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The D-Rank is Equal to the R-Rank

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ABSTRACT. Complete and self-contained proofs to theorems of Shelah as well as some new results are presented:

Theorem 0.1. *Let Δ be a set of formulas closed under Boolean operations and let p be a finite set of formulas. If p is Δ-stable then for $\lambda \geq (2^{|T|})^{++}$ we have that $R[p, \Delta, \lambda] = D[p, \Delta, \lambda]$.*

Theorem 0.2. $R[p, \Delta, \lambda] = R[p, \Delta, \infty]$ *for $\lambda \geq (2^{|T|})^{++}$.*
When $R[p, \Delta, \infty] < \infty$ then $R[p, \Delta, \infty] < |T|^{+}$.

Theorem 0.1 is a generalization of Th II 3.11 from [Sha]. The proofs use the concepts of *p is Δ-stable* and *p is Δ-superstable*, which appear only implicitly in Shelah's work. We use the above functions to characterize *p is Δ-superstable* (when $p = \{x = x\}$ and $\Delta = L$, p is Δ-superstable iff T is superstable).

The proofs presented here are simpler than Shelah's original presentation. We still use Shelah's ideas but in a different form.

Let $S1$ be the "geometric" rank from the ACFA paper of Chatzidakis and Hrushovski.

Theorem 0.3. *Let p be a finite type. If p is stable then*
$$S1[p] = D[p, L, \infty] = R[p, L, \infty].$$

INTRODUCTION

Rank functions are important tools in studying properties of theories. In 1965, Michael Morley published a proof to the categoricity theorem for countable similarity types. That answered to a particular case of Łǒs's conjecture [Lo] about categoricity of first-order theories, the most important conjecture in model theory of that time. To prove the categoricity theorem, Michael Morley [Mo] introduced a rank function called *Morley's rank* often denoted by $RM(p)$, where p is a set of formulas not necessarily a complete type $(RM(p) := R[p, L, \aleph_0]$ see Definition 1.1). This prompted Shelah, Baldwin, and Lascar to introduce several additional rank functions.

In 1970 Saharon Shelah (see [Sh31]) solved Łǒs's conjecture in full generality. For his solution, Shelah identified an important class of first-order theories, the superstable theories, and developed several tools. Namely, in [Sh 10] he introduced the rank $Deg[p]$ $(Deg[p] := Deg[p, L, \infty]$ see Definition 2.2), suitable to deal with superstable theories and a rank suitable to deal with stable theories, the function $R[p, \Delta, \aleph_0]$, both for p not necessarily complete and Δ a finite set of formulas.

John Baldwin proved in his Ph.D. thesis that an $\aleph_1$-categorical theory has finite Morely's rank (see [Ba1]). The main device was a function named later $D[\mathbf{x} = \mathbf{x}, L, \infty]$ by Shelah. Among other results Baldwin showed that

$D[\mathbf{x} = \mathbf{x}, L, \infty] = $ Morley's rank(p) for $\aleph_1$-categorical theories (this was a predecessor of Theorem 0.1).

Lascar in [La2] introduced *Lascar rank* which is another function suitable for superstable theories $U(p)$ is defined only when p is a complete type (the same function is denoted by $L(p)$ in [Sha]). Lascar's rank is different than the ranks we discuss here (even for differentially closed fields, see [HS]).

In 1974, Baldwin and Blass introduced the basic axiomatic properties of the rank functions in [BaBl]; the connected property is from Lascar [La2]. Later (in his book [Sha] from 1978), Shelah presented two other families of rank functions $D[p, \Delta, \lambda]$ and $R[p, \Delta, \lambda]$. The first is a generalization of $Deg[p]$ (since $Deg[p] = D[p, L, \infty]$) and the latter is a generalization of Morley's rank (since $RM[p] = R[p, L, \aleph_0]$). The functions $D[p, L, \infty]$ and $R[p, L, \infty]$ (often denoted also by ∞-rank$[p]$) are well understood and their basic properties appear in several monographs (e.g. John Baldwin's book [Ba2], Steve Buechler's book [Bu] and Anand Pillay's [Pi]). However, some deep results in Section 3 of Chapter II [Sha] relating the rank functions $R[p, \Delta, \lambda]$ and $D[p, \Delta, \lambda]$ were not covered by any of the subsequent expositions known to us. (E. g. *if T is superstable, then* $R[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$.)

In the early nineties Ehud Hrushovski [Hr] introduced the function $S1$ he used to study simple unstable groups. Hrushovski's $s1$ was defined only for finite values in the same unpublished paper he introduced and proved as well as the independence theorem for theories with finite $S1$ rank f (this theorem was later generalize renamed by Kim and Pillay [KP] as the amalgamation theorem for all simple theories). Some of Hrushovski's early results appeared in a restricted form in his papers with Pillay [HP1] and later with Chatzidakis [CH].

The purpose of this article is to present the more difficult contents of Section 3 of Chapter II [Sha] in a complete, simplified, and more conceptual way. (As we note below, Shelah's original proofs are not complete and contain several errors.) To our surprise after writing up proofs to Shelah's theorems we realized that the main combinatorial dividing line is an implicit use of the $S1$ function that was introduced explicitly only more than 20 years later in a geometric context by Ehud Hrushovski.

In our paper we show that Hrushovski's $S1$-rank is equal to $R[p, L, \infty]$ and we use this to give a more conceptual proof of Theorems 0.1 and 0.2 than Shelah did. As a byproduct of the hardest argument in this paper (Theorem 3.1), it is shown that if $R[p, L, \infty] = D[p, L, \infty]$, then also $D[p, L, \infty] = S1[p, L]$. This was shown to be true by Kim and Pillay [KP] under the assumption that T is simple when either $D[p, L, \infty]$ or $S1[p]$ are finite. Here we get equality also for infinite valued rank, but we assume local stability.

We introduce a rank function $Deg[p, L, \lambda]$ (which generalizes Shelah's function from [Sh 10]). While for some trivial cases $D[p, \Delta, \lambda] \neq Deg[p, \Delta, \lambda]$, it is not

difficult to see that for interesting cases either

$$D[p, \Delta, \lambda] = Deg[p, \Delta, \lambda]$$

or at least

$$D[p, \varphi, \lambda] < \omega \text{ for all } \varphi \Leftrightarrow Deg[p, \varphi, \lambda] < \omega \text{ for all } \varphi.$$

This helps us to lower the complexity of Shelah's original treatment. Also from Shelah's proofs we extract relativized notions of Δ-stability and Δ-superstability that clarify the arguments significantly (compare with Harnik and Harrington [HH], Bouscaren [Bo] and with Grossberg and Lessmann [GrLe]).

Lastly a function $S2$ is introduced, it relates to the $S1$-rank in a similar way to the relationship between $Deg[p, \Delta, \lambda]$ and $R[p, \Delta, \lambda]$. It is shown that when $S1[p] = R[p, L, \infty]$ then also $S1[p] = S2[p]$.

To the expert: You may wonder as of what are the differences between this paper and other expositions. More precisely, did not Baldwin, Buechler or Pillay in thier books prove the same theorems? The answer is no. Baldwin [Ba2] deals with superstable theories via what he calls *the continuous rank*. Pillay [Pi] is using the same rank function. (We denote this function by $Deg[p, L, (2^{|T|})^{++}]$.) Buechler in [Bu] is using a different rank to study superstable theories (the infinity rank). Here, in addition to showing that the functions are equal (together with some additional information on improved bounds) we deal with the local case. We do localize not only from the theory to a realizations of a single formula (and sometime a type) but we have another degree of localization. Namely, we allow to replace the entire set of formulas of T with a relatively small set Δ. The more general results require significantly more delicate treatment and more sophisticated combinatorial set theory. For example, we present the necessary machinery to apply Shelah's non-structure technology to show that for an unsuperstable theory T the class $\mathcal{K} := PC(T_1, T)$ has 2^λ pairwise unembeddable models of cardinality λ (for any $T_1 \supseteq T$ and any λ regular greater than $|T_1|$). The devices presented in Baldwin's book are sufficient to get the same conclusion under additional assumptions (he requires the class $\mathcal{K}$ to be an elementary class $\text{Mod}(T)$ where T is an unsuperstable *and* stable first-order theory).

Another natural question to ask is: Did not Shelah (in [Sha]) prove all these theorems? The answer is yes and no. All the theorems with the exceptions of the results connecting to the $S1$ rank are stated in Shelah's book. However there are several inaccuracies and some of the proofs in our opinion are incomplete. Here are some examples:

(1) Lemma II.3.5(1) for the case $X = D$ is false (when ΔL), this is one of the reasons we introduced $Deg[p, \Delta, \lambda]$ which is an interpolant to the continuous rank and $D[p, \Delta, \lambda]$ (references are to [Sha]).

(2) In the center of page 51 Shelah states Lemma B.1 without proof, we prove it in the second appendix.

(3) Shelah's argument that for superstable theories the infinity rank is bounded by $|T|^+$ (see Theorem II.3.13) is incomplete. In the center of page 53 Shelah's presents an outline of a proof, he writes "We can apply this construction to $p \cup r_i$, etc. Hence we can define ...". Following this strategy will require writing about 10 pages of complicated arguments that he does not supply us with. Instead following his outline we present an alternative approach that depends on the fact that we have shown already that $Deg[\cdots] = R[\cdots]$ and we use the tree characterization (and the normalization) lemmas we have obtained for $Deg[\cdots]$.

The structure of this paper:

Section 1: Contains the definition of $R[p, \Delta, \lambda]$ and basic properties. For the sake of completness we have given concise proofs of the connections of $R[\cdots]$ with the local order-property and local stability.

Section 2: The ranks $D[p, \Delta, \lambda]$, $Deg[p, \Delta, \lambda]$ and thier basic properties are introduced as well as the tree characterization-lemma for $Deg[p, \Delta, \lambda]$, the weak tree property and the normalization lemma.

Section 3: Is dedicated to the main theorem (Theorem 3.1) where the equality

$$S1[p, \Delta] = Deg[p, \Delta, \mu^+] = D[p, \Delta, \mu^+] = R[p, \Delta, \mu^+].$$

is derived from the appropriate assumptions (see also Corollary 3.12).

Section 4: Here we characterize local superstability and derive the strong bound on the ∞-rank, namely: In Theorem 4.2 (1) $\implies$ (5) it is shown that if T is superstable then $R[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$.

Section 5: It is shown that the continuous rank is equal to the infinity rank; Namely $R[p, \Delta, (2^{|T|})^{++}] = R[p, \Delta, \infty]$ and the last step in the proof of Theorem 3.1 is carried out in Claims 5.6 and 5.7.

Section 6: The function $S2$ which is an easy generalization of $S1$ is introduced. Its relationship to $S1$ is analog to the relationship between R and Deg.

Appendix A: Contains a proof of Claim 3.10.

Appendix B: Contains a proof to the end-homogeneity lemma we use in the proof of Claim 5.6.

An effort was made to make this presentation as self contained as possible. The notation is standard. Throughout the paper, T denotes a complete first-order theory without finite models. The language of T is denoted $L(T)$. The monster model is denoted by $\mathfrak{C}$.

We thank John Baldwin and Olivier Lessmann for reading a preliminary version of this paper and offering us thier comments and to Ehud Hrushovski for clarifying the history of the $S1$-rank.

1. Local stability and the R-rank

Definition 1.1. Let p be a set of formulas in $\mathbf{x}$, $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in Fml(L(T))\}$ and λ a cardinality (can be finite) or ∞.

(1) $R[p, \Delta, \lambda] \geq \alpha$ is defined by induction on α:

(a) $R[p, \Delta, \lambda] \geq 0$, if p is consistent;

(b) for α limit, $R[p, \Delta, \lambda] \geq \alpha$ if $R[p, \Delta, \lambda] \geq \beta$ for every $\beta < \alpha$

(c) $R[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$ and every $\mu < \lambda$ there are $\{q_i \mid i \leq \mu\}$ *explicitly contradictory* Δ-*types* such that $R[q \cup q_i, \Delta, \lambda] \geq \alpha$ for every $i \leq \mu$. (When $\lambda = \infty$ we interpret it as no restriction on μ).

(2) For an ordinal α denote by $R[p, \Delta, \lambda] = \alpha$ the statement

$$R[p, \Delta, \lambda] \geq \alpha \quad \text{and} \quad R[p, \Delta, \lambda] \not\geq \alpha + 1.$$

(3) We write $R[p, \Delta, \lambda] = \infty$ if $R[p, \Delta, \lambda] \geq \alpha$ for every ordinal α.

Remarks 1.2. (1) $R[p, L, \aleph_0]$ is *Morley's rank* often denoted by $RM[p]$.

(2) $R[p, L, \infty]$ is often called *infinity rank*, Buechler (in [Bu]) denotes it by $R^\infty[p]$.

(3) $R[p, \Delta, \aleph_0]$ is denoted in Pillay ([Pi]) by $R^\Delta_{\aleph_0}[p]$.

We mention some of the basic properties of the *R*-rank:

Lemma 1.3 (Invariance). *For any set of formulas p and $f \in \text{Aut}(\mathfrak{C})$,* $R[p, \Delta, \lambda] = R[f(p), \Delta, \lambda]$.

Proof. Immediate.   ⊣

Lemma 1.4 (Monotonicity).   (1) $p \vdash q$ *implies* $R[p, \Delta, \lambda] \leq R[q, \Delta, \lambda]$,

(2) $\mu < \lambda$ *implies* $R[p, \Delta, \lambda] \leq R[p, \Delta, \mu]$ *and*

(3) $\Delta_1 \subseteq \Delta_2$ *implies that* $R[p, \Delta_1, \lambda] \leq R[p, \Delta_2, \lambda]$

Proof. (1) By induction on α show that $R[p, \Delta, \lambda] \geq \alpha \Rightarrow R[q, \Delta, \lambda] \geq \alpha$. (2), (3) Immediate.   ⊣

Lemma 1.5 (Finite character). *Given* $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in Fml(L(T))\}$, *a cardinal* λ, *and a set of formulas p, there is a finite subset $q \subseteq p$, such that* $R[p, \Delta, \lambda] = R[q, \Delta, \lambda]$.

Proof. By definition and Lemma 1.4(1).   ⊣

Lemma 1.6 (Ultrametric property). *For* $\lambda \geq \aleph_0$

$$R[p \cup \{ \bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda] = \text{Max}_{1 \leq l \leq n} R[p \cup \{\psi_l\}, \Delta, \lambda].$$

Proof. $\text{Max}_{1 \leq l \leq n} R[p \cup \{\psi_l\}, \Delta, \lambda] \leq R[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda]$ follows from Lemma 1.4.

By induction on α show that

$$R[p \cup \{ \bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda] \geq \alpha \Rightarrow \text{Max}_{1 \leq l \leq n} R[p \cup \{\psi_l\}, \Delta, \lambda] \geq \alpha.$$

                                                                                                    ⊣

Lemma 1.7 (Extension property). *For* $\lambda \geq \aleph_0$, *a set of formulas p and a set $A \supseteq \text{dom}(p)$, there exists a complete type $q \supseteq p$ with domain A such that* $R[p, \Delta, \lambda] = R[q, \Delta, \lambda]$.

Proof. Let $\alpha := R[p, \Delta, \lambda]$, use Lemma 1.6 and apply the compactness theorem to the set

$$p \cup \{\psi(\mathbf{x}; \mathbf{a}) \mid R[p \cup \{\neg\psi(\mathbf{x}; \mathbf{a})\}, \Delta, \lambda] < \alpha, \mathbf{a} \in A\}.$$

⊣

Theorem 1.8 (Connected). *Let p be a finite set of formulas in $\mathbf{x}$. If $R[p, \Delta, \lambda] = \alpha < \infty$ then for every $\beta < \alpha$ there exists a Δ-type q such that $R[p \cup q, \Delta, \lambda] = \beta$.*

Proof. Suppose for the sake of contradiction that there are a finite p and ordinals $\beta < \alpha$ such that $R[p, \Delta, \lambda] = \alpha < \infty$ and

$$(*) \quad R[p \cup q, \Delta, \lambda] \geq \beta \implies R[p \cup q, \Delta, \lambda] \geq \beta + 1$$
$$\text{for every } \Delta\text{-type } q.$$

Using (*) we will contradict the hypothesis that $R[p, \Delta, \lambda] < \infty$ by showing that for every Δ-type q and every $\gamma \geq \beta$

$$R[p \cup q, \Delta, \lambda] \geq \beta \text{ implies } R[p \cup q, \Delta, \lambda] \geq \gamma.$$

We proceed by induction on $\gamma \geq \beta$.

For $\gamma = \beta$ there is nothing to prove. For γ limit use the inductive hypothesis.

For $\gamma = \zeta + 1 > \beta$, let q be a Δ-type such that $R[p \cup q, \Delta, \lambda] \geq \beta$. By (*) we have that $R[p \cup q, \Delta, \lambda] \geq \beta + 1$. Using the definition of R, for every finite subset r_0 of q, for every $\mu < \lambda$, there exists $\{q_i \mid i < \mu\}$ a set of explicitly contradictory Δ-types such that

$$(**) \quad R[p \cup r_0 \cup q_i, \Delta, \lambda] \geq \beta \quad \text{for every } i < \mu.$$

By the finite character choose finite $q_i^* \subseteq q_i$ such that $R[p \cup r_0 \cup q_i^*, \Delta, \lambda] = R[p \cup r_0 \cup q_i, \Delta, \lambda]$. An application of the induction hypothesis to (**) gives us that

$$R[p \cup r_0 \cup q_i^*, \Delta, \lambda] \geq \zeta \quad \text{for every } i < \mu.$$

This, using the definition of R again, gives us that $R[p \cup q, \Delta, \lambda] \geq \zeta + 1$.   ⊣

Parts (1) and (2) of the following definition were influenced by Harnik and Harrington [HH], Bouscaren [Bo], Grossberg and Lessmann [GrLe].

Definition 1.9. Let p be a type in $\mathbf{x}$ and let Δ be a set of formulas such that for all $\varphi \in \Delta$, $\varphi = \varphi(\mathbf{x}; \mathbf{y})$.

(1) A type $q \in S_\Delta^{\ell(\mathbf{x})}(A)$ is called a (p, Δ)-*type* if $p \cup q$ is consistent. The set of all (p, Δ)-types is denoted by $S_{p,\Delta}(A)$;

(2) p is called (Δ, λ)-*stable* if for all A, $|A| \leq \lambda$, we have $|S_{p,\Delta}(A)| \leq \lambda$;

(3) p is called Δ-*stable* if there is λ such that p is (Δ, λ)-stable;

(4) p is called Δ-*superstable* if there is λ such that p is (Δ, μ)-stable for all $\mu \geq \lambda$;

(5) We say that $\varphi(\mathbf{x}; \mathbf{y})$ *has the order property* over p if there is a set $\{\mathbf{a}_n \mid n < \omega\}$ with $\ell(\mathbf{a}_n) = \ell(\mathbf{y})$ such that $p \cup \{\varphi(\mathbf{x}; \mathbf{a}_n)^{\text{if } k \leq n} \mid n < \omega\}$ is consistent for all $k < \omega$;

(6) We say that Δ *has the order property over p* if there is
$\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ that has the order property over p.

Lemma 1.10. *Let λ be an infinite cardinal and let $\varphi(\mathbf{x}; \mathbf{y})$ be a formula. If there is a set A such that $|A| \leq \lambda$ and $|S_{p,\{\varphi\}}(A)| > \lambda$, then φ has the order property over p.*

Proof. The proof is similar to that of Theorem I.2.10 of [Sha]. $\dashv$

The following is a generalization of Theorem II 3.13 and Lemma II.2.14 of [Sha], with essentially the same proof.

Theorem 1.11. *Let p be a type in $\mathbf{x}$ and let Δ be a set of formulas. The following are equivalent:*

(1) *For all $\lambda \geq \aleph_0$ such that $\lambda = \lambda^{|\Delta|}$, p is (Δ, λ)-stable;*

(2) *p is Δ-stable;*

(3) *Δ does not have the order property over p.*

Proof. (1) $\Rightarrow$ (2) is immediate from the definition of stability.

(2) $\Rightarrow$ (3) Suppose there is $\varphi \in \Delta$ that has the order property over p. Given $\lambda \geq |\Delta| + \aleph_0$, let $\mu := \mathrm{Min}\{\mu \mid 2^\mu > \lambda\}$. Using the order property and the Compactness Theorem, we obtain $\{a_\eta \mid \eta \in {}^{\mu \geq}2\}$ such that $p \cup \{\varphi(\mathbf{x}; a_\eta)^{\mathrm{if}\,\eta <_{lex} \nu} \mid \eta \in {}^{\mu \geq}2\}$ is consistent for all $\nu \in {}^{\mu \geq}2$. Let $A := \{a_\eta \mid \eta \in {}^{\mu >}2\}$; since $2^{<\mu} \leq \lambda$, $|A| \leq \lambda$. However, $|S_{p,\{\varphi\}}(A)| \geq 2^\mu > \lambda$ and since $\varphi \in \Delta$, $|S_{p,\Delta}(A)| \geq |S_{p,\{\varphi\}}(A)| > \lambda$. We get a contradiction to the condition (2).

(3) $\Rightarrow$ (1) Suppose for contradiction that (1) fails. Let $\lambda \geq \aleph_0$ be such that $\lambda = \lambda^{|\Delta|}$ and p is not (Δ, λ)-stable. Let A be a set of cardinality λ such that $|S_{p,\Delta}(A)| > \lambda$.

Let $\{\varphi_i \mid i < |\Delta|\}$ be an enumeration of Δ and let $S := \prod_{i<|\Delta|} S_{p,\{\varphi_i\}}(A)$. Clearly, the mapping

$$q \in S_{p,\Delta}(A) \mapsto \bar{q} := \langle q\varphi_0, q\varphi_1, \ldots \rangle \in S$$

is an injection. Therefore, $|S| \geq |S_{p,\Delta}(A)| > \lambda$. So, there exists a $\bar{\varphi} \in \Delta$ such that $|S_{p,\{\bar{\varphi}\}}(A)| > \lambda$. (Otherwise $|S| = |\prod_{i<|\Delta|} S_{p,\{\varphi_i\}}(A)| \leq |\prod_{i<|\Delta|} \lambda| = \lambda^{|\Delta|}$.) Finally, applying Lemma 1.10, we get the order property for $\bar{\varphi} \in \Delta$, that contradicts (3). $\dashv$

The R-rank can be used to characterize local stability of types. To see that, we will need the tree characterization of the R-rank.

Definition 1.12. Let p be a type in $\mathbf{x}$, let $\varphi(\mathbf{x}; \mathbf{y})$ be a formula, and let α be an ordinal. Denote

$$\Gamma_p(\varphi, \alpha) := \{\varphi(\mathbf{x}_\eta; \mathbf{y}_{\eta\beta})^{\eta[\beta]} \mid \beta < \alpha, \ \eta \in {}^\alpha 2\} \cup \bigcup \{p(\mathbf{x}_\eta) \mid \eta \in {}^\alpha 2\}.$$

The following is a generalization of Theorem II.2.2 of [Sha]. For the sake of completeness we include a proof.

Lemma 1.13. *Let p be a type in $\mathbf{x}$ and let $\varphi(\mathbf{x}; \mathbf{y})$ be a formula. Then, for every $n < \omega$, the following are equivalent:*

(1) $\Gamma_p(\varphi, n)$ *is consistent;*

(2) $R[p, \varphi, 2] \geq n.$

Proof. By the compactness theorem we may assume that p is finite.

If $n = 0$, the assertion is trivial. Suppose that for all finite types p, $R[p, \varphi, 2] \geq n$ is equivalent to the consistency of $\Gamma_p(\varphi, n)$. We now prove the equivalence for $n + 1$.

If $R[p, \varphi, 2] \geq n + 1$, there are two explicitly contradictory $\{\varphi\}$-types q_1 and q_2 such that $R[p \cup q_1, \varphi, 2] \geq n$ and $R[p \cup q_2, \varphi, 2] \geq n$. Since the types are explicitly contradictory, there is an $\mathbf{a}$ such that $\varphi(\mathbf{x}; \mathbf{a}) \in q_1$ and $\neg\varphi(\mathbf{x}; \mathbf{a}) \in q_2$. By monotonicity, $R[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \varphi, 2] \geq n$ and $R[p \cup \{\neg\varphi(\mathbf{x}; \mathbf{a})\}, \varphi, 2] \geq n$.

Given $\eta \in {}^{n+1}2$, construct $\mathbf{b}_\eta$ and $\{\mathbf{a}_{\eta l} \mid l < n + 1\}$ such that $\mathbf{b}_\eta \models p(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{a}_{\eta l})^{\eta[l]} \mid l < n + 1\}$.

Case 1: $\eta = \nu^\frown\langle 0 \rangle$, $\nu \in {}^n 2$. Apply the induction hypothesis to $p(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{a})\}$ to get $\mathbf{b}_\nu$ such that $\mathbf{b}_\nu \models p(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{a})\} \cup \{\varphi(\mathbf{x}; \mathbf{a}_{\nu l})^{\eta[l]} \mid l < n\}$. Then clearly $\mathbf{b}_\eta := \mathbf{b}_\nu$, $\mathbf{a}_{\eta l} = \mathbf{a}_{\nu l}$, $l < n$, and $\mathbf{a}_\eta := \mathbf{a}$ are exactly as required.

Case 2: $\eta = \nu^\frown\langle 1 \rangle$, $\nu \in {}^n 2$. Here we apply the induction hypothesis to the type $p(\mathbf{x}) \cup \{\neg\varphi(\mathbf{x}; \mathbf{a})\}$ to produce $\mathbf{b}_\eta$ and $\{\mathbf{a}_{\eta l} \mid l < n + 1\}$ as needed.

This completes the induction step. Notice that all the implications in the above proof can be reversed.   ⊣

Theorem 1.14. *Suppose p is a set of formulas in $\mathbf{x}$ and $\varphi(\mathbf{x}; \mathbf{y})$ is a formula. The following are equivalent*

(1) $R[p, \varphi, 2] < \omega,$

(2) $R[p, \varphi, 2] < \infty,$

(3) $R[p, \varphi, \aleph_0] < \infty,$

(4) $R[p, \varphi, \infty] < \infty,$

(5) *the formula φ does not have the order property over p,*

(6) *p is $\{\varphi\}$-stable.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ follow from the fact that $R[\cdot, \cdot, \lambda]$ is antitone in λ. Notice that $(5) \Leftrightarrow (6)$ is one of the statements in Theorem 1.11.

Let us show $(1) \Rightarrow (5)$. If φ has the order property over p, we obtain the consistency of $\Gamma_p(\varphi, \omega)$ by extending the order property to an appropriately chosen linear ordering on ${}^{\omega \geq} 2$ (using compactness), which, by Lemma 1.13, implies $R(p, \varphi, 2) \geq \omega$.

For $(6) \Rightarrow (1)$, if $R[p, \varphi, 2] \geq \omega$, then, by Lemma 1.13, $\Gamma_p(\varphi, \omega)$ is consistent and gives rise to many $(p, \{\varphi\})$-types, which contradicts the local stability assumption on p.

All that is left to show is $(4) \Rightarrow (1)$. By Proposition 4.1, there is a cardinality λ such that $R[p, \varphi, \infty] = R[p, \varphi, \lambda]$. If $R[p, \varphi, 2] \geq \omega$, Lemma 1.13, along with the Compactness Theorem, implies that $\Gamma_p(\varphi, \lambda)$ is consistent.

Let $\{\mathbf{a}_\eta \mid \eta \in {}^{\lambda >} 2\}$ be realizations of $\mathbf{y}_\eta$'s from the tree. Thus, for every $\eta \in {}^{\lambda >} 2$, $p_\eta := p \cup \{\varphi(\mathbf{x}; \mathbf{a}_{\eta\beta})^{\eta[\beta]} \mid \beta < \ell(\eta)\}$ is consistent. Since $R[p, \varphi, \lambda] < \infty$, $\alpha_0 := \min\{R[p_\eta, \varphi, \lambda] \mid \eta \in {}^{\lambda >} 2\}$ is a well-defined ordinal. Let $\eta_0 \in {}^{\lambda >} 2$ be such that $R[p_{\eta_0}, \varphi, \lambda] = \alpha_0$.

For $\gamma < \lambda$, consider $q_\gamma := p_{\eta_0}{}^\frown \bar{0}^\gamma{}^\frown 1$, where $\bar{0}^\gamma$ is a sequence of 0's of length γ. Obviously q_{γ_1} contradicts q_{γ_2} for $\gamma_1 \neq \gamma_2 < \lambda$ and by minimality of α_0, for every $\gamma < \lambda$, $R[q_\gamma, \varphi, \lambda] \geq \alpha_0$. This implies that $R[p_{\eta_0}, \varphi, \lambda] \geq \alpha_0 + 1$, which is a contradiction to the choice of p_{η_0}. $\dashv$

Corollary 1.15. *Suppose p and Δ are sets of formulas in* x. *If $R[p, \Delta, \infty] < \infty$, then p is Δ-stable.*

Proof. Suppose $R[p, \Delta, \infty] < \infty$. By monotonicity, $R[p, \varphi, \infty] < \infty$ for every $\varphi \in \Delta$. Theorem 1.14 implies that for all $\varphi \in \Delta$, φ does not have the order property over p. Thus Δ does not have the order property. By Theorem 1.11 we get that p is Δ-stable. $\dashv$

Remark 1.16. In fact $R[p, \Delta, \infty] < \infty$ implies that p is Δ-superstable (see (6) $\implies$ (1) in Theorem 4.2)

Thus, the R-rank can be used to characterize superstability. Namely, the following theorem holds:

Theorem 1.17. $R[\mathbf{x} = \mathbf{x}, L, \infty] < \infty$ *iff T is superstable.*

This theorem will be derived from a more general result (Theorem 4.2).

2. Deg-RANK AND LOCAL SUPERSTABILITY

The following was introduced by Shelah (in [Sha])to study superstable theories

Definition 2.1. $D[p, \Delta, \lambda] \geq 0$ if p is consistent.
$D[p, \Delta, \lambda] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $D[p, \Delta, \lambda] \geq \beta$.
$D[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$, for every $\mu < \lambda$, there are a finite $r \supseteq q$, $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$, and $\{\mathbf{a}_i : i \leq \mu\}$ such that:

(1) set $\{\varphi(\mathbf{x}; \mathbf{a}_i) : i \leq \mu\}$ is n-contradictory over r;
(2) for every $i \leq \mu$, $D[r \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, \lambda] \geq \alpha$.

In this section, we introduce the Deg-rank, which is a simplification of the D-rank, that will be used to characterize local superstability of types, to find bounds for the D-rank and eventually prove the equality of R and D under certain conditions. It is an interpolant of Shelah's ranks $D[p, \Delta, \lambda]$ (from [Sha]) and $Deg[p]$ (from [Sh 10]).

Definition 2.2. $Deg[p, \Delta, \lambda] \geq 0$ if p is consistent.
$Deg[p, \Delta, \lambda] \geq \delta$, for δ limit, if for every $\alpha < \delta$, $Deg[p, \Delta, \lambda] \geq \alpha$.
$Deg[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$, for every $\mu < \lambda$, there exist $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$ and $\{\mathbf{a}_i : i \leq \mu\}$ such that:

(1) set $\{\varphi(\mathbf{x}; \mathbf{a}_i) : i \leq \mu\}$ is n-contradictory over q;
(2) for every $i \leq \mu$, $Deg[q \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, \lambda] \geq \alpha$.

Remarks 2.3. (1) When $\Delta = L$, then trivially $Deg[p, \Delta, \lambda] = D[p, \Delta, \lambda]$. When Δ is a proper subset of the set of formulas of L than the ranks D and Deg are different.

(2) $Deg[p, L, (2^{|T|})^{++}]$ is what Baldwin in [Ba2] calls *the continuous rank*, denoted by $R_C[p]$. the same rank is denoted by Pillay (in page 72 of [Pi]) as $D(p)$.

(3) Notice that Buechler [Bu] denotes by $R^\infty[p]$ the function $R[p, L, \infty]$ from Definition 2.1.

The D and Deg ranks obey most of the basic properties of the R-rank.

Lemma 2.4 (Invariance). *For any set of formulas p and $f \in \text{Aut}(\mathfrak{C})$,*
$D[p, \Delta, \lambda] = D[f(p), \Delta, \lambda]$ *and* $Deg[p, \Delta, \lambda] = Deg[f(p), \Delta, \lambda]$

Lemma 2.5 (Ultrametric property). *For $\lambda \geq \aleph_0$*

$$Deg[p \cup \{ \bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda] = Max_{1 \leq l \leq n} Deg[p \cup \{\psi_l\}, \Delta, \lambda].$$

Proof. Similar to the proof of 1.6.   ⊣

Lemma 2.6 (Extension property). *For $\lambda \geq \aleph_0$, a set of formulas p and a set $A \supseteq \text{dom}(p)$, there exists a complete type $q \supseteq p$ with domain A such that $Deg[p, \Delta, \lambda] = Deg[q, \Delta, \lambda]$.*

Proof. Similar to 1.7 using Lemma 2.5 instead of Lemma 1.6.   ⊣

Lemma 2.7 (Monotonicity).   (1) $p \vdash q$ *implies* $D[p, \Delta, \lambda] \leq D[q, \Delta, \lambda]$ *and* $Deg[p, \Delta, \lambda] \leq Deg[q, \Delta, \lambda]$;

(2) $\mu < \lambda$ *implies* $D[p, \Delta, \lambda] \leq D[p, \Delta, \mu]$ *and* $Deg[p, \Delta, \lambda] \leq Deg[p, \Delta, \mu]$;

(3) $\Delta_1 \subseteq \Delta_2$ *implies that* $D[p, \Delta_1, \lambda] \leq D[p, \Delta_2, \lambda]$ *and* $Deg[p, \Delta_1, \lambda] \leq Deg[p, \Delta_2, \lambda]$.

Lemma 2.8 (Finite character). *Given $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in Fml(L(T))\}$, a cardinal λ, and a set of formulas p, there is a finite subset $q \subseteq p$, such that $D[p, \Delta, \lambda] = D[q, \Delta, \lambda]$ and $Deg[p, \Delta, \lambda] = Deg[q, \Delta, \lambda]$.*

Theorem 2.9 (Connected). *Let p be a finite set of formulas in $\mathbf{x}$.*

(1) *If $Deg[p, \Delta, \lambda] = \alpha < \infty$ then for every $\beta < \alpha$ there exists a Δ-type q such that $Deg[p \cup q, \Delta, \lambda] = \beta$.*

(2) *Suppose $\Delta = L$. If $D[p, L, \lambda] = \alpha < \infty$ then for all $\beta < \alpha$ and every A, there is $q \in S(A)$ such that $D[p \cup q, L, \lambda] = \beta$.*

Proofs of the above facts are similar to those of the corresponding properties of the R-rank.

Remark 2.10. If $\Delta \neq L$, then the D-rank may fail to have the property (2) in Theorem 2.9.

The ranks R, D, and Deg always satisfy the following relations:

Theorem 2.11. *For every p, Δ and $\lambda \geq \aleph_1$, $Deg[p, \Delta, \lambda] \leq D[p, \Delta, \lambda] \leq R[p, \Delta, \lambda]$.*

Proof. The fact that $Deg[p, \Delta, \lambda] \leq D[p, \Delta, \lambda]$ is trivial from definitions and is valid, of course, for any cardinal λ.

We show by induction on α that

$$D[p, \Delta, \lambda] \geq \alpha \quad \text{implies} \quad R[p, \Delta, \lambda] \geq \alpha.$$

By the finite character, without loss of generality, we assume that p is finite. For $\alpha = 0$, we are done by the definitions of D and R.

For α limit, use the induction hypothesis.

For $\alpha = \beta + 1$, let $\mu < \lambda$, $\mu \geq \aleph_0$. By the definition of D for some finite $q \supseteq p$ there are $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$ and a set $\{\mathbf{b}_i \mid i \leq \mu\}$ such that

(1) $D[q \cup \{\psi(\mathbf{x}; \mathbf{b}_i)\}, \Delta, \lambda] \geq \beta$ for every $i \leq \mu$ and
(2) the set $\{\psi(\mathbf{x}; \mathbf{b}_i) \mid i \leq \mu\}$ is n-contradictory over q.

By the inductive hypothesis we have that

$$R[q \cup \{\psi(\mathbf{x}; \mathbf{b}_i)\}, \Delta, \lambda] \geq \beta \quad \text{for every } i \leq \mu.$$

By the extension property (Lemma 1.7) for every $i \leq \mu$ there exists

$$p_i \in S(\operatorname{dom} q \cup \{\mathbf{b}_i \mid i \leq \mu\}) \text{ with } q \cup \{\psi(\mathbf{x}; \mathbf{b}_i)\} \subseteq p_i \text{ and}$$

$$R[q \cup p_i \Delta, \Delta, \lambda] \geq R[p_i, \Delta, \lambda] = R[q \cup \{\psi(\mathbf{x}; \mathbf{b}_i)\}, \Delta, \lambda] \geq \beta.$$

Since the set $\{\psi(\mathbf{x}; \mathbf{b}_i) \mid i \leq \mu\}$ is n-contradictory over q and $\{p_i \mid i \leq \mu\}$ are all complete types over the same set of parameters, for any $\{p_{i_1}, \ldots, p_{i_n}\}$, $i_1, \ldots, i_n \leq \mu$, p_{i_1} cannot possibly contain all of the $\psi(\mathbf{x}; \mathbf{b}_{i_j})$'s, for $j \in \{2, \ldots, n\}$, so there is a $j \in \{2, \ldots, n\}$ such that $\psi(\mathbf{x}; \mathbf{b}_{i_j}) \notin p_{i_1}$, i. e., $\neg \psi(\mathbf{x}; \mathbf{b}_{i_j}) \in p_{i_1}$. Thus, among any n many $p_i \Delta$'s, there are two explicitly contradictory Δ-types. Since μ is infinite, there exists $S \subseteq \mu + 1$ of cardinality μ such that $i \neq j \in S \Rightarrow p_i \Delta \neq p_j \Delta$ and by the definition of R we get that $R[q, \Delta, \lambda] \geq \beta + 1$. But $q \vdash p$ implies $R[p, \Delta, \lambda] \geq \beta + 1 = \alpha$. $\dashv$

It is well-known that the D-rank is not equal to R-rank in some situations. The following example shows that the ranks D and Deg do not necessarily coincide.

Consider $L = \langle P_1, P_2 \rangle$, where P_1 is a unary and P_2 is a binary predicate. Let T be the following theory in L:

(1) $\forall x \forall y (x = y \leftrightarrow ((P_1(x) \leftrightarrow P_1(y)) \wedge P_2(x, y)))$ (every element is completely determined by P_1 and P_2);
(2) $\forall x (P_1(x) \rightarrow \exists y (\neg P_1(y) \wedge P_2(x, y)) \wedge (\neg P_1(x) \rightarrow \exists y (P_1(y) \wedge P_2(x, y))))$ (there is a bijection between $\{x \mid P_1(x)\}$ and $\{x \mid \neg P_1(x)\}$);
(3) $\exists^\infty x P_1(x)$ (in fact, this is a set of countably many axioms of the form $\exists x_0 \ldots \exists x_n (\bigwedge_{i \leq n} P_1(x_i) \wedge \bigwedge_{i < j \leq n} x_i \neq x_j)$.

Claim 2.12. *The theory T is consistent and categorical in every infinite cardinality.*

Proof. Let λ be an infinite ordinal and let $|M_\lambda| := \{(a, b) \mid a \in \{0, 1\}, b \in \lambda\}$. If we define $P_1^{M_\lambda} := \{(0, b) \mid b \in \lambda\}$ and $P_2^{M_\lambda} := \{((a_1, b), (a_2, b)) \mid a_1, a_2 \in \{0, 1\}, b \in \lambda\}$, then $M_\lambda := \langle |M_\lambda|, P_1^{M_\lambda}, P_2^{M_\lambda} \rangle$ is a model for T.

Clearly, if M is a model of T of cardinality κ, we can construct an isomorphism between M and M_κ. $\dashv$

Thus, T is a complete and superstable theory.

Claim 2.13. *If* $\psi(x,y) \equiv P_1(x) \vee (x = y)$, *then*

(1) $Deg[x = x, \psi(x,y), \aleph_0] = 0;$

(2) $D[x = x, \psi(x,y), \aleph_0] \geq 1.$

(In fact, $D[x = x, \psi(x,y), \aleph_0] = 1$ here.)

Proof. (1) Clearly $Deg[x = x, \psi(x,y), \aleph_0] \geq 0$. On the other hand, for all $a \in |M|$, $\psi(x,a)$ is realized by all b with $P_1(b)$. Therefore, we are not able to find parameters a_i such that $\{\psi(x, a_i)\}$ are contradictory over $x = x$.
So, $Deg[x = x, \psi(x,y), \aleph_0] = 0$.

(2) We use $\varphi(x) \equiv \neg P_1(x)$ as the extension of $x = x$. In conjunction with φ, the formula $\psi(x,y)$ becomes equivalent to $x = y$. Let $\{a_i \mid i < \omega\}$ be different elements in $\neg P_1$, then obviously

(1) $\{\psi(x, a_i) : i \leq \omega\}$ are 2-contradictory over φ;

(2) for every $i \leq \omega$, $D[\varphi(x) \wedge \psi(x,y), \Delta, \lambda] \geq 0$.

Therefore, $D[x = x, \psi(x,y), \aleph_0] \geq 1$. It is also easy to see that
$D[\varphi(x) \wedge \psi(x,y), \Delta, \lambda] = 0$, so in fact $D[x = x, \psi(x,y), \aleph_0] = 1$.   ⊣

The following concept is the key to show that $I(\lambda, T_1, T) = 2^\lambda$ (for $\lambda > |T_1|$ and $T_1 \supseteq T$).

Definition 2.14. (1) Let λ be an infinite cardinality (when $|T|$ is uncountable, λ can be less than $|T|$). We say that T has *the λ-weak tree property* if there are $\{\varphi_n(\mathbf{x}; \mathbf{y}_n) \mid n < \omega\} \subseteq L(T)$ and $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}\lambda\}$ such that the set $\{\varphi_n(\mathbf{x}; \mathbf{a}_{\eta n}) \mid n < \omega\}$ is consistent for all $\eta \in {}^\omega\lambda$, and for all $\eta \in {}^{\omega>}\lambda$ and all infinite $S \subseteq \lambda$ the set $\{\varphi_{\ell(\eta)+1}(\mathbf{x}; \mathbf{a}_{\eta^\frown \alpha}) \mid \alpha \in S\}$ is inconsistent.

(2) Let p be a type. We say that p has *the λ-weak tree property over Δ* if there are $\{\varphi_n(\mathbf{x}; \mathbf{y}_n) \in \Delta \mid n < \omega\} \subseteq L(T)$ and $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}\lambda\}$ such that the set $p \cup \{\varphi_n(\mathbf{x}; \mathbf{a}_{\eta n}) \mid n < \omega\}$ is consistent for all $\eta \in {}^\omega\lambda$, and for all $\eta \in {}^{\omega>}\lambda$ and all infinite $S \subseteq \lambda$ the set $p \cup \{\varphi_{\ell(\eta)+1}(\mathbf{x}; \mathbf{a}_{\eta^\frown \alpha}) \mid \alpha \in S\}$ is inconsistent.

Proposition 2.15. (1) *Let λ be an infinite cardinality. The following are equivalent*

(1) *T has the λ-weak tree property,*

(2) *T has the μ-weak tree property for every $\mu \geq \aleph_0$ and*

(3) *T has the μ-weak tree property for some $\mu \geq \aleph_0$.*

(2) *Let λ be an infinite cardinality. The following are equivalent*

(1) *p has the λ-weak tree property over Δ,*

(2) *p has the μ-weak tree property over Δ for every $\mu \geq \aleph_0$ and*

(3) *p has the μ-weak tree property over Δ for some $\mu \geq \aleph_0$.*

Proof. Use the Compactness Theorem.   ⊣

Proposition 2.16. (1) *If T has the μ-weak tree property for some $\mu \geq \aleph_0$ then T is not superstable.*

(2) *If p has the μ-weak tree property over Δ for some $\mu \geq \aleph_0$ then p is not Δ-superstable.*

Proof. For every λ there exists $\mu \geq \lambda$ of cofinality $\aleph_0$. By Zermelo-König's theorem $\mu^{\aleph_0} > \mu$. Use the μ-weak tree property (over Δ) to construct a set A of cardinality μ such that $|S(A)| = \mu^{\aleph_0} > \mu$ ($|S_{p,\Delta}(A)| = \mu^{\aleph_0} > \mu$).   ⊣

The Deg-rank can be used to characterize superstability and local superstability, under the assumption of stability. To show that, we need the tree-characterization and normalization lemmas for Deg-rank. In order to state the tree-characterization lemma, we need to introduce the appropriate kind of trees.

Definition 2.17. h *is* (Δ, α)-*function for* $\mathbf{x}$ *iff* $h : ds(\alpha) \to \Delta \times \omega$, *where*

$$ds(\alpha) := \{\eta \in {}^{\omega>}\alpha \; : \; \forall \ell < \ell(\eta) - 1 \, (\eta[\ell] > \eta[\ell + 1])\}$$

and

$$h(\eta) = \langle \psi_\eta(\mathbf{x}; \mathbf{y}), n_\eta \rangle.$$

For $U \subseteq ds(\alpha)$ define

$$\Gamma_\mu^U(\theta(\mathbf{x}; \mathbf{a}), h) :=$$
$$\{\exists \mathbf{x}[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{0 < \ell \leq \ell(\eta)} [\psi_{\eta\ell}(\mathbf{x}; \mathbf{y}_{\eta\ell, \nu\ell})] \; : \; \eta \in U \cup \{\langle\rangle\}, \nu \in {}^{\ell(\eta)}\mu\} \cup$$
$$\{\neg \exists \mathbf{x}[\bigwedge_{i \in w} \psi_\eta(\mathbf{x}; \mathbf{y}_{\eta, \nu^\frown i}) \wedge \theta(\mathbf{x}; \mathbf{a})] \; : \; \eta \in U, w \subseteq \mu, |w| = n_\eta, \nu \in {}^{\ell(\eta)-1}\mu\}.$$

The following is an important technical property of the previous trees:

Proposition 2.18. *Let* α, μ, $\theta(\mathbf{x}; \mathbf{a})$, Δ *and* h *be as in the previous definition.*

$\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent* $\iff$ $\Gamma_\lambda^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent* $\forall \lambda \geq \aleph_0$.

Proof. Use the Compactness Theorem.   ⊣

Lemma 2.19 (Tree characterization lemma for Deg). *Let* $\theta(\mathbf{x}; \mathbf{a})$ *be given and let* cf $\mu > |\alpha| + |T|$. *The following are equivalent:*

(1) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha$;

(2) *There exists a* (Δ, α)-*function* h *in* $\mathbf{x}$ *such that* $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent.*

Proof. (2) $\Rightarrow$ (1): Let $\{\mathbf{b}_{\eta, \nu} : \eta \in ds(\alpha), \nu \in {}^{\ell(\eta)}\mu\} \models \Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$. For $\eta \in ds(\alpha) \cup \{\langle\rangle\}$ and $\nu \in {}^{\ell(\eta)}\mu$ define

$$p_{\eta, \nu} = \{\theta(\mathbf{x}; \mathbf{a})\} \cup \{\psi_{\eta\ell}(\mathbf{x}; \mathbf{b}_{\eta\ell, \nu\ell}) : 0 < \ell \leq \ell(\eta)\}.$$

Claim 2.20. *For all* $\eta \in ds(\alpha)$, $\nu \in {}^{\ell(\eta)}\mu$, $Deg[p_{\eta, \nu}, \Delta, \mu^+] \geq \eta[\ell(\eta) - 1]$.

Proof. By induction on $\gamma \leq \eta[\ell(\eta) - 1]$ show $Deg[p_{\eta, \nu}, \Delta, \mu^+] \geq \gamma$. When $\gamma = 0$, $Deg[p_{\eta, \nu}, \Delta, \mu^+] \geq 0$ since $p_{\eta, \nu}$ is consistent by the first part of the tree. The case when γ is limit trivially follows from the induction hypothesis.

If $\gamma + 1 \leq \eta[\ell(\eta) - 1]$, then $\eta^\frown\gamma \in ds(\alpha)$. So by the induction hypothesis, for all $i < \mu$ we have that $Deg[p_{\eta^\frown\gamma, \nu^\frown i}, \Delta, \mu^+] \geq \gamma$. Furthermore, by the second part of the tree, we have that $\{\psi_{\eta^\frown\gamma}(\mathbf{x}; \mathbf{b}_{\eta^\frown\gamma, \nu^\frown i}) : i < \mu\}$ is $n_{\eta^\frown\gamma}$-contradictory over $\theta(\mathbf{x}; \mathbf{a})$; in particular, the set is $n_{\eta^\frown\gamma}$-contradictory over $p_{\eta, \nu}$. Additionally, for

every $i < \mu$, $Deg[p_{\eta,\nu} \cup \psi_{\eta^\frown\gamma}(\mathbf{x}; \mathbf{b}_{\eta^\frown\gamma,\nu^\frown i}), \Delta, \mu^+] = Deg[p_{\eta^\frown\gamma,\nu^\frown i}, \Delta, \mu^+] \geq \gamma$. By the definition of Deg we have $Deg[p_{\eta,\nu}, \Delta, \mu^+] \geq \gamma + 1$. $\dashv$

This is enough to conclude $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha$. When α is a limit ordinal, the claim shows that for all $\beta < \alpha$, $i < \mu$, $Deg[p_{\langle\beta\rangle,\langle i\rangle}, \Delta, \mu^+] \geq \beta$. Since $p_{\langle\beta\rangle,\langle i\rangle} \vdash \theta(\mathbf{x}; \mathbf{a})$, we have that $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta$ for every $\beta < \alpha$. Thus $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha$. When $\alpha = \beta + 1$, from the claim we get that for every $i < \mu$, $Deg[p_{\langle\beta\rangle,\langle i\rangle}, \Delta, \mu^+] \geq \beta$. ¿From the second part of the tree, we get that $\{\psi_{\langle\beta\rangle}(\mathbf{x}; \mathbf{b}_{\langle\beta\rangle,\langle i\rangle}) : i < \mu\}$ is $n_{\langle\beta\rangle}$-contradictory over $\theta(\mathbf{x}; \mathbf{a})$, so $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta + 1 = \alpha$.

(1) $\Rightarrow$ (2): By induction on α, we show that for all $\theta(\mathbf{x}; \mathbf{a})$
$Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha$ implies that there exists a (Δ, α)-function h such that $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent.

The case when $\alpha = 0$ is trivial.

For α limit, by induction hypothesis we have that for every $\beta < \alpha$ there is a (Δ, β)-function h_β such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}), h_\beta)$ is consistent. Define a (Δ, α)-function h by $h(\eta) := h_{\eta[0]+1}(\eta)$, for $\eta \in ds(\alpha)$. Now if for every $\beta < \alpha$, $\{\mathbf{b}_{\eta,\nu}^\beta\} \models \Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}), h_\beta)$, it is easy to check that the assignment $\mathbf{b}_{\eta,\nu} := \mathbf{b}_{\eta,\nu}^{\eta[0]+1}$ for $\eta \in ds(\alpha)$ and $\nu \in {}^{\ell(\eta)}\mu$ realizes $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$.

Now suppose that $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha = \beta + 1$. There are $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$ and $\{\mathbf{a}_i : i \leq \mu\}$ such that $\{\psi(\mathbf{x}; \mathbf{a}_i) : i \leq \mu\}$ is n-contradictory over $\theta(\mathbf{x}; \mathbf{a})$ and $\forall i \leq \mu$ $D[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{a}_i), \Delta, \mu^+] \geq \beta$. By the induction hypothesis there are (Δ, β)-functions h^* and h_i such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}), h^*)$ and $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{a}_i), h_i)$ are consistent for all $i \leq \mu$.

For every $i \leq \mu$ define a (Δ, α)-function h^i by

$$h^i(\eta) = \begin{cases} h^*(\eta) & \text{if } \eta \in ds(\beta); \\ h_i(\xi) & \text{if } \eta = \langle\beta\rangle^\frown\xi; \\ \langle\psi, n\rangle & \text{if } \eta = \langle\beta\rangle. \end{cases}$$

It is enough to show that there is an $i \leq \mu$ such that $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h^i)$ is consistent. Suppose not. For every $i \leq \mu$ there is a finite $u(i) \subseteq ds(\alpha)$ such that $\Gamma_\mu^{u(i)}(\theta(\mathbf{x}; \mathbf{a}), h^i)$ is inconsistent. Apply the pigeonhole principle to the mapping $i \mapsto u(i)$ (from μ to a set of cardinality $\leq |\alpha| + \aleph_0$) and get $S \subseteq \mu$, and finite $u \subseteq ds(\alpha)$ such that $|S| = \mu$ and for every $i \in S$, $u(i) = u$. Pick $U \subseteq ds(\alpha)$ such that $u \subseteq U$ and U is closed under taking initial segments. Now apply the pigeonhole principle to $i \mapsto h^i U$ (from μ to the set of cardinality $\leq |T|$) and get $S' \subseteq S$, and h such that $|S'| = \mu$ and $i \in S'$ implies $h^i U = h$. But we will check that $\Gamma_\mu^U(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent, which will be a contradiction to the choice of U:

If, for all $i \in S'$,

$$\{\mathbf{b}_{\eta,\nu}^i : \eta \in ds(\beta), \nu \in {}^{\ell(\eta)}\mu\} \models \Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{a}_i), h_i), \text{ and}$$
$$\{\mathbf{b}_{\eta,\nu}^* : \eta \in ds(\beta), \nu \in {}^{\ell(\eta)}\mu\} \models \Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}), h^*),$$

then if we put

$$\mathbf{b}_{\eta,\nu} := \begin{cases} \mathbf{a}_i & \text{if } \eta = \langle\beta\rangle, \nu = \langle i\rangle; \\ \mathbf{b}^i_{\xi,\zeta} & \text{if } \eta = \langle\beta\rangle^\frown\xi, \nu = \langle i\rangle^\frown\zeta; \\ \mathbf{b}^*_{\eta,\nu} & \text{if } \eta \in ds(\beta), \end{cases}$$

we have (after renumerating, we may assume $S' = \mu$) that
$\{\mathbf{b}_{\eta,\nu}\} \models \Gamma^U_\mu(\theta(\mathbf{x};\mathbf{a}), h)$. $\dashv$

Remark 2.21. In the previous theorem, the requirement $\operatorname{cf}\mu > |\alpha| + |T|$ is not needed for (2) $\Rightarrow$ (1).

Corollary 2.22. *Let* $\mu > |T| + \aleph_0$ *be a regular cardinality.*
If $Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \mu$ *then there exists a* (Δ, μ)-*function* h *such that*
$\Gamma^{ds(\mu)}_\mu(\theta(\mathbf{x};\mathbf{a}), h)$ *is consistent.*

Proof. Since μ is a limit ordinal we have that $Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \mu$ implies

$$(*) \quad Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \alpha \quad \text{for every} \quad \alpha < \mu.$$

Since μ is regular and greater than $|T|$ we have that $\operatorname{cf}\mu > |\alpha| + |T|$.

Thus Lemma 2.19 applied to $(*)$ gives that for every $\alpha < \mu$ there exists a (Δ, α)-function h_α such that $\Gamma^{ds(\alpha)}_\mu(\theta(\mathbf{x};\mathbf{a}), h_\alpha)$ is consistent.

For $\eta \in ds(\mu)$, let $h(\eta) := h_{\eta[0]+1}(\eta)$.

One can show that $\Gamma^{ds(\mu)}_\mu(\theta(\mathbf{x};\mathbf{a}), h)$ is consistent using a similar argument as in the proof of the tree characterization lemma. $\dashv$

The following lemma asserts that when $\alpha \geq |T|^+$, the tree of formulas given by h in the Lemma 2.19 depends only on the length of η.

Lemma 2.23 (Normalization Lemma). *Let* $\theta(\mathbf{x};\mathbf{a})$ *be given.*
$Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ *iff there exists* $\{\psi_k, n_k \mid 0 < k < \omega\}$ *such that for every* α *the set* $\Gamma^{ds(\alpha)}_{|T|^+}(\theta(\mathbf{x};\mathbf{a}), h)$ *is consistent for a* (Δ, α)-*function* h *satisfying*

$$\forall \eta \in ds(\alpha) \quad h(\eta) = \langle \psi_{\ell(\eta)}(\mathbf{x};\mathbf{y}), n_{\ell(\eta)}\rangle.$$

Proof. $(\Rightarrow)$: Denote by μ the cardinality $|T|^+$. Suppose that

$$Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \mu.$$

Since μ is regular, by the above corollary the tree $\Gamma^{ds(\mu)}_\mu(\theta(\mathbf{x};\mathbf{a}), h)$ is consistent for some (Δ, μ)-function h.

By induction on $0 < k < \omega$ define ψ_k and $n_k < \omega$ and a set $\{\eta^i_k \in ds(\mu) \cap {}^k\mu \mid i < \mu\}$ such that for every $i < \mu$:

(1) $\eta^i_k[k-1] > i$, and

(2) for every $1 \leq l \leq k$ we have that $h(\eta^i_k l) = \langle \psi_l, n_l\rangle$ for all $i < \mu$.

For $k = 1$: Consider the mapping $i \mapsto h(\langle i\rangle)$. Since $h(\eta) = \langle \psi_\eta(\mathbf{x};\mathbf{y}), n_\eta\rangle$, the range of the above mapping has cardinality $\leq |\Delta| + \aleph_0 \leq |T| + \aleph_0$ and its domain has cardinality $|T|^+$. So, there are ψ_1 and n_1 and a set $S \subseteq \mu$ of cardinality μ such that $h(\langle i\rangle) = \langle \psi_1(\mathbf{x};\mathbf{y}), n_1\rangle$ for all $i \in S$. Since S is unbounded in μ, for every $i < \mu$ we can pick $j_i \in S$ greater than i and define η^i_1 as $\langle j_i\rangle$.

For $k + 1$: Suppose that $\{\langle\psi_1, n_1\rangle, \ldots, \langle\psi_k, n_k\rangle\}$ and $\{\eta_k^i \in ds(\mu) \cap {}^k\mu \; : \; i < \mu\}$ are defined.

Apply the pigeonhole principle to the mapping $\alpha \mapsto h(\eta_k^{\alpha\frown}\langle\alpha\rangle)$ from $\mu = |T|^+$ into $\Delta \times \omega$ to get $S \subseteq \mu$ of cardinality μ and $\langle\psi_{k+1}, n_{k+1}\rangle$ such that for every $\alpha \in S$, $h(\eta_k^{\alpha\frown}\langle\alpha\rangle) = \langle\psi_{k+1}, n_{k+1}\rangle$. Now for $i < \mu$ let $\eta_{k+1}^i := \eta_k^{\alpha\frown}\langle\alpha\rangle$, where $\alpha > i, \alpha \in S$.

Let $\tau := \{\eta_k^i \mid k < \omega, i < \mu\}$. Clearly for every $\eta \in \tau$ we have that $|\{\alpha < \mu \; : \; \eta^\frown\langle\alpha\rangle \in \tau\}| = \mu$. Thus pick $f : ds(\mu) \to \tau$ a bijection that is level-preserving and if $\eta^\frown\langle\alpha\rangle \in ds(\mu)$ then $f(\eta^\frown\langle\alpha\rangle) = f(\eta)^\frown\langle\beta_\alpha\rangle$ for some $\beta_\alpha > \alpha$.

Let $h' := h \circ f$. By the construction h' is a (Δ, μ)-function satisfying that for every $\eta \in ds(\mu)$ we have that

$$h'(\eta) = \langle\psi_{\ell(\eta)}(\mathbf{x}; \mathbf{y}), n_{\ell(\eta)}\rangle.$$

Now let α be a given infinite ordinal. By the property of h',

$$\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h') = \{\exists\mathbf{x}[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{0<\ell\le\ell(\eta)} \psi_\ell(\mathbf{x}; \mathbf{y}_{\ell,\nu\ell}) \mid \nu \in {}^{\ell(\eta)}\mu\} \bigcup$$

$$\{\neg\exists\mathbf{x}\Big(\bigwedge_{i\in w} \psi_\ell(\mathbf{x}; \mathbf{y}_{\ell,\nu\frown i})\Big) \mid \ell < \omega, w \subseteq \mu, |w| = n_\ell, \nu \in {}^{\ell-1}\mu\}.$$

Namely the sets $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h')$ and $\Gamma_\mu^{ds(\mu)}(\theta(\mathbf{x}; \mathbf{a}), h')$ are equal (we are using here only the assumption that α is infinite).

($\Longleftarrow$) : Use the hypothesis for $\alpha := |T|^+$. Apply (2) $\Rightarrow$ (1) from Lemma 2.19. $\dashv$

Corollary 2.24. *Let $\theta(\mathbf{x}; \mathbf{a})$ be given.*

(1) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \ge |T|^+$ *iff* $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$.

(2) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$ *iff* *there exists* $\{\langle\psi_k, n_k\rangle \mid 0 < k < \omega\} \subseteq \Delta \times \omega$ *and there are* $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}|T|^+\}$ *such that*

$$\forall\eta \in {}^\omega|T|^+ \quad \{\theta(\mathbf{x}; \mathbf{a})\} \cup \{\psi_k(\mathbf{x}; \mathbf{a}_{\eta k}) \mid k < \omega\}$$

is consistent and for every $0 < k < \omega$ and for every $\eta \in {}^{k-1}|T|^+$ we have that

$$\{\psi_k(\mathbf{x}; \mathbf{a}_{\eta\frown i}) \mid i < |T|^+\} \quad \text{is } n_k\text{-contradictory over } \theta(\mathbf{x}; \mathbf{a}).$$

Proof. (1) Suppose that $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \ge |T|^+$. By the Normalization Lemma there is h such that for every α the set $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent. Using now the Compactness Theorem and
Lemma 2.19 we get that
$Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \ge \alpha$ for every α, namely $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$. The converse is trivial.

(2) Apply the Normalization Lemma to $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \ge |T|^+$ to produce a $(\Delta, |T|^+)$-function h as there. Let $\{\mathbf{a}_\eta\} \models \Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$.

For the converse, use $h(\eta) := \langle\psi_{\ell(\eta)}, n_{\ell(\eta)}\rangle$ to show that the set $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent for every infinite α and apply Lemma 2.19. $\dashv$

Corollary 2.25. *Let* $\theta(\mathbf{x};\mathbf{a})$ *be given. If* $Deg[\theta(\mathbf{x};\mathbf{a}),\Delta,|T|^{++}] \geq |T|^{+}$ *then* $\theta(\mathbf{x};\mathbf{a})$ *has the* λ-*weak tree property over* Δ *for every* $\lambda \geq \aleph_0$.

Proof. Take $\lambda = |T|^{+}$. Application of Corollary 2.24 gives us $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}\lambda\}$. The formulas ψ_k obtained from Corollary 2.24 will witness the λ-weak tree property of $\theta(\mathbf{x};\mathbf{a})$ over Δ. By Proposition 2.15, $\theta(\mathbf{x};\mathbf{a})$ has the λ-weak tree property over Δ for every $\lambda \geq \aleph_0$.   ⊣

Corollary 2.26.   (1) *If* $Deg[\mathbf{x} = \mathbf{x}, L, |T|^{++}] \geq |T|^{+}$ *then* T *is not superstable.*

   (2) *If* $Deg[\theta(\mathbf{x};\mathbf{a}),\Delta,|T|^{++}] \geq |T|^{+}$ *then* $\theta(\mathbf{x};\mathbf{a})$ *is not* Δ-*superstable.*

Proof. By the previous corollary and Proposition 2.16.   ⊣

3. Main Theorem

Theorem 3.1 (Main Theorem). *Let* p *be a finite type. Suppose that* μ *is regular satisfying* $\mu \geq |T|^{+}$ *and* Δ *is a set of formulas which is closed under Boolean operations.*

If

   (1) $R[p, \Delta, \mu^{+}] < \infty$ *or*

   (2) p *is* Δ-*stable and for every* $\{\mu_i \mid i < |\Delta| + \aleph_0\}$ *cardinalities all less than* μ, $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$ *holds (e.g.* $\mu = (2^{|T|})^{+}$ *is such a cardinality),*

then

$$Deg[p,\Delta,\mu^{+}] = D[p,\Delta,\mu^{+}] = R[p,\Delta,\mu^{+}].$$

Before proving this theorem, we present another rank function. The rank function $S1$ was introduced By Hrushovski in the early nineties (in [Hr] unpublished notes) see [HP1] and [CH]; its definition (see below) is motivated by algebraic dimension theory. Here, it will facilitate the proof of the Main Theorem. We find it surprising that in the quest to settle a combinatorial problem one naturally discovers such a geometric object.

Definition 3.2. $S1[p,\Delta] \geq 0$ if p is consistent.

$S1[p,\Delta] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $S1[p,\Delta] \geq \beta$.

$S1[p,\Delta] \geq \alpha + 1$ if for every finite $p_0 \subset p$ there exists $\psi(\mathbf{x};\mathbf{y}) \in \Delta$ and $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\mathrm{dom}(p_0)$ such that

   (1) $S1[p_0 \cup \{\psi(\mathbf{x};\mathbf{b}_n)\}, \Delta] \geq \alpha$ for every $n < \omega$ and

   (2) $S1[p_0 \cup \{\psi(\mathbf{x};\mathbf{b}_n)\} \cup \{\psi(\mathbf{x};\mathbf{b}_m)\}, \Delta] < \alpha$ for $m \neq n < \omega$.

Lemma 3.3. *Given sets of formulas* p *and* q *and* $\Delta_1, \Delta_2 \subseteq \{\varphi(\mathbf{x};\mathbf{y}) \mid \varphi \in Fml(L(T))\}$:

   (1) *(Invariance of* $S1$ *rank) For* $f \in \mathrm{Aut}(\mathfrak{C})$, $S1[f(p),\Delta_1] = S1[p,\Delta_1]$.

   (2) *(Monotonicity of* $S1$ *rank) If* $q \vdash p$ *and* $\Delta_1 \subseteq \Delta_2$, *then* $S1[q,\Delta_1] \leq S1[p,\Delta_2]$.

   (3) *(Finite Character for* $S1$ *rank) There exists* $p_0 \subseteq_{fin} p$ *such that* $S1[p,\Delta] = S1[p_0,\Delta]$.

(4) *(Ultrametric property for S1 rank)* For $n < \omega$ and $\{\psi_l \mid l \leq n\}$ a set of formulas, $S1[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta] = \text{Max}\{S1[p \cup \{\psi_l\}, \Delta] \mid 1 \leq l \leq n\}$.

(5) *(Extension property for S1 rank)* Given a set $A \supseteq \text{dom}(p)$, there exists $p' \in S^{\ell(\mathbf{x})}(A)$ such that $S1[p, \Delta] = S1[p', \Delta]$.

Proof. (1) Immediate.

(2) By induction on α show that $S1[q, \Delta_1] \geq \alpha \Rightarrow S1[p, \Delta_2] \geq \alpha$.

(3) By definition and (2).

(4) $\text{Max}\{S1[p \cup \{\psi_l\}, \Delta] : 1 \leq l \leq n\} \leq S1[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta]$ follows from (2).

By induction on α show that

$$S1[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta] \geq \alpha \Rightarrow \text{Max}\{S1[p \cup \{\psi_l\}, \Delta] : 1 \leq l \leq n\} \geq \alpha.$$

(5) By the Ultrametric property, (4).

⊣

Theorem 3.4. *Let Δ be a set of formulas in $\mathbf{x}$ closed under Boolean operations; and let λ be an infinite cardinal. If $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable, then $S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \leq Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda]$.*

Proof. By induction on ordinals α, we show for all finite p that

$$S1[p, \Delta] \geq \alpha \Rightarrow Deg[p, \Delta, \lambda] \geq \alpha. \qquad (*)_\alpha$$

If $\alpha = 0$, then $(*)_\alpha$ holds by the definitions of the ranks. For α a limit ordinal, $(*)_\alpha$ follows from the continuity of the ranks and the induction hypothesis. Let $\alpha = \beta + 1$ be such that $S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \geq \alpha$ and $(*)_\beta$ is true. By the definition of the $S1$ rank, there are $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\mathbf{a}$ and a formula $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$ as in the definition. By invariance of the $S1$ rank and Compactness Theorem, there is an indiscernible sequence $\{\mathbf{b}_i \mid i < \lambda\}$ over $\mathbf{a}$, such that

(1) $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i), \Delta] \geq \beta$ for every $i < \lambda$ and

(2) $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i) \wedge \psi(\mathbf{x}; \mathbf{b}_j), \Delta] < \beta$ for $i \neq j < \lambda$.

Since $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable, by Theorem 1.11 no formula in the set Δ has the order property over $\theta(\mathbf{x}; \mathbf{a})$. Therefore, there is a number $n < \omega$ witnessing the failure of the order property for $\psi(\mathbf{x}; \mathbf{y})$, i. e. , such that for no $\{\mathbf{d}_l \mid l < n\}$

$$\bigwedge_{l < n} \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k < n} \psi(\mathbf{x}; \mathbf{d}_k)^{\text{if }(k \geq l)} \right]$$

holds.

For each $i < \lambda$, define

$$\psi^*(\mathbf{x}, \mathbf{c}_i) := \bigwedge_{l < n} \neg \psi(\mathbf{x}; \mathbf{b}_l) \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}).$$

Since $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}), \Delta] \geq \beta$ and $\psi(\mathbf{x}; \mathbf{b}_{i+n})$ is logically equivalent to

$$\psi^*(\mathbf{x}; \mathbf{c}_i) \vee \left(\left[\bigvee_{l < n} \psi(\mathbf{x}; \mathbf{b}_l) \right] \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}) \right),$$

by the Ultrametric property, Lemma 3.3, we obtain

$$\begin{aligned}
\beta \; &\leq S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}), \Delta] \\
&= \max\{S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi^*(\mathbf{x}; \mathbf{c}_i), \Delta], \\
&\qquad S1[\theta(\mathbf{x}; \mathbf{a}) \wedge [\bigvee_{l<n} \psi(\mathbf{x}; \mathbf{b}_l)] \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}), \Delta]\}.
\end{aligned}$$

Notice that the second member in the above maximum is less then β. Indeed, applying the Ultrametric property again, by the choice of $\psi(\mathbf{x}; \mathbf{y})$ and parameters $\mathbf{b}_i$, we get

$$\begin{aligned}
&S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \Big[\bigvee_{l<n} \psi(\mathbf{x}; \mathbf{b}_l)\Big] \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}), \Delta] \\
&= \max\{S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_l) \wedge \psi(\mathbf{x}; \mathbf{b}_{i+n}), \Delta] \mid l < n\} < \beta.
\end{aligned}$$

Therefore, $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi^*(\mathbf{x}; \mathbf{c}_i), \Delta] \geq \beta$ and by the induction hypothesis, we have

$$\forall i < \lambda \;\; Deg[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi^*(\mathbf{x}; \mathbf{c}_i), \Delta, \lambda] \geq \beta.$$

Next, the set $\{\psi^*(\mathbf{x}; \mathbf{c}_i) \mid i < \lambda\}$ is n-contradictory over $\theta(\mathbf{x}; \mathbf{a})$. Otherwise, there are $\{i(l) \mid l < n\}$ such that

$$\models \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{l<n} \psi^*(\mathbf{x}; \mathbf{c}_{i(l)})\right],$$

i. e., by the definition of ψ^* and $\mathbf{c}_{i(l)}$,

$$\models \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{l<n} \neg\psi(\mathbf{x}; \mathbf{b}_l) \wedge \bigwedge_{l<n} \psi(\mathbf{x}; \mathbf{b}_{i(l)+n})\right].$$

In particular, for every $k \leq n$,

$$\models \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{l<k} \neg\psi(\mathbf{x}; \mathbf{b}_l) \wedge \bigwedge_{l=k}^{n-1} \psi(\mathbf{x}; \mathbf{b}_{i(l)+n})\right].$$

And by indiscernibility of $\{\mathbf{b}_i \mid i < \lambda\}$ over $\mathbf{a}$, we get

$$\models \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{l<k} \neg\psi(\mathbf{x}; \mathbf{b}_l) \wedge \bigwedge_{l=k}^{n-1} \psi(\mathbf{x}; \mathbf{b}_l)\right],$$

which is a contradiction to the choice of n, witnessing the failure of the order property over $\theta(\mathbf{x}; \mathbf{a})$ for ψ.

Thus, we have found a formula $\psi^*(\mathbf{x}; \mathbf{y}) \in \Delta$ and a sequence of parameters $\{\mathbf{c}_i \mid i < \lambda\}$ such that

(1) $Deg[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi^*(\mathbf{x}; \mathbf{c}_i), \Delta, \lambda] \geq \beta$ for all $i < \lambda$ and
(2) the set $\{\psi^*(\mathbf{x}; \mathbf{c}_i) \mid i < \lambda\}$ is n-contradictory over $\theta(\mathbf{x}; \mathbf{a})$.

Therefore, $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda] \geq \beta + 1 = \alpha$. $\qquad\dashv$

The following fact now trivially follows from Theorems 3.4 and 2.11.

Corollary 3.5. *Let Δ be a set of formulas in $\mathbf{x}$ closed under Boolean operations; and let $\lambda \geq \aleph_1$. If $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable, then*

$$S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \leq Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda] \leq D[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda] \leq R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda].$$

In order to prove the main theorem, we need to show that $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \lambda] \leq S1[\theta(\mathbf{x}; \mathbf{a}), \Delta]$ under appropriate assumptions.

Lemma 3.6. *Suppose that μ is a regular cardinal satisfying $\mu \geq |T|^+$ and Δ is a set of formulas which is closed under Boolean operations. If $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta + 1$ and*

(1) $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] < \infty$ *or*

(2) $\theta(\mathbf{x}; \mathbf{a})$ *is Δ-stable and for every set $\{\mu_i \mid i < |\Delta| + \aleph_0\}$ of cardinalities all less than μ, we have $\prod_{i < |\Delta| + \aleph_0} \mu_i < \mu$ (e.g. $\mu = (2^{|T|})^+$ is such a cardinality),*

then there is a formula $\varphi^*(\mathbf{x}, \mathbf{y}) \in \Delta$ and a set $\{\mathbf{c}_i \mid i < \mu\}$ such that

(1) $R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}, \mathbf{c}_i), \Delta, \mu^+] \geq \beta$ *for all $i < \mu$;*

(2) $R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}, \mathbf{c}_i) \wedge \varphi^*(\mathbf{x}, \mathbf{c}_j), \Delta, \mu^+] < \beta$ *for $i \neq j < \mu$.*

Proof. Since $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta + 1$, there are $\{p_i \mid i \leq \mu\}$ explicitly contradictory Δ-types such that $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup p_i, \Delta, \mu^+] \geq \beta$ for all $i \leq \mu$. Using the hypothesis of the lemma, we get

Claim 3.7. *There are a set A, a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{p_i \mid i < \mu\} \subseteq S_\Delta(A)$ such that $p_i \varphi \neq p_j \varphi$ for every $i \neq j$ and $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup p_i, \Delta, \mu^+] \geq \beta$ holds for every $i < \mu$.*

Proof. The argument is by cases corresponding to the hypothesis of the lemma.

(1) Suppose that $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] < \infty$. Using Theorem 1.8 (by replacing the p_i's by extensions) we may assume that $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup p_i, \Delta, \mu^+] = \beta$ for all $i \leq \mu$. By the finite character there are $q_i \subseteq_{finite} p_i$ such that $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup q_i, \Delta, \mu^+] = \beta$ for all $i \leq \mu$.

Subclaim 3.8. *For every $i < \mu$ the set $\{j < \mu \mid q_i \subseteq p_j\}$ is bounded.*

Proof. Otherwise, there is $S \subseteq \mu$ of cardinality μ such that there exists i_0 satisfying $q_{i_0} \subseteq p_j$ for all $j \in S$. Namely by monotonicity we have that $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup q_{i_0} \cup p_j, \Delta, \mu^+] = \beta$ for every $j \in S$. This, by the definition of R, implies that $R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup q_{i_0}, \Delta, \mu^+] = \beta + 1$ in contradiction to the choice of q_i.   ⊣

Thus by induction on $i < \mu$ we can define an increasing sequence $\{j(i) < \mu \mid i < \mu\}$ such that $q_\xi \nsubseteq p_{j(i)}$ for all $\xi \leq i$. By renumerating the set $\{p_{j(i)} \mid i < \mu\}$ we may assume that $\{p_i \mid i < \mu\}$ also satisfies $q_i \nsubseteq p_j$ for all $i < j$. Since Δ is closed under Boolean operations, the formula $\varphi_i := \bigwedge q_i$ is a Δ-formula. Consider the mapping $i \mapsto \varphi_i$. Since the domain is a regular cardinal larger than $|T|$, there are a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and a set $S \subseteq \mu$ of cardinality μ such that $\varphi_i = \varphi$ for all $i \in S$. Since $q_i \nsubseteq p_j$ for

$j > i$, we have that $\varphi_i \not\subseteq p_j$, and since the types are complete this entails that $\neg\varphi_i \in p_j$.

We have shown that $i \neq j$ implies that $p_i\varphi \neq p_j\varphi$.

(2) For this part we do not use the local stability assumption; we just use that $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$ for every $\{\mu_i < \mu \mid i < |\Delta| + \aleph_0\}$. If there is no set of μ many types as required then let

$$\mu_\varphi := |\{p_i\varphi \; : \; i < \mu\}| < \mu \quad \text{for every} \quad \varphi \in \Delta.$$

However, the mapping $i \mapsto (p_i\varphi)_{\varphi\in\Delta}$ is an injection from μ into $\prod_{\varphi\in\Delta}\{p_i\varphi \; : \; i < \mu\}$, which contradicts the above cardinal arithmetic assumption.

$\dashv$

Claim 3.9. $R[\theta(\mathbf{x}; \mathbf{a}), \varphi(\mathbf{x}; \mathbf{y}), 2] < \omega$.

Proof. We have two arguments according to the two assumptions of Theorem 3.1:

(1) By monotonicity, $R[\theta(\mathbf{x}; \mathbf{a}), \varphi, \infty] \leq R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+]$, and by Theorem 1.14 (1), $R[\theta(\mathbf{x}; \mathbf{a}), \varphi, 2] < \omega$.

(2) Since $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable, it is $\{\varphi\}$-stable for every $\varphi \in \Delta$. By Theorem 1.14 this entails $R[\theta(\mathbf{x}; \mathbf{a}), \varphi, 2] < \omega$.

$\dashv$

By the above claim, $R[\theta(\mathbf{x}; \mathbf{a}), \varphi(\mathbf{x}; \mathbf{y}), 2]$ is a natural number. Let $n_0 := R[\theta(\mathbf{x}; \mathbf{a}), \varphi, 2]$.

Claim 3.10. *If the set* $\Phi := \{q \in S_\varphi(A) \mid \theta(\mathbf{x}; \mathbf{a})\cup q \text{ is consistent}\} = S_{\{\theta\},\{\varphi\}}(A)$ *has cardinality* $\kappa \geq \aleph_0$ *then there exists* $\{r_i \subseteq p_i \in \Phi \mid i < \kappa\}$ *such that*

(1) $|r_i| = n_0 + 2$,

(2) *for every* $i < \kappa$ *if* $q \in \Phi$ *and* $r_i \subseteq q$ *then* $q = p_i$, *and*

(3) $i \neq j$ *implies* $p_i \neq p_j$.

We present the proof of Claim 3.10 in the Appendix.

We know that for $\Phi := \{q \in S_\varphi(A) \mid R[\theta(\mathbf{x}; \mathbf{a}) \cup q, \Delta, \mu^+] \geq \beta\}$, $|\Phi| \geq \mu$ holds, since every $p_i\varphi \in \Phi$, $i < \mu$. Apply the previous claim to Φ and get $\{r_i : i \leq \mu\}$ as in the claim.

For $i < \mu$ define $\varphi^*(\mathbf{x}; \mathbf{c}_i) := \bigwedge r_i$ (since $\varphi^*(\mathbf{x}; \mathbf{y}) = \bigwedge_{n<n_0+2}\varphi(\mathbf{x}; \mathbf{y}_n)$ it does not depend on i).

Since $r_i \subseteq p_i$, for all $i < \mu$, and $R[\theta \cup p_i, \Delta, \mu^+] \geq \beta$, we get

$$R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i), \Delta, \mu^+] \geq \beta.$$

However, if $i \neq j$, then

$$R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_j), \Delta, \mu^*] < \beta.$$

Otherwise, by the Extension Property there would be a $q \in S_\varphi(A)$ extending $\varphi^*(\mathbf{x}; \mathbf{c}_i) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_j)$ such that $R[\theta(\mathbf{x}; \mathbf{a})\cup q, \Delta, \mu^+] \geq \beta$, which would contradict the uniqueness clause (2) from the construction of r_i's from Claim 3.10. $\dashv$

Theorem 3.11. *Suppose that μ is regular cardinal satisfying $\mu \geq (2^{|T|})^+$ and Δ is a set of formulas which is closed under Boolean operations.*
If

(1) $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] < \infty$ *or*
(2) $\theta(\mathbf{x}; \mathbf{a})$ *is Δ-stable and for every $\{\mu_i \mid i < |\Delta| + \aleph_0\}$ cardinalities all less than μ, $\prod_{i < |\Delta| + \aleph_0} \mu_i < \mu$*

holds, then $S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \geq R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+]$.

Proof. We show that for all ordinals α,

$$(*)_\alpha \qquad R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha \Rightarrow S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \geq \alpha.$$

We proceed by induction on α. For $\alpha = 0$, $(*)_\alpha$ holds by the definitions of ranks. For α a limit ordinal, $(*)_\alpha$ follows from the continuity of the ranks and the induction hypothesis. Suppose $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu] \geq \alpha = \beta + 1$ and that $(*)_\beta$ holds. Then by Lemma 3.6 we have a formula $\varphi^*(\mathbf{x}; \mathbf{y}) \in \Delta$ and a set $\{\mathbf{c}_i, i < \mu\}$ such that

(1) $R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i), \Delta, \mu^+] \geq \beta$ for all $i < \mu$;
(2) $R[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i) \wedge \varphi^*(\mathbf{x}, \mathbf{c}_j), \Delta, \mu^+] < \beta$ for $i \neq j < \mu$.

By $(*)_\beta$ this gives us $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i), \Delta] \geq \beta$. Consider the mapping taking $i \mapsto \text{tp}(\mathbf{c}_i, \mathbf{a}, \mathfrak{C})$. By our choice of μ and the pigeon-hole principle, there exists $I \subset \mu$ of cardinality ω such that $\{\mathbf{c}_i \mid i \in I\}$ are indiscernible over a. By Corollary 3.5 and 2 from above, we get $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_i) \wedge \{\varphi^*(\mathbf{x}; \mathbf{c}_j)\}, \Delta] < \beta$ for $i \neq j \in I$. By definition of $S1$ rank, we have $S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \geq \alpha$, completing the induction. $\dashv$

Corollary 3.12. *Suppose that μ is regular satisfying $\mu \geq (2^{|T|})^+$, p is a finite type, and Δ is a set of formulas which is closed under Boolean operations.*
If

(1) $R[p, \Delta, \mu^+] < \infty$ *or*
(2) p *is Δ-stable and for every $\{\mu_i : i < |\Delta| + \aleph_0\}$ cardinalities all less than μ, $\prod_{i < |\Delta| + \aleph_0} \mu_i < \mu$*

holds, then

$$S1[p, \Delta] = Deg[p, \Delta, \mu^+] = D[p, \Delta, \mu^+] = R[p, \Delta, \mu^+].$$

Proof. Follows from Theorems 2.11, 3.4, and 3.11. $\dashv$

Now, in particular, we obtain Theorem 0.1 which was announced in the Abstract.

4. CHARACTERIZATION OF LOCAL SUPERSTABILITY

Proposition 4.1. *For every type p and set of formulas Δ there exists a cardinality $\lambda_{\Delta,p}$ such that for every $\mu \geq \lambda_{\Delta,p}$ we have that $R[p, \Delta, \mu] = R[p, \Delta, \infty]$. Moreover, there exists a cardinality $\lambda_R(T)$ such that*

$$R[p, \Delta, \mu] = R[p, \Delta, \infty] \quad \text{for all } \mu \geq \lambda_R(T), \text{ for every } p \text{ and every } \Delta.$$

Similarly for the D-rank and Deg-rank, there exist $\lambda_D(T)$ and $\lambda_{Deg}(T)$ such that

$$D[p, \Delta, \mu] = D[p, \Delta, \infty] \quad \text{for all } \mu \geq \lambda_D(T) \text{ and}$$
$$Deg[p, \Delta, \mu] = Deg[p, \Delta, \infty] \quad \text{for all } \mu \geq \lambda_{Deg}(T).$$

Proof. Since $\mu > \lambda$ implies that $R[p, \Delta, \mu] \leq R[p, \Delta, \lambda]$, one can show that there exists $\lambda = \lambda_{\Delta,p}$ such that

$$R[p, \Delta, \mu] = R[p, \Delta, \infty] \quad \text{for every } \mu > \lambda_{\Delta,p}.$$

To obtain $\lambda_R(T)$, use finite character and invariance properties together with the fact that $|D(T)| \leq 2^{|T|}$.

The arguments for D and Deg-ranks are analogous.   ⊣

An interesting (and difficult) problem is finding the least $\lambda_R(T)$ and $\lambda_{Deg}(T)$ as in the above Proposition. In the next section, we will establish in particular that an upper bound on $\lambda_R(T)$ turns out to be $|T|^{++}$. This is much harder to prove than just the existence of $\lambda_R(T)$.

Theorem 4.2. *Let p be a finite type and let Δ be a set of formulas closed under Boolean operations. The following conditions are equivalent:*

(1) *p is Δ-superstable,*

(2) *p does not have the λ-weak tree property over Δ for some $\lambda \geq \aleph_0$,*

(3) *p is Δ-stable and $Deg[p, \Delta, |T|^{++}] < |T|^+$,*

(4) *p is Δ-stable and $Deg[p, \Delta, \infty] < |T|^+$,*

(5) *$R[p, \Delta, \infty] < |T|^+$,*

(6) *$R[p, \Delta, \infty] < \infty$,*

(7) *p is Δ-stable and $Deg[p, \Delta, \infty] < \infty$.*

Proof. (1) $\Rightarrow$ (2) By Proposition 2.16, if Δ has the λ-weak tree property over p for some $\lambda \geq \aleph_0$, then p is not Δ-superstable.

(2) $\Rightarrow$ (3) If p is not Δ-stable, then of course p has the λ-weak tree property over Δ for all infinite λ. The fact $Deg[p, \Delta, |T|^{++}] < |T|^+$ follows from Corollary 2.25.

(3) $\Rightarrow$ (4) Follows from monotonicity of the Deg-rank.

(4) $\Rightarrow$ (5) Suppose $Deg[p, \Delta, \infty] < |T|^+$. Let $\lambda_R(T)$ and $\lambda_{Deg}(T)$ be as in Proposition 4.1; let $\lambda := \max\{\lambda_R(T), \lambda_{Deg}(T)\}$ and let $\mu := (2^\lambda)^+$.

By the choice of μ, we have $Deg[p, \Delta, \mu^+] = Deg[p, \Delta, \infty] < |T|^+$. In addition, μ satisfies cardinal-arithmetic assumptions of the second clause of Corollary 3.12 and we are assuming that p is Δ-stable. Therefore according to the Corollary 3.12, $R[p, \Delta, \mu^+] = Deg[p, \Delta, \mu^+] < |T|^+$. By the choice of μ, we get $R[p, \Delta, \infty] = R[p, \Delta, \mu^+] < |T|^+$.

(5) $\Rightarrow$ (6) Immediate.

(6) $\Rightarrow$ (1) Let $\alpha := R[p, \Delta, \infty]$. By Proposition 4.1 there exist λ such that $\alpha := R[p, \Delta, \mu]$ for every $\mu \geq \lambda$. We will show that p is Δ-stable in every $\mu \geq |\Delta| + \lambda + |\alpha|$. Suppose for the sake of contradiction that this is not the case. Then by instability in μ there exists a set A of cardinality μ such that the cardinality of the set of (p, Δ)-types over A is greater than μ. Let $\{p_i \mid i < \mu^+\} \subseteq S_\Delta(A)$ be an enumeration of this set.

By the finite character of R for every i there is a finite $q_i \subseteq p_i$ such that $R[p \cup q_i, \Delta, \mu] = R[p \cup p_i, \Delta, \mu]$. Consider the mapping $i \mapsto q_i$. The domain is μ^+, which is greater than the cardinality of the range, which is

$$|\{\varphi_l(\mathbf{x}; \mathbf{a}_l) \ : \ l < k < \omega, \mathbf{a}_l \in A, \varphi_l(\mathbf{x}; \mathbf{y}_l) \in \Delta\}| = \mu.$$

There exists $S \subseteq \mu^+$ of cardinality μ^+ and there is a finite (p, Δ)-type q^* (with parameters from A) such that for every $i \in S$ we have that $q_i = q^*$. To summarize, we have that for every $i \in S$,

$$\alpha^* := R[p \cup q^*, \Delta, \mu] = R[p \cup q_i, \Delta, \mu] = R[p \cup p_i, \Delta, \mu].$$

Thus for every $\kappa < \mu$ there are $\{r_j \mid j \leq \kappa\} \subseteq \{p_i \mid i \in S\}$ which are distinct pairwise contradictory complete (p, Δ)-types extending q^* with $R[p \cup q^* \cup r_j, \Delta, \mu] = \alpha^*$ for every $j \leq \mu$. This by the definition of R implies that $R[p \cup q^*, \Delta, \mu] \geq \alpha^* + 1$ which is a contradiction.

Thus the statements (1)–(6) are proven to be equivalent. To conclude the proof, we show two implications:

(4) $\Rightarrow$ (7) Trivial.

(7) $\Rightarrow$ (6) Shown analogously to the implication (4) $\Rightarrow$ (5).

$\dashv$

Replacing p by $\mathbf{x} = \mathbf{x}$ and Δ by L, we get

Corollary 4.3. *The following conditions are equivalent:*

(1) T *is superstable,*

(2) T *does not have the λ-weak tree property for some $\lambda \geq \aleph_0$,*

(3) T *is stable and* $Deg[\mathbf{x} = \mathbf{x}, L, |T|^{++}] < |T|^+$,

(4) T *is stable and* $Deg[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$,

(5) $R[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$,

(6) $R[\mathbf{x} = \mathbf{x}, L, \infty] < \infty$,

(7) T *is stable and* $Deg[\mathbf{x} = \mathbf{x}, L, \infty] < \infty$.

Remark 4.4. Even more is true in Theorem 4.2 and Corollary 4.3: we can add conditions $R[p, \Delta, (2^{|T|})^{++}] < |T|^+$ and $R[\mathbf{x} = \mathbf{x}, L, (2^{|T|})^{++}] < |T|^+$, respectively; see Proposition 5.4.

5. BOUNDS ON RANKS

Proposition 5.1. *For any set of formulas Δ and cardinal λ,*

(1) *if* $Deg[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < \infty$, *then* $Deg[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < (2^{|T|})^+$;

(2) *if* $R[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < \infty$, *then* $R[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < (2^{|T|})^+$.

Proof. Use Theorems 1.8 and 2.9, finite character and the invariance together with the fact that $|D(T)| \leq 2^{|T|}$. $\dashv$

In this section we will show several improvements of the above proposition.

Proposition 5.2. *Let p be a set of formulas. For every $\lambda \geq |T|^{++}$*

$$Deg[p, \Delta, \lambda] = Deg[p, \Delta, \infty].$$

Proof. By finite character, we may assume that p is a finite type. By monotonicity of $Deg[p, \Delta, \cdot]$ and Proposition 4.1, it suffices to show that for all $\mu \geq \lambda$

$$Deg[p, \Delta, \lambda] \geq \alpha \implies Deg[p, \Delta, \mu^+] \geq \alpha \text{ for every } \alpha.$$

Suppose $Deg[p, \Delta, \lambda] \geq \alpha$. Since $\lambda > |T|^+$, by monotonicity

$$Deg[p, \Delta, |T|^{++}] \geq \alpha.$$

There are two possibilities: $\alpha < |T|^+$ or $\alpha \geq |T|^+$.

In the first case we use the Tree Characterization Lemma to obtain a (Δ, α)-function h such that $\Gamma_{|T|^+}^{ds(\alpha)}(\theta, h)$ is consistent, where $\theta = \bigwedge p$. Using Proposition 2.18 we get that also $\Gamma_\mu^{ds(\alpha)}(p, h)$ is consistent and the Tree Characterization Lemma gives us that $Deg[p, \Delta, \mu^+] \geq \alpha$.

In the second case we use the Normalization Lemma to get that for every α there exists an (Δ, α)-function h such that

$$\Gamma_{|T|^+}^{ds(\alpha)}(\theta, h) \quad \text{is consistent and} \quad h(\eta) = \langle \psi_{\ell(\eta)}, n_{\ell(\eta)} \rangle.$$

Since h does not depend on α using the compactness theorem and the consistency of $\Gamma_{|T|^+}^{ds(\alpha)}(\theta, h)$ we can show that

$$\Gamma_{\mu^+}^{ds(\alpha)}(\theta, h) \quad \text{is consistent for every } \alpha.$$

Thus by the Tree Characterization Lemma we have $D[p, \Delta, \mu^+] \geq \alpha$.   ⊣

Theorem 5.3.   (1) *If Δ is a set of formulas closed under Boolean operations and*

$$R[p, \Delta, \infty] < \infty, \text{ then } R[p, \Delta, \infty] < |T|^+.$$

(2) *If $Deg[p, \Delta, \infty] < \infty$, then $Deg[p, \Delta, \infty] < |T|^+$.*

Proof. (1) By finite character, there is a finite $p_0 \subset p$ such that $R[p_0, \Delta, \infty] < \infty$. Apply Theorem 4.2 to get $R[p_0, \Delta, \infty] < |T|^+$. By monotonicity, $R[p, \Delta, \infty] < |T|^+$.

(2) Suppose $Deg[p, \Delta, \infty] < \infty$, but $Deg[p, \Delta, \infty] \geq |T|^+$. By monotonicity, we get $Deg[p, \Delta, |T|^{++}] \geq |T|^+$. By Corollary 2.24, we get $Deg[p, \Delta, |T|^{++}] = \infty$ and from Proposition 5.2 we conclude $Deg[p, \Delta, \infty] = \infty$, a contradiction to our hypothesis.   ⊣

Proposition 5.4. *Let p be a type and let Δ be a set of formulas closed under Boolean operations. For every $\lambda \geq (2^{|T|})^{++}$*

$$R[p, \Delta, \lambda] = R[p, \Delta, \infty].$$

Proof. By monotonicity, $R[p, \Delta, \lambda] \geq R[p, \Delta, \infty]$. If $R[p, \Delta, \infty] = \infty$, we have nothing to prove. Otherwise, by Theorem 4.2 we get that p is Δ-superstable; in particular, p is Δ-stable and we are in the conditions of Corollary 3.12, clause (2). Taking into account Proposition 5.2, by the choice of λ we get

$$R[p, \Delta, \lambda] = Deg[p, \Delta, \lambda] = Deg[p, \Delta, \infty] = R[p, \Delta, \infty].$$

   ⊣

Remark 5.5. This proposition enables us to prove Theorem 0.2 from Abstract, because when Δ is finite, the claim follows from Theorem 1.14 if $R[p, \Delta, \lambda] \geq \omega$, and from Lemma II.2.9 in [Sha] (using compactness) if $R[p, \Delta, \lambda] < \omega$.

Thus we may assume Δ is infinite, $|\Delta| = |T|$, and by Corollary II.1.8 in [Sha] without loss of generality Δ is closed under Boolean operations, so we are able to apply Proposition 5.4.

Our goal now is to obtain the equality of R and Deg-ranks under weaker assumptions than those of Corollary 3.12. Namely, we prove our Main Theorem 3.1:

Proof. Denote by $\theta(\mathbf{x}; \mathbf{a})$ the formula $\bigwedge p$. By Theorem 2.11 it is enough to show that for every α

$$(*)_\alpha \quad R[p, \Delta, \mu^+] \geq \alpha \implies Deg[p, \Delta, \mu^+] \geq \alpha.$$

We show that it suffices to prove $(*)_\alpha$ just for $\alpha < |T|^+$:

Claim 5.6. *Suppose that $(*)_\alpha$ holds for every $\alpha < |T|^+$.*
If $R[p, \Delta, \mu^+] \geq |T|^+$ then $Deg[p, \Delta, \mu^+] = \infty$

Proof. Since $R[p, \Delta, \mu^+] \geq |T|^+$ we have by the definition of R that $R[p, \Delta, \mu^+] \geq \alpha$ for every $\alpha < |T|^+$. By $(*)_\alpha$ and the definition of Deg this gives $Deg[p, \Delta, \mu^+] \geq |T|^+$. By an application of Corollary 2.24(1) we are done.   ⊣

If $\alpha = 0$, then $(*)_\alpha$ is true by the definitions of the ranks. In the case when α is limit, use the definition of R, the inductive hypothesis and the definition of Deg.

For $\alpha = \beta + 1$, we have $R[p, \Delta, \mu^+] \geq \beta + 1$ and we are in the conditions of Lemma 3.6. Thus, we get a formula $\varphi^*(\mathbf{x}, \mathbf{y}) \in \Delta$ and a set $\{\mathbf{c}_i \mid i < \mu\}$ such that

(1) $R[p \cup \{\varphi^*(\mathbf{x}, \mathbf{c}_i)\}, \Delta, \mu^+] \geq \beta$ for all $i < \mu$;
(2) $R[p \cup \{\varphi^*(\mathbf{x}, \mathbf{c}_i) \wedge \varphi^*(\mathbf{x}, \mathbf{c}_j)\}, \Delta, \mu^+] < \beta$ for $i \neq j < \mu$.

From the conditions of the theorem it follows that the $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable. By local stability of $\theta(\mathbf{x}; \mathbf{a})$, as in Theorem 3.4 we conclude that there is $n_1 < \omega$ witnessing the failure of the order property for $\varphi^*(\mathbf{x}; \mathbf{y})$, i. e. , for no $\{\mathbf{d}_l : l < n_1\}$

$$\bigwedge_{l < n_1} \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k < n_1} \varphi^*(\mathbf{x}; \mathbf{d}_k)^{\text{if } (k \geq l)} \right]$$

holds.

For every $\bar{\imath} = \langle \bar{\imath}[0], \dots, \bar{\imath}[n_1] \rangle \in [\mu]^{n_1+1}$ define:

$$\psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}}) := \bigwedge_{l < n_1} \neg \varphi^*(\mathbf{x}; \mathbf{c}_{\bar{\imath}[l]}) \wedge \varphi^*(\mathbf{x}; \mathbf{c}_{\bar{\imath}[n_1]}).$$

Using the Ultrametric property of the R-rank, by the choice of $\varphi^*(\mathbf{x}; \mathbf{y})$ we get $R[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}}), \Delta, \mu^+] \geq \beta$ for every $\bar{\imath} \in [\mu]^{n_1+1}$. (The argument is similar to that of Theorem 3.4.)

Applying the induction hypothesis to $\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}})$, we get

$$\forall \bar{\imath} \in [\mu]^{n_1+1} \quad Deg[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}}), \Delta, \mu^+] \geq \beta.$$

By the Tree Characterization Lemma for Deg, for every $\bar{\imath}$ there is a (Δ, β)-function $h_{\bar{\imath}}$ such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}}), h_{\bar{\imath}})$ is consistent.

Also, since $R[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta$, by the induction hypothesis, $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \beta$. So there is a (Δ, β)-function h_* such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x}; \mathbf{a}), h_*)$ is consistent.

Now for every $\bar{\imath}$ define a (Δ, α)-function $h^{\bar{\imath}}$:

$$\text{for } \eta \in ds(\alpha) \quad h^{\bar{\imath}}(\eta) := \begin{cases} h_*(\eta) & \text{if } \eta \in ds(\beta); \\ h_{\bar{\imath}}(\nu) & \text{if } \eta = \langle\beta\rangle^\frown\nu; \\ \langle\psi, n_1\rangle & \text{if } \eta = \langle\beta\rangle. \end{cases}$$

It is enough to show that there is an $\bar{\imath} \in [\mu]^{n_1+1}$ such that $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h^{\bar{\imath}})$ is consistent; or using compactness, it is enough to show $\Gamma_{\aleph_0}^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h^{\bar{\imath}})$ is consistent.

Suppose not; then for each $\bar{\imath}$ there is a finite $u(\bar{\imath}) \subseteq ds(\alpha)$ such that $\Gamma_{\aleph_0}^{u(\bar{\imath})}(\theta(\mathbf{x}; \mathbf{a}), h^{\bar{\imath}})$ is inconsistent. Define, for $\bar{\imath} \in [\mu]^{n_1+1}$,

$$F(\bar{\imath}) := \langle u(\bar{\imath}), h^{\bar{\imath}} u(\bar{\imath})\rangle..$$

Now apply the Combinatorial Lemma from the Appendix B to the function F, $|\text{dom}(F)| = \mu$ (regular $\geq |T|^+$), $|\text{rge}(F)| \leq |T|$ (since $|\alpha| \leq |T|$), and get $\delta < \mu$ and an increasing sequence $\{\gamma(k) < \delta : k < \omega\}$ such that for every $l_0 < \cdots < l_{n_1} < \omega$,

$$F(\gamma(l_0), \ldots, \gamma(l_{n_1-1}), \gamma(l_{n_1})) = F(\gamma(l_0), \ldots, \gamma(l_{n_1-1}), \delta).$$

Claim 5.7. *The set* $\{\psi(\mathbf{x}; \mathbf{c}^{\langle\gamma(0), \ldots, \gamma(n_1-1), \gamma(n_1+l)\rangle}) : l < \omega\}$ *is* n_1-*contradictory over* $\theta(\mathbf{x}; \mathbf{a})$.

Proof. By Theorem I. 2. 4. in [Sha] we may assume that $\{\mathbf{c}_{\gamma(l)} \mid l < \omega\}$ is a Δ^*-n_1-indiscernible sequence over $\mathbf{a}$, where $\Delta^* := \{\psi_n(\mathbf{y}_0, \ldots, \mathbf{y}_{n_1}; \mathbf{z}) \mid n \leq n_1\}$ and

$$\psi_n(\mathbf{y}_0, \ldots, \mathbf{y}_{n_1}; \mathbf{z}) := \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{z}) \wedge \bigwedge_{k \leq n_1} \varphi^*(\mathbf{x}; \mathbf{y}_n)^{\text{if } (k \geq n)} \right].$$

If $\{\psi(\mathbf{x}; \mathbf{d}_l) \mid l < \omega\}$ is not n_1-contradictory over $\theta(\mathbf{x}; \mathbf{a})$, then there are $\{l(k) \mid k < n_1\}$ such that

$$\models \exists \mathbf{x} \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k < n_1} \psi(\mathbf{x}; \mathbf{d}_{l(k)}) \right],$$

i. e. , using the definition of ψ and $\mathbf{d}_{l(k)}$,

$$\models \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k < n_1} \neg\varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(k)}) \wedge \bigwedge_{k < n_1} \varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(n_1+l(k))}) \right].$$

In particular, for every $n \leq n_1$,

$$\models \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k<n} \neg \varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(k)}) \wedge \bigwedge_{k=n}^{n_1-1} \varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(n_1+l(k))}) \right].$$

Now by the Δ^*-n_1-indiscernibility of $\{\mathbf{c}_{\gamma(l)} \mid l < \omega\}$ over $\mathbf{a}$, we get

$$\models \left[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{k<n} \neg \varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(k)}) \wedge \bigwedge_{k=n}^{n_1-1} \varphi^*(\mathbf{x}; \mathbf{c}_{\gamma(k)}) \right],$$

which is a contradiction to the choice of n_1, witnessing the failure of the order property over $\theta(\mathbf{x}; \mathbf{a})$ for φ^*. ⊣

But then, for

$$\langle u, h \rangle := F(\gamma(l_0), \dots, \gamma(l_{n_1-1}), \delta) = F(\gamma(l_0), \dots, \gamma(l_{n_1-1}), \gamma(n_1 + l)),$$

for every $l < \omega$, we may assume that u is closed under taking initial segments and $\Gamma^u_{\aleph_0}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent:

if $\{\mathbf{b}^*_{\eta,\nu} : \eta \in ds(\beta), \nu \in {}^{\ell(\eta)}\aleph_0\} \models \Gamma^{ds(\beta)}_{\aleph_0}(\theta(\mathbf{x}; \mathbf{a}), h_*)$, and
$\{\mathbf{b}^i_{\eta,\nu} : \eta \in ds(\beta), \nu \in {}^{\ell(\eta)}\aleph_0\} \models \Gamma^{ds(\beta)}_{\aleph_0}(\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}^{\bar{\imath}}), h_{\bar{\imath}})$,

then by putting $\bar{\imath}_l := \langle \gamma(0), \dots, \gamma(n_1 - 1), \gamma(n_1 + l) \rangle$, $\mathbf{d}_l := \mathbf{c}^{\bar{\imath}_l}$, $l < \omega$, and

$$\mathbf{b}_{\eta,\nu} := \begin{cases} \mathbf{d}_l & \text{if } \eta = \langle \beta \rangle, \nu = \langle l \rangle; \\ \mathbf{b}^{\bar{\imath}_l}_{\xi,\zeta} & \text{if } \eta = \langle \beta \rangle^\frown \xi, \nu = \langle l \rangle^\frown \zeta; \\ \mathbf{b}^*_{\eta,\nu} & \text{if } \eta \in ds(\beta), \end{cases}$$

for $\eta \in u$, $\nu \in {}^{\ell(\eta)}\aleph_0$, we have $\{\mathbf{b}_{\eta,\nu}\} \models \Gamma^u_{\aleph_0}(\theta(\mathbf{x}; \mathbf{a}), h)$. This contradicts the choice of u and we are done. ⊣

6. $S2$-RANK

The following is a generalization of the $S1$ rank.

Definition 6.1. Let p be a set of formulas in $\mathbf{x}$.
$S2[p, \Delta] \geq 0$ if p is consistent.
$S2[p, \Delta] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $S2[p, \Delta] \geq \alpha$.
$S2[p, \Delta] \geq \alpha + 1$ if for every finite $p_0 \subset p$ there exists $\psi(\mathbf{x}, \mathbf{y}) \in \Delta$, $l < \omega$ and $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\text{dom}(p_0)$ such that

(1) $S2[p_0 \cup \{\psi(\mathbf{x}, \mathbf{b}_n)\}, \Delta] \geq \alpha$ for every $n < \omega$ and
(2) $S2[p_0 \cup \{\bigwedge_{n\in u} \psi(\mathbf{x}, \mathbf{b}_n)\}, \Delta] < \alpha$ whenever $u \in {}^l\omega$.

Remark 6.2. Given $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in Fml(L(T))\}$, closed under Boolean operations, and a set of formulas p in $\mathbf{x}$ such that p is Δ-stable, one can show that $S2[p, \Delta] = S1[p, \Delta]$.

The inequality $S1[p, \Delta] \leq S2[p, \Delta]$ is trivial and holds without any restrictions on Δ and p.

One then can show that $S2[p, \Delta] \leq Deg[p, \Delta, \lambda]$, where λ is infinite cardinal, using a similar argument to that which was used to prove Theorem 3.4.

APPENDIX A. PROOF OF CLAIM 3.10

Proof. (Of the Claim 3.10) Define r_i, p_i by induction on $i < \kappa$. Suppose we have defined $\{r_j, p_j : j < i\}$. Now define an equivalence relation E_i:

$$\text{for } \mathbf{a}, \mathbf{b} \in {}^{\ell(\mathbf{y})}A, \quad \mathbf{a}E_i\mathbf{b} \Leftrightarrow (\forall j < i)[\varphi(\mathbf{x}; \mathbf{a}) \in p_j \Leftrightarrow \varphi(\mathbf{x}; \mathbf{b}) \in p_j].$$

Subclaim A.1. $|{}^{\ell(\mathbf{y})}A / E_i| < \kappa.$

Proof. For every $j < i$, let $\mathbf{c}_j \models p_j$. Suppose for the sake of contradiction that $\{\mathbf{a}_k/E_i \mid k < \kappa\}$ are distinct equivalence classes.

Let $q_k := tp_\varphi(\mathbf{a}_k/ \cup_{j<i} \mathbf{c}_j)$. Let $k \neq l < \kappa$ be given. Then $\mathbf{a}_k/E_i \neq \mathbf{a}_l/E_i$. By the definition of E_i, there is a $j < i$ such that $\neg(\varphi(\mathbf{x}; \mathbf{a}_k) \in p_j \leftrightarrow \varphi(\mathbf{x}; \mathbf{a}_l) \in p_j)$. Without loss of generality we may assume that $\varphi(\mathbf{x}; \mathbf{a}_k) \in p_j$ and $\neg\varphi(\mathbf{x}; \mathbf{a}_l) \in p_j$. Since $\mathbf{c}_j \models p_j$, we have that $\varphi(\mathbf{c}_j; \mathbf{y}) \in q_k$ and $\neg\varphi(\mathbf{c}_j; \mathbf{y}) \in q_l$. Thus for $k \neq l < \kappa$, $q_k \neq q_l$.

Since $i < \kappa$, we get that $|\cup_{j<i} \mathbf{c}_j| < \kappa$. But $|S_\varphi(\cup_{j<i}\mathbf{c}_j)| \geq \kappa$. By Lemma 1.10 this contradicts our stability assumption. $\dashv$

Subclaim A.2. $|\{q \in \Phi : \mathbf{a}E_i\mathbf{b} \Rightarrow [\varphi(\mathbf{x}; \mathbf{a}) \in q \Leftrightarrow \varphi(\mathbf{x}; \mathbf{b}) \in q]\}| < \kappa.$

Proof. Let

$$S := \{q \in \Phi \mid \forall \mathbf{a}, \mathbf{b} \in {}^{\ell(\mathbf{y})}A \ (\mathbf{a}E_i\mathbf{b} \Rightarrow (\varphi(\mathbf{x}; \mathbf{a}) \in q \Leftrightarrow \varphi(\mathbf{x}; \mathbf{b}) \in q))\}.$$

For the sake of contradiction, suppose that $|S| = \kappa$. Let $\bar{A} := {}^{\ell(\mathbf{y})}A /E_i$. By subclaim A.1, we have that $|\bar{A}| < \kappa$; but, we have an obvious injection $S \to S_\varphi(\bar{A})$, so $|S_\varphi(\bar{A})| \geq \kappa$, which is a contradiction to the stability assumption. $\dashv$

By induction on $l \leq n_0$ define $p^l \in \Phi$, $\mathbf{a}_{l+1}^i$ and $t(l) \in \{0, 1\}$ such that:

- $r_l^i := \{\varphi(\mathbf{x}; \mathbf{a}_0^i), \varphi(\mathbf{x}; \mathbf{a}_1^i)^{t(0)}, \dots, \varphi(\mathbf{x}; \mathbf{a}_{l+1}^i)^{t(l)}\} \subseteq p^l$;
- $R[\theta(\mathbf{x}; \mathbf{a}) \cup r_l^i, \varphi, 2] \leq n_0 - l$ or $\exists! q \in \Phi, q \supseteq r_l^i$.

For $l = 0$, by Subclaims A.1 and A.2 there exists p^0 and $\mathbf{a}_0^i, \mathbf{a}_1^i$ such that $\mathbf{a}_0^i E_i \mathbf{a}_1^i$ and $\varphi(\mathbf{x}; \mathbf{a}_0^i) \in p^0$, $\neg\varphi(\mathbf{x}; \mathbf{a}_1^i) \in p^0$. Put $t(0) := 1$.

Suppose we have defined everything for l. If $\exists! q \in \Phi, q \supseteq r_l^i$, let $\mathbf{a}_{l+1}^i = \mathbf{a}_l^i$, $t(l+1) = t(l)$ and $p^{l+1} = q$. If $|\{q \in \Phi : r_l^i \subseteq q\}| > 1$, then, since $\Phi \subseteq S_\varphi(A)$, for some $\mathbf{a}_{l+1}^i \in {}^{\ell(\mathbf{y})}A$ there are $q_0, q_1 \in \Phi$, $r_l^i \cup \{\varphi(\mathbf{x}; \mathbf{a}_{l+1}^i)^t)\} \subseteq q_t$. ¿From definition of R, there is $t(l+1) \in \{0, 1\}$ such that

$$R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup r_l^i \cup \{\varphi(\mathbf{x}; \mathbf{a}_{l+1}^i)^{t(l+1)}\}, \varphi, 2] < R[\{\theta(\mathbf{x}; \mathbf{a})\} \cup r_l^i, \varphi, 2] \leq n_0 - l$$

and put $p^{l+1} := q_{t(l+1)}$.

Now, it is clear from the construction that $r_{n_0+1}^i$ has a unique extension in Φ and $|r_{n_0+1}^i| \leq n_0 + 2$. Put $r_i := r_{n_0+1}^i$, adding eventually some formulas from the unique extension of $r_{n_0+1}^i$ to satisfy the requirement $|r_i| = n_0 + 2$. $\dashv$

Appendix B. A Combinatorial Theorem

Claim B.1 (A combinatorial lemma). *Let $\kappa \geq \aleph_0$ and suppose that $\mu > \kappa$ is regular. For every $n < \omega$ and every $F : [\mu]^{n+1} \to \kappa$ there exists a limit ordinal $\delta < \mu$ such that for every $\xi < \mu$ satisfying $\xi \geq \delta$ there exists an increasing $\{\gamma(k) \mid k < \omega\} \subseteq \delta$ such that for any $\ell_0 < \cdots < \ell_n < \omega$ we have that*

$$F(\gamma(\ell_0), \ldots, \gamma(\ell_n)) = F(\gamma(\ell_0), \ldots, \gamma(\ell_{n-1}), \xi).$$

Proof. Let $\chi > \aleph_0$ be a regular cardinal large enough such that $\{\mu, F\} \subseteq H(\chi)$. Let

$$\mathfrak{B} := \langle H(\chi), \in, \mu, F, \alpha \rangle_{\alpha < \kappa}.$$

Where μ stands for a unary predicate interpreted by the set of ordinals less than μ, F is interpreted by the function F and α is an individual constant interpreted by the corresponding ordinal.

By the Downward Löwenheim-Skolem-Tarski theorem pick an increasing and continuous elementary chain $\{\mathfrak{B}_i \prec \mathfrak{B} \mid i < \mu\}$ satisfying

(1) $\|\mathfrak{B}_i\| < \mu$ and
(2) $i \subseteq \mu^{\mathfrak{B}_i}$ for all $i < \mu$.

Since we have that $\mu = \bigcup_{i < \mu} \mu^{\mathfrak{B}_i}$, there is a closed unbounded subset C of the set $\{\delta < \mu \mid \delta = \mu^{\mathfrak{B}_\delta}\}$.

Pick $\delta \in C$. We show that this δ is as required in the claim. Since μ is a limit ordinal we have that

$$\mathfrak{B} \models \forall x[x \in \mu \to x + 1 \in \mu].$$

Since $\mathfrak{B}_\delta \prec \mathfrak{B}$ we have

$$\mathfrak{B}_\delta \models \forall x[x \in \mu \to x + 1 \in \mu].$$

Since $\delta \in C$ we get that

$$\mathfrak{B}_\delta \models \forall x[x \in \delta \to x + 1 \in \delta].$$

Thus δ is limit.

The definition of $\{\gamma(k) \mid k < \omega\}$ is by induction on k:

Fix any $\gamma(0) < \gamma(1) < \cdots < \gamma(n-1) < \delta$. Suppose $\{\gamma(j) \mid j \leq k\}$ are defined (for $k \geq n - 1$). Let $\psi(x)$ be the following formula (with parameters $\gamma(0), \ldots, \gamma(k)$):

$$\bigwedge \{F(\gamma(\ell_0), \ldots, \gamma(\ell_{n-1}), x) = \alpha \mid \ell_0 < \cdots < \ell_{n-1} \leq k, \alpha < \kappa,$$
$$\mathfrak{B} \models F(\gamma(\ell_0), \ldots, \gamma(\ell_{n-1}), \xi) = \alpha\}.$$

Since $\mathfrak{B} \models \psi(\xi)$, we have that $\mathfrak{B} \models \exists x \psi(x)$. Since all the parameters of ψ are in $\mathfrak{B}_\delta$, $\mathfrak{B}_\delta \prec \mathfrak{B}$ and $\xi > \gamma(k)$ there exists $\gamma(k+1) < \delta$ such that $\gamma(k+1) > \gamma(k)$ and $\mathfrak{B} \models \psi(\gamma(k+1))$. $\dashv$

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