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Revisiting the Focal Conic Structures in Smectic A

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ABSTRACT

Smectic A configurations in equilibrium display complicated focal conic textures. In the planar case, we view these configurations as extremals of a constrained Ginzburg-Landau Equation. This gives rise to a system governed by a variational principle that is subject to simple rules.

INTRODUCTION

In the hierarchy of mesophases, smectic A appears between nematic and smectics of lesser symmetry or the solid. It is characterized by the properties

- presence of a layer structure of nearly constant width and nearly incompressible
- within each layer, centers of gravity show no long range order and each layer is a two dimensional fluid
- the system is optically uniaxial with optic axis n normal to the layer
- n and –n are equivalent

In equilibrium, thin samples often show a complicated focal conic texture, or Dupin cyclide structure [8]. Friedel [5] recognized this to be indicative of the layer properties above and also derived the rules we revisit here. Indeed, we take this opportunity to discuss our first thoughts on these issues. These configurations, in the plane, may be interpreted as singular solutions of a Ginzburg-Landau system, studied extensively by Bethuel, Brezis, and Helein [1], subject to constraints. An interesting consequence of the Ginzburg-Landau viewpoint is the tendency of the material to nucleate new smectic domains in response to defects in preference to deforming to accomodate them.

The simple conic configurations arise from a conservation condition and other properties follow from a stability condition. We do not yet have a satisfactory explanation of the assembly of many ellipses between hyperbolic arcs, but we are able to show that this configuration is consistent with our rules. Moreover, a simple symmetry property characterizes elliptical domains. All of our discussion is completely elementary.

Basic references for us have been Chandrasekhar
Some addtional papers of interest are Guerst [6], Kl
and Sethna and Huang [12].

Jerry Ericksen is our mentor and our friend. He h
convinced of the predictive authority of mathematic

1. KINEMATICS AND LOCAL EQUILIBRIUM

A simplified continuum theory may be based on the
for which we establish constraints. Following [2],[

$$\frac{1}{d}\int_{AB} n \cdot d\bar{x} \;=\; \text{number of layers tr}$$

d a typical layer spacing,

should be independent of path. This gives rise to th

$$|n| = 1 \quad \text{and} \quad \operatorname{curl} n = 0.$$

Given a small region D of defect free material, we i

$$n(x) = \nabla f(x), \quad x \in D.$$

We interpret f as the layer density of the material.
general principles that f is a distance function, althc
point. From (1.2) we have that

$$\nabla n\, n = 0 \quad \text{and} \quad \nabla n \;\; \text{is symmetr}$$

so that

$$\nabla n = -\kappa\, t \otimes t, \quad \text{where } t \cdot n = 0,$$

Supose that (t,n) is a right handed system. Solve th

$$\gamma\colon\ \frac{dp}{ds} = t(p), \ p(0) = x_0, \ |s| \text{ sr}$$

Then

$$\frac{df}{ds} = \nabla f \cdot \frac{dp}{ds} = n \cdot t = 0,$$

so γ is a level curve of f. By using the Frenet Formulas, one sees easily that κ in (1.4) is the curvature of γ. Using a variation of this idea, Virga and Fournier [13] introduce confocal coordinates based on the fields (t,n) which serves also to illustrate the equally spaced layer property of smectic A.

The elementary condition for equilibrium is that

$$\delta\int_D (\operatorname{div} n)^2\, dx = 0 \quad \text{subject to} \quad |n| = 1, \tag{1.5}$$

hence

$$\nabla \operatorname{div} n \parallel n \quad \text{or} \quad t \cdot \nabla \operatorname{div} n = 0.$$

After some manipulation, and writing $\kappa = \kappa(s)$ for the curvature of γ, we see that the equation above is equivalent to

$$\frac{d}{ds}\kappa(s) = 0. \tag{1.6}$$

Hence in unloaded equilibrium, κ is constant on each level surface of f. We conclude that local equilibria are characterized by circular arcs or straight segments, namely,

$$n_*(x) = \frac{x-a}{|x-a|} \quad \text{with} \quad f_*(x) = |x-a|, \quad \text{or} \tag{1.7}$$
$$n(x) = n_0 \quad \text{with} \quad f(x) = n_0 \cdot x, \; n_0 \text{ constant.}$$

2. Energy and Ginzburg-Landau Formulation

We briefly discuss an appealing Ginzburg-Landau approximation as a means of accomodating elementary defect structures. For a mapping $u: \Omega \to \mathbb{R}^2$, let

$$E_\varepsilon(u) = \frac{K}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{\varepsilon^2}\int_\Omega F(u)\, dx, \tag{2.1}$$

where $F(\xi)$ is a smooth non-negative function which vanishes precisely if $|\xi| = 1$. The usual choice is

$$F(\xi) = \frac{1}{4}(1 - |\xi|^2)^2 .$$

For any such u,

$$|\nabla u|^2 = (\operatorname{div} u)^2 + (\operatorname{curl} u)^2 + 2$$

If u satisfies the constraints (1.2), then $|\nabla u|^2 =$

$$E_\varepsilon(u) = \frac{K}{2}\int_\Omega (\operatorname{div} u)^2\, dx,$$

which is the basic energy mentioned in (1.5) for a d
the Ginzburg-Landau formulation is that we know m
and how it can be used to systematically account
Bethuel, Brezis, and Helein [1]. A sequence of mi
Dirichlet boundary condition u_0 of degree M o
relabelled, such that

$$u_\varepsilon \to u^* \quad \text{in } \Omega \setminus \{a_1, \ldots, a_M\}$$

where $a_1, \ldots, a_M$ are special points of Ω, cf. [1], and

$$u^*(z) = \frac{z-a_1}{|z-a_1|} \cdots \frac{z-a_M}{|z-a_M|} e^{ih(z)} \quad \text{i}$$

$$\Delta h = 0 \text{ in } \Omega \text{ and } h \text{ is a real valu}$$

Given $b_1, \ldots, b_M \in \Omega$ and $\rho > 0$, let $B_\rho(b_j)$ den
ρ, $\Omega_\rho = \Omega \setminus \cup B_\rho(b_j)$, and

$$A_\rho(b) = \{v \in H^1(\Omega_\rho, \mathbb{S}^1) : \deg(v, \partial B_\rho(b_j)) =$$

where deg denotes the topological degree or windin

$$\min_{A_\rho(b)} \frac{K}{2}\int_{\Omega_\rho} |\nabla v|^2\, dx = KM\pi |\log$$

where the first term on the right reflects the presence
refer to as an excess energy. Also,

$$\inf_A E_\varepsilon(v) = KM\pi |\log \varepsilon| + KU(a), \; A = \{v \in$$

where $U(a) = \lim U_\rho(a)$ and a is the set of defects
equilibrium positions in (2.2).

Now if u^* is also, locally, an equilibrium smectic A configuration, then in (2.2) $M = 1, h = 0$, and $u^* = n_a$ as in (1.7).

3. The ellipse and the hyperbola

In view of the conclusion above, we may anticipate difficulties in seeking configurations with more than one defect or even configurations with a single defect not satisfying the special condition detailed in [1]. Assume initially a configuration in equilibrium with a single defect at $a \in \Omega$,

$$n(z) = n_a(z), \quad z \in \Omega, \tag{3.1}$$

and that this defect is displaced nearby to $b \in \Omega$ without alteration of the loading environment. Thus

$$\min_{A_\rho(a)} \frac{K}{2} \int_{\Omega_\rho} |\nabla v|^2 dx \; < \; \min_{A_\rho(b)} \frac{K}{2} \int_{\Omega_\rho} |\nabla v|^2 dx \tag{3.2}$$

and a convergent subsequence of minimizers (u_ε) of E_ε converges to n_a.

We may envision two scenarios. First the system may fail to be in equilibrium with the displaced defect b because of (3.2). Second, we may nucleate a region D, with $a,b \in D$, so that, for example, the resulting configuration is given by

$$n(z) = \begin{cases} n_b(z) & z \in D \\ n_a(z) & z \in \Omega \setminus D \end{cases}. \tag{3.3}$$

What sort of region can D be? Assume that $\Gamma = \partial D$ is a simple closed curve. Let us simply impose the condition (1.1) on Γ, that the number of traversed layers is the same on any subarc on approach from D and from $\Omega \setminus D$. Accounting for orientation, and reverting to real notation,

$$\int_{z_1 z_2} n_b(z)\cdot d\bar{x} = -\int_{z_1 z_2} n_a(z)\cdot d\bar{x} \quad \text{or} \quad f_b(z_1) - f_b(z_2) = -(f_a(z_1) - f_a(z_2)). \tag{3.4}$$

Hence

$$f_b(z) + f_a(z) = f_b(z_1) + f_a(z_1) = C \quad \text{for all } z \in \Gamma, \tag{3.5}$$

and Γ is an ellipse with foci at a and b. Morever

symmetry of the ellipse that

$$\frac{K}{2} \int_{D \setminus B_\rho(b)} |\nabla n_b|^2 dx = \frac{K}{2} \int_{D \setminus B_\rho(a)} |\nabla n_a|^2 d$$

whence

$$\frac{K}{2} \int_{\Omega \setminus B_\rho(b)} |\nabla n|^2 dx = \frac{K}{2} \int_{\Omega \setminus B_\rho(a)} |\nabla n_a|^2 dx = i$$

$$= K\pi \,|$$

where $U(a)$ is the minimum possible value of the exc

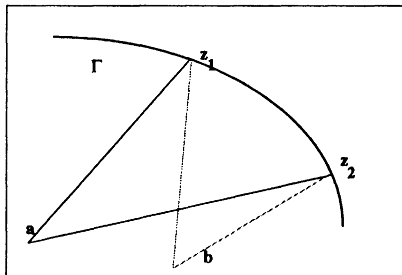

Figure 1. Traversing the layers on the arc $z_1 z_2$

So,
movi
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so that (3.2) holds, with, say, $a \in \Omega$ achieving the

response of the system is to nucleate an elliptical domai

energy.

Consider again a configuration with a defect in Ω

environment, two defects are now seen. By a small ch

the boundary condition remains of topological degree

inconsistent with the presence of two defects, but we m

them which meets $\partial\Omega$. Imposition of the condition (1

conclusion that Γ is an arc of a hyperbola with foci at t

Interestingly, from the viewpoint of free boundary pr

martensitic like materials, the coherence condition (1.1

particular form of boundary. The solution to the nuclea of confocal elliptical domains, one focus governing the nucleated region and the other focus the exterior, or (b) a family of confocal hyperbolic arcs separating a and b. Finally, the point of view given here is not completely novel but may be viewed as a somewhat more systematic formulation of considerations already present in the literature, cf. de Gennes [7].

4. Local Stability

Applying the coherence condition in the form (3.4) to two equilibrium domains Ω_a and Ω_b in contact on an arc α leads to $(n_a + n_b)\cdot\nu = 0$ on α, where n_a and n_b denote the respective directors and ν is the external normal referred to one of the domains. If the region of contact α now shrinks to a point z, we obtain the stability condition

$$(n_a + n_b)\cdot\nu = 0 \quad \text{at } z. \tag{4.1}$$

This leads to one of Friedel's rules, cf. [7] p. 468. If Ω_a and Ω_b are ellipses tangent at z, then z is the intersection of the straight lines joining their foci. Just recall that the

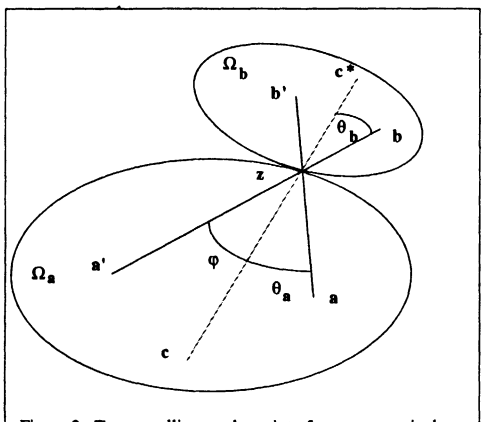

Figure 2. Tangent ellipses: the point of tangency z is the intersection of ab' and a'b.

normal to the ellipse at z bisects the angle $\angle$aza'.
= θ_b so that

$$\theta_b = \theta_a = \varphi.$$

Since the normal is common to both domains, czc*

This places stringent restrictions on the placemer example, unless the point of tangency lies on the axi a disc.

More generally, if two domains of arbitrary shape the visible focus of one domain, (4.1) determines location of b or b' nor which of the two is the visi

5. Many defects and the *plages à éventai*

In an equilibrium configuration with two defects, stability condition (4.1), so in general, once two d theory, infinitely many. This leads to the problem of and $a_j \in D_j$ such that

$$n = \sum n_{a_j} \chi_{D_j} \quad \text{satisfies } (n_{a_j} + n$$
$$\Omega = \cup \bar{D}_j$$

and of establishing a suitable variational criterion. W best way to do this, that is how to pack the domains believe that it will involve the DeGiorgi Γ-limit conditions, [3].

One solution may always be found by choosing fo covering of Ω by discs with centers $\{ a_j \}$. In corresponds to an ensemble of tori, which are deg argues that these are energetically unfavorable and, rarely seen.

Ellipses may be characterized by a simple symmetr two solutions n_a and n_b,

$$(n_a - n_b)\cdot\nu = 0, \quad \nu \text{ normal to } \partial D,$$

and D is symmetric about the perpendicular bisector ellipse.

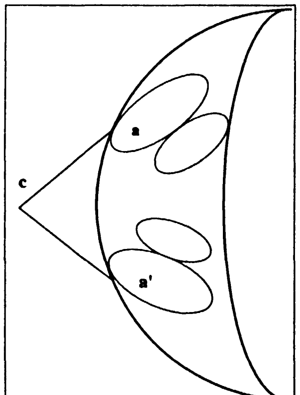

Figure 3. Simplified *texture à éventails*

A frequently observed configuration is the lenticular region between two hyperbolas. This arrangement has the property that the fine structure, represented by infinitely many defects, is limited to the lens while outside the possible defects are the foci of the hyperbola. To construct this solution requires verifying that (4.1) can be satisfied by ellipses tangent to the hyperbolas and each other. We elucidate the constraints and show how this is possible in a subsequent and more complete paper.

An interesting feature is that the general form depicted in Figure 9.1,[7], is a consequence of the construction: the ellipses above the segment joining the foci of the hyperbola point downwards. The lines joining the tangential points of the ellipses intersect at a focus of the hyperbola, no surprise in view of the discussion in §4, but these segments do not extend to the visible foci of the ellipses. We have drawn Figure 3 to illustrate the situation; its geometric configuration is very special just to give the idea.

6. Remarks

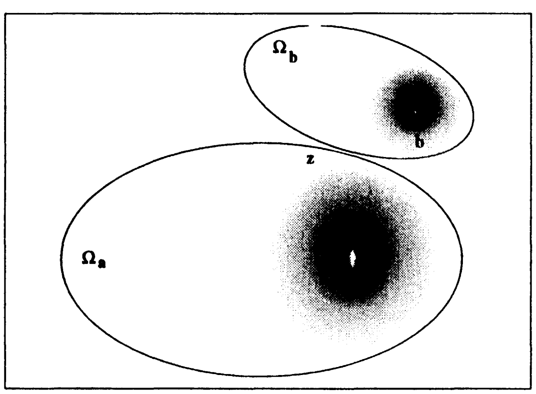

Figure 4 Visible foci lie on different segments joining all the foci of the ellipses

In Figure 4, we depict the appearance of two ellipses
question, not completely resolved in our minds, of
segments joining the foci of the ellipses. We are inv
fluctuations in the texture and the stability condition
weak solutions, but at this writing have not been abl

Acknowledgements

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