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Dynamics of labyrinthine pattern formation in magnetic fluids: a mean–field theory

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Abstract

We are interested in the flow of a droplet of viscous ferrofluid in the Hele–Shaw cell under a transverse magnetic field. The (two–dimensional) phase configuration is observed to evolve into a labyrinthine pattern. We show that the conventional model for this flow has the form of a gradient flux w.r.t. an energy functional, which is the sum of magnetic and surface energy. In particular, we are interested in the behaviour of this flow problem in the regime of large magnetization $M^2 \gg 1$. In this regime, the details of the pattern evolution are observed to be highly sensitive to changes in the initial configuration. This is reflected in the linear stability analysis of the circular phase configuration, a stationary point of the dynamics which is more and more unstable in the limit $M^2 \uparrow \infty$. In order to capture the "generic" behaviour of the dynamical system in this regime, we need a selection principle which dismisses those non–generic solutions. We propose a selection principle for the limit $M^2 \uparrow \infty$ which is based on the natural implicit discretization in time of our gradient flux formulation. We prove that this approach leads (in an appropriate scaling) to the equation

$$\partial_t s - \Delta s^2 = 0$$

for $s(t,x) \in [0,1]$, the local spatial average of the phase configuration $\chi(t,x) \in \{0,1\}$ ($\chi(t,x) = 1$ if $x \in \mathbb{R}^2$ lies in the two–dimensional cross–section of the fluid at time t, $\chi(t,x) = 0$ else). Thus this quantity, which contains information on the "microstructured zone", evolves deterministically, although χ is essentially unpredictable.

AMS code: 76W05, 76E25, 35R35, 58F39

Introduction and summary

We are interested in the following experiment: A droplet of viscous ferrofluid (a collodial suspension of magnetic particles) is trapped between two narrowly spaced horizontal glass plates (the Hele–Shaw cell). Because of this special geometry, the motion of the viscous fluid is strongly overdamped; it is assumed to be governed by Darcy's law [17]. In absence of a magnetic field, the effect of surface tension at the ferrofluid–water interface is such that the droplet is at rest if its cross–section is circular. Now a vertical magnetic field of constant and uniform strength is applied; the ferrofluid is assumed to have constant and uniform magnetization [18]. Due to fringe field effects [18], the circular cross–section undergoes a fingering instability; the fingers are observed to grow into an labyrinthine pattern [18, 12]. In the regime of large magnetization, the details of the pattern evolution are very sensitive to perturbations of the initial configuration. Nevertheless, the envelope of the growing labyrinthine pattern seems to evolve deterministically, see [13, 18]. We propose a deduction of the equation governing the evolution of this microstructured zone and its envelope.

Let us give a short overview of this paper's content. First, we will introduce the energy of a given ferrofluid configuration, which we assume to be a slab of the thickness of the Hele–Shaw cell and thus determined by its two–dimensional cross–section. The energy E will be the sum of a surface energy and a magnetic energy. E coincides with the energy given in [18]. Next, we will introduce a general principle for deriving the dynamics of the flow of a viscous fluid in a Hele–Shaw cell. This principle states that the two–dimensional flow map $\Phi(t,x)$ satisfies a gradient flux on the manifold $\mathcal{M}$ of volume–preserving diffeomorphisms endowed with a metric tensor g and w.r.t. the appropriate energy functional E. We show that this principle is equivalent to the dynamics based on Darcy's law, which were proposed in [17]. Those dynamics lead to an evolution problem for the phase configuration $\chi(t,x)$ ($\chi(t,x)=1$ if $x \in I\!R^2$ lies in the two–dimensional cross–section of the fluid at time t, $\chi(t,x)=0$ else). The formulation as a gradient flux (a special form of dynamical system) provides us with a systematic method for carrying out a linear stability analysis of stationary points of E. This analysis will be performed for the circular phase configuration, quantifying the competition between the stabilizing forces coming from surface energy and the destabilizing ones from the magnetic energy. Our result coincides with the one in [17].

In particular, we are interested in the regime of large "magnetic Bond number" (essentially the ratio between squared magnetization and surface tension) and large "aspect ratio" (the ratio between the square root of the cross–section's area and the cell width). We rescale and thereby nondimensionalize length and time in an way which is appropriate for this regime. In this regime, the evolution of the phase configuration is observed to be very sensitive to perturbations in the initial configuration. This is reflected in the linear stability analysis of the circular phase configuration, which is a stationary point of E and thus a stationary solution of the dynamics. We would like to capture the "generic" behaviour of this delicate dynamical system. But the dynamical system allows for non–generic solutions like the circular phase configuration. Hence we need a selection principle, which dismisses these non–generic solutions. We propose a dynamic selection principle, which is based on the three following steps

1. A gradient flux on a Riemannian manifold $(\mathcal{M}, g)$ w.r.t. a functional E has a natural discretization. This "scheme" consists of a sequence of variational problems, which only involve E and the induced distance *dist*, but not the metric tensor g itself. We apply this to our gradient flux (on an infinite–dimensional Riemannian manifold) and so obtain a scheme for the time–discrete flow map $\{\Phi^{(k)}\}_{k\in N}$. We express this scheme in terms of the time–discrete phase configuration $\{\chi^{(k)}\}_{k\in N}$ and replace *dist* by the more tractable Wasserstein distance d; these distances agree infinitesimally. In Theorem 1 we show that in the limit of vanishing time–step size h, the interpolated $\{\chi^{(k)}\}_{k\in N}$ converges strongly to a weak solution of the original evolution problem for the phase configuration $\{(0,\infty) \ni t \mapsto \chi(t)\}$.

2. For fixed $h > 0$, we let the magnetic Bond number and the aspect ratio tend to infinity in the scheme for the phase configuration $\{\chi^{(k)}\}_{k\in N}$. Theorem 2 states that any solution $\{\chi^{(k)}\}_{k\in N}$ converges weakly to $\{s^{(k)}\}_{k\in N}$, the unique solution of a new scheme which consists of convex variational problems.

3. We let h tend to zero and show in Theorem 3 that the interpolated

$\{s^{(k)}\}_{k\in N}$ converges strongly to the unique weak solution $\{(0,\infty) \ni t \mapsto s(t)\}$ of the nonlinear parabolic evolution equation

$$\partial_t s \; - \; \Delta s^2 \; = \; 0 \, . \tag{1}$$

In view of the above derivation, we may interpret $s(t,x) \in [0,1]$ as the local spatial average of the actual phase configuration $\chi(t,x) \in \{0,1\}$, which is — due to its sensitivity to perturbations of the initial configuration in the considered regime — essentially unpredictable. (1) is well–known as (a special form of) the porous medium equation, see for instance [28]. In particular, it preserves the property of having compact support and thus determines the evolution of the free boundary $\partial\{s(t) > 0\}$. We would like to identify this free boundary with the envelope of the more and more convoluted free boundary $\partial\{\chi(t) = 1\}$ of the original problem.
We also would like to interpret our selection principle in the following way: The "energy landscape" apparently has a fine structure which lives on a scale tending to zero when the magnetic Bond number and the aspect ratio tend to infinity. Our selection principle consists in allowing for "fluctuations" which are large enough to permit the dynamics to ignore this fine structure.
We intend, in future works, to compare these findings to the experimental data of [13].

The energy functional

Our assumptions are those of [18]. Let b be the plate spacing. We assume that the ferrofluid fills a slab $\Omega\times(0,b) \subset \mathbb{R}^3$. Thus its configuration is entirely described by the two–dimensional cross–section $\Omega \subset \mathbb{R}^2$ of this slab. Whenever it is more convenient, we will use the characteristic function χ of Ω

$$\chi(x) \; = \; \left\{ \begin{array}{l} 1 \;\; \text{for } x \in \Omega \\ 0 \;\; \text{else} \end{array} \right\}$$

instead of Ω to describe the phase configuration. Anticipating the fact that the dynamics will preserve the volume of the incompressible ferrofluid, we restrict our attention to phase configurations with given volume: $\int_{R^2} \chi = \alpha$ for some fixed $\alpha > 0$. The energy E of a phase configuration is assumed to be the sum of the surface energy E_s and the magnetic energy E_m. We suppose that the surface energy is given by

(surface tension) × (area of the ferrofluid–water interface).

Thus in terms of χ

$$E_s(\chi) = \sigma b \int |\nabla\chi|,$$

where σ denotes the surface tension. $\int |\nabla\chi|$ is, in the notation of geometric measure theory, the length of the boundary $\partial\Omega$ of Ω, written as the total variation of χ

$$\int |\nabla\chi| := \sup\left\{ \int_{\mathbb{R}^2} \chi \, \mathrm{div}\xi \,\Big|\, \xi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2) \text{ with } |\xi| \leq 1 \right\}. \qquad (2)$$

For the subsequent analysis it is convenient to have this more robust concept of the perimeter of a set. See [16, chapter 1] for an introduction.

The magnetic energy is assumed to be the energy of the magnetic field induced by a uniform magnetization of the slab $\Omega\times(0,b)$ in vertical direction and of magnitude M^2. Thus E_m can be written as

$$E_m(\chi) = (4\pi M)^2 \int_{\mathbb{R}^3} |\Gamma(\bar{\chi}\vec{e})|^2 = (4\pi M)^2 \int_{\mathbb{R}^3} (\bar{\chi}\vec{e})\cdot\Gamma(\bar{\chi}\vec{e}), \qquad (3)$$

where $\bar{\chi}$ is the characteristic function of the slab $\Omega\times(0,b)$, i.e.

$$\bar{\chi}(x,z) = \left\{ \begin{array}{ll} \chi(x) & \text{if } z \in (0,b) \\ 0 & \text{else} \end{array} \right\},$$

$\vec{e}$ the upwards pointing vertical vector of unit length and Γ the orthogonal projection on the gradient fields in $L^2(\mathbb{R}^3)^3$. It is well–known that Γ can be written as convolution operator with the strongly singular kernel

$$\mathrm{D}^2 G(x,z) \quad \text{where} \quad G(x,z) = -\frac{1}{4\pi}\frac{1}{(|x|^2+z^2)^{\frac{1}{2}}} \quad \text{for } (x,z) \in \mathbb{R}^2\times\mathbb{R} = \mathbb{R}^3.$$

This allows us to rewrite the r.h.s. of (3) in terms of χ rather than $\bar{\chi}$:

$$E_m(\chi) = (4\pi M)^2 b \int_{\mathbb{R}^2} \chi K_b \chi,$$

where K_b is the convolution operator with the weakly singular kernel

$$k_b(x) = \frac{1}{b^2} k\left(\frac{x}{b}\right) \quad \text{where} \quad k(x) = \frac{1}{2\pi}\left(\frac{1}{|x|} - \frac{1}{(|x|^2+1)^{\frac{1}{2}}}\right) \quad \text{for } x \in I\!R^2 .$$

Observe that K_b is a symmetric, positive semidefinite bounded operator on $L^2(I\!R^2)$. This and the information on its kernel

$$k_b(x) = \frac{1}{b^2} k\left(\frac{x}{b}\right) \quad \text{where } k \text{ satisfies } k \geq 0 \text{ and } \int_{I\!R^2} k = 1 \tag{4}$$

are the essential properties of K_b we need to derive the mean–field equation. In the physics literature, E_m is sometimes written as a double integral over the perimeter; the reader will find all the usual representations of E_m and a proof of their equivalence in [17].

Thus the total energy E is given by

$$E(\chi) = \sigma b \int |\nabla \chi| + (4\pi M)^2 b \int_{I\!R^2} \chi K_b \chi .$$

To gain some insight into the competition between the surface tension effects and magnetic effects arising from E_s resp. E_m, let us for a moment consider the static problem of minimizing the energy E among all admissible phase configurations in some bounded domain G, i.e. among all $\chi: I\!R^2 \rightarrow \{0,1\}$ with $\int_{I\!R^2} \chi = \alpha$ and support in G. On one hand, E_s wants the minimizer χ of this static problem to have a small perimeter. On the other hand, E_m wants the minimizer χ to be close to the constant $\frac{\alpha}{|G|}$, as we shall argue below. Because of $\chi \in \{0,1\}$ and $\frac{\alpha}{|G|} \in (0,1)$, this can only be achieved in the weak topology and at the expense of a large perimeter.
Let us now argue that E_m wants the minimizer χ to be close to the constant $\frac{\alpha}{|G|}$. The infimum of E_m on the set of all admissible phase configurations in G coincides with the minimum of E_m on the set of all $s: I\!R^2 \rightarrow [0,1]$ with $\int_{I\!R^2} s = \alpha$ and support in G. For small b, the unique minimizer s^* of this relaxed variational problem is close to the constant $\frac{\alpha}{|G|}$ in the strong topology of $L^2(I\!R^2)$. Thus any minimizing sequence $\{\chi_N\}_{N\uparrow\infty}$ of E_m on the set of all admissible phase configurations with support in G converges weakly in $L^2(I\!R^2)$ to s^*.
Later, we will compute the second variation of the functional E with respect to deformations of the circular phase configuration, which is a stationary

point of E. This will provide us with some quantitative information about the competition between E_s and E_m.

The dynamics

Let us now introduce a flexible principle for deriving dynamics for a (single–phase) viscous flow in a Hele–Shaw cell. It is inspired by our prior work on the multiphase flow in porous media [21, 22]. We shall not attempt to provide an entirely rigorous derivation. Let

$$(0,\infty) \ni t \to \Omega(t) \quad \text{resp.} \quad (0,\infty) \ni t \to \chi(t)$$

be the evolution of the phase configuration with initial data Ω^0 resp. χ^0. It is a conventional assumption on the flow of a viscous fluid in a Hele–Shaw cell that the horizontal components of the Eulerian velocity have a parabolic velocity profile in the perpendicular direction, which vanishes on the plate boundaries [1]. For given time $t \in (0,\infty)$ and $x \in \Omega(t)$ let $\vec{u}(t,x) \in I\!R^2$ denote the average of those horizontal components in perpendicular direction. The principle is best stated with help of the (two–dimensional) flow map $\Phi(t,x) \in I\!R^2$, which is related to $\Omega(t)$ by

$$\Omega(t) \;=\; \Phi(t,\Omega^0) \quad \text{for } t \in (0,\infty) \tag{5}$$

and to $\vec{u}(t,x)$ by

$$\partial_t \Phi(t) \;=\; \vec{u}(t)\circ\Phi(t) \;\text{ for all } t \in (0,\infty) \quad \text{and} \quad \Phi(0) \;=\; \mathrm{id}\,. \tag{6}$$

Reflecting the incompressibility of the ferrofluid, one assumes that

$$\Phi(t) \;\in\; \mathcal{M} \quad \text{for all } t \in (0,\infty)\,,$$

where $\mathcal{M}$ is the manifold of all volume preserving diffeomorphisms Φ from Ω^0 onto some Ω. The tangent space $T_\Phi\mathcal{M}$ of $\mathcal{M}$ in a point Φ can be identified with

$$T_\Phi\mathcal{M} \;=\; \Big\{\xi\circ\Phi \,\Big|\, \xi \in C_0^\infty(I\!R^2, I\!R^2) \text{ with } \operatorname{div}\xi = 0\Big\}\,.$$

We endow $\mathcal{M}$ with the Riemannian metric g

$$\begin{aligned} g_\Phi(w_1,w_2) \;&=\; \tfrac{12\mu}{b^2}\int_{\Omega^0} w_1(x)\cdot w_2(x)\,dx \\ &\text{for } w_1, w_2 \in T_\Phi\mathcal{M} \text{ and } \Phi \in \mathcal{M}\,, \end{aligned} \tag{7}$$

where μ is the ferrofluid's viscosity. This Riemannian manifold $(\mathcal{M}, g)$ has been introduced by Arnol'd [5] to study another problem of incompressible flow: He points out that geodesics on $(\mathcal{M}, g)$ satisfy the Euler equations for an inviscid and incompressible fluid, yielding a least action principle for this flow problem (in that application, of course, the constant $\frac{12\mu}{b^2}$ must be replaced). Our flow problem is just of opposite nature: due to the viscosity of the fluid combined with the geometry of the flow domain, the motion is highly dissipative. In fact, as we will see below, for a given "curve" $[0,1] \ni t \mapsto \Phi(t) \in \mathcal{M}$ the quantity

$$\frac{1}{2}\int_0^1 g_{\Phi(t)}(\partial_t\Phi(t), \partial_t\Phi(t))\, dt \tag{8}$$

can be interpreted as the kinetic energy dissipated by friction in one unit of time when the viscous fluid moves in the narrow gap of the Hele–Shaw cell with (two–dimensional) velocity $\vec{u}$ given by (6). Let us also point out that this derivation tacitely assumes that the viscosity of the other phase, in our case water, is much smaller than μ and therefore negligible, as it is supposed in [17].

The initial phase configuration Ω^0 being fixed, each element $\Phi \in \mathcal{M}$ defines a new phase configuration $\Phi(\Omega^0)$, thus the energy functional E is naturally defined on $\mathcal{M}$ by

$$E(\Phi) \;=\; E(\Phi(\Omega^0))\,.$$

Our principle can be formulated as follows

$$\begin{array}{l}(0,\infty) \ni t \mapsto \Phi(t) \in \mathcal{M} \quad \text{is the gradient flux} \\ \text{on } (\mathcal{M}, g) \text{ with respect to } E\,.\end{array} \tag{9}$$

This means in formulas

$$\begin{array}{l} g_{\Phi(t)}(\,\partial_t\Phi(t)\,,\, w\,) \;=\; -\; <\mathrm{d}E(\Phi(t))\,,\, w> \\ \text{for all } w \in T_{\Phi(t)}\mathcal{M} \text{ and } t \in (0,\infty)\,, \end{array} \tag{10}$$

where the linear form $\mathrm{d}E(\Phi)$ on $T_\Phi\mathcal{M}$ is the differential of E in Φ.

Let us now reformulate (9) in terms of the phase configuration $\Omega(t)$ and the velocity $\vec{u}$. Due to (5,6) we have the following kinematic condition: The normal velocity V of the interface $\partial\Omega(t)$ is given by the normal component of $\vec{u}$

$$V \;=\; \vec{u}\cdot\nu\,. \tag{11}$$

Next notice that (10) can be rewritten as

$$\begin{aligned}&\tfrac{12\mu}{b^2}\int_{\Omega(t)} u(t,x)\cdot\xi(x)\,dx \;=\; -\,\partial_\tau\left[E(\Phi_\xi(\tau,\Omega(t)))\right]_{\tau=0}\\ &\text{for all } \xi\in C_0^\infty(I\!R^2,I\!R^2) \text{ with } \operatorname{div}\xi=0 \text{ and all } t\in(0,\infty)\,,\end{aligned}$$

where the one–parameter family $\{\Phi_\xi(\tau)\}_{\tau\in I\!R}$ of volume preserving diffeomorphisms of $I\!R^2$ is defined by

$$\partial_\tau\Phi_\xi(\tau) \;=\; \xi\circ\Phi_\xi(\tau) \;\text{ for all } \tau\in I\!R \quad\text{and}\quad \Phi_\xi(0) \;=\; \mathrm{id}\,.$$

The computation of the first variation $\partial_\tau\left[E(\Phi_\xi(\tau,\Omega))\right]_{\tau=0}$ for an Ω with smooth boundary $\partial\Omega$ is classical:

$$\begin{aligned}&\partial_\tau\left[E(\Phi_\xi(\tau,\Omega))\right]_{\tau=0} \;=\; \sigma\,b\int_{\partial\Omega}\kappa\,\xi\cdot\nu \;+\; 2\,(4\,\pi\,M)^2\,b\int_{\partial\Omega}K_b\chi\,\xi\cdot\nu\\ &\text{for all } \xi\in C_0^\infty(I\!R^2,I\!R^2) \text{ with } \operatorname{div}\xi=0\,,\end{aligned}$$

where ν is the outer normal and κ the curvature of $\partial\Omega$. Observe that the gradient fields on Ω form the orthogonal complement of the set of fields ξ with $\operatorname{div}\xi=0$ in Ω and $\xi\cdot\nu=0$ on $\partial\Omega$ w.r.t. to the scalar product of $L^2(\Omega)$. Thus we see that (10) is equivalent to the existence of a pressure p. More precisely,

$$\left.\begin{aligned}&\text{For all } t\in(0,\infty) \text{ there exists } p(t)\colon\Omega(t)\to I\!R \text{ s.t.}\\ &\vec{u}(t) \;=\; -\tfrac{b^2}{12\,\mu}\,\nabla p(t) \qquad\qquad\qquad\quad \text{in } \Omega(t)\,,\\ &p(t) \;=\; \sigma\,b\,\kappa(t) \;+\; 2\,(4\,\pi\,M)^2\,b\,K_b\chi(t) \;\;\text{on } \partial\Omega(t)\,.\end{aligned}\right\} \tag{12}$$

On the other hand, we have, due to the assumption that $\Phi(t)\in\mathcal{M}$ for all $t\in(0,\infty)$

$$\operatorname{div}\vec{u}(t) \;=\; 0 \quad \text{in } \Omega(t) \;\text{ for all } t\in(0,\infty)\,. \tag{13}$$

Observe that at each time t, $p(t)$ (up to an additive constant) and therefore $u(t)$, is determined by $\Omega(t)$ via (12,13). Together with the kinematic condition (11), (12,13) defines an evolutionary free boundary problem for the interface $\partial\Omega(t)$. We observe that in the case of no magnetization ($M=0$), (11,12,13) is the model of [9] and [26] for the flow of a viscous fluid in a Hele–Shaw cell of width b. In these papers, the first part of (12) is derived with help of Darcy's law, which itself can be obtained in the limit $b \downarrow 0$ by averaging pressure and velocity in the three–dimensional Stokes equation in the vertical coordinate. This approach has been extended to the case of nonzero magnetization ($M^2 > 0$) in [17]; the result coincides with (11,12,13). This justifies the interpretation of (8) as kinetic energy dissipated by friction. To the best of our knowledge, a proof of global existence (in an appropriate weak form) for this free boundary problem is not available, even for the case $M = 0$ — although there are very interesting partial results [10].

Regime under consideration and appropriate scaling

Guided by [25], we identify two dimensionless parameters, the "magnetic Bond number" and the "aspect ratio"

$$\frac{2\,M^2\,b}{\sigma} \quad \text{resp.} \quad \frac{R_0}{b},$$

where $R_0 := (\frac{\alpha}{\pi})^{\frac{1}{2}}$ is the radius of the ball with area α. $\frac{2M^2b}{\sigma}$ is a measure of the relative strength of the magnetic effects w.r.t. the surface tension effects, whereas $\frac{R_0}{b}$ is the ratio between the typical horizontal and vertical length. We are interested in the regime of large magnetic Bond number and large aspect ratio

$$\frac{2\,M^2\,b}{\sigma} \gg 1 \quad \text{and} \quad \frac{R_0}{b} \gg 1. \tag{14}$$

It is natural to nondimensionalize length by measuring the horizontal length in units of R_0. In these new units, the metric tensor and the energy functional can be normalized to

$$\begin{aligned} g_\Phi(w_1, w_2) &= 12\,\mu \int_{\Omega^0} w_1(x) \cdot w_2(x)\,dx\,, \\ E(\chi) &= \sigma\,\left(\tfrac{b}{R_0}\right)^3 \int |\nabla\chi| \;+\; (4\,\pi\,M)^2\,b\,\left(\tfrac{b}{R_0}\right)^2 \int_{R^2} \chi\,K_{\frac{b}{R_0}}\chi\,. \end{aligned}$$

In order to obtain a nontrivial limit, we have to rescale (and thereby nondimensionalize) time by measuring it in units of

$$8\,\pi^2\,\frac{2\,M^2\,b}{\sigma}\left(\frac{b}{R_0}\right)^2\frac{\sigma}{12\,\mu}.$$

In these units, the metric tensor and the energy functional can be normalized to

$$\left.\begin{array}{lcl} g_{\Phi}(w_1, w_2) & = & \int_{\Omega^0} w_1(x) \cdot w_2(x)\, dx\,, \\ E(\chi) & = & \tilde{\sigma} \int |\nabla\chi| + \int_{\mathbb{R}^2} \chi K_{\tilde{b}} \chi\,, \end{array}\right\} \tag{15}$$

with

$$\tilde{\sigma} = \frac{1}{8\pi^2} \frac{\sigma}{2M^2 b} \frac{b}{R_0} \quad \text{and} \quad \tilde{b} = \frac{b}{R_0}.$$

Observe that in the regime (14), we have a convenient separation of scales

$$\tilde{\sigma} \ll \tilde{b} \ll 1. \tag{16}$$

This is essential in the derivation of the mean–field behaviour. Henceforth, we will use the rescaled metric tensor and energy functional in (15).

Linear stability analysis for the circular phase configuration

To gain some insight into the features of the dynamics, let us consider their linearization for an initial circular phase configuration

$$\Omega^0 = \{ x \in \mathbb{R}^2 \,|\, |x| < 1 \}.$$

This analysis has been carried through in [17]. Nevertheless, we choose to display our calculation (which yields the same result), as the formulation of the dynamics as a gradient flow on $(\mathcal{M}, g)$ w.r.t. E provides us with a systematic method of linearization. As we will show below, id is a stationary point of E, i.e. the differential $\mathrm{d}E(\mathrm{id})$, a linear form on $T_{\mathrm{id}}\mathcal{M}$, vanishes. By regarding (9) as a dynamical system on $\mathcal{M}$, we see that its linearization around the sationary point id is given by the linear operator A on $T_{\mathrm{id}}\mathcal{M}$ defined by

$$g_{\mathrm{id}}(A\xi, w) = \mathrm{d}^2E(\mathrm{id})(\xi, w) \quad \text{for } \xi, w \in T_{\mathrm{id}}\mathcal{M},$$

where the symmetric bilinear form $\mathrm{d}^2E(\mathrm{id})$ on $T_{\mathrm{id}}\mathcal{M}$ is the Hessian of E in the stationary point id (see for instance [23, 9.4.5.]). We will do a spectral analysis of the symmetric operator A. Let us start by deriving a formula for

$d^2E(\mathrm{id})$. This is accomplished by computing the first and second variation of E in id. For given $\xi \in T_{\mathrm{id}}\mathcal{M}$ we introduce $\{\Phi_\xi(\tau)\}_{\tau\in R} \subset \mathcal{M}$ by

$$\partial_\tau \Phi_\xi(\tau) \;=\; \xi \circ \Phi_\xi(\tau) \;\text{ for all } \tau \in I\!R \quad\text{and}\quad \Phi_\xi(0) = \mathrm{id}\,.$$

We consider E_s and E_m separately. Let ν and $\nu^\perp$ denote outer normal and counter–clockwise tangential of $\partial\Omega^0$. According to [27, chapter 2, §9], we have for all $\xi \in T_{\mathrm{id}}\mathcal{M}$

$$\begin{aligned} <\mathrm{d}E_s(\mathrm{id})\,,\,\xi> \;&=\; \partial_\tau \left[E_s(\Phi_\xi(\tau))\right]_{\tau=0} \\ &=\; \tilde\sigma \int_{\partial\Omega^0} \nu^\perp\!\cdot\!\mathrm{D}\xi\,\nu^\perp \\ &=\; \tilde\sigma \int_{\partial\Omega^0} \xi\!\cdot\!\nu \;=\; 0 \end{aligned}$$

and thus (see for instance [23, 9.4.3])

$$\begin{aligned} \mathrm{d}^2E_s(\mathrm{id})\,(\xi,\xi) \;&=\; \partial_\tau^2 \left[E_s(\Phi_\xi(\tau))\right]_{\tau=0} \\ &=\; \tilde\sigma \int_{\partial\Omega^0} \left\{\nu^\perp\!\cdot\!\mathrm{D}(\mathrm{D}\xi\,\xi)\,\nu^\perp \;+\; (\nu\!\cdot\!\mathrm{D}\xi\,\nu^\perp)^2\right\} \\ &=\; \tilde\sigma \int_{\partial\Omega^0} \left\{\nu\!\cdot\!\mathrm{D}\xi\,\xi \;+\; (\nu\!\cdot\!\mathrm{D}\xi\,\nu^\perp)^2\right\}. \end{aligned}$$

Observe that the following identities hold for all $\xi, w \in T_{\mathrm{id}}\mathcal{M}$

$$\begin{aligned} &\tilde\sigma \int_{\partial\Omega^0} \left\{\nu\!\cdot\!\mathrm{D}\xi\,\nu \;-\; \partial_{\nu^\perp}(\nu\!\cdot\!\mathrm{D}\xi\,\nu^\perp)\right\}\,(w\!\cdot\!\nu) \\ &= \tilde\sigma \int_{\partial\Omega^0} \left\{\nu\!\cdot\!\mathrm{D}\xi\,w \;+\; (\nu\!\cdot\!\mathrm{D}\xi\,\nu^\perp)\,(\nu\!\cdot\!\mathrm{D}w\,\nu^\perp)\right\} \\ &= \tilde\sigma \int_{\Omega^0} \mathrm{tr}(\mathrm{D}\xi\,\mathrm{D}w^t) \;+\; \tilde\sigma \int_{\partial\Omega^0} (\nu\!\cdot\!\mathrm{D}\xi\,\nu^\perp)\,(\nu\!\cdot\!\mathrm{D}w\,\nu^\perp)\,. \end{aligned} \tag{17}$$

The first identity follows from

$$\partial_{\nu^\perp}(w\!\cdot\!\nu) \;=\; \nu\!\cdot\!\mathrm{D}w\,\nu^\perp \;+\; w\!\cdot\!\nu^\perp\,,$$

whereas the second is a consequence of

$$\mathrm{div}(\mathrm{D}\xi\,w) \;=\; (\nabla\mathrm{div}\xi)\!\cdot\!w \;+\; \mathrm{tr}(\mathrm{D}\xi\,\mathrm{D}w^t)\,.$$

Thus (17) defines a symmetric bilinear form on $T_{\rm id}\mathcal{M}$, which coincides with the symmetric bilinear form $\mathrm{d}^2E_s(\mathrm{id})$ on the diagonal. Hence both forms are identical

$$\mathrm{d}^2E_s(\mathrm{id})\,(\xi,w) \;=\; \tilde{\sigma}\int_{\partial\Omega^0}\left\{\nu{\cdot}\mathrm{D}\xi\,\nu \;-\; \partial_{\nu^\perp}(\nu{\cdot}\mathrm{D}\xi\,\nu^\perp)\right\}(w{\cdot}\nu)\,. \tag{18}$$

Now consider the first and second variation of E_m. Using $k_{\tilde{b}}(-x) = k_{\tilde{b}}(x)$, we obtain for all $\xi \in T_{\rm id}\mathcal{M}$

$$\begin{aligned} <\mathrm{d}E_m(\mathrm{id})\,,\,\xi> \;&=\; \partial_\tau\left[E_m(\Phi_\xi(\tau))\right]_{\tau=0} \\ &=\; 2\int_{\partial\Omega^0}\int_{\Omega^0} k_b(x-\tilde{x})\,d\tilde{x}\,\xi(x){\cdot}\nu(x)\,dx \;=\; 0\,, \end{aligned}$$

where the last identity follows from the fact that $\int_{\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,d\tilde{x}$ is radially symmetric in x. Thus we have

$$\begin{aligned} &\mathrm{d}^2E_m(\mathrm{id})\,(\xi,\xi) \\ &\quad= \partial_\tau^2\left[E_m(\Phi_\xi(\tau))\right]_{\tau=0} \\ &\quad= 2\int_{\partial\Omega^0}\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,(\xi(\tilde{x})-\xi(x)){\cdot}\nu(\tilde{x})\,d\tilde{x}\,\xi(x){\cdot}\nu(x)\,dx\,. \end{aligned}$$

As $\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,\nu(\tilde{x})\,d\tilde{x}$ is parallel to $\nu(x)$,

$$2\int_{\partial\Omega^0}\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,(\xi(\tilde{x})-\xi(x)){\cdot}\nu(\tilde{x})\,d\tilde{x}\,w(x){\cdot}\nu(x)\,dx$$

defines a symmetric bilinear form on $T_{\rm id}\mathcal{M}$, which coincides with the symmetric bilinear form $\mathrm{d}^2E_m(\mathrm{id})$ on the diagonal. Hence both forms are identical

$$\begin{aligned} &\mathrm{d}^2E_m(\mathrm{id})\,(\xi,w) \\ &\quad= 2\int_{\partial\Omega^0}\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,(\xi(\tilde{x})-\xi(x)){\cdot}\nu(\tilde{x})\,d\tilde{x}\,w(x){\cdot}\nu(x)\,dx\,. \end{aligned} \tag{19}$$

We now determine the eigenvalues λ and corresponding eigenvectors ξ of A, that is

$$\lambda\,g_{\rm id}(\xi,w) \;=\; \mathrm{d}^2E(\mathrm{id})\,(\xi,w) \quad \text{for all } w \in T_{\rm id}\mathcal{M}\,. \tag{20}$$

It is obvious from (18,19) that the linear subspace

$$X \;:=\; \left\{\xi \in C^\infty(\overline{\Omega^0},\mathbb{R}^2)\,\middle|\,\mathrm{div}\,\xi = 0 \text{ in } \Omega^0 \text{ and } \xi{\cdot}\nu = 0 \text{ on } \partial\Omega^0\right\}$$

of $T_{\mathrm{id}}\mathcal{M}$ consists of eigenvectors corresponding to the eigenvalue 0. It thus remains to consider the orthogonal complement $X^\perp$ of X in $(T_{\mathrm{id}}\mathcal{M}, g_{\mathrm{id}})$, which can be written as

$$X^\perp = \left\{ \xi \in C^\infty(\overline{\Omega^0}, \mathbb{R}^2) \,\middle|\, \text{there exists a harmonic } p \text{ with } \xi = \nabla p \right\}.$$

In view of (7,18,19), (20) becomes for $\xi = \nabla p \in X^\perp$

$$\begin{aligned} \lambda & \int_{\partial\Omega^0} p\, w\cdot\nu \\ &= \tilde{\sigma} \int_{\partial\Omega^0} \left\{ \mathrm{D}^2 p\,(\nu,\nu) \;-\; \partial_{\nu^\perp}(\mathrm{D}^2 p\,(\nu,\nu^\perp)) \right\}\,(w\cdot\nu) \\ &\quad + 2 \int_{\partial\Omega^0}\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,(\nabla p(\tilde{x}) - \nabla p(x))\cdot\nu(\tilde{x})\,d\tilde{x}\,(w\cdot\nu)(x)\,dx\,. \end{aligned}$$

The reduced eigenvalue problem can thus be reformulated as

$$\left.\begin{aligned} -\Delta p &= 0 \quad \text{in } \Omega^0 \\ \lambda\, p(x) &= \tilde{\sigma}\,\left(\mathrm{D}^2 p\,(\nu,\nu) \;-\; \partial_{\nu^\perp}(\mathrm{D}^2 p\,(\nu,\nu^\perp))\right)(x) \\ &\quad + \; 2\int_{\partial\Omega^0} k_{\tilde{b}}(x-\tilde{x})\,(\nabla p(\tilde{x}) - \nabla p(x))\cdot\nu(\tilde{x})\,d\tilde{x} \\ &\quad \text{for all } x \in \partial\Omega^0\,. \end{aligned}\right\} \tag{21}$$

The boundary condition in (21) is conveniently expressed in polar coordinates (r,θ)

$$\begin{aligned} \lambda & p(1,\theta) \\ &= \tilde{\sigma}\,(p_{,rr} - p_{,r\theta\theta} + p_{,\theta\theta})(1,\theta) \\ &\quad + 2\int_0^{2\pi} \tfrac{1}{b^2}\, k\left(\tfrac{2}{b}\,|\sin(\tfrac{\theta-\tilde{\theta}}{2})|\right)\,(p_{,r}(1,\tilde{\theta}) - p_{,r}(1,\theta))\,d\tilde{\theta} \\ &\quad + 2\int_0^{2\pi} \tfrac{1}{b^2}\, k\left(\tfrac{2}{b}\,\sin(\tfrac{\tilde{\theta}}{2})\right)\,(1-\cos\tilde{\theta})\,d\tilde{\theta}\,p_{,r}(1,\theta) \\ &\quad \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

It is now easily checked that for $n \in \mathbb{N} \cup \{0\}$

$$\lambda_n = \tilde{\sigma}\,n\,(n^2-1) \;-\; 2\,n\,(\kappa_n - \kappa_1) \tag{22}$$

is an eigenvalue of (21) with eigenspace spanned by the Fourier modes

$$p(r,\theta) = r^n \cos(n\,\theta) \quad \text{and} \quad p(r,\theta) = r^n \sin(n\,\theta)\,, \tag{23}$$

where the strictly monotone increasing sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of positive numbers is given by

$$\begin{aligned}\tilde{b}\,\kappa_n &= \int_0^\pi \tfrac{4}{\tilde{b}}\,k\left(\tfrac{2}{\tilde{b}}\sin\theta\right)\sin^2(n\,\theta)\,d\theta \\ &= \int_0^\pi \frac{1}{\pi}\left(\frac{1}{\sin\theta}-\frac{1}{(\sin^2\theta+(\frac{\tilde{b}}{2})^2)^{\frac{1}{2}}}\right)\sin^2(n\,\theta)\,d\theta\,.\end{aligned}$$

As the gradients of the functions (23) form a Hilbert basis of $X^{\perp}$, the entire spectrum of A is given by (22). The representation (22) visualizes and quantifies the competition between the stabilizing effect of surface tension and the destabilizing effect of magnetic forces. By elementary real variable analysis, it can be shown that (provided $\tilde{b}\le\frac{1}{2}$, say)

$$c\,g(\tilde{b}n)\;\le\;\tilde{b}\,\kappa_n\;\le\;C\,g(\tilde{b}n)\,, \tag{24}$$

where g is given by

$$g(z) := \left\{\begin{array}{ll} z^2\,(1-\ln(z)) & \text{for } z\le 1 \\ 1+\ln(z) & \text{for } z\ge 1\end{array}\right\}$$

and $0<c<C<\infty$ denote universal constants. In particular, the spectrum of A is bounded by below.

A dynamic selection principle

Let us interpret the above linear stability analysis for the circular phase configuration in the regime (14): From (22,24) we obtain

$$\lambda_n \longrightarrow 0 \quad \text{for all } n\in\mathbb{N} \quad \text{but} \quad \inf_{n\in\mathbb{N}}\lambda_n \longrightarrow -\infty$$

in the limit $\tilde{b},\frac{\tilde{\sigma}}{\tilde{b}}\downarrow 0$. Hence the circular phase configuration becomes more and more unstable with respect to high mode perturbation. This reflects the experimental observation that the circular phase configuration is very sensitive to small scale perturbations of the initial data in this regime [13, 18]. As we are interested in a mean field theory for the regime (14), we need a selection principle which dismisses those in the limit $\frac{2M^2b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ more and more unstable solutions of (11,12,13). We propose a selection principle which is based on an implicit discretization of the gradient flux (10). More precisely, we consider the natural discretization of (10) for time step size $h > 0$, first let both $\frac{2M^2b}{\sigma}$ and $\frac{R_0}{b}$ tend to infinity and then h to zero. The limit $\frac{2M^2b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ for fixed $h > 0$ will dramatically simplify the "energy landscape". We believe that this procedure is essential in capturing the "generic" behaviour of the gradient flux in the regime (14). This procedure also has the more technical advantage that the natural implicit discretization of the gradient flux gives us a sequence of variational problems, whose behaviour for $\frac{2M^2b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ can be analyzed within the framework of Γ–convergence. In the case of no magnetization, this approach of writing the dynamics as a gradient flux (in an Eulerian and thus less natural formulation than ours) and considering its implicit discretization (which differs from ours) has independently been followed in [3].

Let us now introduce this natural implicit discretization of the gradient flux (9). For fixed $h > 0$ it is given by the following scheme for the time–discrete flow map $\{\Phi^{(k)}\}_{k\in N}$

$$\left.\begin{array}{l} \Phi^{(k)} \text{ is minimizer of} \\ \quad \frac{1}{2}\, dist(\Phi^{(k-1)}, \Phi)^2 \; + \; h\, E(\Phi) \\ \text{among all } \Phi \in \mathcal{M}, \end{array}\right\} \qquad (25)$$

where $\Phi^{(0)} = \mathrm{id}$ and *dist* denotes the induced distance on $(\mathcal{M}, g)$, i.e.

$$dist(\Phi_0, \Phi_1)^2 \; = \; \inf\Big\{ \int_0^1 g_{\Phi(t)}(\partial_t\Phi(t), \partial_t\Phi(t))\, dt \,\Big|\; \Phi\colon [0,1] \to \mathcal{M} \text{ with } \Phi(0) = \Phi_0 \text{ and } \Phi(1) = \Phi_1 \Big\}.$$

We then think of $(0,\infty) \ni t \mapsto \Phi_h(t) \in \mathcal{M}$, which is given by the interpolation

$$\Phi_h(t) \; = \; \Phi^{(k)} \quad \text{for } t \in [k\,h, (k+1)\,h) \text{ and } k \in I\!N \cup \{0\},$$

as an approximation of $(0,\infty) \ni t \mapsto \Phi(t) \in \mathcal{M}$.

In order to formulate our principle (9) in a compact way and to derive a natural discretization, the description in terms of the flow map is more convenient. We now return to a description with help of the phase configuration.

In terms of the time–discrete phase configuration $\{\chi^{(k)}\}_{k\in N}$, which is related to $\{\Phi^{(k)}\}_{k\in N}$ by

$$\chi^{(k)}(x) \;=\; \left\{ \begin{array}{l} 1 \text{ if } x \in \Phi^{(k)}(\Omega^0) \\ 0 \text{ else} \end{array} \right\},$$

the scheme (25) reads

$$\left.\begin{array}{l} \chi^{(k)} \text{ is minimizer of} \\ \quad \frac{1}{2}\, dist(\chi^{(k-1)},\chi)^2 \;+\; h\,E(\chi) \\ \text{among all } \chi\colon \mathbb{R}^2 \to \{0,1\} \text{ with } \int_{R^2} \chi = \alpha \end{array}\right\} \qquad (26)$$

with $\chi^{(0)} = \chi^0$. Here $dist(\chi_0,\chi_1)^2$ now stands for

$$\begin{aligned} & dist(\chi_0,\chi_1)^2 \qquad (27) \\ &\quad = \inf\Big\{ \int_0^1\int_{\Omega_0} |\partial_t \Phi(t,x)|^2\,dx\,dt \,\Big|\, \Phi(t) \text{ is a volume preserving} \\ &\qquad\quad \text{diffeomorphism from } \Omega_0 \text{ onto } \Phi(t,\Omega_0) \text{ for all } t \in [0,1]\,, \\ &\qquad\quad \Phi(0) = \text{id and } \Phi(1,\Omega_0) = \Omega_1 \Big\}. \end{aligned}$$

For technical purposes, it is more convenient to replace the induced distance $dist(\chi_0,\chi_1)$ by the Wasserstein distance $d(\chi_0,\chi_1)$ between χ_0 and χ_1 (considered as Lebesgue densities). Let us now define the notion of Wasserstein distance in the generality we need in this paper (see [24] for an exhaustive overview of distance functions on spaces of measures). We consider the Wasserstein distance on the set $\tilde{K}_R$ of all Lebesgue densities with integral α and support in $\{|x| \le R\}$

$$\begin{aligned} \tilde{K}_R \;:=\; \Big\{ & s\colon \mathbb{R}^2 \to [0,\infty) \text{ measurable} \,\Big| \\ & \int_{R^2} s = \alpha \quad\text{and}\quad s = 0 \text{ on } \{|x| > R\} \Big\}. \end{aligned}$$

For $s_0, s_1 \in \tilde{K}_R$ we introduce the set of nonnegative Borel measures on the product space $\mathbb{R}^2 \times \mathbb{R}^2$ with marginals $s_0\, d\mathcal{L}^2$ and $s_1\, d\mathcal{L}^2$

$$
\begin{aligned}
P(s_0, s_1) \;:=\; \Big\{ p \,\Big|\, & p \text{ is a nonnegative Borel measure on } \mathbb{R}^2 \times \mathbb{R}^2 \text{ with} \\
& \int \zeta(x)\, p(dx\, dy) = \int_{\mathbb{R}^2} s_0(x)\, \zeta(x)\, dx \text{ for all } \zeta \in C_0^0(\mathbb{R}^2) \\
& \int \zeta(y)\, p(dx\, dy) = \int_{\mathbb{R}^2} s_1(y)\, \zeta(y)\, dy \text{ for all } \zeta \in C_0^0(\mathbb{R}^2) \Big\}.
\end{aligned}
$$

The Wasserstein distance $d(s_0, s_1)$ between the measures $s_0\, d\mathcal{L}^2$ and $s_1\, d\mathcal{L}^2$ is given by

$$
d(s_0, s_1)^2 \;=\; \inf \Big\{ \int |x-y|^2\, p(dx\, dy) \,\Big|\, p \in P(s_0, s_1) \Big\}. \tag{28}
$$

For the moment, we only need d on the set S of all phase configurations, i.e.

$$
S \;:=\; \Big\{ \chi \colon \mathbb{R}^2 \to \{0,1\} \text{ measurable with bounded support} \,\Big|\, \int_{\mathbb{R}^2} \chi = \alpha \Big\}.
$$

It will follow from Lemma 1 (Appendix) that we actually have for $\chi_0, \chi_1 \in S$

$$
\begin{aligned}
d(\chi_0, \chi_1)^2 \;=\; \min \Big\{ \int_{\Omega_0} |x - \Phi(x)|^2\, dx \,\Big|\, & \Phi \text{ is a measure preserving map} \\
& \text{from } \Omega_0 \text{ onto } \Omega_1 \text{ in the sense of} \\
& \int_{\Omega_1} \zeta(y)\, dy = \int_{\Omega_0} \zeta(\Phi(x))\, dx \text{ for } \zeta \in C_0^0(\mathbb{R}^2) \Big\}.
\end{aligned}
$$

This relates (28) to (27).

From now on we claim mathematical rigor. We consider the scheme

$$
\left.
\begin{aligned}
& \chi^{(k)} \text{ is minimizer of} \\
& \quad \tfrac{1}{2} d(\chi^{(k-1)}, \chi)^2 \;+\; h\, E(\chi) \\
& \text{among all } \chi \in S,
\end{aligned}
\right\} \tag{29}
$$

where $\chi^{(0)} = \chi^0$. We will show in Proposition 1 that (29) admits a solution. Let us remark that this formulation of the implicit discretization allows for changes in topology of the phase configuration (in contrast to the formulation in [3]). It therefore might also be of interest in the study of singularity formation — for instance if the initial phase configuration has the form of a dumbbell with a thin neck. In the case of no magnetization, this topic has been extensively studied (see for instance [3]). In order to convince the reader that despite the passage to the more tractable Wasserstein metric, (29) is still a discretization of (11,12,13), we state a contingent convergence result whose proof will be presented elsewhere in a broader context.

THEOREM 1. *For given $h > 0$ let $\{\chi_h^{(k)}\}_{k\in N}$ be a solution of (29). We consider the interpolation*

$$\chi_h(t) \;=\; \chi_h^{(k)} \quad \text{for } t \in [k\,h, (k+1)\,h) \text{ and } k \in \mathbb{N} \cup \{0\}.$$

Then there exist a measurable $\chi\colon (0,\infty)\times\mathbb{R}^2 \to \{0,1\}$ and a $\vec{u} \in L^2((0,\infty) \times\mathbb{R}^2)$ satisfying

$$\chi_h \to \chi \quad \text{in } L^1((0,T)\times\mathbb{R}^2) \text{ for all } T < \infty,$$

for a subsequence, and

$$\begin{aligned} &-\int_{(0,\infty)\times R^2} \chi\left\{\partial_t\zeta + \vec{u}\cdot\nabla\zeta\right\} = \int_{R^2} \chi^0\,\zeta(0) \\ &\text{for all } \zeta \in C_0^\infty((-\infty,\infty)\times\mathbb{R}^2). \end{aligned} \tag{30}$$

If the length of the interface does not drop in the limit $h \downarrow 0$, i.e. if

$$\limsup_{h\downarrow 0} \int_0^T \int |\nabla\chi_h(t)|\,dt \;\le\; \int_0^T \int |\nabla\chi(t)|\,dt \quad \text{for all } T < \infty, \tag{31}$$

we have in addition for a.e. $t \in (0,\infty)$

$$\begin{aligned} -\int_{R^2} \chi(t)\,\vec{u}(t)\cdot\xi \;&=\; \bar{\sigma}\int \left\{\operatorname{div}\xi - \nu(t)\cdot \mathrm{D}\xi\,\nu(t)\right\} |\nabla\chi(t)| \\ &\quad + \; 2\int K_{\bar{b}}\chi(t)\,\xi\cdot\nu(t)\,|\nabla\chi(t)| \\ &\qquad \text{for all } \xi \in C_0^\infty(\mathbb{R}^2,\mathbb{R}^2) \text{ with } \operatorname{div}\xi = 0, \end{aligned} \tag{32}$$

where $\nu(t) = \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$ is the Radon–Nykodym derivative of the vector valued measure $\nabla\chi(t)$ w.r.t. its total variation $|\nabla\chi(t)|$, see [16, chapter 3].

Observe that (30) is a weak formulation of (11,13) and that (32) is a weak formulation of (12). The proof of Theorem 1 relies on many techniques introduced in [19]. A condition of the form (31) is also typical for convergence results of more conventional implicit discretizations of geometrical evolution problems which can be written as a gradient flux w.r.t. the area functional, as for instance mean–curvature flow [20].

The main result: the limits $\frac{2M^2 b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ and $h \downarrow 0$

As motivated in the previous section, our selection principle consists in first passing to the limit $\frac{2M^2 b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ and then considering the limit $h \downarrow 0$. The rigorous convergence results are stated in Theorem 2 resp. Theorem 3, which are the main mathematical results of this paper. Throughout the sequel, let the initial configuration $\chi^0 \in S$ satisfy $\int |\nabla \chi^0| < \infty$.

According to (16), the limit $\frac{2M^2 b}{\sigma}, \frac{R_0}{b} \uparrow \infty$ corresponds to $\tilde{b}, \frac{\tilde{\sigma}}{\tilde{b}} \downarrow 0$. Theorem 2 states that any solution of the scheme (29) converges weakly to the solution of a new scheme. The new scheme (33) retains the form of time–discretization of a gradient flux w.r.t. the Wasserstein metric d and an energy functional. The new energy functional

$$\int_{R^2} s^2 \quad \text{for } s \in K$$

(K is defined below) is the Γ–limit under weak convergence of the original energy functional (15)

$$\tilde{\sigma} \int |\nabla \chi| + \int_{R^2} \chi K_{\tilde{b}} \chi \quad \text{for } \chi \in S.$$

We observe that, in contrast to (29), (33) consists of convex variational problems.

THEOREM 2. *For given $\tilde{b}, \tilde{\sigma} > 0$ let $\{\chi^{(k)}_{\tilde{b},\tilde{\sigma}}\}_{k\in N}$ be a solution of (29). Then we have for $\tilde{b}, \frac{\tilde{\sigma}}{\tilde{b}} \downarrow 0$*

$$\chi^{(k)}_{\tilde{b},\tilde{\sigma}} \stackrel{w*}{\rightharpoonup} s^{(k)} \quad \text{in } L^\infty(\mathbb{R}^2) \text{ for all } k \in \mathbb{N},$$

where $\{s^{(k)}\}_{k\in N}$ is the unique solution of

$$\left.\begin{array}{l} s^{(k)} \text{ minimizes} \\ \quad \frac{1}{2} d(s^{(k-1)}, s)^2 + h \int_{\mathbb{R}^2} s^2 \\ \text{among all } s \in K \end{array}\right\} \tag{33}$$

with $s^{(0)} = \chi^0$. Here K is defined by

$$K := \left\{ s\colon \mathbb{R}^2 \to [0,1] \text{ measurable with bounded support} \,\middle|\, \int_{\mathbb{R}^2} s = \alpha \right\}.$$

Theorem 3 states that the solution of scheme (33), appropriately interpolated in time, converges strongly to the solution of the nonlinear diffusion equation (34).

THEOREM 3. *For given $h > 0$ let $\{s_h^{(k)}\}_{k\in N}$ be the solution of (33). Then the interpolation $s_h\colon (0,\infty)\times\mathbb{R}^2 \to [0,1]$ given by*

$$s_h(t) = s_h^{(k)} \quad \text{for } t \in [k\,h, (k+1)\,h) \text{ and } k \in \mathbb{N} \cup \{0\}$$

converges for $h \downarrow 0$:

$$s_h \longrightarrow s \quad \text{in } L^1((0,T)\times\mathbb{R}^2) \quad \text{for all } T < \infty,$$

where s is the unique weak solution of

$$\partial_t s - \Delta s^2 = 0 \tag{34}$$

with initial data χ^0.

The proof of Theorem 2 and Theorem 3

All Lebesgue integrals are — if not otherwise stated — to be taken over $\mathbb{R}^2$.

PROPOSITION 1.

i) *There exists a solution* $\{\chi^{(k)}\}_{k\in N}$ *of (29).*

ii) *There exists* $\{R^{(k)}\}_{k\in N} \subset (0,\infty)$ *depending only on* α *and* χ^0 *(not on* $\tilde{\sigma}$ *and* $\tilde{b}$*) s.t. any solution* $\{\chi^{(k)}\}_{k\in N}$ *of (29) satisfies* $\chi^{(k)} \in \tilde{K}_{R^{(k)}}$ *for all* $k \in \mathbb{N}$.

Proof of Proposition 1
We divide the proof into two parts.

- First we will show with help of familiar compactness and lower semi-continuity arguments that for given $\chi^{(0)} \in S$ and $R < \infty$ there exists a minimizer of

$$F(\chi^{(0)},\chi) := \tfrac{1}{2}\, d(\chi^{(0)},\chi)^2 + h\left\{\tilde{\sigma}\int|\nabla\chi| + \int \chi K_{\tilde{b}}\chi\right\}.$$

among all $\chi \in S \cap \tilde{K}_R$.

- Then we will demonstrate the harder part, namely that for given $R^{(0)} < \infty$ and $\chi^{(0)} \in S \cap \tilde{K}_{R^{(0)}}$ there exists an $R^{(1)} < \infty$ depending only on α,

$$h\left\{\tilde{\sigma}\int|\nabla\chi^{(0)}| + \int \chi^{(0)} K_{\tilde{b}}\chi^{(0)}\right\} \quad \text{and} \quad R^{(0)}$$

s.t.

$$\inf_{S\cap\tilde{K}_{R^{(1)}}} F(\chi^{(0)},\cdot) \;<\; F(\chi^{(0)},\chi) \quad \text{for all}\; \chi \in S - \tilde{K}_{R^{(1)}},$$

Proposition 1 obviously follows from these two parts by iteration.

Let $\{\chi_N\}_{N\uparrow\infty}$ be a minimizing sequence of $F(\chi^{(0)},\cdot)$ in $S \cap \tilde{K}_R$. Because $\{\int|\nabla\chi_N|\}_{N\uparrow\infty}$ is uniformly bounded, $\{\chi_N\}_{N\uparrow\infty}$ is compact in $L^1(\mathbb{R}^2)$, see for instance [16, 1.19 Theorem]. Thus there exists a $\chi^{(1)} \in L^1(\mathbb{R}^2)$ s.t. we have for a subsequence

$$\chi_N \longrightarrow \chi^{(1)} \quad \text{in } L^1(\mathbb{R}^2).$$

The strong convergence assures that we have again $\chi^{(1)} \in S \cap \tilde{K}_R$. It is a straightforward consequence of the definition (28) that $d(\chi^{(0)}, \cdot)^2$ is lower semicontinuous w.r.t. weak–$*$ convergence in $C_0^0(\mathbb{R}^2)^*$:

$$d(\chi^{(0)}, \chi^{(1)})^2 \;\leq\; \liminf_{N\uparrow\infty} d(\chi^{(0)}, \chi_N)^2 .$$

(In fact, it follows from Lemma 2 that $d(\chi^{(0)}, \cdot)^2$ is continuous w.r.t. weak–$*$ convergence in $C_0^0(\mathbb{R}^2)^*$). Likewise, it is an immediate consequence of the definition (2) that the total variation is lower semicontinuous w.r.t. weak–$*$ convergence in $C_0^0(\mathbb{R}^2)^*$:

$$\int |\nabla \chi^{(1)}| \;\leq\; \liminf_{N\uparrow\infty} \int |\nabla \chi_N| .$$

Because $K_{\tilde{b}}$ is a symmetric and positive semidefinite operator in $L^2(\mathbb{R}^2)$, the corresponding quadratic form is lower semicontinuous w.r.t. weak convergence in $L^2(\mathbb{R}^2)$:

$$\int \chi^{(1)} K_{\tilde{b}} \chi^{(1)} \;\leq\; \liminf_{N\uparrow\infty} \int \chi_N K_{\tilde{b}} \chi_N .$$

(Actually, the fact that $K_{\tilde{b}}$ is a convolution operator yields continuity of the quadratic form w.r.t. weak convergence in $L^2(\mathbb{R}^2)$). Thus $\chi^{(1)}$ is a minimizer. Observe that the strong convergence of the minimizing sequence $\{\chi_N\}_{N\uparrow\infty}$ only was required to assure that the nonconvex constraint $\chi_N \in \{0,1\}$ is preserved in the limit.

Let us now demonstrate the second part. Let $R < \infty$ be s.t. $\chi \in S - \tilde{K}_R$. We consider χ_R given by

$$\chi_R(y) := \left\{ \begin{array}{cl} \chi(y) & \text{for } |y| < R \\ 0 & \text{for } |y| \geq R \end{array} \right\}$$

and record that

$$\int |\nabla \chi_R| \leq \int |\nabla \chi| \quad \text{and} \quad \int \chi_R K_{\tilde{b}} \chi_R \leq \int \chi K_{\tilde{b}} \chi . \tag{35}$$

Because of

$$\int \chi_{R,\tau_R} K_{\tilde{b}} \chi_{R,\tau_R}$$
$$= \int\int \chi_R(y)\, k_{\tilde{b}}(\Phi_{\tau_R}(y) - \Phi_{\tau_R}(\tilde{y}))\, \chi_R(\tilde{y}) \det D\Phi_{\tau_R}(y) \det D\Phi_{\tau_R}(\tilde{y})\, d\tilde{y}\, dy$$

and

$$|\Phi_{\tau_R}(y) - \Phi_{\tau_R}(\tilde{y})| \ \geq\ \frac{1}{\sqrt{1+\tau_R}} |y - \tilde{y}|,$$

we obtain the estimate

$$\int \chi_{R,\tau_R} K_{\tilde{b}} \chi_{R,\tau_R}$$
$$\leq\ (1 + C\,\tau_R) \int \chi_R K_{\tilde{b}} \chi_R \ \leq\ (1 + C\,(1 - \frac{\alpha_R}{\alpha})) \int \chi_R K_{\tilde{b}} \chi_R \qquad (39)$$

for some universal constant $C < \infty$. Let us now estimate $d(\chi^{(0)}, \chi_{R,\tau_R})^2$ in terms of $d(\chi^{(0)}, \chi)^2$. To this end, we will construct a $p_R \in P(\chi^{(0)}, \chi_{R,\tau_R})$ for a given $p \in P(\chi^{(0)}, \chi)$:

$$\int \zeta(x,y)\, p_R(dx\, dy)$$
$$= \int_{\{|y|\leq R\}} \zeta(x, \Phi_{\tau_R}(y))\, p(dx\, dy)$$
$$+ \frac{1}{\alpha - \alpha_R} \int_{\{|\tilde{y}|\leq R\}} \chi(\tilde{y})\, (\det D\Phi_{\tau_R}(\tilde{y}) - 1) \int_{\{|y|>R\}} \zeta(x, \Phi_{\tau_R}(\tilde{y}))\, p(dx\, dy)\, d\tilde{y}.$$

In particular we have, using (37),

$$\int |x - y|^2\, p_R(dx\, dy)$$
$$= \int_{\{|y|\leq R\}} |x - \Phi_{\tau_R}(y)|^2\, p(dx\, dy)$$
$$+ \frac{\tau_R}{\alpha - \alpha_R} \int_{\{|\tilde{y}|\leq \bar{R}^{(1)}\}} \chi(\tilde{y}) \int_{\{|y|>R\}} |x - \Phi_{\tau_R}(\tilde{y})|^2\, p(dx\, dy)\, d\tilde{y}.$$

Let us now estimate the two terms on the r.h.s. using the fact that $|x| \leq R^{(0)}$ p–a.e.

$$\int_{\{|y|\leq R\}} |x - \Phi_{\tau_R}(y)|^2\, p(dx\, dy)$$

$$
\begin{aligned}
\leq\ & \int |x-y|^2\, p(dx\,dy) \;-\; \int_{\{|y|>R\}} (|y|-|x|)^2\, p(dx\,dy) \\
& + 2\int_{\{|y|\leq R\}} (|x|+|y|)|y-\Phi_{\tau_R}(y)|\, p(dx\,dy) \\
& + \int_{\{|y|\leq R\}} |y-\Phi_{\tau_R}(y)|^2\, p(dx\,dy) \\
\leq\ & \int |x-y|^2\, p(dx\,dy) \;-\; (\alpha-\alpha_R)\,(R-R^{(0)})^2 \\
& + 2\,\alpha_R\,(R+R^{(0)})\,\tfrac{1}{2}\,\tau_R\,\tilde{R}^{(1)} \\
& + \alpha_R\,(\tfrac{1}{2}\,\tau_R\,\tilde{R}^{(1)})^2,
\end{aligned}
$$

whereas the second term is estimated by

$$
\alpha_R\,\tau_R\,(\sqrt{1+\tau_R}\,\tilde{R}^{(1)} + R^{(0)})^2 .
$$

Thus we obtain with some universal constant C

$$
\begin{aligned}
& d(\chi^{(0)}, \chi_{R,\tau_R})^2 \\
& \quad \leq\ d(\chi^{(0)},\chi)^2 \;+\; (\alpha-\alpha_R)\,\Big(-(R-2\,\tilde{R}^{(1)})^2 + C\,(\tilde{R}^{(1)})^2\Big)\,. \qquad (40)
\end{aligned}
$$

Collecting (35), (38), (39) and (40), we gather

$$
\begin{aligned}
& F(\chi^{(0)}, \chi_{R,\tau_R}) \;\leq\; F(\chi^{(0)},\chi) \,+ \\
& (\alpha-\alpha_R)\left[\tfrac{1}{2}\Big(-(R-2\,\tilde{R}^{(1)})^2 + C\,(\tilde{R}^{(1)})^2\Big) \;+\; \frac{h}{\alpha}\left\{\tilde{\sigma}\int|\nabla\chi| + \int \chi\, K_{\tilde{b}}\chi\right\}\right] \;\Box
\end{aligned}
$$

PROOF OF THEOREM 2

Roughly speaking, Theorem 2 follows from

- the fact that the functional

  $$\int s^2 \quad \text{for } s\in K$$

  is the Γ–limit under weak–∗ convergence in $L^\infty(\mathbb{R}^2)$ of

  $$\tilde{\sigma}\int|\nabla\chi| \;+\; \int \chi\, K_{\tilde{b}}\chi \quad \text{for } \chi\in S\,,$$

  which is a consequence of the separation of scales (16),

- and the continuity of d^2 w.r.t. weak-$*$ convergence in $C_0^0(\mathbb{R}^2)^*$, which is shown in Lemma 2 (Appendix).

First observe that the scheme (33) admits at most one solution. Indeed, the map $K \ni s \mapsto d(s_0, s)^2$ is convex (an immediate consequence of the definition (28)), the map $K \ni s \mapsto \int s^2$ is strictly convex and the space K is convex.

Fix a $k \in \mathbb{N}$. Assume we had already shown that

$$\chi_{\tilde{b},\tilde{\sigma}}^{(k-1)} \overset{w*}{\rightharpoonup} s^{(k-1)} \quad \text{in } L^\infty(\mathbb{R}^2) \tag{41}$$

for some $s^{(k-1)} \in K$. Thanks to Proposition 1 ii) we know that $\chi_{\tilde{b},\tilde{\sigma}}^{(k)} \in S \cap \tilde{K}_{R^{(k)}}$. Thus there exists an $s^{(k)} \in K$ s.t. we have for a subsequence

$$\chi_{\tilde{b},\tilde{\sigma}}^{(k)} \overset{w*}{\rightharpoonup} s^{(k)} \quad \text{in } L^\infty(\mathbb{R}^2)\,. \tag{42}$$

Let us prove that $\chi_{\tilde{b},\tilde{\sigma}}^{(k)}$ being a solution of the variational problem in (29) implies that $s^{(k)}$ is a solution of the variational problem in (33). In the spirit of Γ–convergence we do this by showing, on one hand, that (41,42) implies

$$\begin{aligned} &\tfrac{1}{2}\, d(s^{(k-1)}, s^{(k)})^2 \;+\; h\int (s^{(k)})^2 \\ &\le\; \liminf \left\{ \tfrac{1}{2}\, d(\chi_{\tilde{b},\tilde{\sigma}}^{(k-1)}, \chi_{\tilde{b},\tilde{\sigma}}^{(k)})^2 \;+\; h\left[\tilde{\sigma}\int |\nabla \chi_{\tilde{b},\tilde{\sigma}}^{(k)}| + \int \chi_{\tilde{b},\tilde{\sigma}}^{(k)} K_{\tilde{b}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)}\right]\right\} \end{aligned} \tag{43}$$

and by constructing, on the other hand, for given $s \in K$ a sequence $\{\chi_{\tilde{b},\tilde{\sigma}}\}_{\tilde{b},\tilde{\sigma}}$ in S s.t.

$$\chi_{\tilde{b},\tilde{\sigma}} \overset{w*}{\rightharpoonup} s \quad \text{in } L^\infty(\mathbb{R}^2) \tag{44}$$

and

$$\begin{aligned} &\tfrac{1}{2}\, d(s^{(k-1)}, s)^2 \;+\; h\int s^2 \\ &=\; \lim \left\{ \tfrac{1}{2}\, d(\chi_{\tilde{b},\tilde{\sigma}}^{(k-1)}, \chi_{\tilde{b},\tilde{\sigma}})^2 \;+\; h\left[\tilde{\sigma}\int |\nabla \chi_{\tilde{b},\tilde{\sigma}}| + \int \chi_{\tilde{b},\tilde{\sigma}} K_{\tilde{b}} \chi_{\tilde{b},\tilde{\sigma}}\right]\right\}. \end{aligned} \tag{45}$$

Let us derive (43). Because d^2 is lower semicontinuous w.r.t. weak-$*$ convergence in $C_0^0(\mathbb{R}^2)^*$ (this is an straightforward consequence of (28)), it remains to show

$$\liminf \int \chi_{\tilde{b},\tilde{\sigma}}^{(k)} K_{\tilde{b}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)} \geq \int (s^{(k)})^2 .$$

We prove this by an argument of functional analysis: $K_{\tilde{b}}$ is a symmetric, positive semidefinite bounded linear operator on the Hilbert space $L^2(\mathbb{R}^2)$. Hence there exists a (unique) symmetric, positive semidefinite bounded linear operator $K_{\tilde{b}}^{\frac{1}{2}}$ s.t. $K_{\tilde{b}} = K_{\tilde{b}}^{\frac{1}{2}} K_{\tilde{b}}^{\frac{1}{2}}$ (see for instance [6, Chapter 21.1]). On the other hand, we have due to (4)

$$K_{\tilde{b}} s \to s \quad \text{in } L^2(\mathbb{R}^2) \text{ for all } s \in L^2(\mathbb{R}^2) .$$

From the definition of $K_{\tilde{b}}^{\frac{1}{2}}$ we infer that this implies

$$K_{\tilde{b}}^{\frac{1}{2}} s \to s \quad \text{in } L^2(\mathbb{R}^2) \text{ for all } s \in L^2(\mathbb{R}^2) .$$

Together with (42) this yields

$$K_{\tilde{b}}^{\frac{1}{2}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)} \stackrel{w}{\rightharpoonup} s^{(k)} \quad \text{in } L^2(\mathbb{R}^2)$$

and thus

$$\int (s^{(k)})^2 \leq \liminf \int K_{\tilde{b}}^{\frac{1}{2}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)} K_{\tilde{b}}^{\frac{1}{2}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)} = \liminf \int \chi_{\tilde{b},\tilde{\sigma}}^{(k)} K_{\tilde{b}} \chi_{\tilde{b},\tilde{\sigma}}^{(k)} .$$

Let us now for given $s \in K$ construct a sequence $\{\chi_{\tilde{b},\tilde{\sigma}}\}_{\tilde{b},\tilde{\sigma}} \subset S$ s.t. (44,45) holds. Because of the separation of scales (16), we may choose a λ s.t.

$$\tilde{\sigma} \ll \lambda \ll \tilde{b} \ll 1 .$$

We partition $\mathbb{R}^2$ into squares of length λ. Let C be such a square; we partition C into two rectangles C_0 and C_1 s.t.

$$|C_1| = \int_C s$$

and define $\chi_{\tilde{b},\tilde{\sigma}}$ by

$$\chi_{\tilde{b},\tilde{\sigma}}(x) = \left\{ \begin{array}{l} 1 \text{ if } x \in C_1 \\ 0 \text{ if } x \in C_0 \end{array} \right\} \quad \text{for } x \in C \text{ and all } C .$$

(44) follows from $\lambda \ll 1$. Due to $\tilde{\sigma} \ll \lambda$ we obtain

$$\lim \tilde{\sigma} \int |\nabla \chi_{\tilde{b},\tilde{\sigma}}| = 0$$

and $\lambda \ll \tilde{b}$ yields

$$K_{\tilde{b}} \chi_{\tilde{b},\tilde{\sigma}} \to s \quad \text{in } L^1(I\!R^2)\,.$$

From this we deduce (45), noticing that (41) and (44) imply

$$\left| d(\chi_{\tilde{b},\tilde{\sigma}}^{(k-1)}, \chi_{\tilde{b},\tilde{\sigma}}) - d(s^{(k-1)}, s) \right| \le d(\chi_{\tilde{b},\tilde{\sigma}}^{(k-1)}, s^{(k-1)}) + d(\chi_{\tilde{b},\tilde{\sigma}}, s) \to 0$$

by Lemma 2 (Appendix) □

PROPOSITION 2. *Let $\{s^{(k)}\}_{k \in N}$ be the solution of (33). We then have for all $k \in I\!N$:*

$$\int |\nabla (s^{(k)})^2|^2 \le (\tfrac{1}{h})^2 \, d(s^{(k-1)}, s^{(k)})^2\,, \tag{46}$$

$$\left| \int \left\{ \tfrac{1}{h} (s^{(k)} - s^{(k-1)}) \, \zeta + \nabla (s^{(k)})^2 . \nabla \zeta \right\} \right| \le \tfrac{1}{2} \sup_{R^2} |D^2 \zeta| \, \tfrac{1}{h} \, d(s^{(k-1)}, s^{(k)})^2 \quad \textit{for all } \zeta \in C_0^\infty(I\!R^2)\,. \tag{47}$$

REMARK. A finer analysis yields that $s^{(k)}$ is Lipschitz continuous and that we have an "energy estimate"

$$\int (s^{(k)})^2 + h \int s^{(k)} \, |\nabla s^{(k)}|^2 \le \int (s^{(k-1)})^2$$

and an "entropy estimate"

$$\int s^{(k)} \ln s^{(k)} + h \int |\nabla s^{(k)}|^2 \le \int s^{(k-1)} \ln s^{(k-1)}\,.$$

Proof of Proposition 2
Proposition 2 is a consequence of the first variation of the variational problems in (33). Considering the fact that (33) was derived from the Lagrangian point of view, it is natural to investigate the first variation w.r.t. the independent variables.

Before investigating the first variation, we observe that the constraint $s \leq 1$ in the definition of admissible functions in (33) can be neglected. This amounts to prove a maximum principle for (33): Let $R < \infty$ be so large that $s^{(k)}, s^{(k-1)} \in \tilde{K}_R$. Let us show

$$\left.\begin{array}{l} s^{(k)} \text{ minimizes} \\ \quad \frac{1}{2}\, d(s^{(k-1)}, s)^2 \;+\; h \int s^2 \\ \text{among all } s \in \tilde{K}_R\,. \end{array}\right\} \tag{48}$$

By the elementary compactness and lower semicontinuity arguments of the first part of the proof of Proposition 1, this enlarged variational problem has a solution s_*. We have to show that $s_* \leq 1$. Assume that this is not the case. Let $p_* \in P(s^{(k-1)}, s_*)$ be optimal in the definition of $d(s^{(k-1)}, s_*)^2$ in (28); the existence of such an "optimal transfer plan" is assured by standard compactness and lower semicontinuity arguments. Define $v^{(k-1)}$ and v_* by

$$\begin{aligned} \int v^{(k-1)}(x)\, \zeta(x)\, dx &= \int_{\{s_*(y)>1\}} \zeta(x)\, p(dx\, dy) \quad \text{for all } \zeta \in C_0^0(I\!R^2)\,, \\ \int v_*(y)\, \zeta(y)\, dy &= \int_{\{s_*(y)>1\}} \zeta(y)\, p(dx\, dy) \\ &= \int_{\{s_*(y)>1\}} s_*(y)\, \zeta(y)\, dy \quad \text{for all } \zeta \in C_0^0(I\!R^2)\,. \end{aligned}$$

Then for $t \in [0,1]$,

$$\begin{aligned} \int \zeta(x,y)\, p_t(dx\, dy) &= \int_{\{s_*(y)\leq 1\}} \zeta(x,y)\, p(dx\, dy) \\ &+ (1-t) \int_{\{s_*(y)>1\}} \zeta(x,y)\, p(dx\, dy) \\ &+ t \int_{\{s_*(y)>1\}} \zeta(x,x)\, p(dx\, dy) \\ &\quad \text{for all } \zeta \in C_0^0(I\!R^2 \times I\!R^2) \end{aligned}$$

defines a $p_t \in P(s^{(k-1)}, s_{*,t})$ with

$$s_{*,t} \;:=\; s_* \;-\; t\,(v_* - v^{(k-1)}) \;\in\; \tilde{K}_R\,.$$

We then have

$$\tfrac{1}{2}\, d(s^{(k-1)}, s_{*,t})^2 \;+\; h \int s_{*,t}^2$$

$$\leq \tfrac{1}{2}\, d(s^{(k-1)}, s_*)^2 \;+\; h \int s_*^2$$
$$-\, 2t \int s_*\,(v_* - v^{(k-1)}) \;+\; t^2 \int (v_* - v^{(k-1)})^2$$

and because of $0 \leq v^{(k-1)} \leq 1$

$$\int s_*\,(v_* - v^{(k-1)}) \;=\; \int (s_* - 1)\,(v_* - v^{(k-1)}) \;>\; 0$$

— a contradiction for small $t > 0$.

Let us now consider the first variation of (48) w.r.t. the independent variables. More precisely, let a smooth vector field with bounded support, $\xi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, be given. Consider the corresponding flux, a one–parameter family $\{\Phi_\xi(\tau)\}_{\tau \in R}$ of diffeomorphisms, given by

$$\partial_\tau\, \Phi_\xi(\tau) \;=\; \xi \circ \Phi_\xi(\tau) \;\text{ for all } \tau \in \mathbb{R} \quad\text{and}\quad \Phi_\xi(0) \;=\; \mathrm{id}\,.$$

For any $\tau \in \mathbb{R}$,

$$\det \mathrm{D}\Phi_\xi(\tau)\, s_\tau \circ \Phi_\xi(\tau) \;=\; s$$

defines an $s_\tau \in \tilde{K}_R$, if $R < \infty$ was chosen large enough. Thus we have

$$\tfrac{1}{2}\, d(s^{(k-1)}, s_\tau)^2 \;+\; h \int s_\tau^2 \;\geq\; \tfrac{1}{2}\, d(s^{(k-1)}, s^{(k)})^2 \;+\; h \int (s^{(k)})^2\,. \tag{49}$$

Using

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}[\det \mathrm{D}\Phi_\xi(\tau)]_{\tau=0} \;=\; \mathrm{div}\xi\,,$$

we easily obtain

$$\lim_{\tau \to 0} \tfrac{1}{\tau} \left(\int s_\tau^2 \;-\; \int (s^{(k)})^2 \right) \;=\; -\int (s^{(k)})^2\, \mathrm{div}\xi\,. \tag{50}$$

We now argue that

$$\limsup_{\tau \downarrow 0} \tfrac{1}{\tau} \left(\tfrac{1}{2}\, d(s^{(k-1)}, s_\tau)^2 \;-\; \tfrac{1}{2}\, d(s^{(k-1)}, s^{(k)})^2 \right) \;\leq\; \int (y - x)\cdot\xi(y)\, p(dx\, dy)\,, \tag{51}$$

where $p \in P(s^{(k-1)}, s^{(k)})$ is optimal in the definition of $d(s^{(k-1)}, s^{(k)})^2$ in (28). Indeed,

$$\int \zeta(x, y)\, p_\tau(dx\, dy) \;=\; \int \zeta(x, \Phi_\xi(\tau, y))\, p(dx\, dy)$$
for $\zeta \in C_0^0(\mathbb{R}^2 \times \mathbb{R}^2)$

defines a $p_\tau \in P(s^{(k-1)}, s_\tau)$ and thus we have

$$\begin{aligned} &\tfrac{1}{\tau}\left(\tfrac{1}{2}\,d(s^{(k-1)}, s_\tau)^2 \;-\; \tfrac{1}{2}\,d(s^{(k-1)}, s^{(k)})^2\right) \\ &\quad\le\; \int \tfrac{1}{\tau}\left(\tfrac{1}{2}\,|\Phi_\xi(\tau, y) - x|^2 - \tfrac{1}{2}\,|x - y|^2\right) p(dx\,dy)\,, \end{aligned}$$

which entails (51). We infer from (49,50,51) (and the symmetry in $\xi \to -\xi$) that

$$\left.\begin{array}{l} \int (y - x)\cdot\xi(y)\, p(dx\,dy) \;-\; h\int (s^{(k)})^2\,\mathrm{div}\xi \;=\; 0 \\ \text{for all } \xi \in C_0^\infty(I\!R^2, I\!R^2)\,, \end{array}\right\} \tag{52}$$

which is the first variation.

Because of

$$\begin{aligned} \left|\int (y - x)\cdot\xi(y)\, p(dx\,dy)\right| \;&\le\; \left(\int |y - x|^2\, p(dx\,dy)\right)^{\frac{1}{2}} \left(\int s^{(k)}\,|\xi|^2\right)^{\frac{1}{2}} \\ &\le\; d(s^{(k-1)}, s^{(k)}) \left(\int |\xi|^2\right)^{\frac{1}{2}}, \end{aligned}$$

we deduce from (52) that $\nabla(s^{(k)})^2 \in L^2(I\!R^2)$ and the estimate (46). Finally we observe that

$$\begin{aligned} &\left|\int (s^{(k)} - s^{(k-1)})\,\zeta \;-\; \int (y - x)\cdot\nabla\zeta(y)\, p(dx\,dy)\right| \\ &\quad=\; \left|\int (\zeta(y) - \zeta(x) \;-\; (y - x)\cdot\nabla\zeta(y))\; p(dx\,dy)\right| \\ &\quad\le\; \tfrac{1}{2}\sup_{R^2} |\mathrm{D}^2\zeta| \int |y - x|^2\, p(dx\,dy) \\ &\quad=\; \tfrac{1}{2}\sup_{R^2} |\mathrm{D}^2\zeta|\, d(s^{(k-1)}, s^{(k)})^2 \qquad \text{for all } \zeta \in C_0^\infty(I\!R^2)\,. \end{aligned}$$

This and (52) obviously implies (47).

PROOF OF THEOREM 3

Let us first show that $\{s_h\}_{h\downarrow 0}$ is compact in $L^1((0,T)\times I\!R^2)$ for all $T < \infty$. Starting point is the a priori estimate

$$\sum_{k\in N} \tfrac{1}{h}\, d(s_h^{(k-1)}, s_h^{(k)})^2 \;\le\; 2\int (\chi^0)^2 \;=\; 2\,\alpha. \tag{53}$$

First we infer

$$\begin{aligned}\int s_h^{(N)}(y)\,\tfrac{1}{2}|y|^2\,dy &\leq d(\chi^0,s_h^{(N)})^2 + \int \chi^0(x)\,|x|^2\,dx\\ &\overset{(73)}{\leq} N\sum_{k=1}^{N} d(s_h^{(k-1)},s_h^{(k)})^2 + \int \chi^0(x)\,|x|^2\,dx\\ &\overset{(53)}{\leq} 2\,\alpha\,N\,h + \int \chi^0(x)\,|x|^2\,dx\,,\end{aligned}$$

which translates into the following thightness property

$$\int s_h(T,y)\,\tfrac{1}{2}|y|^2\,dy \leq 2\,\alpha\,T + \int \chi^0(x)\,|x|^2\,dx\,. \tag{54}$$

From (46) and (53) we deduce compactness in space

$$\int_{(h,\infty)\times R^2} |\nabla s_h^2|^2 \leq 2\,\alpha\,. \tag{55}$$

Let us now consider compactness in time. Compactness in time follows from equicontinuity in the Wasserstein metric

$$\begin{aligned} d(s_h^{(k)},s_h^{(k+N)}) &\overset{(73)}{\leq} \sum_{\ell=1}^{N} d(s_h^{(k+\ell-1)},s_h^{(k+\ell)})\\ &\leq (N\,h)^{\frac{1}{2}}\left(\sum_{\ell=1}^{N}\tfrac{1}{h}\,d(s_h^{(k+\ell-1)},s_h^{(k+\ell)})^2\right)^{\frac{1}{2}}\\ &\leq (N\,h\,2\,\alpha)^{\frac{1}{2}} \quad \text{for all } k,N\in I\!N\end{aligned}$$

and compactness in space, in the form of (55) for instance, by some elementary interpolation argument. However, we derive a more explicit estimate with help of Lemma 3 (Appendix). Fix $k,N \in I\!N$ and let $R<\infty$ so large that $s_h^{(k)}, s_h^{(k+N)} \in \tilde{K}_R$. Let $(\phi_*,\psi_*) \in \mathcal{F}_R$ be optimal in the dual representation of $d(s_h^{(k)},s_h^{(k+N)})$ in Lemma 1; we have

$$d(s_h^{(k)},s_h^{(k+N)})^2 = \int s_h^{(k+N)}(y)\,|y-\nabla\psi_*(y)|^2\,dy \quad \text{and} \tag{56}$$

$$\int s_h^{(k)}(x)\,\zeta(x)\,dx = \int s_h^{(k+N)}(y)\,\zeta(\nabla\psi_*(y))\,dy \quad \text{for all } \zeta\in C_0^0(I\!R^2)\,, \tag{57}$$

which we will use for $\zeta := (s_h^{(k+N)})^2 - (s_h^{(k)})^2$. We estimate the following quantity

$$\begin{aligned}
&\int (s_h^{(k+N)} - s_h^{(k)})\,((s_h^{(k+N)})^2 - (s_h^{(k)})^2) \\
&\overset{(57)}{=} \int s_h^{(k+N)}(y)\,(\zeta(y) - \zeta(\nabla\psi_*(y)))\,dy \\
&\leq \int_0^1 \left(\int s_h^{(k+N)}(y)\,|\nabla\zeta((1-t)\,y + t\,\nabla\psi_*(y))|^2\,dy\right)^{\frac{1}{2}} dt \\
&\quad \times \left(\int s_h^{(k+N)}(y)\,|y - \nabla\psi_*(y)|^2\,dy\right)^{\frac{1}{2}} \\
&\overset{(76)}{\leq} \left(\int |\nabla\zeta(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int s_h^{(k+N)}(y)\,|y - \nabla\psi_*(y)|^2\,dy\right)^{\frac{1}{2}} \\
&\leq \left[\left(\int |\nabla(s_h^{(k+N)})^2(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int |\nabla(s_h^{(k)})^2(x)|^2\,dx\right)^{\frac{1}{2}}\right] \\
&\quad \times \left(\int s_h^{(k+N)}(y)\,|y - \nabla\psi_*(y)|^2\,dy\right)^{\frac{1}{2}} \\
&\overset{(46,56)}{\leq} \frac{1}{h}\left[d(s_h^{(k+N-1)}, s_h^{(k+N)}) + d(s_h^{(k-1)}, s_h^{(k)})\right] d(s_h^{(k)}, s_h^{(k+N)}) \\
&\overset{(73)}{\leq} \frac{1}{h}\left[d(s_h^{(k+N-1)}, s_h^{(k+N)}) + d(s_h^{(k-1)}, s_h^{(k)})\right] \sum_{\ell=1}^{N} d(s_h^{(k+\ell-1)}, s_h^{(k+\ell)})\,.
\end{aligned}$$

We sum over all $k \in \mathbb{N}$ and obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} h \int (s_h^{(k+N)} - s_h^{(k)})\,((s_h^{(k+N)})^2 - (s_h^{(k)})^2) \\
&\quad \leq \left[\left(\sum_{k\in\mathbb{N}} d(s_h^{(k+N-1)}, s_h^{(k+N)})^2\right)^{\frac{1}{2}} + \left(\sum_{k\in\mathbb{N}} d(s_h^{(k-1)}, s_h^{(k)})^2\right)^{\frac{1}{2}}\right] \\
&\quad \times \left(\sum_{k\in\mathbb{N}} N \sum_{\ell=1}^{N} d(s_h^{(k+\ell-1)}, s_h^{(k+\ell)})^2\right)^{\frac{1}{2}} \\
&\quad \leq 2N \sum_{k\in\mathbb{N}} d(s_h^{(k-1)}, s_h^{(k)})^2 \;\leq\; 4\alpha N h\,.
\end{aligned}$$

This translates into

$$\int_0^\infty \int (s_h(t+\tau) - s_h(t))\,(s_h(t+\tau)^2 - s_h(t)^2)\,dt \;\leq\; 4\alpha\tau \quad \text{for all } \tau \geq 0\,. \tag{58}$$

Let us remark that (55) and (58) are precisely the estimates one obtains for the conventional implicit scheme for (34), which is stated in the variable $u = s^2$ and is given by

$$\begin{aligned} & u^{(k)} \text{ minimizes} \\ & \int \tfrac{3}{2} u^{\frac{3}{2}} - \int s^{(k-1)} u + h \tfrac{1}{2} \int |\nabla u|^2 \\ & \text{among all measurable } u \colon \mathbb{R}^2 \to [0,\infty), \end{aligned}$$

see [4, 1.7. Existence Theorem]. (54), (55) and (58) yield (if the support of χ^0 is contained in $\{|x| \leq R^0\}$)

$$\begin{aligned} & \int_{\{|y| \geq R\}} s_h(T,y)\,dy \;\leq\; \tfrac{2\alpha}{R^2}\,(2T + (R^0)^2) \quad \text{for all } T \in (0,\infty) \text{ and } R < \infty, \\ & \int_h^\infty \int |s_h(t,x+e) - s_h(t,x)|^4\,dx\,dt \;\leq\; 2\alpha\,|e|^2 \quad \text{for all } e \in \mathbb{R}^2, \\ & \int_0^\infty \int |s_h(t+\tau,x) - s_h(t,x)|^3\,dx\,dt \;\leq\; 4\,\alpha\,\tau \quad \text{for all } \tau \geq 0. \end{aligned}$$

According to [2, 2.21 Theorem], this implies the desired compactness.

Now let s be s.t. we have for a subsequence

$$s_h \longrightarrow s \quad \text{in } L^1((0,T)\times\mathbb{R}^2) \quad \text{for all } T < \infty, \tag{59}$$

and let us show that s is a weak solution of (34) with initial data χ^0, i.e.

$$\begin{aligned} & s \in L^\infty((0,\infty), L^1(\mathbb{R}^2)) \quad \text{and} \quad \nabla s^2 \in L^2((0,\infty)\times\mathbb{R}^2), \\ & s \geq 0 \quad \text{a.e. on } (0,\infty)\times\mathbb{R}^2 \quad \text{and} \\ & -\int_{(0,\infty)\times\mathbb{R}^2} \{ s\,\partial_t\zeta - \nabla s^2 \cdot \nabla\zeta \} = \int_{\mathbb{R}^2} \chi^0(x)\,\zeta(0,x)\,dx \\ & \text{for all } \zeta \in C_0^\infty((-\infty,\infty)\times\mathbb{R}^2). \end{aligned}$$

Of course we have

$$s \in [0,1] \text{ a.e. on } (0,\infty)\times\mathbb{R}^2 \quad \text{and} \quad \int s(t) = \alpha \text{ for all } t \in (0,\infty),$$

thus $s \in L^\infty((0,\infty), L^1(\mathbb{R}^2))$. From (55,59) we infer that $\nabla s^2 \in L^2((0,\infty)\times\mathbb{R}^2)$ and

$$\nabla s_h^2 \stackrel{w}{\longrightarrow} \nabla s^2 \quad \text{in } L^2((0,\infty)\times\mathbb{R}^2). \tag{60}$$

Now let $\zeta \in C_0^\infty((-\infty,\infty)\times\mathbb{R}^2)$ be given; (47) translates into

$$\begin{aligned}
&\Big| -\int_0^\infty \int s_h(t)\,\tfrac{1}{h}\,(\zeta(t+h)-\zeta(t))\,dt \\
&\quad + \int_h^\infty \int \nabla s_h(t)^2 \cdot \nabla\zeta(t)\,dt \; - \; \tfrac{1}{h}\int_0^h \int \chi^0\,\zeta(t)\,dt \Big| \\
&\le \tfrac{1}{2} \sup_{(-\infty,+\infty)\times\mathbb{R}^2} |\mathrm{D}^2\zeta| \sum_{k\in N} d(s^{(k-1)}, s^{(k)})^2 \\
&\overset{(54)}{\le} \alpha \sup_{(-\infty,+\infty)\times\mathbb{R}^2} |\mathrm{D}^2\zeta|\,h\,.
\end{aligned}$$

We pass to the limit with help of (59) and (60). Owing to the uniqueness result for weak solutions of (34) (see for instance [28]), the entire sequence converges to the unique weak solution of (34) with initial data χ^0 □

Appendix: some properties of the Wasserstein metric

The Wasserstein distance has a dual representation which will be given in Lemma 1; we need some more notation

$$\begin{aligned}
\mathcal{F}_R \;:=\; \Big\{ (\phi,\psi) \;\Big|\; & \phi \text{ and } \psi \text{ are convex on } \mathbb{R}^2 \text{ with Lipschitz constant } R\,, \\
& \phi(0)=0\,,\ |\psi(0)| \le R^2 \text{ and} \\
& \phi(x)+\psi(y) \ge x\cdot y \text{ for } (x,y)\in\mathbb{R}^2\times\mathbb{R}^2 \text{ s.t. } |x|,|y|\le R \Big\}.
\end{aligned}$$

LEMMA 1. *Let $s_0, s_1 \in \tilde{K}_R$ be given.*

i) *We have the identity*

$$\begin{aligned}
&\inf\Big\{ \tfrac{1}{2}\int |x-y|^2\,p(dx\,dy) \;\Big|\; p\in P(s_0,s_1) \Big\} \\
&= \sup\Big\{ \int s_0(x)\,\tfrac{1}{2}|x|^2\,dx + \int s_1(y)\,\tfrac{1}{2}|y|^2\,dy \\
&\qquad - \int s_0(x)\,\phi(x)\,dx - \int s_1(y)\,\psi(y)\,dy \;\Big|\; (\phi,\psi)\in\mathcal{F}_R \Big\}.
\end{aligned}$$

ii) *Minimizers p_* and maximizers (ϕ_*, ψ_*) are related by*

$$\begin{aligned}\int \zeta(x,y)\,p_*(dx\,dy) &= \int s_0(x)\,\zeta(x,\nabla\phi_*(x))\,dx \\ &= \int s_1(y)\,\zeta(\nabla\psi_*(y),y)\,dy \\ &\qquad \textit{for all } \zeta \in C_0^0(\mathbb{R}^2\times\mathbb{R}^2)\,.\end{aligned}$$

Proof of Lemma 1

As we have adapted this result of Brenier for our purpose, we give a sketch of Brenier's proof [7, Proposition 3.1] with the simplification by Gangbo [15, Lemma 2.4]. Observe that the elements of $P(s_0, s_1)$ have mass bounded by α. Thus $P(s_0, s_1)$ is compact with respect to the weak–$*$–convergence and hence there always exists a minimizer p_*. According to Arzela–Ascoli, $\mathcal{F}_R$ is compact with respect to uniform convergence on bounded domains; hence there always exists a maximizer (ϕ_*, ψ_*).

Let (ϕ_*, ψ_*) be such a maximizer of

$$\begin{aligned}F(\phi,\psi) \;:=\; & \int s_0(x)\,\tfrac{1}{2}|x|^2\,dx + \int s_1(y)\,\tfrac{1}{2}|y|^2\,dy \\ & - \int s_0(x)\,\phi(x)\,dx - \int s_1(y)\,\psi(y)\,dy\end{aligned}$$

on $\mathcal{F}_R$. Let us argue that (ϕ_*, ψ_*) is actually maximizer of F on the larger class

$$\begin{aligned}\mathcal{F}(s_0,s_1) \;:=\; \Big\{ (\phi,\psi) \,\Big|\, & \phi \text{ and } \psi \text{ are lower semicontinuous and} \\ & \phi(x)+\psi(y) \ge x\cdot y \text{ for all } (x,y)\in\Omega_0\times\Omega_1 \Big\},\end{aligned}$$

where $\{\Omega_i\}_{i\in\{0,1\}}$ is given by

$$\Omega_i \;:=\; \Big\{ z\in\mathbb{R}^2 \,\Big|\, z \text{ is Lebesgue point of } s_i \text{ with } s_i(z)>0 \Big\} \;\subset\; \{|z|\le R\}\,.$$

Indeed, observe that for $(\tilde\phi,\tilde\psi)\in\mathcal{F}(s_0,s_1)$,

$$\begin{aligned}\phi(x) &:= \sup\{\, x\cdot y - \tilde\psi(y) + c \,|\, y\in\Omega_1 \,\}, \\ \psi(y) &:= \sup\{\, y\cdot x - \phi(x) \,|\, |x|\le R \,\},\end{aligned}$$

where $c := \inf_{\Omega_1} \tilde{\psi}$, defines a $(\phi, \psi) \in \mathcal{F}_R$ with

$$\phi \le \tilde{\phi} + c \text{ on } \Omega_0 \quad \text{and} \quad \psi \le \tilde{\psi} - c \text{ on } \Omega_1 .$$

Instead of working with (ϕ_*, ψ_*), it will be more convenient to deal with $(\bar{\phi}_*, \bar{\psi}_*) \in \mathcal{F}(s_0, s_1)$ defined by

$$\begin{aligned} \bar{\phi}_*(x) &:= \sup\{\, x{\cdot}y - \psi_*(y) \,|\, y \in \Omega_1 \,\}, \\ \bar{\psi}_*(y) &:= \sup\{\, y{\cdot}x - \bar{\phi}_*(x) \,|\, x \in I\!R^2 \,\}. \end{aligned} \tag{61}$$

Observe that now

$$\bar{\phi}_* \text{ and } \bar{\psi}_* \text{ are Legendre transforms of each other} \tag{62}$$

and as above

$$\begin{aligned} &\bar{\phi}_* \text{ is Lipschitz continuous on } I\!R^2 \text{ with constant } R \\ &\bar{\phi}_* \le \phi_* \text{ on } \Omega_0 \quad \text{and} \quad \bar{\psi}_* \le \psi_* \text{ on } \Omega_1 . \end{aligned} \tag{63}$$

Thus $(\bar{\phi}_*, \bar{\psi}_*)$ is also a maximizer of F on $\mathcal{F}(s_0, s_1)$ and we have

$$\bar{\phi}_* = \phi_* \text{ on } \Omega_0 \quad \text{and} \quad \bar{\psi}_* = \psi_* \text{ on } \Omega_1 . \tag{64}$$

Referring back to (61), we infer that

$$\bar{\phi}_*(x) := \sup\{\, x{\cdot}y - \bar{\psi}_*(y) \,|\, y \in \Omega_1 \,\}. \tag{65}$$

Let us now consider the first variation: For given $\zeta \in C_0^0(I\!R^2)$ and $t \in I\!R$ we compare $(\bar{\phi}_*, \bar{\psi}_*)$ to $(\phi_t, \psi_t) \in \mathcal{F}(s_0, s_1)$ given by

$$\begin{aligned} \psi_t(y) &= \bar{\psi}_*(y) + t\,\zeta(y), \\ \phi_t(x) &= \sup\left\{\, x{\cdot}y - \psi_t(y) \,\middle|\, y \in \Omega_1 \,\right\}. \end{aligned} \tag{66}$$

We deduce from (65) and (66)

$$\left| \tfrac{1}{t} \left(\phi_t(x) - \bar{\phi}_*(x) \right) \right| \le \sup |\zeta| \quad \text{for all } x \in I\!R^2 . \tag{67}$$

It is well known that, because of (63), $\bar{\phi}_*$ is differentiable for a.e. $x \in \mathbb{R}^2$ (see for instance [27, 5.2 Theorem]). Let us show that for those x

$$\limsup_{t\downarrow 0} \tfrac{1}{t}\,(\phi_t(x) - \bar{\phi}_*(x)) \;\leq\; -\zeta(\nabla\bar{\phi}_*(x))\,. \tag{68}$$

Indeed, let $\{t_N\}_{N\uparrow\infty} \subset (0,\infty)$ be an arbitrary sequence converging to zero. According to (66), there exists $\{y_N\}_{N\uparrow\infty} \subset \Omega_1$ s.t.

$$\phi_{t_N}(x) \;\leq\; x{\cdot}y_N - \psi_{t_N}(y_N) + t_N^2\,. \tag{69}$$

Because Ω_1 is bounded, there exists a $y \in \mathbb{R}^2$ s.t. we have for a subsequence $y_N \to y$. From (69) and (67) and the lower semicontinuity of $\bar{\psi}_*$ we infer

$$\bar{\phi}_*(x) + \bar{\psi}_*(y) \;\leq\; x{\cdot}y\,,$$

which according to (62) yields $y \in \partial\bar{\phi}_*(x)$ (∂ denoting the subgradient of a convex function) and thus $y = \nabla\bar{\phi}_*(x)$. Hence we have shown

$$y_N \;\longrightarrow\; \nabla\bar{\phi}_*(x)\,.$$

But on the other hand we have

$$\tfrac{1}{t_N}\,(\phi_{t_N}(x) - \bar{\phi}_*(x)) \overset{(65)}{\leq} \tfrac{1}{t_N}\,(\phi_{t_N}(x) - x{\cdot}y_N + \bar{\psi}_*(y_N)) \overset{(69)}{\leq} -\zeta(y_N) + t_N\,,$$

which proves (68). With help of (67) and (68) we obtain from the first variation

$$\int s_1(y)\,\zeta(y)\,dy \;\geq\; \int s_0(x)\,\zeta(\nabla\bar{\phi}_*(x))\,dx\,.$$

Replacing ζ with $-\zeta$ we end up with

$$\int s_1(y)\,\zeta(y)\,dy \;=\; \int s_0(x)\,\zeta(\nabla\bar{\phi}_*(x))\,dx \quad \text{for all } \zeta \in C_0^0(\mathbb{R}^2)\,. \tag{70}$$

Let $p \in P(s_0,s_1)$ and $(\phi,\psi) \in \mathcal{F}(s_0,s_1)$ be given. Because of the relation

$$\tfrac{1}{2}\,|x-y|^2 \;=\; \tfrac{1}{2}\,|x|^2 - x{\cdot}y + \tfrac{1}{2}\,|y|^2 \;\geq\; \tfrac{1}{2}\,|x|^2 - \phi(x) + \tfrac{1}{2}\,|y|^2 - \psi(y)$$
for all $(x,y) \in \Omega_0{\times}\Omega_1$

we always have

$$\tfrac{1}{2}\int |x-y|^2\, p(dx\,dy) \;\geq\; F(\phi,\psi)\,,$$

which establishes the "$\geq$"–part in Lemma 1 i). Thanks to (62) we have the identity

$$\bar{\phi}_*(x) + \bar{\psi}_*(\nabla\bar{\phi}_*(x)) \;=\; x\cdot\nabla\bar{\phi}_*(x) \quad \text{for a.e. } x\in I\!R^2\,. \tag{71}$$

According to (70),

$$\int \zeta(x,y)\,\bar{p}_*(dx,dy) \;=\; \int s_0(x)\,\zeta(x,\nabla\bar{\phi}_*(x))\,dx \quad \text{for all } \zeta\in C_0^0(I\!R^2\times I\!R^2)$$

defines a $\bar{p}_*\in P(s_0,s_1)$ which satisfies

$$\tfrac{1}{2}\int |x-y|^2\,\bar{p}_*(dx\,dy) \;\overset{(70,71)}{=}\; F(\bar{\phi}_*,\bar{\psi}_*)\,.$$

This proves the "$\leq$"–part in Lemma 1 i).

Thanks to (62) we have for a.e. $x\in I\!R^2$ and all $y\in I\!R^2$

$$\begin{array}{l}\bar{\psi}_*(y) - \bar{\psi}_*(\nabla\bar{\phi}_*(x)) - x\cdot(y-\nabla\bar{\phi}_*(x)) \;\geq\; 0 \\ \text{with equality if and only if } y=\nabla\bar{\phi}_*(x)\,.\end{array} \tag{72}$$

Because the minimizer p_* has marginals which have Lebesgue densities, (72) also holds for p_*–a.e. $(x,y)\in I\!R^2\times I\!R^2$. Together with the identity

$$\begin{aligned} 0 \;&\geq\; \tfrac{1}{2}\int |x-y|^2\,p_*(dx\,dy) \;-\; \tfrac{1}{2}\int |x-y|^2\,\bar{p}_*(dx\,dy) \\ &\overset{(70)}{=}\; \int \left\{\bar{\psi}_*(y) - \bar{\psi}_*(\nabla\bar{\phi}_*(x)) - x\cdot(y-\nabla\bar{\phi}_*(x))\right\} p_*(dx\,dy)\,, \end{aligned}$$

this yields

$$y \;=\; \nabla\bar{\phi}_*(x) \quad \text{for } p_*\text{–a.e. } (x,y)\in I\!R^2\times I\!R^2\,,$$

and thus

$$\int \zeta(x,y)\,p_*(dx\,dy) \;=\; \int s_0(x)\,\zeta(x,\nabla\bar{\phi}_*(x))\,dx$$
$$\text{for all } \zeta\in C_0^0(I\!R^2\times I\!R^2)\,.$$

But according to (63) and (64), $\nabla\bar{\phi}_*$ and $\nabla\phi_*$ coincide a.e. on $\{s_0>0\}$ □

LEMMA 2.

i) *d is a distance function on the set $\tilde{K}_R$, i.e. we have for all $s_0, s_1, s_2 \in \tilde{K}_R$*

$$\begin{aligned} d(s_0,s_2) &\le d(s_0,s_1) + d(s_1,s_2)\,, \qquad (73)\\ d(s_0,s_1) &= d(s_1,s_0)\,,\\ d(s_0,s_1) &= 0 \quad \textit{if and only if} \quad s_0 = s_1\,. \end{aligned}$$

ii) *Let us endow $\tilde{K}_R$ with the linear structure and the weak-$*$-topology of $C^0_0(\mathbb{R}^2)^*$. Then d is compatible with this structure in the following sense: For every $s \in \tilde{K}_R$ and $\{s_N\}_{N\uparrow\infty} \subset \tilde{K}_R$ we have*

$$\begin{aligned} &d(s,\cdot)^2 \quad \textit{is strictly convex on } \tilde{K}_R\,,\\ &d(s,s_N)^2 \to 0 \quad \textit{if and only if} \quad s_N \overset{w*}{\rightharpoonup} s\,. \end{aligned}$$

Proof of Lemma 2

Part i): The only property that is not obvious from the definition is (73). Let $p_{01} \in P(s_0,s_1)$ and $p_{12} \in P(s_1,s_2)$ be given. It is well–known that there exist parametrized probability measures $\{\mu^0_y\}_{y\in\mathbb{R}^2}$ and $\{\mu^2_y\}_{y\in\mathbb{R}^2}$ on $\mathbb{R}^2$, weakly measurable in y, s.t.

$$\begin{aligned} \int \zeta(x,y)\,p_{01}(dx\,dy) &= \int s_1(y) \int \zeta(x,y)\,\mu^0_y(dx)\,dy\\ \int \zeta(y,z)\,p_{12}(dy\,dz) &= \int s_1(y) \int \zeta(y,z)\,\mu^2_y(dz)\,dy\\ &\qquad \text{for all } \zeta \in C^0_0(\mathbb{R}^2\times\mathbb{R}^2)\,, \end{aligned}$$

see for instance [14, Theorem 10]. Then

$$\begin{aligned} \int \zeta(x,z)\,p_{02}(dx\,dz) &= \int s_1(y) \int\int \zeta(x,z)\,\mu^0_y(dx)\,\mu^2_y(dz)\,dy\\ &\qquad \text{for all } \zeta \in C^0_0(\mathbb{R}^2\times\mathbb{R}^2) \end{aligned}$$

defines a $p_{02} \in P(s_0,s_2)$ which satisfies

$$\Big(\int |x-z|^2\,p_{02}(dx\,dz)\Big)^{\frac12}$$

$$
\begin{aligned}
&\le \left(\int s_1(y) \int\int |x-y|^2 \, \mu_y^0(dx) \, \mu_y^2(dz) \, dy\right)^{\frac{1}{2}} \\
&+ \left(\int s_1(y) \int\int |y-z|^2 \, \mu_y^0(dx) \, \mu_y^2(dz) \, dy\right)^{\frac{1}{2}} \\
&= \left(\int |x-y|^2 p_{01}(dx\,dy)\right)^{\frac{1}{2}} + \left(\int |y-z|^2 p_{12}(dy\,dz)\right)^{\frac{1}{2}} .
\end{aligned}
$$

Part ii): It follows easily from the definition that

$$
\begin{aligned}
&d(s,\cdot)^2 \quad \text{is convex on } \tilde{K}_R , \\
&d(s,s_N)^2 \to 0 \quad \Longrightarrow \quad s_N \overset{w*}{\rightharpoonup} s .
\end{aligned}
$$

For the strict convexity and the "if–part", we have to use the dual representation.

Let us start with strict convexity. For given $s_0, s_1 \in \tilde{K}_R$ and $t \in (0,1)$ we have to show

$$
d(s, t\,s_1 + (1\text{-}t)\,s_0)^2 \;=\; t\,d(s,s_1)^2 + (1\text{-}t)\,d(s,s_0)^2 \quad \Longrightarrow \quad s_1 = s_0 . \quad (74)
$$

We set for convenience $s_t := t\,s_1 + (1\text{-}t)\,s_0$. Let $p_\tau \in P(s,s_\tau)$, $\tau \in \{0,1\}$, be optimal in the definition of (28), i.e.

$$
d(s,s_\tau)^2 \;=\; \int |x-y|^2 p_\tau(dx\,dy) .
$$

Then $p_t := t\,p_1 + (1\text{-}t)\,p_0$ defines a $p_t \in P(s,s_t)$ with

$$
\int |x-y|^2 p_t(dx\,dy) \;=\; t\,d(s,s_1)^2 \;+\; (1\text{-}t)\,d(s,s_0)^2 ,
$$

which according to the l.h.s. of (74) implies that p_t is optimal in (28). Due to Lemma 1, for $\tau \in \{0,t,1\}$, there exists $(\phi_\tau, \psi_\tau) \in \mathcal{F}_R$ s.t.

$$
\int \zeta(x,y)\, p_\tau(dx\,dy) \;=\; \int s(x)\, \zeta(x, \nabla\phi_\tau(x))\, dx \quad \text{for all } \zeta \in C_0^0(I\!R^2 \times I\!R^2) .
$$

Hence we have

$$
\int s(x)\,\zeta(x,\nabla\phi_t(x))\,dx \;=\; \int s(x)\,\{t\,\zeta(x,\nabla\phi_1(x)) + (1\text{-}t)\,\zeta(x,\nabla\phi_0(x))\}\,dx
$$
for all $\zeta \in C_0^0(I\!R^2 \times I\!R^2)$.

In this case we infer that

$$\begin{aligned} \nabla\phi_t(x) &= t\,\nabla\phi_1(x) + (1-t)\,\nabla\phi_0(x) \quad \text{and} \\ |\nabla\phi_t(x)|^2 &= t\,|\nabla\phi_1(x)|^2 + (1-t)\,|\nabla\phi_0(x)|^2 \quad \text{for a.e. } x \in \{s>0\}, \end{aligned}$$

which yields $\nabla\phi_1 = \nabla\phi_0$ a.e. on $\{s>0\}$ and hence $s_1 = s_0$ a.e. on $\mathbb{R}^2$.

Now we prove the continuity property. For fixed $N \in \mathbb{N}$ let $(\phi_N, \psi_N) \in \mathcal{F}_R$ be optimal in the dual representation of $d(s, s_N)^2$. Because $\mathcal{F}_R$ is compact under the topology of locally uniform convergence, there exists $(\phi, \psi) \in \mathcal{F}_R$ s.t. for a subsequence

$$(\phi_N, \psi_N) \longrightarrow (\phi, \psi) \quad \text{uniformly on bounded subsets of } \mathbb{R}^2.$$

We thus obtain

$$\begin{aligned} \tfrac{1}{2}\,d(s, s_N)^2 &= \int s(x)\,\tfrac{1}{2}|x|^2\,dx + \int s_N(y)\,\tfrac{1}{2}|y|^2\,dy \\ &\quad - \int s(x)\,\phi_N(x)\,dx - \int s_N(y)\,\psi_N(y)\,dy \\ &\longrightarrow \int s(x)\,\Big\{|x|^2 - \phi(x) - \psi(x)\Big\}\,dx \;\leq\; 0 \;\square \end{aligned}$$

LEMMA 3. *Let $s \in K$ and $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ be convex s.t.*

$$\int s(y)\,\zeta(\nabla\psi(y))\,dy \leq \int \zeta(x)\,dx \quad \text{for all nonnegative } \zeta \in C_0^0(\mathbb{R}^2). \tag{75}$$

Then we have for all $t \in [0,1]$

$$\int s(y)\,\zeta((1-t)\,y + t\,\nabla\psi(y))\,dy \leq \int \zeta(x)\,dx \quad \text{for all nonnegative } \zeta \in C_0^0(\mathbb{R}^2). \tag{76}$$

Proof of Lemma 3
If ψ is smooth, the result follows immediately from the transformation formula and the fact that

$$\mathrm{Sym}_+ \ni M \mapsto (\det M)^{\frac{1}{2}} \quad \text{is concave}, \tag{77}$$

where Sym_+ is the set of positive semidefinite 2×2–matrices (we learned of this property from [8]).

In the general case, we observe that due to Aleksandrov,

$$\psi \text{ is twice differentiable almost everywhere on } \mathbb{R}^2, \tag{78}$$

see for instance [11, Theorem A.2.]. If y is such a point of differentiability of ψ, we denote by $M(y) \in \mathrm{Sym}_+$ the matrix of second derivatives. Observe that we have

$$\begin{aligned} &\int \psi\, e{\cdot}\mathrm{D}^2\zeta e \;\geq\; \int e{\cdot}Me\,\zeta \\ &\text{for all nonnegative } \zeta \in C_0^\infty(\mathbb{R}^2) \text{ and } e \in \mathbb{R}^2. \end{aligned} \tag{79}$$

Let us now deduce from (75) that

$$\det M \;\geq\; s \quad \text{a.e. on } \mathbb{R}^2. \tag{80}$$

Indeed, let $y_0 \in \mathbb{R}^2$ be s.t. ψ is twice differentiable in y_0 and that s has a Lebesgue point in y_0. For any $\epsilon > 0$ and $\delta > 0$, there exists $R > 0$ s.t. for all $r \in [0, R]$ and a.e. $y \in \mathbb{R}^2$

$$|(M(y_0) + \epsilon\,\mathrm{id})\,(y - y_0)| \;<\; r \;\Longrightarrow\; |\nabla\psi(y) - \nabla\psi(y_0)| \;<\; (1+\delta)\,r.$$

Thus we obtain from (75)

$$\int_{\{|(M(y_0)+\epsilon\,\mathrm{id})\,(y-y_0)|<r\}} s(y)\,dy \;\leq\; \int_{\{|x-\nabla\psi(y_0)|<(1+2\delta)\,r\}} 1\,dx,$$

which after affine transformation reads

$$\begin{aligned} \det\,(M(y_0) + \epsilon\,\mathrm{id})^{-1} \int_{\{|y|<r\}} s\left(y_0 + (M(y_0) + \epsilon\,\mathrm{id})^{-1}y\right)\,dy \\ \leq\; \int_{\{|x|<(1+2\delta)\,r\}} dx. \end{aligned}$$

In the limit $r \downarrow 0$ we obtain

$$\det\,(M(y_0) + \epsilon\,\mathrm{id})^{-1}\, s(y_0) \;\leq\; 1 + 2\,\delta.$$

Fix a nonnegative $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\int \phi = 1$ and introduce for $\epsilon > 0$ the mollification operator J_ϵ by

$$(J_\epsilon\zeta)(x) := \int \tfrac{1}{\epsilon^2}\,\phi(\tfrac{x-\tilde{x}}{\epsilon})\,\zeta(\tilde{x})\,d\tilde{x}\,.$$

First observe that

$$(\det \mathrm{D}^2 J_\epsilon\psi)^{\frac{1}{2}} \overset{(79)}{\geq} (\det J_\epsilon M)^{\frac{1}{2}} \overset{(77)}{\geq} J_\epsilon(\det M)^{\frac{1}{2}} \overset{(80)}{\geq} J_\epsilon s^{\frac{1}{2}}$$

and thus

$$\begin{aligned}(\det \mathrm{D}\,((1-t)\,\mathrm{id} + t\,J_\epsilon\nabla\psi))^{\frac{1}{2}} &\overset{(77)}{\geq} (1-t) + t\,(\det \mathrm{D}^2 J_\epsilon\psi)^{\frac{1}{2}} \\ &\geq (1-t) + t\,J_\epsilon s^{\frac{1}{2}}\,.\end{aligned}$$

Hence we obtain with help of the transformation formula for $t \in (0,1)$ and nonnegative $\zeta \in C_0^0(\mathbb{R}^2)$

$$\int (J_\epsilon s^{\frac{1}{2}}(y))^2\,\zeta((1-t)\,y + t\,J_\epsilon\nabla\psi(y))\,dy \leq \int \zeta(x)\,dx\,,$$

which turns into (76) for $\epsilon \downarrow 0$ □

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