SHELAH'S STABILITY SPECTRUM AND HOMOGENEITY SPECTRUM IN FINITE DIAGRAMS

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ABSTRACT. We present Saharon Shelah's Stability Spectrum and Homogeneity Spectrum theorems, as well as the equivalence between the order property and instability in the framework of Finite Diagrams. Finite Diagrams is a context which generalizes the first order case. Localized versions of these theorems are presented. Our presentation is based on several papers; the point of view is contemporary and some of the proofs are new. The treatment of local stability in Finite Diagrams is new.

1. INTRODUCTION

Saharon Shelah's Finite Diagrams Stable in Power [Sh3], published in 1970, is one of the seminal articles in model theory. It contains a large number of key ideas which have shaped the development of classification theory. The model-theoretic framework of the paper is more general than the first order case. However, while all the particular cases of the results in the first order case can be found in several more recent publications of Saharon Shelah as well as countless expositions, the non first order content of [Sh3] is still not available in a concise form.

The primary purpose of this paper is to present, in this more general framework, most of the stability results of [Sh3], together with the order/stability dichotomy from [Sh12], and the homogeneity spectrum appearing in The Lazy Model Theorist's Guide To Stability [Sh54]. A secondary purpose is to present the necessary background to [Le] and [GrLe2]. This is done in a contemporary and self-contained manner, and includes improvements and techniques from [Sh b], [Sh300], and [Gr1]. Finally, with very little additional work, we *localize* all the theorems and obtain local versions of the Stability Spectrum Theorem and the Homogeneity Spectrum in Finite Diagrams. The study of local stability in more general frameworks has been started in [GrLe1].

The framework introduced by S. Shelah in [Sh3] is the study of classes of models of a finite diagram. These classes are described in more detail below. Such classes are examples of *nonelementary* classes and the results presented in this paper belong to what Shelah calls the *classification theory for nonelementary classes*. The word nonelementary refers to the fact that the compactness theorem fails. While many of the questions of classification theory for first order theories have been solved (see [Sh b]), classification theory for nonelementary classes is still under-developed. This is not to say

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that the subject is small or not interesting. Thousands of pages have been devoted to its questions: See for example [BaSh1],[BaSh2], [BaSh3], [Gr1], [Gr2], [GrHa], [GrLe1], [GrLe2], [GrSh1], [GrSh2], [HaSh], [HySh], [Ke], [Ki], [KlSh], [Le], [MaSh], [Sh3], [Sh48], [Sh87a], [Sh87b], [Sh88], [Sh tape], [Sh299], [Sh300], [Sh394], [Sh472], [Sh576] and Shelah's forthcoming book [Sh h]. The techniques used are usually set-theoretic and combinatorial in nature, although more recently, new ideas coming from geometric stability theory are being imported. The failure of the compactness theorem for a class of models makes their model theory delicate and sometimes sensitive to the axioms of set theory. This is one of the reasons why some additional assumptions are often made; a "monster model", set-theoretic assumptions, amalgamation properties, and so on.

Let us describe briefly what is meant by the class of models of a finite diagram. Two perspectives are given below.

Given a first order theory T and a model M of T, the *finite diagram* of the model M is the set of complete types over the empty set realized in M. Fix a set D of complete T-types and consider the class of models whose finite diagram is a subset of D. Such models are called D*-models* for convenience. In another language, we study the class of models omitting all the types over the empty set which do not belong to D. There are several connections between the class of D-models and the class of models of some theory $T^* \subseteq L_{\lambda^+,\omega}$, for a cardinal λ. First, the class of D-models can be axiomatized by some theory $T \subseteq L_{\lambda^+,\omega}$, provided $\lambda \geq |D(T) \setminus D|$. On the other hand, from the point of view of Shelah's conjecture (see below) for example, the class of models of a countable theory $T^* \subseteq L_{\omega_1,\omega}$ is equivalent to the class of D-models of a countable first order theory T, where D is the set of isolated types over the empty set.

Both in [Sh3] and [Sh54], S. Shelah studied these classes under an additional assumption. Let us say a few words about exactly what this additional assumption is (it takes two equivalent forms in [Sh3] and [Sh54], and yet another equivalent formulation is given here). Since the compactness theorem fails for this class of models, it is crucial to have a good understanding of what the *meaningful* types are, i.e. which types can be realized by D-models. A corollary of the compactness theorem is that given a model M and a type p over a subset A of M, it is possible to find an elementary extension N of M in which p is realized. This fails, in general, for the class just described. There is a natural obstacle why this cannot work in general: Suppose p is a complete type over a set of parameters A, where A is a subset of a D-model M. Suppose there is a D-model N containing M in which p is realized, say by the sequence $\bar{c}$. Then, since $A \cup \bar{c} \subseteq N$ and N is a D-model, necessarily, all the subsequences of the set $A \cup \bar{c}$ realize (over the empty set) types that belong to D. The assumption that Shelah made (although not in those terms) is that this is the only restriction. This class of models, with the additional assumption on types, is the framework that S. Shelah calls *finite diagrams*. Note that when D is the set $D(T)$ of all complete T-types over the empty set, then this is the first order case.

An alternative way of looking at this framework is as follows. Given a theory T, fix a large *homogeneous* model $\mathfrak{C}$ of T. In general, $\mathfrak{C}$ is not saturated. Let D be the diagram of $\mathfrak{C}$. Then, the class of D-models can be assumed to be the class of elementary submodels of $\mathfrak{C}$ and above meaningful types are the ones realized in $\mathfrak{C}$. Note that when $\mathfrak{C}$ is saturated, then this is the first order case.

Using the first order case as a guide, there are four important results in Stability Theory all due to S. Shelah. See [Sh b].

- A theory T is stable if and only if it does not have the order property.
- If a theory T is stable in λ, then given any set of finite sequences I of cardinality λ^+ and a set A of cardinality λ there exists a subset $J \subseteq I$ of cardinality λ^+ indiscernible over A.
- (The Stability Spectrum) For a theory T, either T is not stable or T is stable and there exist cardinals $\kappa(T)$ and $\lambda(T)$ satisfying $\kappa(T) \leq |T|^+$ and $\kappa(T) \leq \lambda(T) \leq 2^{|T|}$ such that T is stable in μ if and only if $\mu \geq \lambda(T)$ and $\mu^{<\kappa(T)} = \mu$.
- (The Saturation Spectrum) A theory T has a λ-saturated model of cardinality λ if and only if $\lambda \geq |D(T)|$ and either $\lambda^{<\lambda} = \lambda$ or T is stable in λ.

This paper contains Shelah's generalizations of above theorems to the class of models of finite diagrams. The first two results use the notion of *splitting* and can be generalized without too much difficulty to this context. As to the last two, the optimal versions rely on the notion of *forking*. However, forking only works in settings where the compactness theorem holds. To remedy this, Shelah introduced the notion of *strong splitting*, which predates forking (and dividing). Since *strong splitting* does not satisfy all the properties of forking, the proofs are more intricate and combinatorial in flavor.

Classes of models of a finite diagram are important also because they provide a natural test-case to generalize ideas from first order logic to more general nonelementary classes. On the one hand, many of the technical difficulties arising from the failure of the compactness theorem are present. On the other hand, the model theory is more manageable as we have a good understanding of types. Note also that, in contrast to other nonelementary contexts, this work is completely done within ZFC. We added a discussion on the strength of the main assumption of Finite Diagrams after Hypothesis 2.5.

The classification theory for finite diagrams has been the focus of some activity recently. The focus of [Sh3] was stable diagrams. In [HySh], Saharon Shelah and Tapani Hyttinen develop a context corresponding to superstability. They prove the existence of types over the realization of which strong-splitting satisfies the axioms of a pregeometry. In [Le], Olivier Lessmann introduced a rank for the $\aleph_0$-stable case. The finite diagrams for which the rank is bounded are called *totally transcendental*. Totally transcendental diagrams behave surprisingly like totally transcendental first order theories; there is a nicely behaved dependence relation, pregeometries and the methods of John T. Baldwin and Alistair Lachlan [BaLa] can be adapted to give *geometric* proofs of categoricity, construct nonisomorphic models, as well as other applications. In a work in preparation [GrLe2], we prove the Main Gap for totally transcendental diagrams. The decomposition theorem is in fact an application of a more general decomposition theorem.

2. The Framework of Finite Diagrams

The notation is standard. Abbreviations like AB stands for $A \cup B$, and $A\bar{b}$ for $A \cup \{\mathrm{ran}(\bar{b})\}$. When M is a model, $\|M\|$ stands for the cardinality of M. The notation $A \subseteq M$ means that A is a subset of the universe of M.

Let T be a first order complete theory in a language L. Denote by $L(T)$ the set of first order formulas in L. Let $\bar{M}$ be the a very saturated model of T. For $\Delta \subseteq L$, $A \subseteq M$, and a (not necessarily finite) sequence $\bar{a} \in M$, define the *Δ-type of $\bar{a}$ over A in M* by

$$\mathrm{tp}_\Delta(\bar{a}/A, M) = \{\phi(\bar{x}, \bar{b}) \mid \bar{b} \in A, \phi(\bar{x}, \bar{y}) \text{ or } \neg\phi(\bar{x}, \bar{y}) \in \Delta, \text{ and } M \models \phi[\bar{a}, \bar{b}]\}.$$

When Δ is $L(T)$ it is omitted and when M is $\bar{M}$, it is omitted also.

Definition 2.1. (1) The *finite diagram* of A is

$$D(A) = \{\mathrm{tp}(\bar{a}/\emptyset) \mid \bar{a} \in A, \bar{a} \text{ finite }\}.$$

Such sets will be denoted by D and called *finite diagrams*.

(2) The set A is a *D-set* if $D(A) \subseteq D$. The model M is a *D-model* if $D(M) \subseteq D$.

(3) We let $S^n_\Delta(A) = \{\mathrm{tp}_\Delta(\bar{c}/A) \mid \bar{c} \in \bar{M}, \ell(\bar{c}) = n\}$, for $\Delta \subseteq L(T)$. When $\Delta = L(T)$ it is omitted. When $n = 1$ it is omitted. A type $p \in S^n(A)$ is called a *D-type* if and only if $A \cup \bar{c}$ is a D-set, for every $\bar{c}$ realizing p.

$S^n_D(A)$ will denote the set of D-types over A in n variables.

When $D = D(T)$, then $S_D(A) = S(A)$.

Definition 2.2. The model M is a *(D, λ)-homogeneous* model if M realizes every $p \in S_D(A)$ for $A \subseteq M$ with $|A| < \lambda$.

When $D = D(T)$, then a model is (D, λ)-homogeneous if and only if it is λ-saturated.

The next lemma shows that if M is (D, λ)-homogeneous, then it is λ-universal for the class of D-models.

Lemma 2.3. *Let M be (D, λ)-homogeneous and A be a D-set of cardinality λ. Let $B \subseteq A$ such that $|B| < \lambda$. Then for every elementary mapping $f \colon B \to M$, there is an elementary mapping $g \colon A \to M$ extending f.*

Proof. Write $A = B \cup \{a_i : i < \alpha \leq \lambda\}$. Construct an increasing sequence of elementary mappings $\langle f_i \mid i < \lambda\rangle$ by induction on $i < \alpha$, such that $f_0 = f$,

$$B \cup \{a_j : j < i\} \subseteq \mathrm{dom}(f_i) \quad \text{and} \quad \mathrm{ran}(f_i) \subseteq M.$$

In case $i = 0$ or i a limit, it is obvious. Assume f_i is constructed. Define $q_i = f_i(\mathrm{tp}(a_i/B \cup \{a_j : j < i\}))$. By induction hypothesis $q_i \in S_D(f_i(B \cup \{a_j : j < i\}))$. Hence, since M is (D, λ)-homogeneous, q_i is realized by some $b_i \in M$. Let $f_{i+1} = f_i \cup \langle a_i, b_i\rangle$. The elementary mapping $g = \bigcup_{i<\alpha} f_i$ is as required. □

Recall from the first order case that a model is *λ-homogeneous*, if for any partial elementary mapping f from M into M with $|\mathrm{dom}(f)| < \lambda$ and $c \in M$, there is an elementary extension g of f from M into M such that $\mathrm{dom}(g) \supseteq \mathrm{dom}(f) \cup c$. The next lemma is an extension of the familiar first order result that a model M is λ-saturated if and only if M is λ-homogeneous and $< \aleph_0$-universal if and only if M is λ-homogeneous and λ-universal.

Lemma 2.4. *M is a (D, λ)-homogeneous model if and only if $D(M) = D$ and M is λ-homogeneous.*

Proof. The only if part follows from the previous lemma. To see the converse, we show that M is (D,μ)-homogeneous for every $\mu \leq \lambda$ by induction on μ.

For the base case, assume that $\mu < \aleph_0$. Let $p \in S_D(\bar{c})$, where $\bar{c} \in M$ is finite. Let a be any element realizing p. By assumption $\mathrm{tp}(a\hat{\ }\bar{c}/\emptyset) \in D$. Since $D(M) = D$, there exist a' and $\bar{c}' \in M$ realizing $\mathrm{tp}(a\hat{\ }\bar{c}/\emptyset)$. Let f be a partial elementary mapping such that $f(\bar{c}) = \bar{c}'$ and $f(a) = a'$. Then, by λ-homogeneity of M, there is a partial elementary mapping g from M to M, extending $f^{-1} \restriction \bar{c}'$, with $\mathrm{dom}(g) \supseteq \bar{c}' \cup a'$. Then we have that a' realizes $f(p)$, and so $g(a')$ realizes $g(f(p)) = p$. Hence, p is realized in M.

By induction, let $C \subseteq M$ of cardinality $\mu < \lambda$ and assume that we have already shown that M is (D,μ)-homogeneous. Let $p \in S_D(C)$ and a be any element realizing p. Then $C \cup a$ is a D-set of cardinality μ, so by (D,μ)-homogeneity of M, using the previous lemma, there exists an elementary mapping f sending $C \cup a$ into M. Hence, by λ-homogeneity of M, there is g, an elementary mapping from M into M, extending $f^{-1} \restriction C$ with $\mathrm{dom}(g) \supseteq f(C) \cup f(a)$. To conclude, notice that since a realizes p, $f(a)$ realizes $f(p)$ and $g(f(a))$ realizes $g(f(p)) = p$. This shows that M realizes p, since $g(f(a)) \in M$, and completes the proof. □

The following hypothesis is made throughout the paper. It is equivalent to Shelah's original assumption in [Sh3] and [Sh54]. Also, the same assumption was made by H. Jerome Keisler in his categoricity theorem [Ke].

Hypothesis 2.5. There exists a $(D,\bar{\kappa})$-homogeneous model $\mathfrak{C}$, with $\bar{\kappa}$ much larger than any cardinality mentioned in this paper.

In view of the preceding lemma, we may assume that any D-set lies in $\mathfrak{C}$. Also, satisfaction is with respect to $\mathfrak{C}$. Notice also that for any D-set A

$$S^n_D(A) = \{\mathrm{tp}(\bar{a}/A,\mathfrak{C}) \mid \bar{a} \in \mathfrak{C}\}.$$

The study of a *finite diagram* D is thus the study of the class of D-models under the additional assumption that there exists a $(D,\bar{\kappa})$-homogeneous model $\mathfrak{C}$, with $\bar{\kappa}$ very large.

Hypothesis 2.5 is a natural assumption to make. Let us say a few words about why we feel this is so. The most outstanding test question in the classification theory for nonelementary classes is a conjecture of S. Shelah, made in the mid-1970s:

Conjecture 2.6 (Shelah). Let T be a countable $L_{\omega_1,\omega}$ theory. If there exists a cardinal $\lambda \geq \beth_{\omega_1}$ such that T is categorical in λ, then T is categorical in every $\mu \geq \beth_{\omega_1}$.

As we mentioned in the introduction, it is equivalent to solve this conjecture for the class of D-models of a countable first order theory, where D is the set of isolated types over the empty set (whence the relevance of this discussion here). Most experts agree that the full conjecture seems currently out of reach. However, several attempts to solve the conjecture since the late 1970s have indicated that categoricity (sometimes in several cardinals and sometimes under additional set-theoretic axioms) implies the existence of various kinds of *amalgamation properties* and the existence of *monster models* (see for example [Sh48], [Sh87a], [Sh87b], [Sh88], or [BaSh3]). By monster model, we mean a

large model with universal or homogeneous properties. By amalgamation properties we mean that the class of models of T satisfies the μ-amalgamation property for a class of cardinals μ. Recall that a class of models $\mathcal{K}$ has the μ-amalgamation property if for every triple of models $M_0, M_1, M_2 \in \mathcal{K}$ of cardinality μ such that $M_0 \prec M_1$, $M_0 \prec M_2$, and $M_0 \subseteq M_1 \cap M_2$, there exist a model $N \in \mathcal{K}$ and embeddings $f_i \colon M_i \to N$ for $i = 1, 2$ such that $f_1 \restriction M_0 = f_2 \restriction M_0$. For example, by Robinson's Consistency Lemma, the class of model of a first order theory T has the μ-amalgamation property, for every cardinal $\mu \geq |T|$.

While Shelah observed from the work of Leo Marcus [Mr], that the existence of a monster model quite as in Hypothesis 2.5 does not follow from the assumption of Shelah's conjecture, it is certainly reasonable to conjecture that it implies the existence of a monster model with a similar flavor. Thus, experience gained in this framework can shed light on the potentially more general framework. These results are additional motivations to develop classification theory either inside a homogeneous model [Sh3], [Sh54], [Gr1], [Gr2], [HySh], [GrLe2], [Le], or for nonelementary classes with amalgamation properties [Sh48], [Sh87a], [Sh87b], [GrHa], [Sh394]. In fact, under *monster model* or *amalgamation properties* several approximations of Shelah conjecture are known: for example [Ke], [Sh48], [Sh87a], [Sh87b] or [Le].

In this vein, the two following conjectures were made by Rami Grossberg in 1989, in an email communication with John T. Baldwin:

Conjecture 2.7. Let T be a countable $L_{\omega_1,\omega}$ theory. If T is categorical is some large enough λ, then there exists a μ_0 such that the class of models of T has the μ-amalgamation property for every μ greater than μ_0.

Amalgamation properties are closely related to monster model hypotheses: When T is a Scott sentence, the conclusion of the previous conjecture implies the existence of arbitrarily large model-homogeneous models

Conjecture 2.8. Let T be a countable $L_{\omega_1,\omega}$ theory such that there exists a μ_0 such that the class of models of T has the μ-amalgamation property for every μ greater than μ_0. If T is categorical in some $\lambda \geq \beth_{\omega_1}$, then T is categorical in every cardinal $\mu \geq \beth_{\omega_1}$.

Before finishing this discussion, we can ask the following related question:

Question 2.9. *Let T be a countable theory in $L_{\omega_1,\omega}$. Is there a cardinal $\mu(T)$ with the property that if the class of models of T has the $\mu(T)$-amalgamation property then it has the λ-amalgamation property for arbitrarily large λ?*

3. Stability and Order in Finite Diagrams

In this section, we present the equivalence between stability and the failure of the order property in the context of finite diagrams (Corollary 3.12).

Definition 3.1. Let D be a finite diagram.

(1) The diagram D is said to be *stable in* λ if for every $A \subseteq \mathfrak{C}$ of cardinality at most λ and for every $n < \omega$ we have $|S^n_D(A)| \leq \lambda$.

(2) We say that D is *stable* if there is a λ such that D is stable in λ.

By the pigeonhole principle, it is enough to consider $n = 1$, i.e. D is stable in λ if and only if for all $A \subseteq \mathfrak{C}$ of cardinality at most λ, we have $|S_D(A)| \leq \lambda$.

Definition 3.2. Let D be a finite diagram.

(1) D has the λ-*order property* if there exist a D-set $\{\bar{a}_i \mid i < \lambda\}$, and a formula $\phi(\bar{x}, \bar{y}) \in L(T)$ such that
$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$
(2) D has the *order property* if D has the λ-order property for every cardinal λ.

Notice that the order property is formulated differently from the order property used by Shelah in [Sh b]. The formulation given here is equivalent to the usual order property in the first order case, and is more natural in nonelementary cases; when it holds there are many nonisomorphic models (see [Sh12], [GrSh1], and [GrSh2]).

Recall some standard definitions. A set of finite sequences $\{\bar{a}_i \mid i < \alpha\}$ is said to be an n-*indiscernible sequence over* A, for $n < \omega$ if $\mathrm{tp}(\bar{a}_0, \dots, \bar{a}_{n-1}/A) = \mathrm{tp}(\bar{a}_{i_0}, \dots, \bar{a}_{i_{n-1}}/A)$. for every $i_0 < \cdots < i_{n-1} < \alpha$. Then $\{\bar{a}_i \mid i < \alpha\}$ is an *indiscernible sequence over* A, if it is an n-indiscernible sequence over A for every $n < \omega$. It is said to be an *indiscernible set*, if in addition, the ordering does not matter. We will not have to distinguish between the two, as in the presence of stability, every indiscernible sequence is, in fact, an indiscernible set (Remark 3.4 and Corollary 3.12). Hence, we will often say indiscernible for indiscernible sequence, or set when they coincide or when it does not matter.

Remark 3.3. If there exists a D-set $\{\bar{a}_i \mid i < \omega\}$, which is an indiscernible sequence, and a formula $\phi(\bar{x}, \bar{y})$ such that
$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \omega,$$
then D has the order property.

Proof. Let λ be an infinite cardinal. Let $\{\bar{c}_i \mid i < \lambda\}$ be new constants. Consider the union of the following sentences:

- $\phi(\bar{c}_i, \bar{c}_j)$, if $i < j < \lambda$;
- $\neg\phi(\bar{c}_i, \bar{c}_j)$, if $i \geq j, i, j < \lambda$;
- $\psi(\bar{c}_{i_0}, \dots, \bar{c}_{i_n})$, for each $\psi(\bar{x}_0, \dots \bar{x}_n) \in \mathrm{tp}(\bar{a}_0, \dots, \bar{a}_n/\emptyset)$, and each $n < \omega$, and each $i_0 < \cdots < i_n < \lambda$.

The above set of sentences is consistent (use $\{\bar{a}_i \mid i < \omega\}$). Let $\bar{b}_i$ be the interpretation of $\bar{c}_i$ in $\bar{M}$, the monster model for T. The last clause implies that $\{\bar{b}_i \mid i < \lambda\}$ is a D-set. By the first two clauses, we have
$$\models \phi[\bar{b}_i, \bar{b}_j] \quad \text{if and only if} \quad i < j < \lambda.$$
Hence, D has the λ-order property. We are done since λ was arbitrary. □

Remark 3.4. Suppose D does not have the order property. Let $\{\bar{a}_i \mid i < \alpha\}$ be an infinite indiscernible sequence over A. Then $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible set over A.

Proof. Suppose that the conclusion fails. Then, there exist an integer $n < \omega$, a permutation $\sigma \in S_n$, and indices $i_0 < \cdots < i_n < \alpha$ such that

$$\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_n/A) \neq \mathrm{tp}(\bar{a}_{i_\sigma(0)}, \ldots, \bar{a}_{i_\sigma(n)}/A).$$

Since $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible sequence over A, we have $\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_n/A) \neq \mathrm{tp}(\bar{a}_{\sigma(0)}, \ldots, \bar{a}_{\sigma(n)}/A)$. Since any permutation is a product of transpositions, we may assume that there exist $k_0 < k_1 \leq n$ such that $\sigma(k_0) = k_1$, $\sigma(k_1) = k_0$ and $\sigma(i) = i$, otherwise. Hence, there exists $\phi(\bar{x}, \bar{y}, \bar{b})$, where $\bar{b} \in A \cup \{\bar{a}_i \mid i \leq n, i \neq k_0, k_1\}$ such that $\models \phi[\bar{a}_{k_0}, \bar{a}_{k_1}, \bar{b}]$ and $\models \neg\phi[\bar{a}_{k_1}, \bar{a}_{k_0}, \bar{b}]$. Then, the D-set $\{\bar{a}_i \hat{} \bar{b} \mid n < i < \alpha\}$ is an infinite indiscernible sequence (over $\emptyset$). Hence $\models \phi[\bar{a}_i, \bar{a}_j, \bar{b}]$ if and only if $n < i < j < \alpha$. This implies that D has the order property by the previous remark. $\square$

The main tool to prove that the failure of the order property implies stability (Theorem 3.9) is *splitting*. Recall the definition.

Definition 3.5. Let Δ_1 and Δ_2 be sets of formulas. Let A be a set and $B \subseteq A$. For $p \in S^n(A)$, we say that p *(Δ_1, Δ_2)-splits over B* if there are $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \Delta_2$ such that $\mathrm{tp}_{\Delta_1}(\bar{b}/B) = \mathrm{tp}_{\Delta_1}(\bar{c}/B)$ with $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$.

When $\Delta_1 = \Delta_2 = L(T)$, we just say that *$p$ splits over B*. When $\Delta_1 = \{\phi(\bar{x}, \bar{y})\}$ and $\Delta_2 = \{\psi(\bar{x}, \bar{y})\}$, we write *$(\phi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$-splits*, omitting the parentheses.

For a statement $\mathbf{t}$ and a formula ϕ, the following convention is made: $\phi^{\mathbf{t}} = \neg\phi$ if the statement $\mathbf{t}$ is false and $\phi^{\mathbf{t}} = \phi$, if the statement $\mathbf{t}$ is true. The same notation is used when $\mathbf{t} \in \{0, 1\}$, where 0 stands for falsehood and 1 stands for truth.

The next two lemmas give sufficient conditions guaranteeing the existence and uniqueness of nonsplitting extensions.

Lemma 3.6. *Let $A \subseteq B \subseteq C$ be sets such that B realizes all the Δ_1-types over A that are realized in C. Assume $p_1, p_2 \in S_{\Delta_2}(C)$ and $p_1 \restriction B = p_2 \restriction B$. If p_1, p_2 do not (Δ_1, Δ_2)-split over A, then $p_1 = p_2$.*

Proof. By symmetry, it is enough to show that $p_1 \subseteq p_2$. Let $\phi(\bar{x}, \bar{b}) \in p_1$. By assumption $\mathrm{tp}_{\Delta_1}(\bar{b}/A)$ is realized by some $\bar{c} \in B$. Hence $\phi(\bar{x}, \bar{c}) \in p_1$ since p_1 does not (Δ_1, Δ_2)-split over A, and $\phi(\bar{x}, \bar{y})^t \in \Delta_2$ for $t = 0$ or 1. Thus $\phi(\bar{x}, \bar{c}) \in p_2$ and so $\phi(\bar{x}, \bar{b}) \in p_2$ also since p_2 does not (Δ_1, Δ_2)-split over A. $\square$

Lemma 3.7. *Let $A \subseteq B \subseteq C$ be D-sets, such that B realizes every D-type over A, which is realized in C. Suppose $p \in S_D(B)$ does not split over A. Then, there is a unique type $q \in S_D(C)$ extending p that does not split over A.*

Proof. Uniqueness was proved in the previous lemma. Hence, it is enough to show existence. Define q explicitly by setting:

$$q := \{\phi(x, \bar{c}) \mid \text{There exists } \bar{b} \in B \text{ realizing } \mathrm{tp}(\bar{c}/A) \text{ and } \phi(x, \bar{b}) \in p\}.$$

This is well-defined. By assumption p does not split over A and so the definition does not depend on the choice of $\bar{b} \in B$.

First notice that q is complete. Suppose $\bar{c} \in C$ and $\phi(x, \bar{y}) \in L(T)$. Suppose $\phi(x, \bar{c}) \notin q$. Let $\bar{b} \in B$ realize $\mathrm{tp}(\bar{c}/A)$. By definition, we have $\phi(x, \bar{b}) \notin p$. Hence,

$\neg\phi(x, \bar{b}) \in p$, since $p \restriction B$ is complete. Thus, $\neg\phi(x, \bar{c}) \in q$, by definition of q. Also, q is consistent. Let $\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n) \in q$. Then $\phi_i(x, \bar{b}_i) \in p$, for $\bar{b}_1\hat{}\dots\hat{}\bar{b}_n \in B$ realizing $\mathrm{tp}(\bar{c}_1\hat{}\dots\hat{}\bar{c}_n/A)$. Since p is consistent, we have

$$\models \exists x[\phi_1(x, \bar{b}_1) \wedge \cdots \wedge \phi_n(x, \bar{b}_n)].$$

Then, by an elementary mapping sending each $\bar{b}_i$ to $\bar{c}_i$ fixing A we conclude that

$$\models \exists x[\phi_1(x, \bar{c}_1) \wedge \ldots \wedge \phi_n(x, \bar{c}_n)].$$

Hence, the set $\{\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n)\}$ is consistent.

Now let us see that$q$ does not split over A. Otherwise, there are $\bar{c}_1$, $\bar{c}_2 \in C$, and $\phi(x, \bar{y})$ such that $\mathrm{tp}(\bar{c}_1/A) = \mathrm{tp}(\bar{c}_2/A)$ and $\phi(x, \bar{c}_1)$, $\neg\phi(x, \bar{c}_2) \in q$. Choose $\bar{b}_1$, $\bar{b}_2 \in B$, such that $\mathrm{tp}(\bar{b}_1/A) = \mathrm{tp}(\bar{b}_2/A) = \mathrm{tp}(\bar{c}_1/A)$. We have $\phi(x, \bar{b}_1)$, $\neg\phi(x, \bar{b}_2) \in p$, by definition of q. Hence p splits over A, contradiction.

Finally, let us show that q is a D-type. Suppose not. Then, there is a realizing q and $\bar{c} \in C$ such that $\mathrm{tp}(a\hat{}\bar{c}/\emptyset) \notin D$. Let $\bar{b} \in B$ realize $\mathrm{tp}(\bar{c}/A)$. Since a realizes p, we have $\mathrm{tp}(a\bar{b}/\emptyset) \in D$. Hence, in particular

$$\mathrm{tp}(a\bar{b}/\emptyset) \neq \mathrm{tp}(a\bar{c}/\emptyset).$$

Hence there is $\phi(x, \bar{y})$, with $\models \phi[a, \bar{b}]$, and $\models \neg\phi[a, \bar{c}]$. This implies that $\phi(x, \bar{b})$, and $\neg\phi(x, \bar{c}) \in q$. This shows that q splits over A,a contradiction. $\square$

We will use the following notational convention: For Δ a set of formulas, we write

$$S_{D,\Delta}(B) = \{\mathrm{tp}_\Delta(c/B, \mathfrak{C}) \mid c \in \mathfrak{C}\}.$$

When $\Delta = \{\phi(\bar{x}, \bar{y})\}$, we write $S_{D,\phi}(B)$ instead of $S_{D,\{\phi\}}(B)$.

Corollary 3.8. *Let $A \subseteq B$ be D-sets. Then*

$$|\{p \in S_{D,\Delta_2}(B) : p \textit{ does not } (\Delta_1, \Delta_2)\textit{-split over } A\}| \leq 2^{|D|^{|A|}}.$$

Proof. Since $|S_D(A)| \leq |D|^{|A|}$, we can find C, with $|C| \leq |D|^{|A|}$ such that C realizes all the types in $S^n_{D,\Delta_1}(A)$. Then, by Lemma 3.6, we have

$$\begin{aligned} &|\{p \in S_{D,\Delta_2}(B) : \ p \text{ does not } (\Delta_1, \Delta_2)\text{-split over } A\}| \leq \\ &\leq |\{p : p \in S_{D,\Delta_2}(C)\}| \leq |D|^{|C|} \leq |D|^{|D|^{|A|}} \leq 2^{|D|^{|A|}}. \end{aligned}$$

$\square$

The proof of the next theorem follows [Gr1].

Theorem 3.9. *Let $\lambda \geq |L(T)|$. If D is not stable in 2^{2^λ}, then D has the λ^+-order property.*

Proof. We first claim that there exist a D-set A of cardinality 2^{2^λ} and a formula $\phi(x, \bar{y})$ such that

$$|S_{D,\phi}(A)| > |A|.$$

Since D is not stable in 2^{2^λ}, there is a D-set A of cardinality 2^{2^λ} such that $|S_D(A)| > |A|$. Define

$$f\colon S_D(A) \to \Pi_{\phi(x,\bar{y})\in L} S_{D,\phi}(A), \quad \text{by} \quad f(p) = (p \restriction \phi)_{\phi\in L}.$$

Then, f is injective and since $\lambda \geq |L(T)|$, by the pigeonhole principle, there must be $\phi(x,\bar{y}) \in L$ such that $|S_{D,\phi}(A)| > |A|$. This proves the claim.

Let A and ϕ be as in the claim, we will show that

$$\psi(x_0, \bar{x}_1, \bar{x}_2, y_0, \bar{y}_1, \bar{y}_2) := \phi(x_0, \bar{y}_1) \leftrightarrow \phi(x_0, \bar{y}_2)$$

demonstrates the order property. For convenience, let $\mu = 2^{2^\lambda}$. Let $\{a_i : i < \mu^+\} \subseteq \mathfrak{C}$ be such that $i \neq j < \mu^+$ implies $\mathrm{tp}_\phi(a_i/A) \neq \mathrm{tp}_\phi(a_j/A)$. This is possible since $|S_{D,\phi}(A)| > |A|$. Let $\chi(\bar{y}, x) = \phi(x, \bar{y})$ and $n = \ell(\bar{y})$. Define an increasing continuous chain of sets $\langle A_i : i < \mu \rangle$ such that:

(1) $A_0 = \emptyset$ and $|A_i| \leq \mu$, $i < \mu$.
(2) For every $B \subseteq A_i$ of cardinality at most λ and every type $p \in S_{D,\phi}(A_i) \cup S^n_{D,\chi}(A_i)$, $p \restriction B$ is realized in A_{i+1}.

This is possible since there are at most $\mu^\lambda = \lambda$ subsets of A_i of cardinality λ and at most $|S_D(B)| \leq |D|^\lambda \leq (2^{|L(T)|})^{|B|} \leq (2^\lambda)^\lambda < \mu$ possible types for each set B.

Claim. For every $j < \mu^+$, there is i with $j < i < \mu^+$ such that for all $l < \lambda^+$ the type $q_i = \mathrm{tp}(a_i, A_l)$ (χ, ϕ)-splits over each $B \subseteq A_l$ of cardinality at most λ.

Proof. Otherwise, there is $j < \mu^+$ such that for every i with $j < i < \mu^+$, there is $l < \lambda$ and $B^i \subseteq A_l$ of cardinality at most λ such that q_i does not (χ, ϕ)-split over B^i. Since $\mu^+ > \lambda$, by the pigeonhole principle, we can find $l < \lambda$ such that μ^+ many q_i's do not (χ, ϕ)-split over a subset of A_l. By a second application of the pigeonhole principle, since $\mu^+ > \mu \geq |A_l|^\lambda = |\{B \subseteq A_l : |B| \leq \lambda\}|$, we can find $\mu^+ > (2^{2^\lambda})$ many types that do not (χ, ϕ)-split over a set of cardinality at most λ. This contradicts Corollary 3.8. Hence, the claim is true. □

Among the i's satisfying the claim, pick one such that $a_i \notin \bigcup_{l<\lambda} A_l$. This is possible since $|\bigcup_{l<\lambda} A_l| \leq \mu$. Then, by construction, for every $l < \lambda^+$, the type $\mathrm{tp}_\phi(a_i/A_l)$ (χ, ϕ)-splits over every $B \subseteq A_l$ of cardinality at most λ. Define $\bar{a}_l$, $\bar{b}_l$ and c_l in A_{2l+2}, as well as $B_l = \cup\{\bar{a}_k, \bar{b}_k, c_k : k < l\}$ by induction on $l < \lambda^+$ such that

(1) $B_l \subseteq A_{2l}$ and $|B_l| \leq \lambda$;
(2) $\mathrm{tp}_\chi(\bar{a}_l/B_l) = \mathrm{tp}_\chi(\bar{b}_l/B_l)$;
(3) Both $\phi(x, \bar{a}_l)$ and $\neg\phi(x, \bar{b}_l)$ belong to $\mathrm{tp}(a_i/A_{2l})$;
(4) $c_l \in A_{2l+1}$ realizes $\phi(x, \bar{a}_l) \wedge \neg\phi(x, \bar{b}_l)$.

This is possible: Set $B_0 = \emptyset$. If B_l is constructed, since $B_l \subseteq A_{2l}$ of cardinality at most λ, $\mathrm{tp}_\phi(a_i/A_{2l})$ (χ, ϕ)-splits over B_l, hence we can find $\bar{a}_l$ and $\bar{b}_l$ in A_{2l} such that $\mathrm{tp}_\chi(\bar{a}_l/B_l) = \mathrm{tp}_\chi(\bar{b}_l/B_l)$ and both $\phi(x, \bar{a}_l)$ and $\neg\phi(x, \bar{b}_j)$ belong to $\mathrm{tp}(a_i/A_{2j})$. Then, by construction of A_{2l+1}, we can find $c_l \in A_{2l+1}$, realizing $\mathrm{tp}_\phi(a_i/A_2) \restriction \{\bar{a}_l, \bar{b}_l\}$ and hence realizing $\phi(x, \bar{a}_l) \wedge \neg\phi(x, \bar{b}_l)$. When l is a limit ordinal, we define B_l by continuity.

Now, let $\bar{d}_l = c_l\hat{\ }\bar{a}_l\hat{\ }\bar{b}_l$. It is easy to see from (2), (3) and (4) that $\{\bar{d}_l : l < \lambda^+\}$ and $\psi(x_0, \bar{x}_1, \bar{x}_2, y_0, \bar{y}_1, \bar{y}_2) = \phi(x_0, \bar{y}_1) \leftrightarrow \phi(x_0, \bar{y}_2)$ together demonstrate the (D, λ^+)-order property. □

The next theorem is a converse of Theorem 3.9. The proof uses Hanf number techniques. For a first order theory T and Γ a set of T-types over the empty set, the class $\mathrm{EC}(T, \Gamma)$ is the class of models of T omitting every type in Γ. For cardinals λ and κ, the *Hanf-Morley number* $\mu(\lambda, \kappa)$ is defined to be the smallest cardinal μ with the property that for every $\mathrm{EC}(T, \Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{EC}(T, \Gamma)$ contains a model of cardinality μ then $\mathrm{EC}(T, \Gamma)$ contains models of arbitrarily large cardinality. Clearly, when $\kappa = 0$, $\mu(\lambda, \kappa) = \aleph_0$; this is the first order case. When $\kappa \geq 1$, the notion of *wellordering number* $\delta(\lambda, \kappa)$ needs to be introduced. For cardinals λ, κ, the number $\delta(\lambda, \kappa)$ is the smallest ordinal δ with the property that for every $\mathrm{EC}(T, \Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{EC}(T, \Gamma)$ contains a model with a predicate of order type δ, then $\mathrm{EC}(T, \Gamma)$ contains a model where this predicate is not wellordered. If $\kappa \geq 1$, it is a standard result that $\mu(\lambda, \kappa) = \beth_{\delta(\lambda, \kappa)}$. (Note that the methods of the proof below show $\mu(\lambda, \kappa) \leq \beth_{\delta(\lambda, \kappa)}$.) A standard result on wellordering numbers states that $\delta(\lambda, \kappa) \leq (2^\lambda)^+$. This will be used in the proof and explains the cardinal $\beth_{(2^{|T|})^+}$ appearing in the statement.

Theorem 3.10. *If D has the λ-order property for every $\lambda < \beth_{(2^{|T|})^+}$, then D is not stable and D has the ω-order property witnessed by an indiscernible sequence.*

Proof. We will show first that D has the ω-order property witnessed by an indiscernible sequence. By assumption, for each $\alpha < (2^{|T|})^+$, we can find a D-set

$$P_\alpha = \{\bar{a}_{\alpha,j} \mid j < (\beth_\alpha)^+\}$$

and a formula ϕ_α witnessing the order property. Hence, by the pigeonhole principle, we may assume that $\phi_\alpha = \phi$ is fixed for all α.

Notice that M is a D-model of T if and only if $M \in \mathrm{EC}(T, \Gamma)$, with $\Gamma = D(T) \setminus D$. But $|D(T) \setminus D| \leq 2^{|T|}$, and so the well-ordering number for this class is at most $\delta(|T|, 2^{|T|}) = (2^{|T|})^+$.

For $\alpha < (2^{|T|})^+$, define $M_\alpha \prec \mathfrak{C}$ containing $\{\bar{a}_{\alpha,j} : j < (\beth_\alpha)^+\}$ of cardinality $(\beth_\alpha)^+$. This is possible by the downward Löwenhweim-Skolem Theorem. Each M_α belongs to $EC(T, \Gamma)$. Define $F\colon (2^{|T|})^+ \to \bigcup_{\alpha < (2^{|T|})^+} M_\alpha$, by $F(\alpha) = M_\alpha$.

Consider the following model

$$M = \langle H(\bar{\chi}), \in, F, (2^{|T|})^+, T, P, \models, \psi \rangle_{\psi \in L},$$

where $\bar{\chi}$ is a regular cardinal big enough so $H(\bar{\chi})$ contains everything that has been mentioned so far in this proof. The predicates $(2^{|T|})^+$ and T are unary predicates whose interpretations are the corresponding sets. The meaning of the binary predicates $\models$ and $\in$ and of the constants ψ, for each $\psi \in L$ is their true meaning in $H(\chi)$. Also F is a unary function and the interpretation of F is the one we just defined. P is a unary predicate, whose interpretation in each M_α is the D-set P_α witnessing the order property. More precisely, we have that

$$M \models \forall \alpha \in (2^{|T|})^+ (\bar{a}_{\alpha,i} \in M_\alpha) \wedge$$
$$\bar{a}_{\alpha,j} \in M_\alpha \wedge P\bar{a}_{\alpha,i} \wedge P\bar{a}_{\alpha,j}] \to (M_\alpha \models \phi[\bar{a}_{\alpha,i}, \bar{a}_{\alpha,j}] \leftrightarrow i \in j).$$

Let $N \prec M$ such that $(2^{|T|})^+ \subseteq N$ of cardinality $(2^{|T|})^+$. Therefore, we can fix a bijection $G\colon |N| \to (2^{|T|})^+$. Define $a < b$ if and only if $G(a) \in G(b)$.

Form $N' = \langle N, <, G\rangle$ an expansion of N. Let $T' = Th(N')$ and for each $\psi(\bar{x}) \in L$ define $\psi'(\bar{x}, y)$ by $\exists \alpha \in (2^{|T|})^+(y = M_\alpha \wedge \bar{x} \in M_\alpha \wedge M_\alpha \models \psi[\bar{x}])$. Let $\Gamma' = \{\{\psi'(\bar{x}, y) : \psi(\bar{x}) \in p\} : p \in \Gamma\}$. Then, we have that $|T'| = |T|$ and $|\Gamma'| = |\Gamma|$, so $\delta(|T'|, 2^{|T'|}) = (2^{|T|})^+$.

We first claim that N' omits every type in Γ'.

Suppose not. There is $p' \in \Gamma'$ such that for some $\bar{c}\hat{\ }a \in N'$ we have that $\models \psi'[\bar{c}, a]$, for all $\psi' \in p'$. But then, by definition $\bar{c}$ is in some M_α and $\bar{c}$ realizes every $\psi(\bar{x})$ in p. But $p \in \Gamma$ so this contradicts the fact that $M_\alpha \in \mathrm{EC}(T, \Gamma)$. Hence, we have a model $N' \in \mathrm{EC}(T', \Gamma')$ well-ordered by $<$ and of order-type $(2^{|T|})^+$. Thus, we can find a model $N'' \in \mathrm{EC}(T', \Gamma')$, whose universe is not wellordered by $<$. Therefore, by taking away elements if necessary, there exists elements $b_n \in N''$ such that $N'' \models b_{n+1} + n + 1 < b_n$ and $N'' \models b_n \in (2^{|T|})^+$ for $n < \omega$.

Define a sequence of sets $\langle X_n \mid n < \omega\rangle$ such that

(1) $N'' \models$ "X_n is an n-indiscernible sequence in M_{b_0} of cardinality $\beth_{b_n}$".
(2) $N'' \models$ "X_n has the D-order property"

This is possible. Construct the X_n by induction on $n < \omega$. For $n = 0$, let $X_0 = \{\bar{a}_{b_0, j} : j < \beth_{b_0}\}$, i.e. the interpretation in N'' of the interpretation of the predicate P in M_α. Then the first requirement is satisfied since X_0 has the right cardinality and there is nothing to check for 0-indiscernibility. The second requirement is also satisfied since M and so N'' knows that they witness the order property.

Assume X_n has already been constructed. Define

$$f : [X_n]^{n+1} \to S^{n+1}_{L(T)}(\emptyset), \quad \text{by} \quad (c_1, \dots, c_{n+1}) \mapsto \mathrm{tp}(c_1, \dots, c_{n+1}/\emptyset).$$

We know by Erdős-Rado that

$$\beth_n^+(\beth_{b_{n+1}}) \to (\beth^+_{b_{n+1}})^{n+1}_{\beth_{b_{n+1}}}$$

and we have $\beth_{b_n} \geq \beth_{b_n + n + 1} \geq \beth_n^+(\beth_{b_{n+1}})$, so we can find a subset X_{n+1} of X_n of cardinality $\beth_{b_{n+1}}$ such that every increasing $(n+1)$-tuple from it has the same type. This implies that X_{n+1} is an $(n+1)$-indiscernible sequence with the right cardinality. Since the second requirement is preserved by renumbering if needed, we are done.

This is enough. Let $\{\bar{c}_i : i < \omega\}$ be a new set of constants. Define T_1 to be the union of the following set of sentences:

- T;
- $\bar{c}_i \neq \bar{c}_j$, whenever $i \neq j$;
- $\phi(\bar{c}_i, \bar{c}_j)^{i<j}$, for every $i, j < \omega$;
- $\chi(\bar{c}_{i_1}, \dots, \bar{c}_{i_n})$, for every $\chi \in \mathrm{tp}(\bar{a}_1, \dots, \bar{a}_n/\emptyset)$, $i_1 < \dots < i_n$ and $n < \omega$;
- $\psi(\bar{c}_{i_1}, \dots \bar{c}_{i_n}) \leftrightarrow \psi(\bar{c}_{j_1}, \dots, \bar{c}_{j_n})$, whenever $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$, $n < \omega$ and $\psi \in L(T)$.

By the Compactness Theorem and the definition of X_n, T_1 has a model N_1. Call $\bar{a}_i = \bar{c}_i^{N_1}$ Notice also that the construction ensures that $\{\bar{a}_i : i < \omega\}$ is a D-set. Hence we have the ω-order property witnessed by indiscernibles.

We will use these to show that D is not stable. Let μ be a given cardinality. Define $\kappa = \min\{\kappa : 2^\kappa > \mu\}$. By compactness, using the indiscernibility of $\{\bar{a}_i : i < \omega\}$, we can get a D-set $\{\bar{a}_\eta : \eta \in {}^{\kappa \geq}2\}$ such that $\models \phi[\bar{a}_\eta, \bar{a}_\nu]$ if and only if $\eta \prec \nu$. Let $A = \bigcup_{\eta \in {}^{\kappa >}2} \bar{a}_\eta$. Then $|A| \leq \mu$, by choice of κ, and for $\eta \neq \nu \in {}^{\kappa}2$, we have that $\mathrm{tp}(\bar{a}_\eta/A) \neq \mathrm{tp}(\bar{a}_\nu/A)$. Indeed, there is a first $i < \kappa$ such that $\eta[i] \neq \nu[i]$, say $\eta[i] = 0$. But then $\psi(\bar{a}_{\eta^\frown 0}, \bar{x}) \in \mathrm{tp}(\bar{a}_\eta/A)$ and $\neg\psi(\bar{a}_{\eta^\frown 0}, \bar{x}) \in \mathrm{tp}(\bar{a}_\nu/A)$. Thus $|S_D(A)| \geq 2^\kappa > \mu$ and so D is not stable in μ. □

The next corollary tells us that if D is stable, we can find $\lambda < \beth_{(2^{|T|})^+}$ demonstrating this. Notice that if $D = D(T)$ we are in the first order case and the bound on the first stability cardinal is actually $2^{|T|}$.

Corollary 3.11. *If D is stable, then there exists $\lambda < \beth_{(2^{|T|})^+}$ such that D is stable in λ.*

Proof. Suppose that D is not stable in any $\lambda < \beth_{(2^{|T|})^+}$. Then, since $\beth_{(2^{|T|})^+}$ is a strong limit, for each $\lambda < \beth_{(2^{|T|})^+}$, we have $2^{2^\lambda} < \beth_{(2^{|T|})^+}$ and so D is not stable in 2^{2^λ}. Hence by Theorem 3.9, D has the λ^+-order property for all $\lambda < \beth_{(2^{|T|})^+}$ and so by Theorem 3.10 D is not stable. □

The next corollary is the order/stability dichotomy.

Corollary 3.12. *D is stable if and only if D does not have the order property.*

Proof. If D is not stable, then it is not stable in 2^{2^λ} for any $\lambda \geq |L(T)|$ so by Theorem 3.9, D has the λ-order property for every cardinal λ. For the converse, we use Theorem 3.10. □

4. The Stability Spectrum for Finite Diagrams

In the first part of this section, combinatorial properties related to splitting are introduced for finite diagrams. They can be used to give another characterization of stability (see Corollary 4.7). In the second part, the focus is on a more delicate tool; *strong splitting*. It is a substitute for the notion of forking. The appropriate cardinal invariant and combinatorial property related to strong splitting are introduced. They are used to derive the Stability Spectrum Theorem (Theorem 4.17).

Definition 4.1. (1) D satisfies $(*\lambda)$ if there exists an increasing continuous chain of D-sets $\{A_i : i \leq \lambda\}$ and $p \in S_D^n(A_\lambda)$ such that

$$p \upharpoonright A_{i+1} \text{ splits over } A_i, \qquad \text{for all } i < \lambda.$$

(2) D satisfies $(B * \lambda)$ if there exists a tree of types $\{p_\eta \in S_D(B_\eta) \mid \eta \in {}^{\lambda >}2\}$, and formulas $\phi_\eta(\bar{x}, \bar{a}_\eta)$ such that $p_\eta \subseteq p_\nu$ if $\eta \prec \nu$ and

$$\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta^\frown 0} \quad \text{and} \quad \neg\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta^\frown 1}.$$

The next two remarks are routine induction using the definition. As an illustration we prove the first one.

Remark 4.2. If there exists a type $p \in S_D(A)$ that splits over every subset of A of cardinality less than λ, then D satisfies $(*\lambda)$.

Proof. Let $p \in S_D(A)$ be such that p splits over every subset B of A of cardinality less than λ. Construct an increasing continuous chain of sets $\{A_i : i \leq \lambda\}$ of cardinality less than λ demonstrating $(*\lambda)$ as follows. Let $A_0 = \emptyset$ and $A_\delta = \bigcup_{i<\delta} A_i$, if δ is a limit ordinal. If A_i is constructed of cardinality less than λ, then by assumption p splits over A_i. Hence, we can find $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y})$ such that $\text{tp}(\bar{b}/A_i) = \text{tp}(\bar{c}/A_i)$ and $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$. Let $A_{i+1} = A_i \cup \bar{b} \cup \bar{c}$. □

Remark 4.3. In the definitions of $(*\lambda)$ and $(B * \lambda)$ we may assume that $|A_i| < |i|^+ + \aleph_0$ and similarly that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$.

Lemma 4.4. *If D satisfies $(*\lambda)$, then D satisfies $(B * \lambda)$.*

Proof. We first show that if $p \in S^n_D(A)$ splits over $B \subseteq A$, then there is a partial elementary mapping f such that $f \restriction B = id_B$ and p and $f(p)$ are contradictory types:

If p splits over B, then there are $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y})$ such that $\text{tp}(\bar{b}/B) = \text{tp}(\bar{c}/B)$ and $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$. Hence there is an elementary mapping f such that $f \restriction B = id_B$ and $f(\bar{b}) = \bar{c}$. Then clearly p and $f(p)$ are contradictory types.

Now assume that D satisfies $(*\lambda)$. By definition, there exists an increasing continuous chain of sets $\{A_i \mid i \leq \lambda\}$ and $p \in S^n_D(A_\lambda)$ such that $p \restriction A_{i+1}$ splits over A_i for $i < \lambda$. By Remark 4.3, we may assume that $|A_i| < |i|^+ + \aleph_0$. By the first paragraph, for each $i < \lambda$ there exists an elementary mapping f_i such that $A_i \subseteq \text{dom}(f_i) \subseteq A_{i+1}$ and $f_i(p \restriction A_{i+1})$ and $p \restriction A_{i+1}$ are contradictory types.

Define G_η, p_η, B_η and F_η by induction on $\eta \in {}^{\lambda \geq}2$ such that:

(1) $p_\eta \in S_D(B_\eta)$.
(2) G_η is an elementary mapping with $\text{dom}(G_\eta) = A_{\ell(\eta)}$ and $\text{ran}(G_\eta) = B_\eta$.
(3) If $\nu \prec \eta$ then $G_\nu \subseteq G_\eta$, $p_\nu \subseteq p_\eta$, $B_\nu \subseteq B_\eta$ and $F_\nu \subseteq F_\eta$, and if $\ell(\eta)$ is a limit ordinal, we set $G_\eta = \bigcup_{i<\ell(\eta)} G_{\eta\restriction i}$, $p_\eta = \bigcup_{i<\ell(\eta)} p_{\eta\restriction i}$, and $B_\eta = \bigcup_{i<\ell(\eta)} B_{\eta\restriction i}$.
(4) $p_\eta = G_\eta(p \restriction A_{\ell(\eta)})$, and the types $p_{\eta\hat{}0}$ and $p_{\eta\hat{}1}$ are explicitly contradictory.
(5) F_η is an elementary mapping extending $G_{\eta\hat{}0} \circ f_{\ell(\eta)} \circ G_{\eta\hat{}1}$ with $\text{dom}(F_\eta) = B_{\eta\hat{}0}$, such that $F_\eta \restriction B_\eta = id_{B_\eta}$ and $F_\eta(p_{\eta\hat{}0}) = p_{\eta\hat{}1}$.

This is enough. The tree of types $\{p_\eta \mid \eta \in {}^{\lambda\geq}2\}$ shows that D satisfies $(B * \lambda)$.

The construction is by induction on $\ell(\eta)$: For $\eta = \langle\rangle$, let $B_{\langle\rangle} = A_0$, $G_{\langle\rangle} = id_{A_0}$ and $p_{\langle\rangle} = p \restriction A_0$. If $\ell(\eta)$ is a limit ordinal use (3). Now assume that G_η, p_η, B_η are constructed for $\ell(\eta) = i$. Let $G_{\eta\hat{}0}$ be an extension of G_η with domain A_{i+1}. Define $B_{\eta\hat{}0} = \text{ran}(G_{\eta\hat{}0})$ and $p_{\eta\hat{}0} = G_{\eta\hat{}0}(p \restriction A_{i+1})$. Now $G_{\eta\hat{}0} \circ f_{\ell(\eta)} \circ G_{\eta\hat{}1}$ is an elementary mapping with domain $\subseteq B_{\eta\hat{}0}$ which is the identity on B_η. Let F_η be an elementary mapping extending it with domain $B_{\eta\hat{}0}$. Set $B_{\eta\hat{}1} = \text{ran}\, F_\eta$ and $p_{\eta\hat{}0} = F_\eta(p_{\eta\hat{}1})$. □

The following theorem shows that the combinatorial properties $(*\lambda)$ and $(B * \lambda)$ contradict stability in λ.

Theorem 4.5. *If D satisfies $(*\lambda)$ or $(B * \lambda)$ then for every $\mu < 2^\lambda$, D is not stable in μ.*

Proof. By the previous lemma, it is enough to show that if D satisfies $(B * \lambda)$ then for every $\mu < 2^\lambda$, D is not stable in μ.

Let $\mu < 2^\lambda$. Let $\kappa = \min\{\kappa \mid 2^\kappa > \mu\}$. Then $\lambda \geq \kappa$ so D satisfies $(B * \kappa)$.

By definition, there exists $p_\eta \in S_D(B_\eta)$ and $\phi_\eta(\bar{x}, \bar{a}_\eta)$ for $\eta \in {}^{\kappa>}2$, such that $p_\eta \subseteq p_\nu$ if $\eta \prec \nu$ and $\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta^\frown 0}$ and $\neg\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta^\frown 1}$. By Remark 4.3, we may assume that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$.

Let $B = \bigcup_{\eta\in {}^{\kappa>}2} B_\eta$. Then $|B| \leq \sum_{\eta\in {}^{\kappa>}2} |B_\eta| \leq \kappa \cdot 2^{<\kappa} \leq \mu$, by choice of κ and assumption on $|A_i|$. Now for each $\eta \in {}^\kappa 2$, let a_η realize p_η. Define $q_\eta = \mathrm{tp}(a_\eta/B)$. Then for $\nu \neq \eta \in {}^\kappa 2$, there is a first $i < \kappa$ such that $\eta[i] \neq \nu[i]$, say $\eta[i] = 0$ and $\nu[i] = 1$. Hence $p_{\eta^\frown 0} \subseteq q_\eta$ and $p_{\eta^\frown 1} \subseteq q_\nu$, so q_η and q_ν are contradictory types. Therefore $|S_D(B)| \geq |\{q_\eta \mid \eta \in {}^\kappa 2\}| = 2^\kappa > \mu$, so D is not stable in μ. □

The next theorem is a sort of converse.

Theorem 4.6. *If there is a D-set A such that*

$$|S_D(A)| > |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$$

then D satisfies $(*\lambda)$.

Proof. Let $\mu_0 = |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$. By Remark 4.2 it is enough to find a type $p \in S_D(A)$ which splits over every subset $B \subseteq A$ of cardinality less than λ.

Such a type p always exists: Otherwise for every $p \in S_D(A)$, there exists $B_p \subseteq A$ of cardinality less than λ such that p does not split over B_p. Since $|S_D(A)| > \mu_0 \geq |A|^{<\lambda}$, by the pigeonhole principle, we can find $S \subseteq S_D(A)$ of cardinality μ_0^+ and B such that p does not split over B, for each $p \in S$. But, by Corollary 3.8,

$$|\{p \in S_D(A) : p \text{ does not split over } B\}| \leq 2^{|D|^{|B|}} \leq \sum_{\mu<\lambda} 2^{|D|^\mu} \leq \mu_0,$$

a contradiction. □

This gives another characterization of instability. This characterization will be used in the Homogeneity Spectrum Theorem (Theorem 5.9). Notice that $(B * \lambda)$ can be used in lieu of $(*\lambda)$ in the following corollary.

Corollary 4.7. *D is not stable if and only if D satisfies $(*\lambda)$, for every cardinal λ.*

Proof. If D satisfies $(*\lambda)$ for every λ, then Theorem 4.5 implies that D is not stable in λ for every λ. Hence D is not stable.

For the converse, suppose that D is not stable and let λ be given. Then D is not stable in 2^{2^λ}. Hence, there exists a D-set A of cardinality 2^{2^λ} such that $|S_D(A)| > 2^{2^\lambda} = |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$. Therefore D satisfies $(*\lambda)$ by the previous theorem. $\square$

For the second part, we will focus on strong splitting.

Definition 4.8. A type $p \in S^n(A)$ *splits strongly* over $B \subseteq A$ if there exists $\{\bar{a}_i : i < \omega\}$ an indiscernible sequence over B and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{a}_0)$ and $\neg\phi(\bar{x}, \bar{a}_1) \in p$.

A combinatorial property similar to $(*\lambda)$ is now defined in terms of strong splitting.

Definition 4.9. D satisfies $(C * \lambda)$ if there exists an increasing continuous chain of sets $\{A_i \mid i \leq \lambda\}$ and $p \in S^n_D(A_\lambda)$ such that

$$p \upharpoonright A_{i+1} \text{ splits strongly over } A_i, \quad \text{for each } i < \lambda.$$

Clearly if D satisfies $(C * \lambda)$, then it satisfies $(*\lambda)$ and similarly to Remark 4.3, we may assume that $|A_i| < |i|^+ + \aleph_0$ in the definition of $(C * \lambda)$.

The next cardinal invariant plays the role of $\kappa(T)$ for the notion of strong splitting. It appears in the Stability Spectrum theorem.

Definition 4.10. Let

$$\kappa(D) = \min\{\kappa : \text{For all } p \in S_D(A) \text{ there is } B \subseteq A, |B| < \kappa \text{ such that } p \text{ does not split strongly over } B\}.$$

If it is undefined, we let $\kappa(D) = \infty$.

Theorem 4.11. *Let D be stable in λ. Then $\kappa(D)$ is well-defined and $\kappa(D) \leq \lambda$.*

Proof. Suppose that $\kappa(D) > \lambda$. Then, by definition of $\kappa(D)$, there exists a D-set A and a type $p \in S_D(A)$ such that p splits strongly over every subset B of A of cardinality at most λ. Similarly to Remark 4.2 this implies that D satisfies $(C * \lambda)$. Hence, D satisfies $(*\lambda)$. By Theorem 4.5 D is not stable in λ, a contradiction. $\square$

To deal with strong splitting, some understanding of indiscernibles is needed. Theorem 4.13 is one of the main results to produce indiscernible sequences in the presence of stability. Recall Lemma I.2.5 of [Sh b].

Fact 4.12. *Let B and let $\{\bar{a}_i \mid i < \alpha\}$ be given. Consider $q_i = \operatorname{tp}(\bar{a}_i/B \cup \{\bar{a}_j \mid j < i\}) \in S_D(B \cup \{\bar{a}_j \mid j < i\})$ and suppose that*

(1) *If $i < j < \alpha$ then $q_i \subseteq q_j$;*
(2) *For each $i < \alpha$ the type q_i does not split over B.*

Then $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible sequence over B.

Theorem 4.13. *Let D be stable in λ. Let I be a set of finite sequences and let A be a set such that $I \cup A$ is a D-set. If $|A| \leq \lambda < |I|$ then there exists a subset of I of cardinality λ^+ which is an indiscernible set over A.*

Proof. By the pigeonhole principle, there exists a subset J of I of cardinality λ^+ and $n < \omega$ such that $\bar{a} \in J$ implies $\ell(a) = n$. Write $J = \{\bar{a}_i : i < \lambda^+\}$.

Claim. There are D-sets B and C, $A \subseteq B \subseteq C$, such that every type in $S_D(B)$ is realized in C, and there exists a type $p \in S^n_D(C)$ such that for every D-set C_1 containing C of cardinality λ, there exists an extension $p_1 \in S^n_D(C_1)$ of p such that p_1 does not split over B and is realized in $J \setminus C$.

Proof of the Claim. Assume that B, C and p as in the claim cannot be found. For each $i < \lambda$ construct D-sets A_i of cardinality at most λ such that every $p \in S_D(A_{i+1})$ which is realized in $J \setminus A_{i+1}$ splits over A_i.

This is possible: Let $A_0 = \emptyset$ and $A_\delta = \bigcup_{i<\delta} A_i$ for δ a limit. Now assume A_i of cardinality at most λ is already constructed. Then $|S_D(A_i)| \leq \lambda$ by stability in λ. Hence, there exists a D-set A^i of cardinality λ, containing A_i, realizing all the types over A_i. Now for any $p \in S^n_D(A^i)$, A_i, A^i and p do not satisfy the assumptions of the claim. Therefore, there exists C_p, a D-set, $C_p \supseteq A^i$ of cardinality λ such that every extension of p in $S^n_D(C_p)$ that is realized in $J \setminus C_p$ splits over A_i. Let $A_{i+1} = \bigcup_{p \in S^n_D(A^i)} C_p$. Then A_{i+1} is a D-set of cardinality at most λ with the desired property.

Let $A_\lambda = \bigcup_{i<\lambda} A_i$. Since J has cardinality λ^+, there is $\bar{a} \in J \setminus A^n_\lambda$. Let $p = \operatorname{tp}(\bar{a}/A_\lambda)$. By construction $p \restriction A_{i+1}$ splits over A_i so D satisfies $(*\lambda)$. Hence, D is not stable in λ by Theorem 4.5, a contradiction. □

Let B, C and $p \in S^n_D(C)$ be as in the claim. Construct $\{\bar{b}_i : i < \lambda^+\} \subseteq J$ by induction on $i < \lambda^+$ as follows. If $\bar{b}_j$ is defined for $j < i$ let $C_i = C \cup \{\bar{b}_j \mid j < i\}$ and $p_i \in S^n_D(C_i)$ be an extension of p which does not split over B and is realized in $J \setminus C^n_i$. Let $\bar{b}_i$ be in $J \setminus C^n_i$ realizing p_i. Then $\{\bar{b}_i \mid i < \lambda^+\}$ is an indiscernible sequence by Fact 4.12. Since D is stable, then it does not have the order property by Corollary 3.12 and hence $\{\bar{b}_i \mid i < \lambda^+\}$ is an indiscernible set, by Remark 3.4. □

The next two theorems prepare for the Stability Spectrum Theorem.

Theorem 4.14. *Let D be stable in λ. Let $\mu \geq \lambda$ be such that $\mu^{<\kappa(D)} = \mu$. Then D is stable in μ.*

Proof. Suppose that D is not stable in μ. Let A be a D-set of cardinality μ such that $|S_D(A)| > |A|$. By assumption, $|S_D(A)| > |A|^{<\kappa(D)}$. Hence $|S_D(A)| \geq \lambda^{++}$. Since D is stable in λ, then that $\kappa(D) \leq \lambda$ by Theorem 4.11. Hence, for each $p \in S_D(A)$ there exists a subset $B_p \subseteq A$ of cardinality less than $\kappa(D)$ such that p does not split strongly over B_p. Since there are $|A|^{<\kappa(D)} = |A|$ such B_p's, by the pigeonhole principle, there exists a set $S \subseteq S_D(A)$ of cardinality λ^{++} and a D-set $B \subseteq A$ of cardinality less than $\kappa(D)$ such that p does not split strongly over B, for each $p \in S$.

Construct $\{\phi_i(x, \bar{a}_i) \mid i < \lambda^+\}$ and $p_i \in S$, for $i < \lambda^+$ such that

(*) $$\{\phi_j(x, \bar{a}_j) : j < i\} \cup \{\neg\phi_i(x, \bar{a}_i)\} \subseteq p_i.$$

To do this, define $S_i \subseteq S$ and $A_i \subseteq A$ for $i < \lambda^+$ such that

(1) $A_0 = \emptyset$, $A_\delta = \bigcup_{i<\delta} A_i$ for δ limit, and $A_i \subseteq A_{i+1}$;

(2) $|A_i| \leq \lambda$, for each $i < \lambda$;
(3) $S_i = \{p \in S \mid p$ is the unique extension of $p \upharpoonright A_i\}$;
(4) A_{i+1} is a subset of A such that if $q \in S_D(A_i)$ has at least two contradictory extensions in S, then it has at least two extensions $q, r \in S$ such that $q \upharpoonright A_{i+1} \neq r \upharpoonright A_{i+1}$.

For $i = 0$ or i a limit ordinal, do (1). For the successor stage: If A_i is constructed and $q \in S_D(A_i)$ has two extensions $q_1, q_2 \in S$, then there is $\phi_q(x, \bar{y})$ and $\bar{a}_q \in A$ such that $\phi_q(x, \bar{a}_q) \in q_1$ and $\neg\phi_q(x, \bar{a}_q) \in q_2$. Since $|S_D(A_i)| \leq \lambda$, A_{i+1} of cardinality λ as in (4) can be found.

Notice that since $|S| = \lambda^{++}$ and $|\bigcup_{i<\lambda^+} S_i| \leq \sum_{i<\lambda^+} |S_D(A_i)| \leq \lambda^+ \cdot \lambda = \lambda^+$, there exists $p \in S \backslash \bigcup_{i<\lambda^+} S_i$. For each $i < \lambda^+$ consider $p \upharpoonright A_i$. Since $p \notin S_i$, by definition of S_i the type $p \upharpoonright A_i$ has at least two contradictory $q, r \in S$. By (4), we may assume that $q \upharpoonright A_{i+1} \neq r \upharpoonright A_{i+1}$. Hence, either $p \upharpoonright A_{i+1} \neq q \upharpoonright A_{i+1}$, or $p \upharpoonright A_{i+1} \neq r \upharpoonright A_{i+1}$. Thus, in either case, there is $p_i \in S$ such that $p \upharpoonright A_{i+1} \neq p_i \upharpoonright A_{i+1}$. Hence, there exist $\bar{a}_i \in A_{i+1}$ and $\phi_i(x, \bar{y}) \in L(T)$ such that $\phi_i(x, \bar{a}_i) \in p$ and $\neg\phi_i(x, \bar{a}_i) \in p_i$. This establishes (*)

Now for each $i < \lambda^+$, let b_i realize p_i. The set $\{b_i\hat{\ }\bar{a}_i : i < \lambda^+\}$ has cardinality λ^+ and B has cardinality less than $\kappa(D) \leq \lambda$, so by Theorem 4.13 there is a subset of $\{b_i\hat{\ }\bar{a}_i \mid i < \lambda^+\}$ of cardinality λ^+ which is indiscernible over B. Without loss of generality, we may assume that $\{b_i\hat{\ }\bar{a}_i \mid i < \lambda^+\}$ is indiscernible over B. By stability in λ we have $|S_D(\bigcup_{k<\lambda} \bar{a}_k)| \leq \lambda$. Hence, by the pigeonhole principle, there exist i and j with $\lambda < j < i < \lambda^+$ such that $p_i \upharpoonright \bigcup_{k<\lambda} \bar{a}_k = p_j \upharpoonright \bigcup_{k<\lambda} \bar{a}_k$. By choice of j, we have $\phi_j(x, \bar{a}_j) \in p_i$ and $\neg\phi_j(x, \bar{a}_j) \in p_j$. Now if $\phi_j(x, \bar{a}_0) \in p_i$ then since $\neg\phi_j(x, \bar{a}_j) \in p_j$, p_j splits strongly over B, since $\{\bar{a}_0, \bar{a}_j, \bar{a}_{j+1}, \ldots\}$ is indiscernible over B. And if $\phi_j(x, \bar{a}_0) \notin p_i$ then $\neg\phi_j(x, \bar{a}_0) \in p_i$, and since $\phi_j(x, \bar{a}_j) \in p_i$ then p_i splits strongly over B, since $\{\bar{a}_j, \bar{a}_0, \bar{a}_1, \ldots\}$ is indiscernible over B. This contradicts the choice of S and B. □

Theorem 4.15. *Let D be stable in λ. Let $\mu \geq \lambda$ be such that $\mu^{<\kappa(D)} > \mu$. Then D is not stable in μ.*

To prove this theorem, a proposition is needed.

Proposition 4.16. *Let D be stable in λ. Let $\chi \leq \lambda$ be a cardinal such that $\lambda^\chi > \lambda$. Let I be an indiscernible sequence. Then, for every $\bar{c} \in \mathfrak{C}$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ either*

$$|\{\bar{a} \in I \; : \models \phi[\bar{a}, \bar{c}]\}| < \chi \quad \textit{or} \quad |\{\bar{a} \in I \; : \models \neg\phi[\bar{a}, \bar{c}]\}| < \chi.$$

Proof. Let I and $\phi(x, \bar{c})$ contradict the conclusion of the proposition. Then, without loss of generality $|I| = \chi$. Write $I = \{\bar{a}_i \mid i < \chi\}$. Since I is indiscernible, there exists $J = \{\bar{a}_i \mid i < \lambda\}$ containing I, indiscernible of cardinality λ. By the pigeonhole principle, either $\{i < \lambda \; : \models \phi[\bar{a}_i, \bar{c}]\}$ or $\{i < \lambda \; : \models \neg\phi[\bar{a}_i, \bar{c}]\}$ has cardinality λ. Without loss of generality, assume that it is the second. Hence, by a re-enumeration (recall that J is necessarily an indiscernible set), define $J_1 = \{\bar{a}_i : i < \chi + \lambda\}$ such that $\models \phi[\bar{a}_i, \bar{c}]$ if and only if $i < \chi$. Let $q = \mathrm{tp}(\bar{c}/J_1)$. Then for any $E \subseteq J_1$ of cardinality χ with complement of cardinality λ we can find a function $f_E \colon J_1 \to J_1$ with $f(\bar{a}_i) \in E$ if and only if $i < \chi$. Then, for two such sets $E_1 \neq E_2$, we have $f_{E_1}(q) \neq f_{E_2}(q)$. Hence $|S_D(J_1)| \geq \lambda^\chi > \lambda$, contradicting the stability in λ. □

Proof of the Theorem. By assumption, there exists $\kappa < \kappa(D)$ such that $\kappa = \min\{\kappa \mid \mu^\kappa > \mu\}$. Let $\chi \leq \lambda$ such that $\chi = \min\{\chi \mid \lambda^\chi > \lambda\}$. Observe that $\mu^\kappa > \chi^\kappa$: Otherwise, $\lambda \leq \mu < \mu^\kappa \leq \chi^\kappa \leq \lambda^\kappa$, and so $\chi \leq \kappa$ by minimality of χ. Hence $\lambda < \mu^\kappa \leq \chi^\kappa = 2^\kappa$. But $(C * \kappa)$ holds and $\lambda \leq 2^\kappa$, so D is not stable in λ by Theorem 4.5, a contradiction.

Now, by definition of $(C * \kappa)$, there exists an increasing, continuous chain of D-sets $\{A_i \mid i \leq \kappa\}$ and a type $p \in S_D(A_\kappa)$ such that $|A_i| \leq |i| + \aleph_0$ and

$$p \restriction A_{i+1} \text{ splits strongly over } A_i, \quad \text{for each } i < \kappa.$$

By definition of strong splitting, for each $i < \kappa$, there exist $\{\bar{a}^i_\alpha \mid \alpha < \omega\}$ indiscernible over A_i and $\phi_i(x, \bar{y}) \in L(T)$ such that both $\phi_i(x, \bar{a}^i_0)$, and $\neg\phi_i(x, \bar{a}^i_1)$ belong to $p \restriction A_{i+1}$.

For each $\eta \in {}^{\kappa>}\mu$, construct a type p_η, a D-set B_η and an elementary mapping G_η, by induction on $\ell(\eta)$ such that:

(1) $p_\eta \in S_D(B_\eta)$ and if $\eta \prec \nu$ then $p_\eta \subseteq p_\nu$ and $B_\eta \subseteq B_\nu$;
(2) G_η is an elementary mapping from $A_{\ell(\eta)}$ onto B_η;
(3) $|B_\eta| \leq \kappa$;
(4) For each $c \in \mathfrak{C}$ the set $\{\alpha < \mu \mid c \text{ realizes } p_{\eta^\frown\alpha}\}$ has cardinality less than χ.

Let $B_{\langle\rangle} = A_0$, $G_{\langle\rangle} = id_{A_0}$ and $p_{\langle\rangle} = p \restriction A_0$. For η such that $\ell(\eta)$ is a limit ordinal, define everything by continuity. For the successor case, suppose that p_η, B_η and G_η have been constructed for η, with $\ell(\eta) = i$. Let F be an elementary mapping extending G_η with domain A_κ. Let $\bar{b}^i_\alpha = F(\bar{a}^i_\alpha)$, for $\alpha < \omega$. Then $\{\bar{b}^i_\alpha \mid \alpha < \omega\}$ is indiscernible over B_η. Hence, we can extend this set to $\{\bar{b}^i_\alpha \mid \alpha < \mu\}$ such that $\{\bar{b}^i_\alpha \mid \alpha < \mu\}$ is also indiscernible over B_η. For $\alpha < \mu$, let $G_{\eta^\frown\alpha}$ be an elementary mapping extending G_η, with domain A_{i+1} such that $G_{\eta^\frown\alpha}(\bar{a}^i_0) = \bar{b}^i_\alpha$ and $G_{\eta^\frown\alpha}(\bar{a}^i_1) = \bar{b}^i_{\alpha+1}$. This is possible by indiscernibility. Let $p_{\eta^\frown\alpha} = G_{\eta^\frown\alpha}(p \restriction A_{i+1})$ and $B_{\eta^\frown\alpha} = \operatorname{ran} G_{\eta^\frown\alpha}$. Hence (1)–(3) are satisfied. To see (4), observe that for each $\alpha < \mu$, both $\phi_i(x, \bar{b}^i_\alpha)$ and $\neg\phi_i(x, \bar{b}^i_{\alpha+1})$ belong to $p_{\eta^\frown\alpha}$. Since $\{\bar{b}^i_\alpha \mid \alpha < \mu\}$ is indiscernible and $\chi \leq \lambda < \lambda^\chi$, (4) follows from the previous proposition.

The construction implies the conclusion. Let $B = \bigcup_{\eta \in {}^{\kappa>}\mu} B_\eta$. Then $|B| \leq \mu^{<\kappa} \cdot \kappa = \mu$, by choice of κ. For each $\eta \in {}^\kappa\mu$, let $p_\eta = \bigcup_{i<\kappa} p_{\eta \restriction i}$. By continuity, each p_η is a D-type and let a_η realize p_η. Then $\operatorname{tp}(a_\eta/B) \in S_D(B)$. By (4), for each $c \in \mathfrak{C}$, the set $\{\eta \in {}^\kappa\mu \mid a_\eta = c\}$ has cardinality at most χ^κ and we observed that $\chi^\kappa < \mu^\kappa$. Hence, $|S_D(B)| > \mu$, so D is not stable in μ. □

We finish this section with the Stability Spectrum Theorem.

Theorem 4.17 (The Stability Spectrum). *Let D be a finite diagram. Either D is not stable, or D is stable and there exist cardinals $\kappa \leq \lambda < \beth_{(2^{|T|})^+}$ such that for every cardinal μ, D is stable in μ if and only if $\mu \geq \lambda$ and $\mu^{<\kappa} = \mu$.*

Proof. If D is not stable, there is nothing to prove. If D is stable, let $\lambda(D)$ be the first cardinal λ for which D is stable λ. Then $\lambda(D) < \beth_{(2^{|T|})^+}$ by Corollary 3.11. Moreover, $\kappa(D)$ is defined and $\kappa(D) \leq \lambda(D)$ by Theorem 4.11.

Let μ be given. If $\mu < \lambda(D)$, then D is not stable in μ by choice of $\lambda(D)$. Suppose that $\mu \geq \lambda(D)$. If $\mu^{<\kappa(D)} = \mu$, then D is stable in μ by Theorem 4.14. If $\mu^{<\kappa(D)} > \mu$, then D is not stable in μ by Theorem 4.15. □

5. The Homogeneity Spectrum

The section is devoted to the proof of the Homogeneity Spectrum Theorem (Theorem 5.9). The proof will proceed by cases, and is broken into several theorems. There are two types of results. On the one hand there are theorems showing the existence of a (D,λ)-homogeneous model of cardinality λ from assumptions like stability in λ and $\lambda^{<\lambda}$. On the other hand, there are results showing that such models do not exist from the failure of these conditions. The combinatorial properties defined in the previous section and parts of the Stability Spectrum Theorem will play a crucial role.

Theorem 5.1. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} = \lambda$. Then there is a (D,λ)-homogeneous model of cardinality λ.*

Proof. First, by Zermelo-König, λ is regular. By the downward Löwenheim-Skolem theorem, define an increasing continuous chain $\langle M_i \mid i < \lambda\rangle$ of D-models of cardinality λ, such that M_{i+1} realizes every D-type over every $A \subseteq M$ of cardinality less than λ. This is possible since we have only $\lambda^{<\lambda} = \lambda$ subsets of A of cardinality less than λ and only $|D|^{|A|} \leq \lambda^{<\lambda} = \lambda$ D-types over A. Let $M = \bigcup_{i<\lambda} M_i$. Then M has cardinality λ, and since λ is regular, M is (D,λ)-homogeneous. □

Theorem 5.2. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} > \lambda$. If D satisfies $(B * \lambda)$ then there is no (D,λ)-homogeneous model of cardinality λ.*

Proof. Suppose $\lambda^{<\lambda} > \lambda \geq |D|$. Assume, by way of contradiction, that there is a (D,λ)-homogeneous model M of cardinality λ. Since D satisfies $(B*\lambda)$ there exist D-types $p_\eta \in S_D(B_\eta)$ and $\phi_\eta(\bar{x},\bar{a}_\eta)$ for $\eta \in {}^{\lambda>}2$ such that $\phi_\eta(\bar{x},\bar{a}_\eta) \in p_{\eta^\frown 0}$ and $\neg\phi_\eta(\bar{x},\bar{a}_\eta) \in p_{\eta^\frown 1}$. In addition $p_\eta \subseteq p_\nu$ when $\eta \prec \nu$. By Remark 4.3, we may assume that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$. Hence, by (D,λ)-homogeneity of M, we may assume that $B_\eta \subseteq M$ for each $\eta \in {}^{\lambda>}2$.

For each $\mu < \lambda$ and $\eta \in {}^{\mu}2$, there are 2^μ types in $S_D(B_\eta)$. Each such type is realized in M, since M is (D,λ)-homogeneous and so $2^\mu \leq \lambda$, since M has cardinality λ. Hence, λ is singular, since otherwise $\lambda^{<\lambda} = \lambda$. Furthermore, λ is a strong limit (if there is $\mu < \lambda$ such that $2^\mu = \lambda$, then $\lambda^{\mathrm{cf}(\lambda)} = 2^{\mu\cdot\mathrm{cf}(\lambda)} \leq \lambda$, contradicting Zermelo-König).

Let $\kappa = \mathrm{cf}(\lambda)$ and let $\lambda_i < \lambda$ for $i < \kappa$ be increasing and continuous such that $\lambda = \sum_{i<\kappa} \lambda_i$. Let $A_i \subseteq M$ of cardinality λ_i for $i < \kappa$ such that $M = \bigcup_{i<\kappa} A_i$.

For each $i < \kappa$, define a sequence $\eta_i \in {}^{\lambda>}2$ and a finite set C_{i+1} such that

(1) If $i < j$ then $\eta_i \prec \eta_j$;
(2) C_{i+1} is a finite subset of $B_{\eta_{i+1}}$;
(3) The type $p_{\eta_{i+1}} \restriction C_i$ is not realized in A_i.

This is enough: Let $p = \bigcup_{i<\kappa} p_{\eta_i}$. Then $p \restriction \bigcup_{i<\kappa} C_i$ is a D-type (by continuity) over a set of cardinality κ, which is not realized in M. This contradicts the (D,λ)-homogeneity of M since $\kappa < \lambda$.

This construction is possible. Define $\eta_0 = \langle\rangle$, and for $\delta < \kappa$ a limit ordinal let $\eta_\delta = \bigcup_{i<\delta} \eta_i$. For the successor case, assume that $\eta_i \in {}^{\lambda>}2$ is constructed. Define $\tau_\alpha =$

$\eta_i\hat{\ }0_\alpha$, where 0_α is a sequence of zeroes of order type α, for $\alpha < 2^{\lambda_i}$. Then $\tau_\alpha \in {}^{\lambda>}2$, since $\lambda_i < \lambda$ and λ is a strong limit.

We claim that there are $\alpha < \beta < (2^{\lambda_i})^+$ such that $\models \phi_{\tau_\alpha}[c, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}]$, for every $c \in A_i$.

Suppose that this is not the case. Let $A_i = \{c_\gamma \mid \gamma < \lambda_i\}$. Then, for every $\alpha < \beta < (2^{\lambda_i})^+$ there exists $\gamma < \lambda_i$ such that $\models \neg(\phi_{\tau_\alpha}[c_\gamma, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}])$. By the Erdős-Rado theorem, there is $\gamma < \lambda_i$ and an infinite set $S \subseteq (2^{\lambda_i})^+$ such that for every $\alpha < \beta$ in S we have $\models \neg(\phi_{\tau_\alpha}[c_\gamma, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}])$. This is an immediate contradiction.

Hence, let $\alpha < \beta$ be as in (*). Let $C_{i+1} = \bar{a}_{\tau_\alpha} \cup \bar{a}_{\tau_\beta}$ and let $\eta_{i+1} = \tau_\alpha\hat{\ }1$. Since $\phi_{\tau_\alpha}(x, \bar{a}_{\tau_\alpha})$ and $\neg\phi_{\tau_\beta}(x, \bar{a}_{\tau_\beta})$ are in $p_{\eta_{i+1}} \upharpoonright C_i$, the type $p_{\eta_{i+1}}$ is omitted in A_i. This finishes the construction and proves the theorem. □

The next theorem is, in particular, an improvement of Proposition 4.16. It allows us to define *averages* (Definition 5.4). Averages are used in Theorem 5.6.

Theorem 5.3. *Let D be stable. Let I be an infinite indiscernible set over A of cardinality at least $\kappa(D)$. Let $\bar{b} \in \mathfrak{C}$. Then there is $J \subseteq I$ with $|J| < \kappa(D)$ such that $I \setminus J$ is indiscernible over $A \cup J \cup \bar{b}$.*

Proof. Let $I = \{\bar{c}_i \mid i < \alpha\}$. Since D is stable, $\kappa(D)$ is defined by Theorem 4.11. Hence, there exists $B \subseteq A \cup I$ of cardinality less than $\kappa(D)$ such that the type $\text{tp}(\bar{b}/A \cup I)$ does not split strongly over B. Let $J = B \setminus A$. Then $J \subseteq I$ has cardinality less than $\kappa(D)$. We will show that $I \setminus J$ is indiscernible over $A \cup J \cup \bar{b}$. Clearly, $I \setminus J$ is indiscernible over $A \cup J$. If $I \setminus J$ is not indiscernible over $A \cup J \cup \bar{b}$, then, there exist an integer $n < \omega$ and indices $i_0 \cdots < i_n$ such that $\text{tp}(\bar{c}_0, \dots, \bar{c}_n/A \cup J \cup \bar{b}) \neq \text{tp}(\bar{c}_{i_0}, \dots, \bar{c}_{i_n}/A \cup J \cup \bar{b})$. Then $\models \phi[\bar{c}_0, \dots, \bar{c}_n, \bar{a}, \bar{b}, \bar{c}]$ and $\models \neg\phi[\bar{c}_{i_0}, \dots, \bar{c}_{i_n}, \bar{a}, \bar{b}, \bar{c}]$, for some formula $\phi \in L(T)$, parameters $\bar{a} \in A$ and $\bar{c} \in J$. Let $\bar{d}_0 = \bar{c}_0\hat{\ }\dots\hat{\ }\bar{c}_n$ and $\bar{d}_1 = \bar{c}_{i_0}\hat{\ }\dots\hat{\ }\bar{c}_{i_n}$. By taking sequences from $I \setminus J$, it is easy to find $\{\bar{d}_i \mid i < \omega\}$ indiscernible over $A \cup J$. Thus $\{\bar{d}_i\hat{\ }\bar{a}\hat{\ }\bar{c} \mid i < \omega\}$ is indiscernible over $A \cup J$. Hence, the type $\text{tp}(\bar{b}/A \cup I)$ splits strongly over $A \cup J$, a contradiction to the choice of B. □

Definition 5.4. Let I be an indiscernible sequence of cardinality at least $\kappa(D)$. Let A be such that $A \cup I$ is a D-set. Define the *average of I over A*, by

$$\begin{aligned}\text{Av}(I, A) = \{\phi(\bar{x}, \bar{a}) \mid \phi(\bar{x}, \bar{y}) \in L(T), \bar{a} \in A, \text{ and } \models \phi[\bar{b}, \bar{a}],\\ \text{for at least } \kappa(D) \text{ elements } \bar{b} \in I\}.\end{aligned}$$

Theorem 5.5. *Let D be stable. Let I be an indiscernible sequence of cardinality at least $\kappa(D)$ and A be such that $A \cup I$ is a D-set. Then $\text{Av}(I, A) \in S_D^n(A)$, where $n = \ell(\bar{a})$ for $\bar{a} \in I$. In addition, if $|I| > |A|$, then $\text{Av}(I, A)$ is realized in I.*

Proof. Averages are complete: Assume $\phi(\bar{x}, \bar{c}) \notin \text{Av}(I, A)$, with $\bar{c} \in A$. Then by definition, the set $J \subseteq I$ of elements realizing $\phi(\bar{x}, \bar{c})$ has cardinality less than $\kappa(D)$. Thus, since $I \setminus J$ has cardinality at least $\kappa(D)$, and all elements in $I \setminus J$ realize $\neg\phi(\bar{x}, \bar{c})$, necessarily $\neg\phi(\bar{x}, \bar{c}) \in \text{Av}(I, A)$. Averages are consistent: Let $\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n) \in \text{Av}(I, A)$. Then, if $\bar{c} = \bar{c}_1\hat{\ }\dots\hat{\ }\bar{c}_n$, by Theorem 5.3, there is $J_{\bar{c}}$, $J_{\bar{c}} \subseteq I$ of cardinality less than $\kappa(D)$

such that $I \setminus J_{\bar{c}}$ is indiscernible over $\bar{c}$. Hence, since each $\phi_i(x, \bar{c}_i)$ was realized by at least $\kappa(D)$ elements of I, we can find one in $I \setminus J_{\bar{c}}$. But then, all elements in $I \setminus J_{\bar{c}}$ realize $\phi_i(x, \bar{c}_i)$ by indiscernibility ($1 \leq i \leq n$), so $\{\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n)\}$ is consistent. The last sentence follows similarly: For any $\bar{c} \in A$, every element of $I \setminus J_{\bar{c}}$ realizes $\mathrm{Av}(I, A) \restriction \bar{c}$, since they realize every formula in it, and so if $|I| > |A|$, we can find $\bar{b} \in I \setminus \bigcup_{\bar{c} \in A} J_{\bar{c}}$ realizing $\mathrm{Av}(I, A)$. It remains to show that $\mathrm{Av}(I, A)$ is a D-type: Notice that if we stretch I to J, $I \subseteq J$ indiscernibles of cardinality greater than $|A|$, we have $\mathrm{Av}(I, A) = \mathrm{Av}(J, A)$. Then $\mathrm{Av}(I, A)$ is realized in J, thus in $\mathfrak{C}$, since J is a D-set, and so $\mathrm{Av}(I, A)$ is a D-type. □

Theorem 5.6. *Let $\lambda \geq |D|$. If D is stable in λ, then there is a (D, λ)-homogeneous model of cardinality λ.*

Proof. Suppose first that λ is regular. Define an increasing continuous chain $\langle M_i \mid i < \lambda \rangle$ of models of cardinality λ, such that M_0 realizes all the types in D, and M_{i+1} realizes all the types over M_i. Such a construction is possible since D is stable in λ and $\lambda \geq |D|$. Let $M = \bigcup_{i<\lambda} M_i$. Then, M has cardinality λ and M is (D, λ)-homogeneous by regularity of λ.

Now suppose that λ is singular. Construct an increasing continuous chain of models $\langle M_i \mid i < \lambda \cdot \lambda \rangle$ as above of length $\lambda \cdot \lambda$. Let $M = \bigcup_{i<\lambda \cdot \lambda} M_i$. Notice that M has cardinality λ. We now show that it is (D, λ)-homogeneous. Let $A \subseteq M$ of cardinality less than λ and $p_0 \in S_D(A)$. We will find I indiscernibles of cardinality greater than $|A|$ with $p_0 = \mathrm{Av}(I, A)$. Let $p \in S_D(M)$ extending p_0 and choose $C \subseteq M$ of cardinality less than $\kappa(D)$ such that p does not split strongly over C. Since D is stable in λ, then $\lambda^{<\kappa(D)} = \lambda$ by Theorem 4.15. Hence, $\mathrm{cf}(\lambda) \geq \kappa(D)$. Thus, considering the sequence $\langle M_{\lambda \cdot i} \mid i < \lambda \rangle$ we can find $i < \lambda$ such that $C \subseteq M_{\lambda \cdot i}$.

We claim that p does not split over $M_{\lambda \cdot i + \lambda}$. Otherwise, there are $\bar{b}$ and $\bar{c}$ in M and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{b}) \in p$, $\neg\phi(\bar{x}, \bar{c}) \in p$ and

$$\mathrm{tp}(\bar{b}/M_{\lambda \cdot i+\lambda}) = \mathrm{tp}(\bar{c}/M_{\lambda \cdot i+\lambda}).$$

Let $q := \mathrm{tp}(\bar{b}/M_{\lambda \cdot i+\lambda})$. Now, since λ is singular, we have $\omega < \lambda$. Consider the following set

$$\{j < \lambda : q \restriction M_{\lambda \cdot i+\omega \cdot (j+1)} \text{ splits over } M_{\lambda \cdot i+\omega \cdot j}\}.$$

Since D is stable in λ, in particular $(*\lambda)$ fails so we can find γ with

$$\lambda \cdot i < \gamma < \gamma + \omega < \lambda \cdot \lambda$$

such that $q \restriction M_{\gamma+\omega}$ does not split over M_γ. For each $n < \omega$, we can choose $\bar{b}_n \in M_{\gamma+n+1}$ realizing $\mathrm{tp}(\bar{b}/M_{\gamma+n})$. Now, $\mathrm{tp}(\bar{b}_n/M_{\gamma+n})$ does not split over M_γ ($\forall n < \omega$) by monotonicity. Hence $\{\bar{b}_n \mid n < \omega\}$ are indiscernible over M_γ, by Fact 4.12. Similarly, both $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}\}$ and $\{\bar{b}_0, \bar{b}_1, \dots, \bar{c}\}$ are indiscernible over M_γ. In fact, since D is stable, D does not have the order property by Corollary 3.12, and thus they are indiscernible sets by Remark 3.4. Now suppose that for some $n < \omega$, the formula $\phi(\bar{x}, \bar{b}_n) \in p$. Then p splits strongly over C since

$$\{\bar{b}_n, \bar{c}, \bar{b}_{n+1}, \dots\} \text{ is indiscernible over } C.$$

Otherwise $\neg\phi(\bar{x}, \bar{b}_0) \in p$. Then p splits strongly over C because

$$\{\bar{b}, \bar{b}_0, \bar{b}_1, \dots\} \text{ is indiscernible over } C.$$

We have a contradiction in both cases, which proves the claim.

We now use the claim to prove the conclusion of the theorem. First, we may assume that $\lambda \cdot i = 0$, so p does not split over M_0. Now for each $\alpha < \lambda \cdot \lambda$, choose $a_\alpha \in M_{\alpha+1}$ realizing $p \restriction M_\alpha$. Since p does not split over M_0 the sequence $I := \{a_\alpha \mid \alpha < \lambda \cdot \lambda\}$ is indiscernible. Let $\phi(x, \bar{a}) \in p_0$. There is $\alpha_0 < \lambda^2$ such that $\phi(x, \bar{a}) \in p_0 \restriction M_{\alpha_0}$, so we have that $\models \phi[a_\alpha, \bar{a}]$ for every $\alpha \geq \alpha_0$. Hence there are $\lambda \geq \kappa(D)$ many elements of I realizing $\phi(x, \bar{a})$, showing that $\phi(x, \bar{a}) \in \mathrm{Av}(I, A)$. So $\mathrm{Av}(I, A) \supseteq p_0$ and since both types are complete, we have $p_0 = \mathrm{Av}(I, A)$. Thus since $|I| > |A|$, there are elements of I realizing p_0. This shows that p_0 is realized in M. Hence M is (D, λ)-homogeneous. $\square$

The next lemma is an improvement of Corollary 3.8. It is needed in the proof of Theorem 5.8.

Lemma 5.7. *Let D be stable. Let $A \subseteq B$ be D-sets such that every D-type over A is realized in B. Fix $n < \omega$ and define*

$$\Gamma := \{p \in S_D^n(B) \mid p \text{ does not split over } A\}.$$

Then, for each $p \in \Gamma$, there is a sequence $\langle \bar{a}_i^p \mid i \leq \omega \rangle$ indiscernibles over A such that

(*) $\qquad p \neq q \in \Gamma \quad$ *implies* $\quad \mathrm{tp}(\langle \bar{a}_i^p : i < \omega \rangle / A) \neq \mathrm{tp}(\langle \bar{a}_i^q : i < \omega \rangle / A).$

Moreover,

$$|\Gamma| \leq |\bigcup_{m<\omega} S_D^m(A)|^{\aleph_0} \leq |D|^{|A|+\aleph_0}.$$

Proof. It is enough to establish (*), since the last statement follows from (*) by a computation.

For each $p \in \Gamma$, define

$$I_p := \langle \bar{a}_i^p : i < \kappa(D) \rangle,$$

by induction on $i < \kappa(D)$ such that $\mathrm{tp}(\bar{a}_i^p / B \cup \{\bar{a}_j^p : j < i\})$ extends p and does not split over A. This is possible by Lemma 3.7. By Fact 4.12 the sequence I_p is indiscernible over A. Hence, it is enough to show that

$$\mathrm{tp}(\langle \bar{a}_i^p : i < \kappa(D) \rangle / A) \neq \mathrm{tp}(\langle \bar{a}_i^q : i < \kappa(D) \rangle / A), \qquad \text{for } p \neq q \in \Gamma.$$

We will use the following claim.

Claim. If $\bar{b} \in B$ and $\bar{b}_1 \in \mathfrak{C}$ such that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}_1/A)$, then

$$|\{i < \kappa(D) : \mathrm{tp}(\bar{b}\frown\bar{a}_0^p / A) \neq \mathrm{tp}(\bar{b}_1\frown\bar{a}_i^p / A)\}| < \kappa(D)$$

Proof of the Claim. To show this, define $\{\bar{a}_i^p : \kappa(D) \leq i < \kappa(D)^+\}$, by induction on i ($\kappa(D) \leq i < \kappa(D)^+$) such that $\mathrm{tp}(\bar{a}_i^p / B \cup \{\bar{a}_j^p : j < i\} \cup \bar{b}_1)$ extends p and does not split over A. Hence, by Fact 4.12, $I' = \{\bar{a}_i^p : i < \kappa(D)^+\}$ is indiscernible. By construction

$$\mathrm{tp}(\bar{b}_1\frown\bar{a}_i^p / A) = \mathrm{tp}(\bar{b}\frown\bar{a}_i^p / A) = \mathrm{tp}(\bar{b}\frown\bar{a}_i^p / A), \quad \text{for } i \geq \kappa(D),$$

since $\bar{b} \in B$ and I_p is indiscernible over B. Thus

$$|\{i \in I' : \mathrm{tp}(\bar{b}\frown\bar{a}_0^p / A) = \mathrm{tp}(\bar{b}_1\frown\bar{a}_i^p / A)\}| > \kappa(D),$$

but then, all $\bar{a}_1 \in I'$ but a subset of cardinality less than $\kappa(D)$ are indiscernibles over $\bar{b} \cup \bar{b}_1$ and so

$$|\{i \in I' : \mathrm{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) \neq \mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)\}| < \kappa(D).$$

The claim follows since $I_p \subset I'$. □

Suppose by way of contradiction that there are $p \neq q \in \Gamma$ with

$$\mathrm{tp}(\langle \bar{a}_i^p : i < \kappa(D)\rangle/A) = \mathrm{tp}(\langle \bar{a}_i^q : i < \kappa(D)\rangle/A).$$

Since $p \neq q$, there is $\bar{b} \in B$ and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{b}) \in q$. By construction, $\models \phi[\bar{a}_i^p, \bar{b}]$ and $\models \neg\phi[\bar{a}_i^q, \bar{b}]$, for all $i < \kappa(D)$. Let f be an elementary mapping such that $f \restriction A = id_A$ and $f(\bar{a}_i^p) = \bar{a}_i^q$ for $i < \kappa(D)$. Clearly, f exists by assumption on p and q. Call $\bar{b}_1 = f^{-1}(\bar{b})$. By applying the claim, we know that $|\{i < \kappa(D) : \mathrm{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) \neq \mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)\}| < \kappa(D)$, hence let $\bar{a}_i^p$, $(i < \kappa(D))$ such that $\mathrm{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) = \mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)$. But, by definition of f, we know that $\mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A) = \mathrm{tp}(\bar{b}\hat{\ }\bar{a}_i^q/A)$. Hence $\mathrm{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) = \mathrm{tp}(\bar{b}\hat{\ }\bar{a}_i^q/A)$. Since $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A)$, we then must have $\models \phi[\bar{a}_i^q, \bar{b}]$, the desired contradiction. □

We now prove the last significant ingredient of the Homogeneity Spectrum Theorem.

Theorem 5.8. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} > \lambda$. Suppose that D is stable but not in λ If D does not satisfy $(*\lambda)$ then there is no (D, λ)-homogeneous model of cardinality λ.*

Proof. By way of contradiction, assume that M is a (D, λ)-homogeneous model of cardinality λ. Let $\{A_\alpha \mid \alpha < \mathrm{cf}(\lambda)\}$ be an increasing continuous chain of sets such that $|A_\alpha| < \lambda$ and $M = \bigcup_{\alpha<\mathrm{cf}(\lambda)} A_\alpha$.

Since D is not stable in λ, there is a D-set B of cardinality λ such that $|S_D(A)| > \lambda$. Then, by Lemma 2.3 we may assume that $B \subseteq M$ since M is (D, λ)-homogeneous. Hence $|S_D(M)| > \lambda$.

We first claim that for each $p \in S_D(M)$, there is $\alpha < \mathrm{cf}(\lambda)$ such that p does not split over A_α.

Suppose not. Let $p \in S_D(M)$ such that p splits over every A_α. If λ is regular, then $\lambda = \mathrm{cf}(\lambda)$ and this implies that D satisfies $(*\lambda)$, a contradiction. Suppose that λ is singular. For each $\alpha < \mathrm{cf}(\lambda)$, choose $\bar{b}_\alpha$, $\bar{c}_\alpha$ in M and $\phi_\alpha(x, \bar{y})$ such that $\mathrm{tp}(\bar{b}_\alpha/A_\alpha) = \mathrm{tp}(\bar{c}_\alpha/A_\alpha)$ and $\phi_\alpha(x, \bar{b}_\alpha) \in p$ and $\neg\phi_\alpha(x, \bar{c}_\alpha) \in p$. Then $p \restriction \{\bar{b}_\alpha, \bar{c}_\alpha\}$ is not realized in A_α. Set $A := \bigcup_{\alpha<\mathrm{cf}(\lambda)} \{\bar{b}_\alpha, \bar{c}_\alpha\}$. Then $p \restriction A$ is not realized in $\bigcup_{\alpha<\mathrm{cf}(\lambda)} A_\alpha = M$. This contradicts the (D, λ)-homogeneity of M since $|A| \leq \mathrm{cf}(\lambda) < \lambda$. This proves the claim.

Now since $|S_D(M)| > \lambda$, by the pigeonhole principle, there exists $\Gamma \subseteq S_D(M)$ of cardinality λ^+ and $\alpha < \mathrm{cf}(\lambda)$, such that if $p \in \Gamma$, then p does not split over A_α. Since $A_\alpha \subseteq M$ of cardinality less than λ and M is (D, λ)-homogeneous, we are in the situation of the previous lemma. Thus for each $p \in \Gamma$ there is $\{\bar{a}_i^p : i \leq \omega\}$ an indiscernible set over A_α such that

$$p \neq q \quad \text{if and only if} \quad \mathrm{tp}(\langle \bar{a}_i^p : i < \omega\rangle/A_\alpha) \neq \mathrm{tp}(\langle \bar{a}_i^q : i < \omega\rangle/A_\alpha).$$

Using the (D, λ)-homogeneity of M and the fact that $|A_\alpha| < \lambda$, construct $\{\bar{b}_i^p : i \leq \omega\} \subseteq M$ for each $p \in \Gamma$ with the following two properties:

(1) $\mathrm{tp}(\langle \bar{b}_j^p : j \leq i \rangle / A_\alpha) = \mathrm{tp}(\langle \bar{a}_j^p : j \leq i \rangle / A_\alpha)$
(2) If $\mathrm{tp}(\langle \bar{a}_j^p : j \leq i \rangle / A_\alpha) = \mathrm{tp}(\langle \bar{a}_j^q : j \leq i \rangle / A_\alpha)$, then $\bar{b}_j^p = \bar{b}_j^q$ for every $j \leq i$.

We now show that

$$(*) \qquad \bar{b}_\omega^p \neq \bar{b}_\omega^q, \qquad \text{if } p \neq q \in \Gamma.$$

Let $p, q \in \Gamma$ such that $p \neq q$. By construction, we have that

$$\mathrm{tp}(\langle \bar{a}_j^p : j < \omega \rangle / A_\alpha) \neq \mathrm{tp}(\langle \bar{a}_j^q : j < \omega \rangle / A_\alpha).$$

Hence, there is a minimal $i < \omega$ such that

$$\mathrm{tp}(\bar{a}_0^p, \dots, \bar{a}_i^p \bar{a}_{i+1}^p / A_\alpha) \neq \mathrm{tp}(\bar{a}_0^q, \dots, \bar{a}_i^q \bar{a}_{i+1}^q / A_\alpha).$$

By minimality of i and (1), we have

$$(**) \qquad \mathrm{tp}(\bar{b}_0^p, \dots, \bar{b}_i^p / A_\alpha) = \mathrm{tp}(\bar{b}_0^q, \dots, \bar{b}_i^q / A_\alpha).$$

Now, we have the following equations

$$\begin{aligned}
\mathrm{tp}(\bar{b}_0^p, \dots, \bar{b}_i^p \bar{b}_\omega^p / A_\alpha) &= \mathrm{tp}(\bar{a}_0^p, \dots, \bar{a}_i^p \bar{a}_\omega^p / A_\alpha) && \text{(by definition (2))} \\
&= \mathrm{tp}(\bar{a}_0^p, \dots, \bar{a}_i^p \bar{a}_{i+1}^p / A_\alpha) && \text{(by indiscernibility)} \\
&\neq \mathrm{tp}(\bar{a}_0^q, \dots, \bar{a}_i^q \bar{a}_{i+1}^q / A_\alpha) && \text{(by choice of } i) \\
&= \mathrm{tp}(\bar{a}_0^q, \dots, \bar{a}_i^q \bar{a}_\omega^q / A_\alpha) && \text{(by indiscernibility)} \\
&= \mathrm{tp}(\bar{b}_0^q, \dots, \bar{b}_i^q \bar{b}_\omega^q / A_\alpha) && \text{(by definition (2))}
\end{aligned}$$

Hence (*) follows from the previous equations and (**).

Therefore (*) implies that we have $|\Gamma|$ many different elements $\bar{b}_\omega^p \in M$. This is a contradiction, since

$$|\Gamma| = \lambda^+ > \lambda = \|M\|.$$

This finishes the proof. □

We can now present the Homogeneity Spectrum Theorem.

Theorem 5.9 (The Homogeneity Spectrum). *There exists a (D, λ)-homogeneous model of cardinality λ if and only if $\lambda \geq |D|$ and either D is stable in λ or $\lambda^{<\lambda} = \lambda$.*

Proof. The proof is divided into 5 cases.

Case 1: $\lambda < |D|$. Then, there can be no (D, λ)-homogeneous model M of cardinality λ, since we require that $D(M) = D$, and there are not enough elements in M to realize all the types in D.

Case 2: $\lambda \geq |D|$ and $\lambda^{<\lambda} = \lambda$. Then, there exists a (D, λ)-homogeneous model M of cardinality λ by Theorem 5.1.

Case 3: $\lambda \geq |D|$ and D is stable in λ. Then, there is a (D, λ)-homogeneous model M of cardinality λ by Theorem 5.6.

Case 4: $\lambda \geq |D|$, $\lambda^{<\lambda} > \lambda$ and D is not stable. Then, by Corollary 4.7, D satisfies $(*\lambda)$. Hence D satisfies $(B * \lambda)$ by Lemma 4.4. Therefore, there is no (D, λ)-homogeneous model M of cardinality λ by Theorem 5.2.

Case 5: $\lambda \geq |D|$, $\lambda^{<\lambda} > \lambda$ and D is stable but not in λ. This case is divided into two sub-cases according to whether D satisfies $(*\lambda)$. If D does satisfy $(*\lambda)$, then D also satisfies $(B * \lambda)$ by Lemma 4.4. Therefore the result follows from Theorem 5.2. If D does not satisfy $(*\lambda)$, then by Theorem 5.8 we have no (D, λ)-homogeneous model of cardinality λ.

The proof is complete. □

6. Local Stability and Local Homogeneity in Finite Diagrams

In this section, we set the necessary definitions to localize the results of this paper. We fix a type and show that all the results of this paper hold inside the set of realizations of this fixed type, with the appropriate local definitions.

Fix $\Sigma(\bar{x})$ a set of $L(T)$-formulas, maybe over a D-set of parameters. We localize the notion of types with respect to Σ. For a model M, denote by $\Sigma(M)$ the set of realizations of $\Sigma(\bar{x})$ in M. In the following definition, Σ is used as a superscript to avoid clashes with the notation set so far.

Definition 6.1. For A a D-set, let

$$S_D^{\Sigma}(A) = \{\mathrm{tp}(c/A) \mid A \cup \bar{c} \text{ is a } D\text{-set and } c \text{ realizes } \Sigma\}.$$

Although the definition makes sense for any $A \subseteq M$, it will only be used when $A \subseteq \Sigma(M)$.

Definition 6.2. A model M is (D, λ, Σ)-*homogeneous*, if M realizes every type in $S_D^{\Sigma}(A)$, for each $A \subseteq \Sigma(M)$ of cardinality less that λ.

We can relax the monster model assumption to:

Hypothesis 6.3. There exists a D-model $\mathfrak{C}$ such that $\mathfrak{C}$ is $(D, \bar{\kappa}, \Sigma)$-*homogeneous*, for some $\bar{\kappa}$ larger than any cardinal needed in this paper.

We will work inside $\Sigma(\mathfrak{C})$. The results of Section 2 hold relativized to realizations of Σ. Thus, $\mathfrak{C}$ can be assumed to contain every D-set $A \subseteq \Sigma(M)$, for any D-model M. And also $\mathfrak{C}$ is homogeneous with respect to subsets of $\Sigma(\mathfrak{C})$. Write $S_D^{\Sigma}(A)$ for $S_D^{\Sigma}(A, \mathfrak{C})$, (note $A \subseteq \Sigma(\mathfrak{C})$ is always assumed).

Here are the local version of stability and order:

Definition 6.4. (1) D is (λ, Σ)-*stable* if $|S_D^{\Sigma}(A)| \leq \lambda$ for every $A \subseteq \Sigma(\mathfrak{C})$ of cardinality λ.

(2) D is Σ-*stable* if D is (λ, Σ)-stable for some cardinal λ.

Definition 6.5. (1) D has the (λ, Σ)-*order property* if there exist a formula $\phi(\bar{x}, \bar{y}) \in L(T)$ and a set $\{\bar{a}_i \mid i < \lambda\} \subseteq \Sigma(\mathfrak{C})$, such that

$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$