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Proof of the Mumford-Shah Conjecture for Two Shaded Image Segmentations

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Abstract

In this paper, we consider minimizing the Mumford-Shah functional over two-valued functions in the plane. Existence of minimizers is straightforward and we show that the edge set of any minimizer is a finite union of C^1 curves.

Keywords: image segmentation, sets of finite perimeter, isoperimetric inequality

AMS Classifications: 49K10, 49N60, 49Q05, 49Q15

1 Introduction

In the variational approach to image segmentation, one seeks minimizers of the Mumford-Shah functional

$$E(u, K) = \int_{\Omega\setminus K} |u - g|^2 dx + \int_{\Omega\setminus K} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K),$$

where $g \in L^\infty(\Omega)$ is the initial image, $u \in C^1(\Omega\setminus K)$, $\mathcal{H}^{N-1}(K)$ is the $N-1$ dimensional Hausdorff measure of the relatively closed set $K \subset \Omega$, and $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain. Minimizers of this functional are close to the initial image due to the first term, smoothed due to the second, and segmented due to all three: if the first term forces any "low energy" u to have large enough gradient along some $N-1$ dimensional surface K, then the image is segmented across K, which relieves u from needing to be smooth across K. The last term ensures that segmentations occur only when necessary.

From the point of view of image processing, the set K corresponds to edges of objects in an image, placed where a smooth grey scale image is forced to change too much too quickly.

Mumford and Shah [MS] conjectured that if $N = 2$ minimizers exist and the edge set K of any minimizer consists of a finite number of C^1 curves. It was shown in [DGCL] that minimizers exist. This was done by reformulating the problem in SBV, a space introduced by De Giorgi and Ambrosio [DGA], so that weak solutions could be shown to exist using a compactness theorem due to Ambrosio [A]. They then proved that any minimizer u is in $C^1(\Omega\setminus K)$, where K is the closure in Ω of the jump set of u, i.e., the set of points that are not Lebesgue for u.

Attention has largely turned to the regularity of K (see, e.g., [DS], [AP], [AFP]). In particular, it was shown in [AP] and [AFP] that, for $\Omega \subset \mathbb{R}^N$, optimal edge sets are $C^{1,\alpha}$ hypersurfaces outside a closed set of $\mathcal{H}^{N-1}$ measure 0. The main idea was to analyze the behavior of $|\nabla u|$ near $x \in K$ that can cause a singularity in K at x.

In this paper, we take a step towards understanding the regularity of K when there are no singularities caused by $|\nabla u|$. We consider minimizing the Mumford-Shah functional only over two-valued functions in the plane (and so also rule out singularities due to triple junctions, i.e., singularities in K that occur when three regions with different values of u meet at a point), which is equivalent to minimizing over constant multiples of characteristic functions. If $S \subset \Omega$, we denote its characteristic function by χ_S, and for $u = C\chi_S$, the edge set K is $\overline{\partial_* S} \cap \Omega$ since $\partial_* S$ is the jump set of $C\chi_S$. Our energy is then

$$E(C\chi_S) = \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(\overline{\partial_* S} \cap \Omega). \tag{1.1}$$

This variational problem corresponds to finding an optimal placement of edges around clusters of overlaping objects.

We first consider the energy

$$E(C\chi_S) := \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(\partial_* S) \tag{1.2}$$

and prove that for a minimizer S one has $\mathcal{H}^1(\overline{\partial_* S} \cap \Omega \backslash \partial_* S) = 0$, so that the last term in the above energy is the same as $\mathcal{H}^1(\overline{\partial_* S} \cap \Omega)$. Since $\partial_* S \subset \overline{\partial_* S} \cap \Omega$, minimizers of (1.2) coincide with minimizers of (1.1). Furthermore, we show that for such an S, there exists an open set $A \subset \Omega$ such that $\mathcal{L}^2(S \Delta A) = 0$ and $A = \bar{A}^\circ$. Next, we prove $A = \cup_{i=1}^m A_i$, where A_i are the connected components of A and the distance between $\overline{\partial_* A_i}$ and $\overline{\partial_* A_j}$ is positive away from $\partial\Omega$, if $i \neq j$. Analogous results are obtained for S^c, the complement of S in Ω, e.g., $\mathcal{L}^2(S^c \Delta \cup_{i=1}^p O_i) = 0$ where O_i are connected, etc. Finally, we conclude that $\overline{\partial_* A_i} \cap \overline{\partial_* O_j} \cap \Omega$ is C^1 for $i = 1, \ldots, m$, $j = 1, \ldots, p$, which proves the Mumford-Shah conjecture for "two shaded" image segmentations.

2 Preliminaries

We consider a bounded, simply connected, Lipschitz domain $\Omega \subset \mathbb{R}^2$, and we define the space of functions of bounded variation $BV(\Omega)$ in the usual way (see, e.g., [EG] and [Z]). For $E \subset \Omega$, χ_E stands for the characteristic function of E. Given two sets A and B, the *symmetric difference* is given by $A \Delta B := (A \backslash B) \cup (B \backslash A)$, and if $D \subset \Omega$ then we define $\text{dist}_D(A, B) := \text{dist}(A \cap D, B \cap D)$. For $A \subset \Omega$, we denote by A^c its complement, $\bar{A}$ its closure, and A° its interior. We write $D \subset\subset \Omega$ if $D \subset \Omega$ is open and $\bar{D} \subset \Omega$.

We say that a set $E \subset \Omega$ has *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$, in which case the *measure theoretic boundary* in Ω is defined as

$$\partial_* E := \left\{ x \in \Omega : \limsup_{\delta \to 0^+} \frac{\mathcal{L}^2(B(x,\delta) \cap E)}{\mathcal{L}^2(B(x,\delta))} > 0 \text{ and } \limsup_{\delta \to 0^+} \frac{\mathcal{L}^2(B(x,\delta) \backslash E)}{\mathcal{L}^2(B(x,\delta))} > 0 \right\},$$

where $B(x, \delta)$ is the open ball in $\mathbb{R}^2$ centered at x with radius δ. We denote by $\nu_E(x)$ the *measure theoretic normal* to E at $x \in \partial_* E$ (for properties of this normal, see [EG]). The *reduced boundary* $\partial^* E$ is the set of $x \in \partial_* E$ such that x is a Lebesgue point for ν_E with respect to the Radon measure $\mathcal{H}^1 \lfloor \partial_* E$.

For $u \in BV(\Omega)$, we write $Du = D_{ac}u + D_s u$, where $D_{ac}u$ and $D_s u$ stand for, respectively, the absolutely continuous and singular parts of Du with respect to $\mathcal{L}^2$. We also consider the set $S(u)$ of points which are not Lebesgue points for u. We use the representation $D_{ac}u = \nabla u \mathcal{L}^2$. We say u is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $Du = \nabla u \mathcal{L}^2 + D_s u \lfloor S(u)$. This space was introduced by De Giorgi and Ambrosio [DGA].

3 Regularity of Edge Sets

Definition 3.1 *For $C > 0$ fixed, we define*

$$E_g^\chi(S) := \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(\partial_* S),$$

where $g \in L^\infty(\Omega)$ and $S \subset \Omega$ is measurable.

It follows from BV compactness in L^1 and the lower semicontinuity of perimeter that E_g^χ has a minimum. Indeed, even if we let C vary there is a minimum. Let $C_n\chi_{S_n}$ be a minimizing sequence, and note that we can assume $|C_n| \leq ||g||_\infty$. χ_{S_n} is bounded in $BV(\Omega)$, so, for a subsequence, $C_n \to C$ and $\chi_{S_n} \to \chi_S$ in $L^1(\Omega)$ for some $S \subset \Omega$. Since perimeter is lower semicontinuous, it follows that $C\chi_S$ is a minimizer.

Lemma 3.2 *Suppose that S minimizes E_g^χ. Then $\mathcal{H}^1(\overline{\partial_* S} \cap \Omega \backslash \partial_* S) = 0$.*

Proof. Note first that if $C\chi_S$ is a minimizer of E, then the conclusion holds by [DGCL]. Here, we need to show that the result is true even if $C\chi_S$ is a minimum only over characteristic functions. The basic strategy follows [AP].

Step 1: We claim that for $D \subset\subset \Omega$, there exists $r_D > 0$ such that if $x \in \overline{\partial_* S} \cap D$ and $r \leq r_D$, then $\mathcal{H}^1(\partial_* S \cap B(x,r)) \geq 2r$. Clearly, it suffices to show this for $x \in \partial_* S \cap D$.

Let $D \subset\subset \Omega$ and $x \in \partial_* S \cap D$ and choose $r_D < \min\{2(C + ||g||_\infty)^{-2}, \operatorname{dist}(D, \partial\Omega)\}$. Suppose that $r \leq r_D$ and $\mathcal{H}^1(\partial_* S \cap B(x,r)) < 2r$. We will show that this leads to a contradiction. Put

$$S_t := S' \cap \partial B(x,t)$$

and

$$T_t := (S^c)' \cap \partial B(x,t),$$

where $S' := \{x \in \Omega : S \text{ has density 1 at } x\}$, and similarly for $(S^c)'$.

Step 1.A: We claim that

$$\mathcal{H}^1(\{t \in (0,r) : \mathcal{H}^1(S_t) = 0 \text{ or } \mathcal{H}^1(T_t) = 0\}) > 0. \tag{3.1}$$

Suppose that

$$\mathcal{H}^1(S_t), \mathcal{H}^1(T_t) > 0 \text{ for } \mathcal{H}^1\text{-a.e. } t \in (0,r).$$

We can choose $\phi_n \in C^\infty(B(x,r))$ such that

$$\phi_n \xrightarrow{L^1} \chi_S \text{ on } B(x,r)$$

and

$$|D\phi_n|(B(x,r)) \to \mathcal{H}^1(\partial_* S \cap B(x,r)). \tag{3.2}$$

It follows that for $\mathcal{H}^1$-a.e. $t \in (0,r)$ we have

$$\int_{\partial B(x,t)} |\phi_n - \chi_{S'}| d\mathcal{H}^1 \to 0. \tag{3.3}$$

For $t \in (0,r)$ such that (3.3) holds and $\mathcal{H}^1(S_t), \mathcal{H}^1(T_t) > 0$, we then have

$$\liminf_{n\to\infty} \int_{\partial B(x,t)} \left|\frac{\partial \phi_n}{\partial \tau}\right| d\mathcal{H}^1 \geq 2,$$

where $\frac{\partial \phi_n}{\partial \tau}$ denotes the tangential derivative of ϕ_n on $\partial B(x,t)$. Hence, by (3.2)

$$\begin{aligned} \mathcal{H}^1(\partial_* S \cap B(x,r)) &= \lim_{n\to\infty} |D\phi_n|(B(x,r)) \\ &\geq \liminf_{n\to\infty} \int_0^r \int_{\partial B(x,t)} \left|\frac{\partial \phi_n}{\partial \tau}\right| d\mathcal{H}^1 dt \\ &\geq 2r. \end{aligned}$$

This concludes the proof of (3.1). Since $\mathcal{H}^1 \lfloor \partial_* S$ is a Radon measure, we can choose $t \in (0,r)$ such that, e.g., $\mathcal{H}^1(S_t) = 0$ and $\mathcal{H}^1(\partial_* S \cap \partial B(x,t)) = 0$. Set

$$T := S' \backslash B(x,t).$$

Step 1.B: Next, we claim that $\mathcal{H}^1(\partial_* T \backslash \partial_* S) = 0$. Note that $\partial_* T \backslash \partial_* S \subset \partial B(x,t)$ and if $y \in \partial_* T \cap \partial B(x,t)$, then

$$\limsup_{r \to 0+} \frac{\mathcal{L}^2(S \cap B(y,r))}{\mathcal{L}^2(B(y,r))} > 0.$$

If in addition S does not have density 1 at y (i.e., $y \notin S_t$), then $y \in \partial_* S$. Thus $\partial_* T \backslash \partial_* S \subset S_t$ and since $\mathcal{H}^1(S_t) = 0$, we have $\mathcal{H}^1(\partial_* T \backslash \partial_* S) = 0$.

Step 1.C: We prove that $E_g^\chi(T) < E_g^\chi(S)$. From the isoperimetric inequality and Step 1.B we have

$$\mathcal{H}^1(\partial_* S) - \mathcal{H}^1(\partial_* T) = \mathcal{H}^1(\partial_*[S \backslash T]) \geq 2\sqrt{\pi}\mathcal{L}^2(S \backslash T)^{\frac{1}{2}}.$$

Since $r \leq r_D$, we know that $r < 2(C + ||g||_\infty)^{-2}$, and so

$$\begin{aligned} \mathcal{L}^2(S \backslash T) &< \pi r^2 \\ &< 4\pi(C + ||g||_\infty)^{-4}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}^2(S \backslash T)(C + ||g||_\infty)^2 &< 2\sqrt{\pi}\mathcal{L}^2(S \backslash T)^{\frac{1}{2}} \\ &\leq \mathcal{H}^1(\partial_* S) - \mathcal{H}^1(\partial_* T). \end{aligned}$$

But this implies that $E_g^\chi(T) < E_g^\chi(S)$ because

$$\int_\Omega |C\chi_T - g|^2 dx - \int_\Omega |C\chi_S - g|^2 dx \leq \mathcal{L}^2(S \backslash T)(C + ||g||_\infty)^2.$$

Since this contradicts S being a minimizer, we have proved the claim in Step 1.

Step 2: Now, following [AP], we set $\mu := \mathcal{H}^1 \lfloor \partial_* S$ and note that

$$\liminf_{r \to 0+} \frac{\mu(B(x,r))}{r} \geq 2$$

for all $x \in \overline{\partial_* S}$. Hence,

$$0 = \mu(\overline{\partial_* S} \backslash \partial_* S) \geq \mathcal{H}^1(\overline{\partial_* S} \backslash \partial_* S).$$

□

Lemma 3.3 *Suppose that S minimizes E_g^χ. Then there is an open set $A \subset \Omega$ such that $\mathcal{L}^2(S \Delta A) = 0$ and $A = \bar{A}^\circ$.*

Proof. Define S' as in the previous lemma, and note that $\chi_{S'}$ has the same total variation measure and jump set as χ_S. We wish to show that we can take $A = S'^\circ$. It is clear that

$$\overline{S'} \supset S' \cup \overline{\partial_* S'}$$

and we claim that $\overline{S} = S' \cup \overline{\partial_* S'}$. Suppose that $x \notin S' \cup \overline{\partial_* S'}$. Then S does not have density 1 at x and we can choose an $r > 0$ such that $B(x,r) \cap \partial_* S' = \emptyset$. Hence, $|D\chi_S|(B(x,r)) = \mathcal{H}^1(\partial_* S \cap B(x,r)) = 0$, and so χ_S is a constant $\mathcal{L}^2$-a.e. in $B(x,r)$. Since S does not have density 1 at x, we know that S has density 0 on $B(x,r)$, and so $B(x,r) \cap S' = \emptyset$ and $x \notin \overline{S'}$.

Now, suppose that $x \in S' \backslash \overline{\partial_* S'}$. Then S has density 1 at x and and we can choose $r > 0$ such that $B(x,r) \cap \partial_* S' = \emptyset$, so S has density 1 on $B(x,r)$, and $x \in S'^\circ$. Clearly, $S'^\circ \subset S' \backslash \overline{\partial_* S'}$, thus

$$S'^\circ = S' \backslash \overline{\partial_* S'}.$$

Since S' minimizes E_g^χ, we know that $\mathcal{H}^1(\partial_* S') < \infty$ and by the previous lemma $\mathcal{H}^1(\overline{\partial_* S'} \cap \Omega) < \infty$, hence

$$\mathcal{L}^2(S \Delta S'^\circ) = 0.$$

Clearly $S'^{\circ} \subset (\overline{S'^{\circ}})^{\circ}$. We also have $\overline{S'^{\circ}} \subset \overline{S'} = S' \cup \overline{\partial_* S'}$. Suppose $B \subset S' \cup \overline{\partial_* S'}$ is open. If $B \cap \overline{\partial_* S'} \neq \emptyset$, then $\mathcal{L}^2(B \backslash S) > 0$. But this is a contradiction since $\mathcal{L}^2(\overline{\partial_* S'}) = 0$. Therefore, $B \subset S'$ which implies $(S' \cup \overline{\partial_* S'})^{\circ} = S'^{\circ}$. So, $(\overline{S'^{\circ}})^{\circ} \subset S'^{\circ}$ and

$$S'^{\circ} = (\overline{S'^{\circ}})^{\circ}.$$

□

Lemma 3.4 *Suppose that S minimizes E^{χ}_{g}. Then we can write $A = \cup_{i=1}^{m} A_i$, where A is the set from Lemma 3.3 and A_i are disjoint, open, and connected sets. Furthermore,*

$$\text{dist}_D(A_i, A_j) > 0 \text{ if } i \neq j \text{ and } D \subset\subset \Omega.$$

Proof. We may write $A = \cup_{i=1}^{\infty} A_i$, where A_i are disjoint, open, and connected sets. We first claim that $\partial^* A \cap \partial^* A_i \cap \partial^*(A \backslash A_i) = \emptyset$. We know (see, e.g., Theorem 5.6.2 of [Z], Theorem 1 in Section 5.7.2 of [EG]) that if $x \in \partial^* A \cap \partial^* A_i \cap \partial^*(A \backslash A_i)$, then

$$\lim_{r \to 0+} \frac{\mathcal{L}^2(A \cap B(x,r))}{\mathcal{L}^2(B(x,r))} = \frac{1}{2},$$

$$\lim_{r \to 0+} \frac{\mathcal{L}^2(A_i \cap B(x,r))}{\mathcal{L}^2(B(x,r))} = \frac{1}{2},$$

and

$$\lim_{r \to 0+} \frac{\mathcal{L}^2([A \backslash A_i] \cap B(x,r))}{\mathcal{L}^2(B(x,r))} = \frac{1}{2},$$

which is a contradiction.

We next claim that

$$\mathcal{H}^1(\partial_* A_i \cap \partial_*[A \backslash A_i]) = 0. \tag{3.4}$$

We first show that $\partial_* A_i, \partial_*(A \backslash A_i) \subset \overline{\partial_* A}$. If $x \notin \overline{\partial_* A}$, then we can choose $r > 0$ such that $B(x,r) \cap \partial_* A = \emptyset$. This implies $|D\chi_A|(B(x,r)) = 0$, and so χ_A is a constant $\mathcal{L}^2$-a.e. on $B(x,r)$. Since $A = S'^{\circ}$, it follows that $B(x,r) \subset A$ or $B(x,r) \subset A^c$, which yields $x \notin \partial_* A_i \cup \partial_*(A \backslash A_i)$. We conclude, using Lemma 3.2, that

$$\begin{aligned}
\mathcal{H}^1(\partial_* A_i \cap \partial_*[A \backslash A_i]) &= \mathcal{H}^1(\overline{\partial_* A} \cap \partial_* A_i \cap \partial_*[A \backslash A_i]) \\
&= \mathcal{H}^1(\partial_* A \cap \partial_* A_i \cap \partial_*[A \backslash A_i]) \\
&= \mathcal{H}^1(\partial^* A \cap \partial^* A_i \cap \partial^*[A \backslash A_i]) \\
&= 0.
\end{aligned}$$

If A_i is removed from A, then $\int_{\Omega} |C\chi_S - g|^2 dx$ is increased by at most $(C + ||g||_{\infty})^2 \mathcal{L}^2(A_i)$. It is clear from the definition of measure theoretic boundary and the proof of (3.4) that

$$\partial_* A \subset \partial_* A_i \cup \partial_*[A \backslash A_i] \subset \overline{\partial_* A}.$$

So,

$$\begin{aligned}
\mathcal{H}^1(\partial_* A) &= \mathcal{H}^1(\partial_* A_i \cup \partial_*[A \backslash A_i]) \\
&= \mathcal{H}^1(\partial_* A_i) + \mathcal{H}^1(\partial_*[A \backslash A_i]) - \mathcal{H}^1(\partial_* A_i \cap \partial_*[A \backslash A_i]) \\
&= \mathcal{H}^1(\partial_* A_i) + \mathcal{H}^1(\partial_*[A \backslash A_i]).
\end{aligned}$$

Therefore, by removing A_i from A, $\mathcal{H}^1(\partial_* A)$ is decreased by $\mathcal{H}^1(\partial_* A_i)$. Due to the minimality of A it follows that

$$\mathcal{H}^1(\partial_* A_i) \leq (C + ||g||_{\infty})^2 \mathcal{L}^2(A_i). \tag{3.5}$$

Although the relative isoperimetric inequality (Theorem 5.4.3 in [Z], Theorem 2, Section 5.6.2 in [EG]) is stated for balls, it is immediate from the proof that a relative isoperimetric inequality holds for any bounded Lipschitz domain. In particular, there exists a constant $k > 0$ such that

$$\min\{\mathcal{L}^2(E), \mathcal{L}^2(\Omega \backslash E)\}^{1-\frac{1}{2}} \leq k\mathcal{H}^1(\partial_* E)$$

for all $E \subset \Omega$ measurable. Let A_i be a connected component of A, and suppose that $\mathcal{L}^2(A_i) \leq \frac{1}{2}\mathcal{L}^2(\Omega)$. It follows from our isoperimetric inequality that

$$\frac{\mathcal{L}^2(A_i)}{\mathcal{H}^1(\partial_* A_i)} \leq k\mathcal{L}^2(A_i)^{\frac{1}{2}}.$$

This, together with (3.5), gives

$$\mathcal{L}^2(A_i) \geq k^{-2}(C + ||g||_\infty)^{-4}.$$

Since Ω is bounded, there are finitely many A_i.

Finally, we prove that $\overline{\partial_* A_i} \cap \overline{\partial_* A_j} = \emptyset$ if $i \neq j$ and so $\text{dist}_D(A_i, A_j) > 0$ for $D \subset\subset \Omega$. Suppose that $x \in \overline{\partial_* A_i} \cap \overline{\partial_* A_j} \cap D$, where $D \subset\subset \Omega$ and $i \neq j$. Then

$$\mathcal{H}^1([\partial_* A_i \cup \partial_* A_j] \cap B(x,r)) \geq 4r \tag{3.6}$$

for $r < r_D$, where we have applied Step 1 in the proof of Lemma 3.2 to $\partial_* A_i$ and $\partial_* A_j$, and we used the fact that, by an argument just like that proving (3.4), we know these sets intersect on a set of $\mathcal{H}^1$ measure 0.

However, note that if for $r > 0$ we take $T := A \backslash B(x,r)$, then

$$\int_\Omega |C\chi_T - g|^2 dx - \int_\Omega |C\chi_A - g|^2 dx$$

is at most $(C + ||g||_\infty)^2 \mathcal{L}^2(B(x,r))$, while

$$\mathcal{H}^1(\partial_* T) - \mathcal{H}^1(\partial_* A) = \mathcal{H}^1(\partial B(x,r) \cap A) - \mathcal{H}^1(\partial_* A \cap B(x,r)).$$

Since $E_g^\chi(T) \geq E_g^\chi(A)$, we have

$$\mathcal{H}^1(\partial_* A \cap B(x,r)) \leq (C + ||g||_\infty)^2 \mathcal{L}^2(B(x,r)) + \mathcal{H}^1(\partial B(x,r) \cap A).$$

A similar argument can be made for $T := A \cup B(x,r)$, and as

$$\min\{\mathcal{H}^1(\partial B(x,r) \cap A), \mathcal{H}^1(\partial B(x,r) \backslash A)\} \leq \pi r,$$

it follows that

$$\mathcal{H}^1(\partial_* A \cap B(x,r)) \leq (C + ||g||_\infty)^2 \mathcal{L}^2(B(x,r)) + \pi r, \tag{3.7}$$

contradicting (3.6) for sufficiently small r.

□

Now, note that

$$\begin{aligned} E_{C-g}^\chi(S^c) &= \int_\Omega |C\chi_{S^c} - (C-g)|^2 dx + \mathcal{H}^1(\partial_* S^c) \\ &= \int_\Omega |C(1-\chi_S) - (C-g)|^2 dx + \mathcal{H}^1(\partial_* S) \\ &= E_g^\chi(S). \end{aligned}$$

Hence, S^c minimizes E_{C-g}^χ if S minimizes E_g^χ, and so we may write

$$\mathcal{L}^2(S^c \triangle \cup_{i=1}^p O_i) = 0,$$

$\text{dist}_D(O_i, O_j) > 0$ if $i \neq j$ and $D \subset\subset \Omega$, and all properties obtained for S and A_i hold also for S^c and O_i.

We will need the following lemma in order to prove the regularity theorem, Theorem 3.6.

Lemma 3.5 *Let $A \subset\subset \Omega$ be a simply connected domain with Lipschitz boundary. Suppose that $E \subset \Omega$ has finite perimeter. Suppose further that there are $a \neq b \in \partial A$ so that the connected components C, D of $\partial A \backslash \{a, b\}$ are such that $E \cap A$ has density 0 $\mathcal{H}^1$-a.e. on C and $A \backslash E$ has density 0 $\mathcal{H}^1$-a.e. on D. Then*

$$\left| \int_{\partial_* E \cap A} \nu_E \cdot e_1 d\mathcal{H}^1 \right| = |b_2 - a_2|,$$

and similarly for e_2 and $|b_1 - a_1|$.

Proof. The proof is a natural generalization of the proofs of equations (6.7) and (6.8) in [L].

□

Theorem 3.6 *Suppose that S minimizes E_g^χ. Then $\overline{\partial_* S} \cap \Omega$ is a finite union of C^1 curves.*

Proof. Set $C_{i,j} := \overline{\partial_* A_i} \cap \overline{\partial_* O_j} \cap \Omega$ and note that $\overline{\partial_* S} \cap \Omega = \cup_{i=1}^m \cup_{j=1}^p C_{i,j}$. We claim that $C_{i,j}$ is a C^1 curve.

In Step 1, for $D \subset\subset \Omega$ we find a constant $\bar{c} \in (0,1)$ depending on D such that for sufficiently small $r > 0$, given any $x \in C_{i,j} \cap D$ we can choose $t \in (\bar{c}r, r)$ with the following property: we can find $a, b \in \partial B(x,t)$ so that one connected component of $\partial B(x,t) \backslash \{a, b\}$ does not intersect A_i, and the other connected component does not intersect O_j. In Step 2, we get an estimate for the maximum distance between the line L connecting these points and $C_{i,j} \cap B(x,t)$. In particular, we find a constant $c' > 0$ such that this maximum is bounded above by $c't^2$. In Step 3, we show that ν_{A_i} is locally uniformly continuous on $\partial^* A_i$, and so it can be extended continuously to $\overline{\partial_* A_i} \cap \Omega$. Step 4 consists of proving that $C_{i,j}$ is locally the graph of a C^1 function, and finally we prove in Step 5 that $C_{i,j}$ are the connected components of $\overline{\partial_* S} \cap \Omega$.

Step 1: Let $D \subset\subset \Omega$ and $x \in C_{i,j} \cap D$, and set

$$m(x,t) := \mathcal{H}^0(C_{i,j} \cap \partial B(x,t))$$

and

$$\alpha(x,r) := \mathcal{H}^1(\{t \in (0,r) : m(x,t) \geq 4\}).$$

Choose $D' \subset\subset \Omega$ such that $D \subset\subset D'$ and set

$$\bar{r}_D := \min\{r_D,\ \mathrm{dist}(D, \partial D'),\ \mathrm{dist}_{D'}(C_{i,j}, \overline{\partial_* S} \backslash C_{i,j})\} > 0.$$

For $r < \bar{r}_D$ we know from the fact that (3.1) led to a contradiction that

$$\mathcal{H}^1(\{t \in (0,r) : \mathcal{H}^1(A_i \cap \partial B(x,t)), \mathcal{H}^1(O_j \cap \partial B(x,t)) > 0\}) = r. \tag{3.8}$$

If $y \in \overline{A_i \cap \partial B(x,t)} \cap \overline{O_j \cap \partial B(x,t)}$, then for all $\delta > 0$, χ_S is not a constant $\mathcal{L}^2$-a.e. on $B(y,\delta)$, and so $y \in C_{i,j}$. For $t \in (0,r)$ such that $\mathcal{H}^1(A_i \cap \partial B(x,t)), \mathcal{H}^1(O_j \cap \partial B(x,t)) > 0$, it is immediate that either $m(x,t) = \infty$ or

$$\mathcal{H}^0(\overline{A_i \cap \partial B(x,t)} \cap \overline{O_j \cap \partial B(x,t)}) \geq 2,$$

and so we have $m(x,t) \geq 2$. By the definition of $\bar{r}_D$, we know that if $r < \bar{r}_D$, then

$$\mathcal{H}^1([C_{i,j} \Delta \overline{\partial_* A_i}] \cap B(x,r)) = 0$$

and

$$\mathcal{H}^1([\overline{\partial_* A_i} \Delta \partial_* A_i] \cap B(x,r)) = 0.$$

By (3.7) we have, for $c := (C + ||g||_\infty)^2$,

$$\begin{aligned} c\pi r^2 + \pi r &\geq \mathcal{H}^1(C_{i,j} \cap B(x,r)) \\ &\geq \int_0^r m(x,t)dt \\ &\geq 4\alpha(x,r) + 2(r - \alpha(x,r)), \end{aligned}$$

which implies that $\alpha(x,r) \leq \frac{1}{2}c\pi r^2 + \left(\frac{\pi}{2} - 1\right) r$. If necessary, we can redefine $\bar{r}_D > 0$ to guarantee that we can find $\bar{c} \in (0,1)$ such that $r - \alpha(x,r) > \bar{c}r$ for all $r \leq \bar{r}_D$. Choose $t \in (\bar{c}r, r)$ such that $m(x,t) \in \{2,3\}$ and, by (3.8), $\mathcal{H}^1(A_i \cap \partial B(x,t)), \mathcal{H}^1(O_j \cap \partial B(x,t)) > 0$. For either value of m, we can choose $a, b \in \overline{A_i \cap \partial B(x,t)} \cap \overline{O_j \cap \partial B(x,t)}$, so that one connected component of $\partial B(x,t) \backslash \{a,b\}$ does not intersect A_i, and the other does not intersect O_j.

Step 2: Let L be the straight line segment connecting a and b, and let l be its length. Assume, without loss of generality, that e_2 is normal to L in the O_j direction. We can consider adding the A_i "side" of L to A_i, and similarly for O_j, which must not reduce E_g^χ. That is, we set

$$T := (A_i \cup \text{ the } A_i \text{ "side" of } L \text{ in } B(x,t)) \backslash \text{ the } O_j \text{ "side" of } L \text{ in } B(x,t)$$

and note that

$$E_g^\chi(T) \leq E_g^\chi(S) - \mathcal{H}^1(C_{i,j} \cap B(x,t)) + l + c\pi t^2.$$

Since $E_g^\chi(T) \geq E_g^\chi(S)$, it follows from Step 1 in Lemma 3.2 that

$$c\pi t^2 + l \geq \mathcal{H}^1(C_{i,j} \cap B(x,t)) \geq 2t.$$

Set

$$d(x,t) := \sup\{\text{dist}(y,L) : y \in C_{i,j} \cap B(x,t)\}.$$

We claim that we can find $c' > 0$ depending only on c and $\bar{r}_D$ such that

$$d(x,t) \leq c't^2. \tag{3.9}$$

We know, for T as above, that

$$E_g^\chi(T) \leq E_g^\chi(S) - \mathcal{H}^1(C_{i,j} \cap B(x,t)) + l + 4cd(x,t)t,$$

so

$$4ctd(x,t) + l \geq \mathcal{H}^1(C_{i,j} \cap B(x,t)) \geq 2t. \tag{3.10}$$

We claim also that

$$4d(x,t)^2 \leq \mathcal{H}^1(C_{i,j} \cap B(x,t))^2 - l^2. \tag{3.11}$$

Since $|\nu_{A_i}| = 1$ $\mathcal{H}^1$-a.e. on $\partial_* A_i$, it follows that

$$\begin{aligned} \mathcal{H}^1(C_{i,j} \cap B(x,t)) &= \int_{C_{i,j} \cap B(x,t)} |\nu_{A_i}|^2 d\mathcal{H}^1 \\ &= \int_{C_{i,j} \cap B(x,t)} (\nu_{A_i} \cdot e_1)^2 d\mathcal{H}^1 + \int_{C_{i,j} \cap B(x,t)} (\nu_{A_i} \cdot e_2)^2 d\mathcal{H}^1. \end{aligned}$$

By Jensen's inequality, we know

$$\fint_{C_{i,j} \cap B(x,t)} (\nu_{A_i} \cdot e_1)^2 d\mathcal{H}^1 \geq \left(\fint_{C_{i,j} \cap B(x,t)} \nu_{A_i} \cdot e_1 d\mathcal{H}^1 \right)^2,$$

and similarly for e_2. Hence, we have

$$\mathcal{H}^1(C_{i,j}\cap B(x,t)) =$$

$$= \frac{\mathcal{H}^1(C_{i,j}\cap B(x,t))}{\mathcal{H}^1(C_{i,j}\cap B(x,t))}\left[\int_{C_{i,j}\cap B(x,t)}(\nu_{A_i}\cdot e_1)^2 d\mathcal{H}^1 + \int_{C_{i,j}\cap B(x,t)}(\nu_{A_i}\cdot e_2)^2 d\mathcal{H}^1\right]$$

$$\geq \frac{1}{\mathcal{H}^1(C_{i,j}\cap B(x,t))}\left[\left(\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_1 d\mathcal{H}^1\right)^2 + \left(\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_2 d\mathcal{H}^1\right)^2\right].$$

So,

$$\mathcal{H}^1(C_{i,j}\cap B(x,t))^2 \geq \left(\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_1 d\mathcal{H}^1\right)^2 + \left(\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_2 d\mathcal{H}^1\right)^2. \quad (3.12)$$

Note that the same holds if $\nu_{A_i}\cdot e_k$ is replaced by $|\nu_{A_i}\cdot e_k|$. From Lemma 3.5, with $E = A_i$ and $A = B(x,t)$, we know that

$$\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_1 d\mathcal{H}^1 = 0 \text{ and } \left|\int_{C_{i,j}\cap B(x,t)}\nu_{A_i}\cdot e_2 d\mathcal{H}^1\right| = l. \quad (3.13)$$

Hence $\int_{C_{i,j}\cap B(x,t)}|\nu_{A_i}\cdot e_2|d\mathcal{H}^1 \geq l$, and to prove the claim (3.11), it is sufficient by (3.12) and (3.13) to prove $2d \leq \int_{C_{i,j}\cap B(x,t)}|\nu_{A_i}\cdot e_1|d\mathcal{H}^1$. Let $\varepsilon > 0$ be given and choose $v \in \partial^* A_i \cap B(x,t)$ such that $\mathrm{dist}(v,L) > d(x,t)-\varepsilon$. Since A_i and O_j are connected, we can find $z \in A_i \cap \partial B(x,t)$, $w \in O_j \cap \partial B(x,t)$, and smooth curves K and M, such that K connects v and z in $\{v\}\cup A_i \cap B(x,t)$ and is normal to $C_{i,j}$ at v and to $\partial B(x,t)$ at z, and M connects v and w in $\{v\}\cup O_j\cap B(x,t)$ and is normal to $C_{i,j}$ at v and to $\partial B(x,t)$ at w. We can then apply Lemma 3.5 to both "sides" of $K\cup M$ in $B(x,t)$, yielding, together with the arbitrariness of ε, $2d \leq \int_{C_{i,j}\cap B(x,t)}|\nu_{A_i}\cdot e_1|d\mathcal{H}^1$.

Now, we have

$$\begin{aligned}
4d(x,t)^2 &\leq \mathcal{H}^1(C_{i,j}\cap B(x,t))^2 - l^2 \text{ (by (3.11))}\\
&= (\mathcal{H}^1(C_{i,j}\cap B(x,t)) - l)(\mathcal{H}^1(C_{i,j}\cap B(x,t)) + l)\\
&\leq (4ctd(x,t))(4ctd(x,t) + 2l) \text{ (by (3.10))}\\
&= 16c^2t^2 d(x,t)^2 + 8ctd(x,t)l\\
&\leq 16c^2t^2 d(x,t)^2 + 16ct^2 d(x,t), \text{ (since } l \leq 2t)
\end{aligned}$$

which gives

$$d(x,t) \leq 4ct^2(1 + c\bar{r}_D).$$

We label this last constant multiplying t^2 by c'.

Step 3: We claim that ν_{A_i} is locally uniformly continuous on $\partial^* A_i$. Let $y \in C_{i,j}\cap B(x,\frac{1}{2}t)$. Let $n \in \mathrm{N}$ and choose $t(y) \in (\bar{c}\frac{1}{2n}t, \frac{1}{2n}t)$ as for x. Choose a and b for y, and denote the normal to $L(a,b)$ by $\nu(y)$. We may then find $t(a) \in (\bar{c}t(y), t(y))$ such that $\partial B(a,t(a))$ intersects $C_{i,j}$ two or three times, with A_i and O_j separated in $\partial B(a,t(a))$ by a' and b', and $b' \in B(y,t(y))$. Since $\mathrm{dist}(b',L(a,b)) \leq c't(y)^2$, similarly for $\mathrm{dist}(a,L(a',b'))$, and $\mathrm{dist}(a,b')$, $\mathrm{dist}(a',a) > \bar{c}t(y)$, we see that

$$|\nu(a) - \nu(y)| \leq ct(y)$$

$$|\nu(a') - \nu(a)| \leq ct(y),$$

for some $c > 0$, where $\nu(a)$ is normal to $L(a, y)$ and $\nu(a')$ is normal to $L(a', a)$.

We may procede similarly n times, each time picking $a^k \in \partial B(a^{k-1}, t(a^{k-1}))$, $t(a^k) \in (\bar{c}t(a), t(a))$, with $\nu(a^k)$ normal to $L(a^k, a^{k-1})$. Since $nt(a) < \frac{1}{2}t$, we know that we stay in $B(x, t)$. It follows that we have

$$|\nu(a^k) - \nu(a^{k-1})| \leq \frac{ct}{2n}.$$

Setting

$$\beta := \nu(y) \cdot e_1,$$

we see that

$$\nu(a^k) \cdot e_1 \geq \beta - \frac{ct}{2}$$

for all $k \in \{1, \ldots, n\}$. We have

$$a_2^n - y_2 = \sum_{k=1}^{n} (a_2^k - a_2^{k-1}),$$

where $a^0 := y$. Assuming $\beta > 0$ without loss of generality, and further assuming $\beta - \frac{ct}{2} > 0$, we also have

$$\begin{aligned} a_2^k - a_2^{k-1} &= (\nu(a^k) \cdot e_1)(a^k, a^{k-1}) \\ &\geq (\beta - \tfrac{ct}{2})\bar{c}^2 t(y), \end{aligned}$$

so

$$\begin{aligned} a_2^n - y_2 &\geq n(\beta - \tfrac{ct}{2})\bar{c}^2 t(y) \\ &\geq \bar{c}^3(\beta - \tfrac{ct}{2})\tfrac{1}{2}t. \end{aligned}$$

But,

$$\begin{aligned} |a_2^n - y_2| &\leq 2d(x, t) \\ &\leq 2c' t^2, \end{aligned}$$

which implies that $\beta \leq t(4\frac{c'}{\bar{c}^2} + \frac{c}{2})$. If $\beta - \frac{ct}{2} \leq 0$, we still have $\beta \leq t\frac{c}{2}$. A similar argument can be made for $\nu(y) \cdot e_2$, so that

$$|\nu(y) - \nu(x)| \leq \tilde{c}t, \tag{3.14}$$

for some $\tilde{c} > 0$. Since, for $x, y \in \partial^* A$, we can choose r small enough so that $\nu(x)$ is arbitrarily close to $\nu_{A_i}(x)$ and n large enough so that $\nu(y)$ is arbitrarily close to $\nu_{A_i}(y)$, (3.14) implies local uniform continuity of ν_{A_i}, and so ν_{A_i} can be extended continuously from $\partial^* A$ to $\overline{\partial_* A} \cap \Omega$. In particular, this shows that $\partial B(x, r)$ intersects $C_{i,j}$ exactly twice for sufficiently small $r > 0$, and furthermore that $\overline{\partial_* A} \cap \Omega = \partial^* A$.

Step 4: We show that $C_{i,j}$ is locally the graph of a C^1 function. Let $x \in C_{i,j}$ be given and by Step 3, choose $r > 0$ such that $C_{i,j}$ intersects $\partial B(x, r)$ twice, at a and b, and $\nu_{A_i}(y) \cdot e_2 > 0$ for all $y \in C_{i,j} \cap B(x, r)$, where $e_2 = \nu(x)$. Let L be the line segment connecting a and b, and let l be its length, and assume that a and b are oriented so that $b - a = l\, e_1$. For $\lambda \in (0, l)$, consider the line L_λ through $a + \lambda e_1$ in the direction e_2. Since $\nu_{A_i}(y) \cdot e_2 > 0$ for all $y \in B(x, r) \cap C_{i,j}$, we know that L_λ intersects $C_{i,j} \cap B(x, r)$ just once. We label the intersection $\gamma(\lambda)$. We therefore can define $f : (0, l) \to \mathbb{R}$ by

$$\lambda \mapsto [\gamma(\lambda) - (a + \lambda e_1)] \cdot e_2$$

and $C_{i,j} \cap B(x, r)$ is the graph of f. Let $\lambda_1 > \lambda_2 \in (0, l)$ and take $|\lambda_1, \lambda_2|$ to be the region in $B(x, r)$ between L_{λ_1} and L_{λ_2}. We will denote $\nu_{A_i}(\gamma(\lambda_1))$ by $\nu_{A_i}(\lambda_1)$. We have

$$
\begin{aligned}
&\left|\frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2}-\frac{\nu_{A_i}(\lambda_1)\cdot e_1}{\nu_{A_i}(\lambda_1)\cdot e_2}\right| = \\
&\quad = \left|\frac{\int_{C_{i,j}\cap B(x,r)\cap|\lambda_1,\lambda_2|}\nu_{A_i}\cdot e_1 d\mathcal{H}^1}{\lambda_1-\lambda_2}-\frac{\nu_{A_i}(\lambda_1)\cdot e_1}{\nu_{A_i}(\lambda_1)\cdot e_2}\right| \\
&\quad = \left|\frac{\mathcal{H}^1(C_{i,j}\cap B(x,r)\cap|\lambda_1,\lambda_2|)(\nu_{A_i}(\lambda_1)\cdot e_1+O(\lambda_1-\lambda_2))}{\lambda_1-\lambda_2}-\frac{\nu_{A_i}(\lambda_1)\cdot e_1}{\nu_{A_i}(\lambda_1)\cdot e_2}\right| \\
&\quad = \left|\frac{\left(\frac{\lambda_1-\lambda_2}{\nu_{A_i}(\lambda_1)\cdot e_2}+o(\lambda_1-\lambda_2)\right)\left(\nu_{A_i}(\lambda_1)\cdot e_1+O(\lambda_1-\lambda_2)\right)}{\lambda_1-\lambda_2}-\frac{\nu_{A_i}(\lambda_1)\cdot e_1}{\nu_{A_i}(\lambda_1)\cdot e_2}\right| \\
&\quad = \tfrac{o(\lambda_1-\lambda_2)}{\lambda_1-\lambda_2}+O(\lambda_1-\lambda_2),
\end{aligned}
$$

where the first equality follows just as Lemma 3.5, the second follows from (3.14), and the third from an argument similar to the proof of (3.9). Hence, $f \in C^1$.

Step 5: Finally, we show that $C_{i,j}$ is a connected component of $\overline{\partial_* S} \cap \Omega$. Let $x, y \in C_{i,j}$. By the regularity of $C_{i,j}$ and the connectedness of A_i and O_j, we may choose smooth curves, one in A_i and one in O_j, that connect x and y and are normal to $C_{i,j}$ at x and y. The union of these curves is Jordan, and so we may consider the interior region, R. Since Ω is simply connected, we have $R \subset\subset \Omega$. We can choose $r \in (0, \bar{r}_R)$ so that, for x and y, and for $z \in R$, we have, e.g., $B(z,r) \cap C_{i,j}$ is the graph of a continuous function on a line segment, and so it is connected. The curve $B(z,r) \cap C_{i,j}$ can be shown to continue, as before, by choosing balls with radius r centered at points in $C_{i,j} \cap \partial B(z,r)$. It can be continued in R as long as these balls stay in R and the curve does not self intersect. But by the choice of these balls, and since ∂R is normal to $C_{i,j}$ at x, the connected curve begun at x cannot self intersect in $R \cup \{x\}$. Since $\mathcal{H}^1(C_{i,j})$ is finite and each ball adds r to $\mathcal{H}^1(C_{i,j})$, the connected curve begun at x must leave R. Because $\partial R \cap C_{i,j} = \{x, y\}$, the connected curve must cross ∂R at y, and so x and y are connected in $C_{i,j}$. Hence, $C_{i,j}$ is connected. Since the $C_{i,j}$ are closed in Ω and mutually disjoint, they are the connected components of $\partial_* A$, and hence each connected component of $\partial_* A$, of which there are finitely many, is C^1.

□

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