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Refinements of Balance Laws
Induced by Microslip**

**David R. Owen
Carnegie Mellon University**

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Structured Deformations and the Refinements of Balance Laws Induced by Microslip

David R. Owen
Department of Mathematics
Carnegie Mellon University

*Dedicated to James Bell, in memory of his contributions
to both the arts and sciences.*

Abstract

Non-classical, "structured deformations" are used to describe microslip and to refine both balance laws and constitutive relations. The main tools employed are identification relations that link the ingredients in structured deformations to geometrical changes at small length scales.

Short title: Structured Deformations and Microslip

Telephone: (412) 268-8481

Fax: (412) 268-6380

E-mail: do04@andrew.cmu.edu

I. Introduction

The purpose of this paper is to employ non-classical deformations (specifically, the structured deformations introduced in DEL PIERO & OWEN, 1993) to describe slip at the microlevel, to derive the refinements of balance laws induced by the presence of microslip, and to propose constitutive relations that reflect the refinements in balance laws. The principal advantage of structured deformations as a basis for describing microslip, as opposed to descriptions that employ classical deformations together with internal variables such as plastic deformation, lies in the fact that each quantity employed in the description of a structured deformation can be shown to be the limit of quantities that describe geometrical changes at smaller and smaller length scales. For example, the tensor measure of deformation without slip introduced here is the limit of deformation gradients computed at points away from slip planes; the vector measure of slip turns out to be a limit of averages of smaller and smaller slips within smaller and smaller regions containing a given point. Besides the obvious advantage of having direct geometrical interpretations within the mathematical context of structured deformations, such quantities and the limiting or "identification relations" (see eqns (3) - (5)) that they obey permit one easily to deduce from information available at the microlevel corresponding information about continuum fields. Thus, for example, transformation properties of the deformation without slip under change of observer and reference configuration follow immediately from those

of the deformation gradients of which it is the limit. Moreover, not only do structured deformations factor globally into “purely microscopic” and “simple” compositional factors, but the macroscopic deformation gradient at each point decomposes additively into the tensor measure of deformation without microslip plus the tensor measure of deformation due to microslip. These two tensors are expressed through identification relations, respectively, as limits of smooth deformations away from slip planes and as limits of averages of small jumps in displacements across slip planes.

The basic facts about structured deformations needed here were developed in earlier articles (DEL PIERO & OWEN, 1993; DEL PIERO & OWEN, 1995) and are summarized in Section II. The specific example of single slip, where all microslips are perpendicular to a single, given direction, is presented in Section III. The principal new developments of this article, the decompositions of flux densities and of fields induced by structured deformations, are described in Section IV. Each term or group of terms in the decomposition of flux densities, eqn (16), obeys an identification relation that permits us to distinguish flux densities due to slip (or, more generally, due to any kind of disarrangements) from flux densities without slip (or without disarrangements). (The derivations of the identification relations eqns (18) and (19) will be presented in a future article.) Examples of decompositions of fluxes are presented in Sections V and VI for contact forces and for contact moments in the presence of microslip, and these examples permit us in Section VII to derive refinements, eqns (35) and (36), of the classical balance of forces and

moments for a continuum. The refined balance laws are related to the classical balance laws, written in terms of the Piola-Kirchhoff stress, in much the same way as the latter balance laws are related to the classical balance laws, written instead in terms of the Cauchy stress, except that the first relation is induced by a purely microscopic structured deformation, while the second is induced by the classical deformation that takes the reference configuration into the deformed configuration. Thus, the refined form of the balance laws reflects the availability of a new reference configuration that differs from the classical one by a purely microscopic deformation. The additional terms that appear in the refined balance laws immediately suggest refinements of constitutive laws. Three such refinements are proposed in Section VII: one for the “response without slip” and two alternative proposals for the “response due to slip”. In one proposal for the response due to slip, it is assumed that the contact forces due to slip are self-equilibrated; in the other, it is assumed that the contact moments due to slip are self-equilibrated. A full analysis of these and other proposals for the response due to slip awaits further research.

II. Structured Deformations

The definition of a *structured deformation* (κ, g, G) from a region \mathcal{A} in space \mathcal{E} rests on two more elementary kinds of deformations: classical deformations and simple deformations. A *classical deformation* from \mathcal{A} is a mapping $f : \mathcal{A} \rightarrow \mathcal{E}$ that extends to all of \mathcal{E} as a C^1 -mapping that is invert-

ible and whose inverse is of class C^1 . Roughly speaking, a simple deformation is a “piecewise-classical deformation”. More precisely, a *simple deformation from \mathcal{A}* is a pair (κ, f) with κ a subset of \mathcal{A} and f a mapping from $\mathcal{A} \setminus \kappa$ into \mathcal{E} such that κ has volume zero, f is one-to-one, and $\mathcal{A} \setminus \kappa$ can be written as a finite union of regions from each of which f is a classical deformation. The simple shearing of a rectangular block illustrates the notion of a classical deformation, whereas the piecewise-rigid “shearing” of a deck of cards provides an example of a simple deformation: here, \mathcal{A} is the region containing all of the cards, κ is the set of points on the interfaces between cards, $\mathcal{A} \setminus \kappa$ is the set of cards (without the interfaces), and f restricted to each card is a rigid translation that can vary discontinuously from one card to the next. For each simple deformation (κ, f) from \mathcal{A} , the set κ is here called the *disarrangement site*, and the mapping f is called the *transplacement* for (κ, f) . (In DELPIERO & OWEN, 1993, 1995, the set κ was called the crack site. The term “disarrangement” was introduced in the context of structured deformations in OWEN, 1995.)

A *structured deformation* (κ, g, G) from \mathcal{A} consists of a simple deformation (κ, g) from \mathcal{A} and a tensor field $G : \mathcal{A} \setminus \kappa \rightarrow \text{Lin}\mathcal{V}$ (with \mathcal{V} the translation space of \mathcal{E} and $\text{Lin}\mathcal{V}$ the set of all linear mappings on \mathcal{V}) such that G is piecewise continuous and there is a positive constant m for which

$$m \leq \det G(x) \leq \det \text{Grad } g(x) \quad (1)$$

for all x in $\mathcal{A} \setminus \kappa$. For example, if g is the simple shear of a rectangular block

and G is the constant field whose only value is the identity tensor I in $\text{Lin}\mathcal{V}$, then the triple (\emptyset, g, I) (with \emptyset denoting the empty set) is a structured deformation in which the simple deformation (\emptyset, g) is actually a classical deformation. In this example,

$$\det \text{Grad } g(x) = \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 = \det G(x), \quad (2)$$

so that (1) holds with $m = 1$ and with equality in both relations.

The definition of structured deformation by itself provides no interpretation for the tensor field G , but this situation is remedied by means of the following “Approximation Theorem” (DEL PIERO & OWEN, 1993): *every structured deformation (κ, g, G) is a limit of simple deformations from \mathcal{A} in the sense that there exists a sequence $m \mapsto (\kappa_m, f_m)$ of simple deformations such that $g = \lim_{m \rightarrow \infty} f_m$ and $G = \lim_{m \rightarrow \infty} \text{Grad } f_m$. Here, the limits are taken in the sense of L^∞ -convergence on \mathcal{A} . (A limit relation between κ and the sequence $m \mapsto \kappa_m$ also can be established but is not needed in the present discussion.) As an illustration of the Approximation Theorem, the structured deformation (\emptyset, g, I) introduced above is a limit of simple deformations (κ_m, f_m) in which f_m is a piecewise-rigid shearing of a deck of cards, m is the number of cards, and in which the relative displacement of adjacent cards is proportional to m^{-1} .*

The relations in the Approximation Theorem

$$g = \lim_{m \rightarrow \infty} f_m \quad (3)$$

$$G = \lim_{m \rightarrow \infty} \text{Grad } f_m \quad (4)$$

justify our calling g the *transplacement* for (κ, g, G) and G the *deformation without disarrangements*: $G(x)$ represents the local deformation at x without including the effects of separation or slip occurring at the disarrangement sites κ_m for the approximating simple deformations (κ_m, f_m) . Thus, by means of structured deformations we may distinguish between the structured simple shear (\emptyset, g, I) and the classical simple shear $(\emptyset, g, \text{Grad } g)$: in the former, the deformation without disarrangements is I , that of a rigid deformation; in the latter, the deformation without disarrangements is $\text{Grad } g$, the deformation gradient of the simple shear g appearing in eqn (2). Moreover, the limit operations in eqns (3) and (4) permit us to interpret the deformations associated with G as occurring in smaller and smaller pieces of the body, just as the disarrangement sites κ_m can be considered to be diffusing throughout the body as m becomes large. We may then call g and $\text{Grad } g$ the *macroscopic transplacement* and *macroscopic local deformation*. These are the quantities one encounters in classical descriptions of the geometrical changes in a continuous body, whereas G reflects geometrical changes at a smaller length scale such as changes in geometry at the microlevel.

The eqns (3) and (4) that identify g and G permit one to deduce an *identification relation for* $M := \text{Grad } g - G$ (DEL PIERO & OWEN, 1995): for each sequence $m \mapsto (\kappa_m, f_m)$ satisfying eqns (3) and (4) and each

$x \in \mathcal{A} \setminus \kappa,$

$$M(x) = \text{Grad } g(x) - G(x) = \lim_{r \rightarrow 0} \lim_{m \rightarrow \infty} \left(\frac{4}{3} \pi r^3 \right)^{-1} \int_{\Gamma(f_m) \cap \mathcal{B}(x,r)} [f_m](y) \otimes \nu(y) dA_y, \quad (5)$$

where $\Gamma(f_m)$ is the set of jump points of f_m , $[f_m](y)$ is the jump of f_m at y , $\nu(y)$ is the unit normal to $\Gamma(f_m)$ at a point y in $\Gamma(f_m)$, and $\mathcal{B}(x, r)$ is the ball centered at x of radius r . As noted in DEL PIERO & OWEN, 1995, the tensor $M = \text{Grad } g - G$ represents through eqn (5) the volume density of deformation due to discontinuities in transplacements at the microlevel, and we shall here call M the *deformation due to microdisarrangements*. (The term “deformation due to microfracture” was used in DEL PIERO & OWEN, 1995). Thus, the trivial algebraic identity

$$\text{Grad } g = G + M \quad (6)$$

has the deeper significance of *an additive decomposition of macroscopic deformation Grad g into the deformation without disarrangements G plus the deformation due to microdisarrangements M*.

Another useful decomposition for structured deformations is *the factorization*

$$(\kappa, g, G) = (\kappa, g, \text{Grad } g) \circ (\emptyset, i, K) \quad (7)$$

of (κ, g, G) into a simple deformation $(\kappa, g, \text{Grad } g)$ that follows a purely microscopic structured deformation (\emptyset, i, K) , with i the identity mapping and $K = (\text{Grad } g)^{-1} G$. The simple deformation (\emptyset, i) in (\emptyset, i, K) indicates

the absence macroscopic displacements and macroscopic disarrangements, so that when $K \neq I$, i.e., when $\text{Grad } g \neq G$, the only geometrical changes associated with (\emptyset, i, K) are due to disarrangements at the microlevel.

We note finally that $\det G$ represents the volume change without disarrangements, whereas $\det \text{Grad } g$ represents the macroscopic volume change. Thus, the inequality (1) expresses the condition that disarrangements can only increase or maintain volumes, the former occurring through the creation of voids. This condition is necessary in order that the transplacements f_m that approximate g be one-to-one and, hence, not cause interpretation of matter. Of course, the equality associated with (1)

$$\det G = \det \text{Grad } g \quad (8)$$

expresses the condition that no volume changes occur through disarrangements.

III. An Example: Single Slip

We now shall identify a collection of structured deformations of the form (\emptyset, g, G) that are intended to describe the simplest type of slip in metallic crystals: the discontinuity surfaces at the microlevel consist of a single family of parallel planes, and all discontinuities in displacement at the microlevel are translations parallel to these planes. To this end, we fix a unit vector u and assume that there is a sequence $m \mapsto (\kappa_m, f_m)$ satisfying eqns (3) and (4) along with the following slip conditions: for all m

s1) the disarrangement site κ_m is a family of planes perpendicular to u ;

s2) at each point y in κ_m , $[f_m](y) \cdot u = 0$.

Condition s1) tells us that the unit normal vector $\nu(y)$ in eqn (5) may be taken to be the unit vector u , so that the identification relation (5) becomes

$$M(x) = s(x) \otimes u, \quad (9)$$

where the vector

$$s(x) = \lim_{r \rightarrow 0} \lim_{m \rightarrow \infty} \left(\frac{4}{3} \pi r^3 \right)^{-1} \int_{\kappa_m \cap \mathcal{B}(x,r)} [f_m](y) dA_y \quad (10)$$

will be called the *slip vector at x* in the structured deformation (\emptyset, g, G) . Note that $s(x)$ is a limit of averages of slips occurring within a small sphere centered at x as the magnitudes of the slips and the radius of the sphere tend to zero. Hence, the dimensionless vector $s(x)$ is a density of slip discontinuities that completely determines the tensor $M(x)$. We shall henceforth call $M = s \otimes u$ the *deformation due to microslip*.

It follows immediately from eqn (10) and s2) that

$$s(x) \cdot u = 0 \quad (11)$$

for all x , so that the slip vector at each point is parallel to the family of slip planes in s1). Consequently, the matrix of $M(x)$ relative to the orthogonal basis $(u, s(x), u \times s(x))$ has the simple form:

$$[M(x)] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

Moreover, the condition (1) together with eqns (6) and (9) implies

$$\begin{aligned}
\det G(x) &\leq \det (G(x) + M(x)) \\
&= \det (G(x) (I + G(x)^{-1} M(x))) \\
&= \det G(x) \det (I + G(x)^{-1} s(x) \otimes u) \quad (13) \\
&= \det G(x) (1 + \text{tr} (G(x)^{-1} s(x) \otimes u)) \\
&= \det G(x) (1 + G(x)^{-1} s(x) \cdot u),
\end{aligned}$$

and relation (13) tells us that G , s and u are subject to the restriction

$$G(x)^{-1} s(x) \cdot u \geq 0. \quad (14)$$

Moreover, equality holds in relation (13) if and only if equality holds in relation (14). Thus, the macroscopic volume change $\det \text{Grad } g(x)$ equals the volume change without slip $\det G(x)$ if and only if $G(x)^{-1} s(x)$ is perpendicular to u .

IV. Fluxes due to Disarrangements and Fluxes without Disarrangements

For a given structured deformation (κ, g, G) from \mathcal{A} , the factorization eqn (7) contains the purely microscopic structured deformation (\emptyset, i, K) , with

$$K = (\text{Grad } g)^{-1} G, \quad (15)$$

and the simple deformation $(\kappa, g, \text{Grad } g)$. Eqn (7) permits us to think of carrying out (κ, g, G) in two steps: first, put in all of the deformations at

the microlevel without any macroscopic changes, and then deform the body classically without further changes at the microlevel. This process leads us to think of a global reference configuration associated with the structured deformation (\emptyset, i, I) in which the body occupies the region \mathcal{A} , of a final global configuration associated with (κ, g, G) in which the body occupies the region $g(\mathcal{A})$, and of an intermediate global configuration associated with (\emptyset, i, K) in which the body *also* occupies the region \mathcal{A} but has undergone geometrical changes only at the microlevel. In this and the following sections, we exploit the difference between the two configurations on \mathcal{A} in order to obtain refined expressions for the volume density of the flux of a vector field defined on \mathcal{A} .

Let $w : \mathcal{A} \rightarrow \mathcal{V}$ be a smooth vector field on \mathcal{A} and consider the identity

$$\det K \operatorname{Div} w = \operatorname{Div} (K^{*T} w) - w \cdot \operatorname{Div} K^* + \operatorname{Grad} w \cdot ((\det K) I - K^*), \quad (16)$$

where K^* is the adjugate of K :

$$K^* = \det K (K^{-1})^T. \quad (17)$$

We easily can verify this identity by noting the relations

$$\operatorname{Div} (K^{*T} w) = \operatorname{Div} K^* \cdot w + \operatorname{Grad} w \cdot K^*$$

and

$$\operatorname{Grad} w \cdot I = \operatorname{Div} w.$$

From the Approximation Theorem we may choose a determining sequence $m \mapsto (\kappa_m, f_m)$ for the purely microscopic deformation (\emptyset, i, K) and obtain

the following *identification relations* for the terms in the eqn (16):

$$\det K \operatorname{Div} w |_{\mathbf{x}} = \lim_{r \rightarrow 0} \lim_{m \rightarrow \infty} r^{-3} \sum_{\mathcal{C} \in \mathbf{C}_m(\mathbf{x})} \int_{f_m(\operatorname{bdy}(\mathcal{C}_r(\mathbf{x}) \cap \mathcal{C}))} w(\mathbf{y}) \cdot \nu(\mathbf{y}) dA_{\mathbf{y}} \quad (18)$$

$$\operatorname{Div} (K^* T w) |_{\mathbf{x}} = \lim_{r \rightarrow 0} \lim_{m \rightarrow \infty} r^{-3} \sum_{\mathcal{C} \in \mathbf{C}_m(\mathbf{x})} \int_{f_m((\operatorname{bdy} \mathcal{C}_r(\mathbf{x})) \cap \mathcal{C})} w(\mathbf{y}) \cdot \nu(\mathbf{y}) dA_{\mathbf{y}}. \quad (19)$$

In eqns (18) and (19) for each m , \mathbf{C}_m is a collection of closed cubes \mathcal{C} that cover the region \mathcal{A} and whose faces together include the disarrangement site κ_m , and $\mathcal{C}_r(\mathbf{x})$ is a cube centered at \mathbf{x} of side r whose faces are disjoint from all the disarrangement sites κ_m . The surface integral in eqn (18) is taken over the image under f_m of all the faces of the parallelepiped $\mathcal{C}_r(\mathbf{x}) \cap \mathcal{C}$, so that the sum in eqn (18) represents the total flux of w across the image of the faces of $\mathcal{C}_r(\mathbf{x})$ and across the image of the faces of cubes \mathcal{C} in \mathbf{C}_m containing the disarrangement sites κ_m inside of $\mathcal{C}_r(\mathbf{x})$. Therefore, the limit on the right-hand side of eqn (18) and, hence, the left-hand side $\det K(\mathbf{x}) (\operatorname{Div} w)(\mathbf{x})$, represents the *volume density of the total flux of w* . The surface integral in eqn (19) is taken over the image under f_m of only those faces of the parallelepiped $\mathcal{C}_r(\mathbf{x}) \cap \mathcal{C}$ that belong to the boundary of $\mathcal{C}_r(\mathbf{x})$ and *not* to the images of faces of cubes \mathcal{C} in \mathbf{C}_m containing the disarrangement sites κ_m inside of $\mathcal{C}_r(\mathbf{x})$. Therefore, the limit of the right-hand side of eqn (19), and also the left-hand side $\operatorname{Div} (K^* T w)(\mathbf{x})$, represents the *volume density of the flux of w without disarrangements*. Consequently, the remaining terms $(-w \cdot \operatorname{Div} K^* + \operatorname{Grad} w \cdot ((\det K) I - K^*)) |_{\mathbf{x}}$ on the right-hand side of eqn (16) represent the *volume density of the flux of w due to disarrangements*.

We note that the volume density of total flux $\det K \operatorname{Div} w$ and the volume density of flux due to disarrangements $-w \cdot \operatorname{Div} K^* + \operatorname{Grad} w \cdot ((\det K) I - K^*)$ need not be the divergence of a vector field, whereas the volume density of flux without disarrangements is the divergence of the vector field $K^{*T} w$. However, in the special case where there is no volume change due to disarrangements, eqns (8) and (15) tell us that $\det K = 1$ and, therefore, eqn (16) becomes

$$\operatorname{Div} w = \operatorname{Div} (K^{*T} w) + \operatorname{Div} ((I - K^{*T}) w), \quad (20)$$

so that the obvious algebraic identity

$$w = K^{*T} w + (I - K^{*T}) w \quad (21)$$

has the deeper significance of a *decomposition of the field w into the field without disarrangements $K^{*T} w$ and the field due to disarrangements $(I - K^{*T}) w$.*

V. Stresses due to Slip and Stresses without Slip

We illustrate the decomposition (21) in the case where the purely microscopic structured deformation (\emptyset, i, K) arises from eqn (15) when (κ, g, G) is the single slip introduced in Section III. In accordance with eqn (8), we assume that equality holds relation (14), i.e.,

$$G^{-1} s \cdot u = 0, \quad (22)$$

so that no volume change is associated with the slip. Eqns (6), (15), (17)

and (9) tell us that

$$\begin{aligned}
K^* &= (K^{-1})^T = ((\text{Grad } g)^{-1} G)^{-T} \\
&= (G^{-1}(G + M))^T = (I + G^{-1}s \otimes u)^T \\
&= I + u \otimes G^{-1}s.
\end{aligned} \tag{23}$$

Let $S : \mathcal{A} \rightarrow \text{Lin } \mathcal{V}$ be the Piola-Kirchhoff stress tensor field associated with a system of contact forces, let $a \in \mathcal{V}$ be an arbitrary unit vector, consider for each region \mathcal{P} included in \mathcal{A} the a -component of the contact force on \mathcal{P}

$$\begin{aligned}
a \cdot \int_{\text{bd}_y \mathcal{P}} S(x) \nu(x) dA_x &= \int_{\text{bd}_y \mathcal{P}} a \cdot S(x) \nu(x) dA_x \\
&= \int_{\text{bd}_y \mathcal{P}} S(x)^T a \cdot \nu(x) dA_x,
\end{aligned}$$

and put $w := S^T a$ in eqn (21) to obtain

$$\begin{aligned}
S^T a &= K^{*T} S^T a + (I - K^{*T}) S^T a \\
&= (SK^*)^T a + (S(I - K^*))^T a \\
&= \{(SK^*)^T + (S(I - K^*))^T\} a.
\end{aligned}$$

Because the unit vector $a \in \mathcal{V}$ was arbitrary, eqn (21) is equivalent to the relation

$$\begin{aligned}
S &= SK^* + S(I - K^*) \\
&= S(I + u \otimes G^{-1}s) - Su \otimes G^{-1}s.
\end{aligned} \tag{24}$$

where eqn (23) was employed in obtaining the second relation in eqn (24).

According to the interpretations obtained in Section IV, $S + Su \otimes G^{-1}s = SK^*$ is the *stress without slip* and $-Su \otimes G^{-1}s$ is the *stress due to slip*. In

view of the identification eqn (19), $S + Su \otimes G^{-1}s$ may be regarded as determining the contact force per unit area on surfaces not affected by slip, whereas $-Su \otimes G^{-1}s$ determines the contribution to the total contact force from the presence of slip planes. The *traction due to slip* at a point x on a surface with normal n is given by

$$\left(-S(x)u \otimes G^{-1}(x)s(x)\right)n = \left(G^{-1}(x)s(x) \cdot n\right)\left(-S(x)u\right) \quad (25)$$

and is parallel to the total traction $S(x)u$ on a plane through x with normal u . The traction due to slip is the contribution to the total traction from slip planes. Eqns (25) and (22) tell us that the traction due to slip vanishes on surfaces parallel to the slip planes and, for a fixed point x , has its largest magnitude on a surface normal to the vector $G^{-1}(x)s(x)$ and, hence, according to eqn (22), on a surface normal to the slip planes.

VI. Contact Moments due to Slip and Contact Moments without Slip

A second illustration of the decompositions eqns (16), (18), and (19) also involves the Piola-Kirchhoff stress S and a structured deformation (\emptyset, g, G) that describes single slip and satisfies eqn (22). In this example, we let a be a unit vector in \mathcal{V} , let o be a point in \mathcal{E} , and consider for each region \mathcal{P} in \mathcal{A}

the a -component of *contact moment on \mathcal{P} about o* :

$$\begin{aligned}
a \cdot \int_{\text{bdy}\mathcal{P}} (g(x) - o) \times S(x) \nu(x) dA_x &= \int_{\text{bdy}\mathcal{P}} a \cdot (g(x) - o) \times S(x) \nu(x) dA_x \\
&= \int_{\text{bdy}\mathcal{P}} a \times (g(x) - o) \cdot S(x) \nu(x) dA_x \\
&= \int_{\text{bdy}\mathcal{P}} S(x)^T (a \times (g(x) - o)) \cdot \nu(x) dA_x.
\end{aligned} \tag{26}$$

The last integral in eqn (26) identifies the vector field

$$w(x) = S(x)^T (a \times (g(x) - o)) \tag{27}$$

for substitution into eqns (16), (18) and (19). In this case, eqns (27), (20) and the arbitrariness of a yield

$$\begin{aligned}
(g - o) \times \text{Div} S - \omega_{\text{Sk}(S(\text{Grad}g)^T)} &= \\
(g - o) \times \text{Div}(SK^*) - \omega_{\text{Sk}(SK^*(\text{Grad}g)^T)} + &\tag{28} \\
(g - o) \times \text{Div}(S(I - K^*)) - \omega_{\text{Sk}(S(I - K^*)(\text{Grad}g)^T)}, &
\end{aligned}$$

where, for each skew tensor $A \in \text{Lin}\mathcal{V}$, ω_A denotes the dual vector for A and, for each tensor $B \in \text{Lin}\mathcal{V}$, $\text{Sk}B$ denotes the skew part of B . In (28), the vector field $(g - o) \times \text{Div}(SK^*) - \omega_{\text{Sk}(SK^*(\text{Grad}g)^T)}$ may be called the *volume density of contact moments without slip*, and the vector field $(g - o) \times \text{Div}(S(I - K^*)) - \omega_{\text{Sk}(S(I - K^*)(\text{Grad}g)^T)}$ may be called the *volume density of contact moments due to slip*.

Eqns (24), (6), (9), and (22) imply the formulas

$$\begin{aligned}
S(I - K^*)(\text{Grad } g)^T &= -(Su \otimes G^{-1}s)(G + s \otimes u)^T \\
&= -(Su \otimes G^{-1}s)(G^T + u \otimes s) \\
&= -(Su \otimes G^{-1}s)G^T - (Su \otimes G^{-1}s)(u \otimes s) \\
&= -Su \otimes G(G^{-1}s) - (Su \otimes s)(u \cdot G^{-1}s) \\
&= -Su \otimes s
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
SK^*(\text{Grad } g)^T &= S(I + u \otimes G^{-1}s)(G + M)^T \\
&= (S + Su \otimes G^{-1}s)(G^T + u \otimes s) \\
&= SG^T + 2Su \otimes s,
\end{aligned} \tag{30}$$

which will be used in our discussion of balance laws in the next section.

VII. The Refinements of Balance Laws and Constitutive Equations Induced by Slip

Classical balance laws for forces and moments in terms of the Piola-Kirchhoff stress S and body force b per unit mass take the local forms:

$$\text{Div } S + \rho_0 b = 0 \tag{31}$$

$$(g - o) \times \text{Div } S - \omega_{Sk(S(\text{Grad } g)^T)} + (g - o) \times \rho_0 b = 0, \tag{32}$$

or, equivalently,

$$\text{Div } S + \rho_0 b = 0 \tag{33}$$

$$Sk(S(\text{Grad } g)^T) = 0. \tag{34}$$

The decompositions, eqns (24) and (28) along with eqns (29) and (30) yield the following refinements of eqns (33) and (34):

$$\text{Div} \left(S \left(I + u \otimes G^{-1} s \right) \right) - \text{Div} \left(Su \otimes G^{-1} s \right) + \rho_0 b = 0 \quad (35)$$

$$Sk \left(SG^T + 2Su \otimes s \right) - Sk \left(Su \otimes s \right) = 0. \quad (36)$$

The balance laws in classical form, eqns (33) and (34), and the balance laws in refined form, eqns (35) and (36), are equivalent, but the refined form displays the individual contributions of contact forces and moments without slip and those due to slip. Just as the balance laws, eqns (31) and (32), are transformed versions of the balance laws

$$\text{div} T + \rho b = 0 \quad (37)$$

$$Sk(T) = 0, \quad (38)$$

with T the Cauchy stress, $\rho = \rho_0 / \det \text{Grad} g$, and with the transformation being induced by the simple deformation $(\emptyset, g, \text{Grad} g)$, the refined balance laws, eqns (35) and (36), are transformed versions of eqns (33) and (34) induced by the purely microscopic deformation (\emptyset, i, K) .

The refined balance laws, eqns (35) and (36), contain quantities that describe the details of the process of slip and that can enter into refined constitutive relations. For example, $S(I + u \otimes G^{-1} s)$ describes the stress without slip, and G describes the deformation without slip. Therefore, it is reasonable to identify the response of the portion of the body away from sites

of slip through a relation between $S(I + u \otimes G^{-1}s)$ and G :

$$S(I + u \otimes G^{-1}s) = \hat{S}_\setminus(G). \quad (39)$$

The function \hat{S}_\setminus may be called the *response without slip*, because it relates the stress without slip to the deformation without slip. When the structured deformation (κ, g, G) is a classical deformation, s is zero and G equals $\text{Grad } g$, so that eqns (33), (34), and (39) reduce to the balance laws and constitutive equation for an elastic body.

Eqns (35), (36), and (39) amount to twelve scalar relations to be satisfied by S, g, G, s , which, in component form, amount to twenty four unknown scalar fields. The decomposition eqn (6) together together with eqn (9) yield the relation

$$\text{Grad } g = G + s \otimes u \quad (40)$$

which accounts for nine additional scalar relations, yielding 21 in total and leaving three more unknowns than equations. Of particular interest here is the fact that the refined balance laws, eqns (35) and (36), point to some reasonable possibilities for additional constitutive equations. For example, a material may respond to slip in such a way that the contact forces due to slip are self-equilibrating:

$$\text{Div}(Su \otimes G^{-1}s) = 0. \quad (41)$$

This situation might occur in a body in which the slip is constrained by the relation

$$s = |s| k \quad (42)$$

in which k is a fixed unit vector normal to u . Relation (42) would be satisfied in a deck of very thin cards in which each card in the deck slips in a fixed, preassigned direction relative to the one below it, so that the cards above and below a given one would pull it in opposite directions. We note that eqn (41) is equivalent to

$$\text{Grad}(Su)(G^{-1}s) = -\text{Div}(G^{-1}s)Su, \quad (43)$$

and eqns (35), (36), (39), (40) and (43) provide twenty-four scalar relations for the twenty-four unknown scalar fields.

Alternatively, a material might respond to slip in such a way that the contact moments due to slip are self-equilibrating:

$$Sk(Su \otimes s) = 0. \quad (44)$$

This situation would require in a deck of cards that the cards above and below a card that slips react to the slip by cooperatively pulling in the same direction. The cooperation required might be available in a material on which slip planes are concentrated in very thin bands separated by regions without slip. Eqn (44) asserts that $Su \otimes s$ is symmetric, which is equivalent to the assertion that Su and s are linearly dependent: for each x there exist scalars $\alpha(x), \beta(x)$ not both zero such that

$$\alpha(x)S(x)u + \beta(x)s(x) = 0. \quad (45)$$

Thus, eqn (44) requires that the slip vector $s(x)$ and the traction $S(x)u$ on the slip plane through x have the same or opposite directions. Eqns (45) and



(11) then imply that

$$\alpha(x) S(x) u \cdot u = 0, \quad (46)$$

so that $\alpha(x) = 0$ or $S(x) u \cdot u = 0$. Suppose that $S(x) u \cdot u \neq 0$. Then $\alpha(x) = 0$ and, therefore, $\beta(x) \neq 0$; we conclude that $s(x) = 0$. Thus, eqn. (44) implies that at a point where the traction $S(x) u$ on a slip plane is not tangential, $s(x) = 0$, i.e., no slip occurs. In other words, *when the moments due to slip are self-equilibrating, slip can occur only on slip planes having purely tangential tractions, i.e., arbitrarily small normal tractions on slip planes will prevent slip from occurring. When slip does occur at x , $s(x)$ and $S(x) u$ are proportional through eqn (45).*

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Department of Mathematics

Carnegie Mellon University

Pittsburgh, PA 15213

USA