(\nabla z_j)|)\, dx \\
&\leq C\int_{\{|\nabla v_j| \leq \lambda\}} \left(|\nabla z_j|^{p-1} + |\nabla v_j|^{p-1}\right)|\nabla v_j|\, dx + C\int_{\{|\nabla v_j| \leq \lambda\}} |\nabla v_j|^p\, dx \\
&+ C\int_{\{|\nabla v_j| \geq \lambda\}} \left(|\nabla z_j|^{p-1} + |\nabla v_j|^{p-1}\right)|\nabla z_j|\, dx + C\int_{\{|\nabla v_j| \geq \lambda\}} |\nabla z_j|^p\, dx,
\end{aligned}$$

and so, using Hölder's inequality, (6.4) and the equiintegrability of $\{|\nabla z_j|^p\}$,

$$\limsup_{j\to\infty}\left|\int_\Omega \theta(x)\,[\psi(\nabla u_j) - \psi(\nabla z_j) - \psi(\nabla v_j)]\, dx\right| = O(\lambda).$$

Letting $\lambda \to 0^+$ and removing the regularity restrictions imposed on ψ as in (5.4)-(5.5), we conclude that the varifold associated to $\{\nabla u_j\}$ is $P\nu + \tilde{\Lambda} = \Lambda$.

<u>*Step 2*</u>. We introduce two sets of measures supported on the unit sphere $\mathcal{S}$ of M, namely

$$\begin{aligned}
A &= \{\mu \in \mathcal{M}(\mathcal{S}) : \mu \geq 0, \langle \mu, \psi\rangle \geq 0 \text{ if } \psi \in \mathcal{H}_p, Q\psi(0) = 0\}, \\
H &= \left\{\overline{\delta_{\nabla u/|\nabla u|} \otimes |\nabla u|^p\, \mathcal{L}^N} : u \in W_0^{1,p}(B)\right\},
\end{aligned}$$

where B is the unit ball in R^N, and the average measures of H are defined by

$$\langle \overline{\delta_{\nabla u/|\nabla u|} \otimes |\nabla u|^p\, \mathcal{L}^N}, \psi\rangle = \frac{1}{|B|}\int_B \psi\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u|^p\, dx$$

for $\psi \in C(\mathcal{S})$. We do not distinguish henceforth a continuous function on $\mathcal{S}$ from its p-homogeneous extension. Note that, in view of (6.1), $\tilde{\lambda}_x \in A$ for $\tilde{\pi}$ a.e. $x \in \Omega$. It is clear that A is weak * closed and $H \subset A$.

Proposition 6.1 *A is the weak * closure of H. Moreover, if $R > 0$ then $A \cap \{\|\mu\| \le R\}$ is the weak * closure of $H \cap \{\|\mu\| \le R\}$.*

Remark. The second statement will be useful in Step 3 where we will use the fact that the weak * topology of $\mathcal{M}(S)$ is metrizable on closed balls.

Proof of Proposition 6.1. The proof is a standard application of the Hahn-Banach Theorem. We start by proving that H is convex. Fix $\theta \in (0,1)$ and let for $i = 1,2$

$$\mu_i = \overline{\delta_{\nabla u_i/|\nabla u_i|} \otimes |\nabla u_i|^p \mathcal{L}^N}, \quad u_i \in W_0^{1,p}(B).$$

Let $x_0 \in B$ be such that $|x_0| = 1/2$ and define

$$\tilde{u}_1(x) = k^{-1+N/p} u_1(kx), \quad \tilde{u}_2(x) = k^{-1+N/p} u_2(k(x - x_0)),$$

where $k \ge 4$. Clearly $\tilde{u}_i \in W_0^{1,p}(B)$, $\tilde{u}_1$ and $\tilde{u}_2$ have disjoint supports, and a change of variables shows that $\tilde{\mu}_i = \mu_i$ for $i = 1,2$. It follows that the function

$$\tilde{u} = \theta^{1/p}\tilde{u}_1 + (1-\theta)^{1/p}\tilde{u}_2 \in W_0^{1,p}(B)$$

generates $\mu = \theta\mu_1 + (1-\theta)\mu_2$ and so $\mu \in H$.

We now show that A cannot be separated from H. Assume that $\psi \in C(S)$ is such that $\langle \nu, \psi \rangle \ge a$ for all $\nu \in H$ and for some $a \in \mathbf{R}$. Hence, extending ψ as p-homogeneous,

$$Q\psi(0) = \inf_{u \in W_0^{1,p}(B)} \frac{1}{|B|} \int_B \psi(\nabla u)\, dx \ge a$$

and so $0 \ge Q\psi(0) \ge a$. We conclude that $Q\psi(0)$ is finite, thus $Q\psi(0) = 0$ by homogeneity, and $0 = Q\psi(0) \ge a$. By definition of A, we have that $\langle \mu, \psi \rangle \ge 0 \ge a$ for all $\mu \in A$. Hence A cannot be separated from H.

Similarly, one shows that $A_R = A \cap \{\|\mu\| \le R\}$ cannot be separated from $H_R = H \cap \{\|\mu\| \le R\}$. Indeed, let $\psi \in C(S)$ be such that $\langle \nu, \psi \rangle \ge a$ for all $\nu \in H_R$ and for some $a \in \mathbf{R}$. We first note that $a \le 0$. In fact, if $a > 0$, since

$$\|\mu\| = \frac{1}{|B|} \int_B |\nabla u|^p\, dx$$

when

$$\mu = \overline{\delta_{\nabla u/|\nabla u|} \otimes |\nabla u|^p \mathcal{L}^N}, \quad u \in W_0^{1,p}(B),$$

then

$$\inf \left\{ \frac{1}{|B|} \int_B \psi(\nabla u)\, dx : u \in W_0^{1,p}(B), \frac{1}{|B|} \int_B |\nabla u|^p \le R \right\} \ge a. \tag{6.5}$$

Hence, if $u \in W_0^{1,p}(B)$ and $C(u) = \frac{1}{|B|} \int_B |\nabla u|^p\, dx > R$, then for $v = (R/C(u))^{1/p} u$, (6.5) and the homogeneity imply

$$\frac{1}{|B|} \int_B \psi(\nabla u)\, dx = \frac{C(u)}{R} \frac{1}{|B|} \int_B \psi(\nabla v)\, dx \ge a$$

which, together with (6.5), yields $Q\psi(0) \ge a$. Therefore $Q\psi(0)$ is finite, and by the homogeneity we have $0 = Q\psi(0) \ge a$, contradicting our assumption. We conclude that $a \le 0$ and so

$$\inf_{u \in W_0^{1,p}(B)} \left\{ \frac{1}{|B|} \int_B \psi(\nabla u)\, dx - \frac{a}{R} \frac{1}{|B|} \int_B |\nabla u|^p\, dx \right\} \ge Q\psi(0) = 0.$$

Thus $Q\left(\psi - \frac{a}{R}|\cdot|^p\right)(0)$ is finite, and by the homogeneity, it vanishes. Therefore if $\mu \in A_R$ then

$$\langle \mu, \psi\rangle - \frac{a}{R}\|\mu\| = \langle \mu, \psi\rangle - \frac{a}{R}\langle \mu, 1\rangle \geq 0$$

yielding $\langle \mu, \psi\rangle \geq a$ because $a \leq 0$. ■

<u>Step 3</u>. Construction of $\{z_j\}$.
Using condition 3 in Theorem 1.1 with $\psi(A) = |A|^p$ we have

$$\int_\Omega \int_{\mathbf{M}} |A|^p \, d\nu_x(A)\, dx \leq \int_\Omega \frac{d\pi}{d\mathcal{L}^N}(x) \int_S d\lambda_x(A)\, dx \leq \pi(\Omega) < \infty,$$

which, together with conditions 1 and 2, and by Theorem 2.3 (see [21] for the proof) implies that ν is a $W^{1,p}$-Young measure. Using the decomposition lemma (Lemma 1.2) (see also Step 2, Section 5) we find a sequence $\{z_j\}$ bounded in $W^{1,p}(\Omega)$ and satisfying (6.2).

<u>Step 4</u>. Construction of $\{v_j\}$ when

$$\tilde{\Lambda} = \sum_{i=1}^{I} c_i \lambda_i \otimes \delta_{x_i}, \quad x_i \in \Omega, \lambda_i \in A, c_i > 0.$$

Here we search for a sequence $\{v_j\}$ bounded in $W_0^{1,p}(\Omega)$ such that (6.3) holds, i.e. $\{\nabla v_j\}$ generates $(\delta_0 \otimes \mathcal{L}^N, \tilde{\Lambda})$ and, in addition,

$$\lim_{j\to\infty} \|\nabla v_j\|^p_{L^p(\Omega)} = \left\|\tilde{\Lambda}\right\|.$$

By Proposition 6.1 and the remark after it, there exist sequences $\left\{w_j^{(i)}\right\}$ in $W_0^{1,p}(B)$ such that

$$\lim_{j\to\infty} \frac{1}{|B|} \int_B \psi(\nabla w_j^{(i)})\, dx = \langle \lambda_i, \psi\rangle$$

for all $\psi \in \mathcal{H}_p$. In particular

$$\|\lambda_i\| = \lim_{j\to\infty} \frac{1}{|B|} \int_B \left|\nabla w_j^{(i)}\right|^p dx.$$

Now

$$v_j(x) = j^{-1+N/p} \frac{1}{|B|^{1/p}} \sum_{i=1}^{I} c_i^{1/p} w_j^{(i)}\left(j(x - x_i)\right)$$

has the desired properties.

<u>Step 5</u>. Construction of $\{v_j\}$ in the general case.
To obtain $\{v_j\}$ satisfying (6.3), we will use the following approximation lemma.

Lemma 6.2 *Let $\tilde{\Lambda}$ be a nonnegative, finite, Radon measure on $\Omega \times S$ with slicing decomposition $\left\{\tilde{\lambda}_x\right\}_{x\in\Omega} \otimes \tilde{\pi}$, let A be a convex set of the set of all nonnegative, finite, Radon measures on S, and suppose that*

$$\tilde{\lambda}_x \in A \text{ for } \tilde{\pi} \text{ a.e. } x \in \Omega.$$

*Then $\bar{\Lambda}$ can be approximated in the weak * topology by measures of the form*

$$\bar{\Lambda}^{(k)} = \sum_{i=1}^{I_k} c_i^{(k)} \lambda_i^{(k)} \otimes \delta_{x_i^{(k)}}, \quad x_i^{(k)} \in \Omega, \lambda_i^{(k)} \in A, c_i^{(k)} > 0,$$

such that

$$\left\| \bar{\Lambda}^{(k)} \right\| \le \left\| \bar{\Lambda} \right\|.$$

Before proving the approximation lemma, we conclude the construction of $\{v_j\}$. By Lemma 6.2 we have

$$\bar{\Lambda} = \text{ w-* limit } \bar{\Lambda}^{(k)}, \bar{\Lambda}^{(k)} = \sum_{i=1}^{I_k} c_i^{(k)} \lambda_i^{(k)} \otimes \delta_{x_i^{(k)}}, \left\| \bar{\Lambda}^{(k)} \right\| \le \left\| \bar{\Lambda} \right\|.$$

Also, Step 4 yields the existence of sequences $\left\{ v_j^{(k)} \right\}$ bounded in $W_0^{1,p}(\Omega)$ generating the YM-V pair $(\delta_0 \otimes \mathcal{L}^N, \bar{\Lambda}^{(k)})$, and such that

$$\lim_{j \to \infty} \left\| \nabla v_j^{(k)} \right\|_{L^p(\Omega)}^p = \left\| \bar{\Lambda}^{(k)} \right\| \le \left\| \bar{\Lambda} \right\|$$

for all k. Separability of $C_0(\Omega)$, $C_0(\mathrm{M})$ and $C(\mathcal{S})$, and a standard diagonalization argument allow us to extract a diagonal subsequence $v_k = v_{j(k)}^{(k)}$ satisfying (6.3) and

$$\sup_k \|\nabla v_k\|_{L^p(\Omega)}^p \le \left\| \bar{\Lambda} \right\| + 1.$$

It remains to prove Lemma 6.2.

Proof of Lemma 6.2. The result is well-known to experts. We include a proof for the convenience of the reader. By Besicovitch's Covering Theorem, for each $k \in \mathbf{N}$ there exists a finite family of disjoint closed balls $B(x_i^{(k)}, r_i^{(k)})$ such that

$$\bar{\pi} \left(\Omega \setminus \bigcup_{i \in I_k} B(x_i^{(k)}, r_i^{(k)}) \right) < \frac{1}{k}, \quad r_i^{(k)} < \frac{1}{k}. \tag{6.6}$$

Set

$$\langle \lambda_i^{(k)}, \psi \rangle = \frac{1}{\bar{\pi} \left(B(x_i^{(k)}, r_i^{(k)}) \right)} \int_{B(x_i^{(k)}, r_i^{(k)})} \langle \bar{\lambda}_x, \psi \rangle \, d\bar{\pi}(x).$$

Since A is convex we have $\lambda_i^{(k)} \in A$, and we define

$$\bar{\Lambda}^{(k)} = \sum_{i=1}^{I_k} c_i^{(k)} \lambda_i^{(k)} \otimes \delta_{x_i^{(k)}}, \quad c_i^{(k)} = \bar{\pi} \left(B(x_i^{(k)}, r_i^{(k)}) \right).$$

Then

$$\left\| \bar{\Lambda}^{(k)} \right\| = \sum_{i=1}^{I_k} c_i^{(k)} \left\| \lambda_i^{(k)} \right\| \le \sum_{i=1}^{I_k} \bar{\pi} \left(B(x_i^{(k)}, r_i^{(k)}) \right) \le \bar{\pi}(\Omega) = \left\| \bar{\Lambda} \right\|.$$

For $\psi \in C(S)$ and $\theta \in W_0^{1,\infty}(\Omega)$ with Lipschitz constant $\mathrm{Lip}(\theta)$ one has

$$\begin{aligned}
&\left|\langle \tilde{\Lambda}^{(k)} - \tilde{\Lambda}, \theta \otimes \psi\rangle\right| = \\
&\left|\int_\Omega \langle \tilde{\lambda}_x, \psi\rangle \theta(x)\, d\tilde{\pi}(x) - \sum_{i=1}^{I_k} c_i^{(k)} \langle \lambda_i^{(k)}, \psi\rangle \theta(x_i^{(k)})\right| \\
&\leq \left|\sum_{i=1}^{I_k} \int_{B(x_i^{(k)}, r_i^{(k)})} \langle \tilde{\lambda}_x, \psi\rangle \theta(x)\, d\tilde{\pi}(x) - \sum_{i=1}^{I_k} \int_{B(x_i^{(k)}, r_i^{(k)})} \langle \tilde{\lambda}_x, \psi\rangle\, d\tilde{\pi}(x) \theta(x_i^{(k)})\right| \\
&\quad + \int_{\Omega \setminus \cup B(x_i^{(k)}, r_i^{(k)})} \left|\langle \tilde{\lambda}_x, \psi\rangle\right| |\theta(x)|\, d\tilde{\pi}(x) \\
&\leq \frac{1}{k} \mathrm{Lip}(\theta) \int_\Omega \left|\langle \tilde{\lambda}_x, \psi\rangle\right| d\tilde{\pi}(x) + \|\psi\|_{L^\infty(S)} \|\theta\|_{L^\infty(\Omega)}\, \tilde{\pi}\left(\Omega \setminus \cup B(x_i^{(k)}, r_i^{(k)})\right),
\end{aligned}$$

and this expression tends to zero as $k \to \infty$. We have used (6.6). The assertion follows since test functions of the above type are dense and $\left\{\left\|\tilde{\Lambda}^{(k)}\right\|\right\}$ is bounded. ■

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