REMARKS ON LOCAL STABILITY AND THE LOCAL ORDER PROPERTY

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ABSTRACT. We continue the study of stability of a type in several directions: (1) Inside a fixed model, (2) for classes of models where the compactness theorem fails and (3) for the first order case. Appropriate localizations of the order property, the independence property, and the strict order property are introduced. We are able to generalize some of the results that were known in the case of local stability for the first order theories, and for stability for nonelementary classes (existence of indiscernibles, existence of averages, stability spectrum, equivalence between order and instability). In the first order case, we also prove the local version of Shelah's Trichotomy Theorem. Finally, as an application, we give a new characterization of stable types when the ambient first order theory is simple.

1. INTRODUCTION

Victor Harnik and Leo Harrington in [HH], while presenting an alternative approach of forking to that of Saharon Shelah [Sh a], started a localized generalization of stability theory extending Saharon Shelah's Unstable Formula Theorem (Theorem II 2.2 [Sh a]). This work was later continued and extended by Anand Pillay in [P]. About ten years later Zoe Chatzidakis and Ehud Hrushovski in their deep study of the model theory of fields with an automorphism [CH] as well as Ehud Hrushovski and Anand Pillay [HP] discovered natural examples of this phenomenon in algebra and obtained results in local stability for first order simple theories.

In parallel, Rami Grossberg and Saharon Shelah continued their study of stability and the order property in contexts where the compactness theorem fails; inside a model and for nonelementary classes (see for example [Gr1], [Gr2], [GrSh1], [GrSh2], [Sh12], and [Sh300]).

The goal of this paper is to continue the study of local stability both in the first order case and in cases where the compactness theorem fails. When possible, we have tried to merge first order local stability with nonelementary stability theory and obtain results improving existing theorems in two directions. Three

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frameworks, listed in decreasing order of generality, are examined: (1) Inside a fixed structure; (2) For a nonelementary class of structures; (3) For the first order case. Hence, the results of (1) hold for (2) and those of (2) hold in (3). We study local versions of stability and the order property in (1) and (2). In (3), we also study local versions of the independence property and the strict order property. By *local*, we mean inside the set of realizations of a fixed type.

In (1) and (2), since the compactness theorem fails, we cannot use the forking machinery or definability of types, as [HH], [P] and [Sh a] do. Hence, the methods used have a combinatorial and set-theoretic flavor. Also, by (2) we mean the study of models of an infinitary logic, or of the class $\mathrm{PC}(T_1, T, \Gamma)$ (see the beginning of Section 3 for a definition). Hence, in addition to the absence of compactness, we have to do without the existence of saturated or even homogeneous models, as such models do not exist in general. Thus, frameworks (1) and (2) are more general than the study of finite diagrams [Sh3], also known as stability inside a homogeneous model, which was recently the focus of some activity, for example by Tapani Hyttinen and Saharon Shelah or Olivier Lessmann. In fact, for a treatment of local stability in the context of finite diagrams including the complete local stability spectrum and the local homogeneity spectrum, see [GrLe].

The basic *structure* assumption will be the impossibility of coding, via a formula in a given logic, a linear order of a certain length inside the set of realizations of a fixed type p. Note that this is slightly different from the terminology used by some authors, but essentially equivalent. (It is equivalent when the complexity of the formula used to code the order is of no importance.) It is a natural assumption to make. It is known that the existence of long orders implies the existence of many nonisomorphic models (see Theorem VIII 3.2 in [Sh a]), even in nonelementary cases (see for example [Sh12] and [GrSh1]).

The paper is organized as follows:

In Section 2, we study stability and order for the realizations of a type p inside a fixed model M. In particular, the model M may omit many types. Denote by $p(M)$ the set of realizations of p in M. We prove that the impossibility of coding a linear order of a certain length inside $p(M)$ implies local stability (Theorem 5). By local stability, we mean the usual definitions in terms of the number of types extending the fixed type p. This is used to prove the existence of indiscernibles (Theorem 9), as well as averages (Theorem 12).

In Section 3, we study these local notions for classes of models that fail to satisfy the compactness theorem. We obtain a characterization of local stability for such a class of models in terms of the failure of the local order property, and a version of the stability spectrum (Theorem 16).

Finally, in Section 4, we particularize our discussion to the first order case. We introduce local version of the independence property and the strict order property. We prove the local version of Shelah's Trichotomy Theorem: the local order property is equivalent to the disjunction of the local independence property and the local strict order property (Corollary 21). We characterize the local independence property in terms of averages (Theorem 23) and give, as an application, a characterization of stable types in terms of averages when the ambient first order theory is simple (Corollary 26).

Credits have been given throughout the text when particular cases of these results were known, either in the local first order case, or the nonlocal nonelementary case.

2. Local notions inside a fixed model

In this section, we work inside a fixed structure M. Denote by $L(M)$ the set of first order formulas in the language of M [1]. We will say formulas for $L(M)$-formulas.

Let p be a fixed set of formulas (maybe with parameters in M) such that p is realized in M. Denote by $p(M)$ the set of elements of M realizing p.

Recall the notion of complete type inside a model. Let $A \subseteq M$, Δ be a set of $L(M)$-formulas and $\bar{c} \in M$. We let

$$\mathrm{tp}_\Delta(\bar{c}/A, M) = \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in \Delta \text{ or } \neg\phi(\bar{x}, \bar{y}) \in \Delta, M \models \phi[\bar{c}, \bar{a}]\}.$$

We omit Δ when $\Delta = L(M)$.

For $A \subseteq M$ and Δ a set of formulas, we let

$$S_{\Delta,p}(A, M) = \{\mathrm{tp}_\Delta(\bar{c}/A, M) \mid \bar{c} \in M \text{ and } \bar{c} \text{ realizes } p\}.$$

We omit Δ when $\Delta = L(M)$.

For a type q and a set A, we denote by $q \upharpoonright A$ the set of formulas in q with parameters in A. For a set of formulas Δ, we denote by $q \upharpoonright \Delta$ the set of instances in q of formulas of Δ.

The next two definitions are the main concept of this paper.

Definition 1. For an infinite cardinal $\lambda \geq |L(M)|$, the model M is said to be (λ, p)-*stable* if $|S_p(A, M)| \leq \lambda$ for each $A \subseteq p(M)$ of cardinality at most λ.

Note that in the above definition we make demands only on subsets of $p(M)$. In fact, throughout the rest of this paper, we will only deal with types $q \in S_p(A, M)$ such that $A \subseteq p(M)$.

[1] This is arbitrary, we may consider for $L(M)$ a fragment of a larger logic, or even a subset with some weak closure properties.

Definition 2. M has the (λ, p)-*order property* if there exists a formula $\phi(\bar{x}, \bar{y}) \in L(M)$ and a set $\{\bar{a}_i \mid i < \lambda\} \subseteq p(M)$, such that

$$M \models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$

The first theorem (Theorem 5) is a local version inside a model of Shelah's Theorem that the failure of the order property implies stability for complete, first order theories. A generalization of Shelah's theorem for nonelementary classes and in the local case will appear in the next section (Theorem 5). Theorem 5 will also be used in a key way to prove existence of indiscernibles (Theorem 9) and averages (Theorem 12) in this section. The technical tool needed to prove it is *splitting*. Recall the definition.

Definition 3. Let $q \in S_p(B, M)$, with $B \subseteq p(M)$. Let $\Delta_1, \Delta_2 \subseteq L(M)$. The type q is said to (Δ_1, Δ_2)-*split over* A, if there exist elements $\bar{b}, \bar{c} \in B$ and a formula $\phi(\bar{x}, \bar{y}) \in \Delta_2$ such that $\mathrm{tp}_{\Delta_1}(\bar{b}/A, M) = \mathrm{tp}_{\Delta_1}(\bar{c}/A, M)$ and both $\phi(\bar{x}, \bar{b})$ and $\neg\phi(\bar{x}, \bar{c})$ belong to q. We simply say *splits* for $(L(M), L(M))$-splits.

The next fact is a variation on Exercise I.2.3 from [Sh a].

Proposition 4. *Let $B \subseteq C \subseteq p(M)$ and let $A \subseteq M$. Suppose that B realizes all the types in $S_{\Delta_2, p}(A)$ that are realized in C. Let $q, r \in S_{\Delta_1, p}(C)$ such that q, r do not (Δ_1, Δ_2)-split over A. If $q \restriction B = r \restriction B$, then $q = r$.*

Proof. Suppose $q \neq r$. Then there exists $\phi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{c} \in C$ such that $\phi(\bar{x}, \bar{c}) \in q$ and $\neg\phi(\bar{x}, \bar{c}) \in r$. Consider $\mathrm{tp}_{\Delta_2}(\bar{c}/A, M)$. By assumption on B, there exists $\bar{b} \in B$ such that $\mathrm{tp}_{\Delta_2}(\bar{b}/A, M) = \mathrm{tp}_{\Delta_2}(\bar{c}/A, M)$. Since neither q, nor r (Δ_1, Δ_2)-split over A, we have $\phi(\bar{x}, \bar{b}) \in q$ and $\neg\phi(\bar{x}, \bar{b}) \in r$. This contradicts the assumption that $q \restriction B = r \restriction B$. □

The following theorem localizes results from [Sh12] and [Gr1]. The proof appearing in [Sh12] uses generalizations of a theorem of Paul Erdős and Michael Makkai appearing in [EM]. The proof given here is simpler and closer to [Gr1].

Theorem 5. *Let μ and λ be cardinals such that $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, and $\lambda \geq 2^{2^\mu}$. If M does not have the (μ^+, p)-order property, then M is (λ, p)-stable.*

Proof. Suppose that M is not (λ, p)-stable. Then, there exists $A \subseteq p(M)$ of cardinality λ such that $|S_p(A, M)| > \lambda$.

For each $q \in S_p(A, M)$, we have $(q \restriction \phi) \in S_{\phi, p}(A, M)$. Define

$$f \colon S_p(A, M) \to \Pi_{\phi \in L} S_{\phi, p}(A, M), \quad \text{by} \quad f(q) = (q \restriction \phi)_{\phi \in L(M)}.$$

Then, f is a well-defined injection. Observe that

$$|\Pi_{\phi \in L(M)} S_{\phi, p}(A, M)| \leq \lambda^{|L(M)|} \leq \lambda^\mu < \lambda^+ \leq |S_p(A, M)|.$$

By the pigeonhole principle, we can find $\phi \in L(M)$ such that $|S_{\phi, p}(A, M)| > \lambda$.

Fix $\phi(\bar{x}, \bar{y})$ as above and choose $\{\bar{a}_i \mid i < \lambda^+\} \subseteq p(M)$ such that $i \neq j$ implies $\mathrm{tp}_\phi(\bar{a}_i/A, M) \neq \mathrm{tp}_\phi(\bar{a}_j/A, M)$.

Write $\chi(\bar{y}, \bar{x}) := \phi(\bar{x}, \bar{y})$. Define $\langle A_i \mid i < \lambda \rangle$ an increasing continuous sequence of subsets of $p(M)$ containing A, each of cardinality at most λ, such that

(*) $\quad A_{i+1}$ realizes every type in $S_p(B, M)$, for each $B \subseteq A_i$ with $|B| \leq \mu$.

This is possible: Having constructed A_i of cardinality at most λ, there are at most $\lambda^\mu = \lambda$ subsets B of A_i of cardinality μ. Further, for each such B, we have $|S_p(B, M)| \leq 2^\mu \leq \lambda$, so we can add the needed realizations in A_{i+1} from $p(M)$ while keeping $|A_{i+1}| \leq \lambda$.

We now claim that (*) allows us to choose, for every $i < \lambda^+$, an index j, with $i < j < \lambda^+$, such that for each $l < \mu^+$ the type $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ (χ, ϕ)-splits over each $B \subseteq A_l$ of cardinality at most μ.

Otherwise, there is $i < \lambda^+$ such that for every index j, with $i < j < \lambda^+$, there exists $l < \mu^+$ and $B^j \subseteq A_l$ of cardinality μ such that $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ does not (χ, ϕ)-split over B^j. By the pigeonhole principle (since $\lambda^+ \geq \mu$) we can find $S \subseteq \lambda^+$ of cardinality λ^+, an ordinal $l < \mu^+$, and $B \subseteq A_{l+1}$ of cardinality μ such that $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ does not (χ, ϕ)-splits over B, for every $j \in S$. By (*) we can choose $C \subseteq A_{l+1}$ of cardinality at most 2^μ such that C realizes every type in $S_{\chi,p}(B, M)$. Then, since $|S_{\phi,p}(C, M)| \leq 2^{2^\mu} < \lambda^+$, by the pigeonhole principle, we may assume that $\mathrm{tp}_\phi(\bar{a}_j/C, M)$ is constant for $j \in S$. By Proposition 4, we must have $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M) = \mathrm{tp}_\phi(\bar{a}_i/A_{l+1}, M)$, for $i, j \in S$. This contradicts the choice of $\bar{a}_i$s and the fact that $A \subseteq A_{l+1}$.

Define $\{\bar{c}_l, \bar{d}_l, \bar{b}_l \mid l < \mu^+\} \subseteq A_{2l+2}$ and $B_l = \bigcup\{\bar{c}_k, \bar{d}_k, \bar{b}_k \mid k < l\}$ such that:

(1) $B_l \subseteq A_{2l}$ and $|B_l| \leq \mu$;
(2) $\mathrm{tp}_\chi(\bar{c}_l/B_l, M) = \mathrm{tp}_\chi(\bar{d}_l/B_l, M)$;
(3) Both $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l)$ belong to $\mathrm{tp}_\phi(\bar{a}_j/A_{2l}, M)$;
(4) $\bar{b}_l \in A_{2l+1}$ realizes both $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l)$.

This is possible: Let $B_0 = \emptyset$ and $B_l = \bigcup_{k<l} B_k$ when l is a limit ordinal. Having constructed $B_l \subseteq A_{2l}$ of cardinality at most μ, the type $\mathrm{tp}_\phi(\bar{a}_j/A_{2l})$ (χ, ϕ)-splits over B_l and hence there are $\bar{c}_l, \bar{d}_l \in A_{2l}$ with $\mathrm{tp}_\chi(\bar{c}_l/B_l, M) = \mathrm{tp}_\chi(\bar{d}_l/B_l, M)$ and $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l) \in \mathrm{tp}_\phi(\bar{a}_j/A_{2l}, M)$. Then, by construction we can find $\bar{b}_l \in A_{2l+1}$ realizing $\mathrm{tp}_\phi(\bar{a}_j/\bar{c}_l\bar{d}_l, M)$ so (4) is automatically satisfied.

Now, the set $\{\bar{b}_l\hat{\ }\bar{c}_l\hat{\ }\bar{d}_l \mid l < \mu^+\} \subseteq p(M)$ and the formula

$$\psi(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{y}_0, \bar{y}_1, \bar{y}_2) := \phi(\bar{x}_0, \bar{y}_1) \leftrightarrow \phi(\bar{x}_0, \bar{y}_2)$$

demonstrate that M has the (μ^+, p)-order property. $\square$

The following definition generalizes the notion of relative saturation.

Definition 6. We say that a set $C \subseteq M$ is *relatively* (λ, p)*-saturated* if C realizes every $q \in S_p(B, M)$ for every $B \subseteq C$ such that $|B| < \lambda$.

The following lemma is a version of $\kappa(T) \leq |T|^+$ for the notion of splitting.

Lemma 7. *Let μ be a cardinal such that $\mu \geq |L(M)|$. Suppose that M does not have the (μ^+, p)-order property. Suppose that $B \subseteq p(M)$ is relatively (μ^+, p)-saturated. Then for each $q \in S_p(B, M)$ there is $A \subseteq B$ of cardinality at most μ such that q does not split over A.*

Proof. Suppose, for a contradiction, that there exist a relatively (μ^+, p)-saturated set B and a type $q \in S_p(B, M)$, such that q splits over every $A \subseteq B$ of cardinality at most μ.

We will show that M has the (μ^+, p)-order property. Construct a sequence of sets $\langle A_i \mid i < \mu^+ \rangle$ such that:

(1) $A_0 = \emptyset$;
(2) $A_i = \bigcup_{j<i} A_j$, when i is a limit ordinal;
(3) $A_i \subseteq B$, for each $i < \mu^+$;
(4) $|A_i| \leq \mu$, for each $i < \mu^+$;
(5) There are $\phi_i \in L(M)$ and $\bar{a}_i, \bar{b}_i \in A_{i+1}$, such that $\text{tp}(\bar{a}_i/A_i, M) = \text{tp}(\bar{b}_i/A_i, M)$ and $\phi(\bar{x}, \bar{a}_i)$ and $\neg\phi(\bar{x}, \bar{b}_i)$ are in q;
(6) A_{i+1} contains $\bar{c}_i$ realizing $q \restriction (A_i \cup \bar{a}_i\bar{b}_i)$.

This is possible: For $i = 0$ or a limit ordinal, it is obvious. Suppose that A_i has been constructed. Since $|A_i| \leq \mu$ and $A_i \subseteq B$, q splits over A_i. Hence, there exist a formula $\phi_i \in L(M)$, and $\bar{a}_i, \bar{b}_i \in B$ demonstrating this. Since B is relatively (μ^+, p)-saturated, and $q \restriction (A_i \cup \bar{a}_i\bar{b}_i) \in S_p(A_i \cup \bar{a}_i\bar{b}_i, M)$, there exists $\bar{c}_i \in B$ realizing $q \restriction (A_i \cup \bar{a}_i\bar{b}_i)$. Let $A_{i+1} = A_i \cup \{\bar{a}_i, \bar{b}_i, \bar{c}_i\}$. All the conditions are satisfied.

This is enough: By the pigeonhole principle, since $\mu \geq |L(M)|$, we may assume that there exists $\phi \in L(M)$ such that $\phi_i = \phi$, for each $i < \mu^+$. Now consider $\{\bar{c}_i \hat{\ } \bar{a}_i \hat{\ } \bar{b}_i \mid i < \mu^+\}$ and the formula

$$\psi(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{y}_0, \bar{y}_1, \bar{y}_2) := \phi(\bar{x}_0, \bar{y}_1) \leftrightarrow \phi(\bar{x}_0, \bar{y}_2).$$

It is easy to see that they demonstrate that M has the (μ^+, p)-order property. □

The following fact is Lemma I.2.5 of [Sh a].

Fact 8. *Let $B \subseteq p(M)$ and let $\{\bar{a}_i \mid i < \alpha\} \subseteq p(M)$ be given. Consider the type $q_i = \text{tp}(\bar{a}_i/B \cup \{\bar{a}_j \mid j < i\}, M) \in S_p(B \cup \{\bar{a}_j \mid j < i\}, M)$ and suppose that*

(1) *If* $i < j < \alpha$ *then* $q_i \subseteq q_j$;
(2) *For each* $i < \alpha$ *the type* q_i *does not split over* B.

Then $\{\bar{a}_i \mid i < \alpha\}$ *is indiscernible over* B.

The next theorem is a generalization of two theorems. (1) When p is stable for every model of a first order theory, a version of this theorem appears in [P]. (2) When $p := \{\bar{x} = \bar{x}\}$, it appears in [Gr1].

Theorem 9. *Let* μ *and* λ *be cardinals such that* $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, *and* $\lambda \geq 2^{2^\mu}$. *If* M *does not have the* (μ^+, p)*-order property, then for every* $I \subseteq p(M)$ *and every* $A \subseteq p(M)$ *such that* $|I| > \lambda \geq |A|$, *there exists* $J \subseteq I$ *of cardinality* λ^+ *indiscernible over* A.

Proof. Let $I = \{\bar{a}_i \mid i < \lambda^+\}$. By the pigeonhole principle, we may assume that $\ell(\bar{a}_i) = \ell(\bar{a}_j)$, for $i, j < \lambda^+$.

Define $\langle A_i \mid i < \lambda^+ \rangle \subseteq p(M)$ such that:

(1) $A_0 = A$;
(2) $A_i = \bigcup_{j<i} A_j$, when j is a limit ordinal;
(3) $A_i \subseteq p(M)$;
(4) $|A_i| \leq \lambda$, for every $i < \lambda^+$;
(5) A_{i+1} contains $\bar{a}_i$;
(6) A_{i+1} realizes every type in $S_p(B, M)$, for each $B \subseteq A_i$ of cardinality at most μ.

This is possible: For $i = 0$ it is clear. If i is a limit ordinal it is easy. Let us concentrate on the successor stage. Assume that A_i of cardinality λ has been constructed. By cardinal assumption, there are $\lambda = \lambda^\mu$ subsets B of A_i of cardinality μ, and for each such B we have $|S_p(B, M)| \leq 2^\mu \leq \lambda$. Hence, A_{i+1} satisfying (3)–(6) can be found.

Consider the following stationary subset of λ^+

$$S = \{i < \lambda^+ \mid cf(i) \geq \mu^+\}.$$

Let $r_i := \text{tp}(\bar{a}_i/A_i, M)$. Then clearly $r_i \in S_p(A_i, M)$. Now, for each $i \in S$, since $\text{cf}(i) \geq \mu^+$, the set A_i is relatively (μ^+, p)-saturated. Hence, by Lemma 7, there exists $B_i \subseteq A_i$ of cardinality at most μ such that r_i does not split over B_i. Furthermore, since $\text{cf}(i) = \mu^+$, there exists $j < i$ such that $B_i \subseteq A_j$.

This shows that the function $f \colon S \to \lambda^+$ defined by

$$f(i) = \min\{j \mid B_i \subseteq A_j\},$$

is regressive. Hence, by Fodor's lemma (see Theorem 22 of [Je]), there is $S' \subseteq S$ of cardinality λ^+ and $i_0 < \lambda^+$ such that for each $i \in S'$ we have $B_i \subseteq A_{i_0}$. Since there are only $\lambda^\mu = \lambda$ subsets of A_{i_0} of size μ, we may assume, by the pigeonhole principle, that there exists a set $B \subseteq A_{i_0}$ such that $B_i = B$ for each $i \in S'$. Now, M does not have the (μ^+, p)-order property, and $\lambda^\mu = \lambda$, so Theorem 5 implies that M is (λ, p)-stable. Hence, $|S_p(A_{i_0}, M)| \leq \lambda$, and thus by the pigeonhole principle, we may further assume that $\text{tp}(\bar{a}_i / A_{i_0}, M) = \text{tp}(\bar{a}_j / A_{i_0}, M)$, for every $i, j \in S'$.

By re-enumerating if necessary, we may assume that $S' \setminus (i_0 + 1) = \lambda^+$. Now let

$$q_i := \text{tp}(\bar{a}_i / A_{i_0} \cup \{\bar{a}_j \mid j < i\}) \in S_p(Ai_0 \cup \{\bar{a}_j \mid j < i\}).$$

By Proposition 4 we have that $q_i \subseteq q_j$ if $i < j$. Thus, all the assumptions of Fact 8 are satisfied, so $J = \{\bar{a}_i \mid i < \lambda^+\}$ is indiscernible over A, since $A \subseteq A_{i_0}$. This finishes the proof. □

In the previous theorem, we demanded that A be a subset of $p(M)$. The next remark summarizes what we can do when $A \subseteq M$ is not necessarily contained in $p(M)$. It follows from the previous theorem by considering an expansion of $L(M)$ with constants for elements in A.

Remark 10. Let $\mu \geq |L(T)|$ be a cardinal. Let $A \subseteq M$ be given and suppose that M does not have the (μ^+, p)-order property even allowing parameters from A. Let $\lambda^\mu = \lambda$ and $\lambda \geq 2^{2^\mu}$. Then, for every $I \subseteq p(M)$ of cardinality λ^+, there exists $J \subseteq I$ of cardinality λ^+ indiscernible over A.

Definition 11. Let I be an infinite set of finite sequences. Let $A \subseteq M$. We define the *average of I over A in M* as follows

$$\begin{aligned} \text{Av}(I, A, M) := \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in L(M), \\ \text{and } M \models \phi[\bar{c}, \bar{a}] \text{ for } |I| \text{ elements } \bar{c} \in I\}. \end{aligned}$$

We will be interested in conditions guaranteeing that averages are well-defined. It is a known fact (see Lemma III 1.7 (1) of [Sh a]) that if M is a model of a complete, first order, stable theory T, then for every infinite set of indiscernibles I and $A \subseteq M$, the average $\text{Av}(I, A, M)$ is a complete type over A. Also, if $|I| > |A| + \kappa(T)$, then the average is realized by an element of I (this is essentially Lemma III 3.9 of [Sh a]). A corresponding local result (Theorem 23) in the the presence of compactness will be given in Section 4. Inside a fixed model, the situation is more delicate. The next theorem is a localization of Conclusion 1.11 in [Sh300]. Notice the similarity with the assumptions of Theorem 9.

Theorem 12. *Let μ and λ be cardinals such that $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, and $\lambda \geq 2^{2^\mu}$. If M does not have the (μ^+, p)-order property, then for every $I \subseteq p(M)$ of cardinality λ^+, there exists $J \subseteq I$ of cardinality λ^+ such that for each $A \subseteq p(M)$ the average $\text{Av}(J, A, M)$ is a complete type over A. Moreover, if $|J| > |A|$, then $\text{Av}(J, A, M) \in S_p(A, M)$.*

Proof. Let $I = \{\bar{a}_\alpha \mid \alpha < \lambda^+\}$. We may assume by the pigeonhole principle that there exists $n < \omega$ such that $\ell(\bar{a}_\alpha) = n$, for each $\alpha < \lambda^+$.

We first essentially repeat the proof of Theorem 9 and construct a sequence $\langle A_\alpha \mid \alpha \leq \lambda^+ \rangle$ such that:

(1) $A_0 = \emptyset$, $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ when δ is limit, and $A_\alpha \subseteq A_{\alpha+1}$.
(2) $A_\alpha \subseteq p(M)$.
(3) $|A_\alpha| \leq \lambda$, for every $\alpha < \lambda^+$.
(4) $A_{\alpha+1}$ contains $\bar{a}_\alpha$.
(5) $A_{\alpha+1}$ realizes all types in $S_p(A_\alpha, M)$.

This is possible: Since M does not have the (μ^+, p)-order property, then M is (λ, p)-stable by Theorem 5. Hence, $|S_p(A_\alpha, M)| \leq \lambda$ inductively, for each $\alpha < \lambda^+$.

Now (5) implies that

(6) If $\mathrm{cf}(\delta) \geq \mu^+$ then A_δ is relatively (μ^+, p)-saturated.

As in the proof of Theorem 9, we can find a set $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \geq \mu^+\}$ of cardinality λ^+ and an ordinal $\alpha(*) = \min S$ such that

(7) For each $\alpha \in S$, the type $\mathrm{tp}(\bar{a}_\alpha / A_\alpha, M)$ does not split over $A_{\alpha(*)}$.
(8) If $\alpha, \beta \in S$ and $\alpha < \beta$ then $\mathrm{tp}(\bar{a}_\alpha / A_\alpha, M) \subseteq \mathrm{tp}(\bar{a}_\beta / A_\beta, M)$.

We claim that the set $J = \{\bar{a}_\alpha \mid \alpha \in S\}$ is as desired. To show this, we will show that

(*) For every $\bar{c} \in p(M)$ and $\phi(\bar{x}, \bar{y}) \in L(M)$, either

$$|\{\alpha \in S \,:\, M \models \phi[\bar{a}_\alpha, \bar{c}]\}| \leq \mu \quad \text{or} \quad |\{\alpha \in S \,:\, M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\}| \leq \mu.$$

This implies the conclusion of the theorem: For $A \subseteq p(M)$, condition (*) implies that $\mathrm{Av}(J, A, M)$ is a consistent set of formulas over A, as each finite subset is realized by all but μ many elements of J. Since $\mathrm{Av}(J, A, M)$ is always complete, we have that $\mathrm{Av}(J, A, M)$ is a complete type over A. For the last sentence, notice that all but $|A| + |L(M)| + \mu$ elements of J realize $\mathrm{Av}(J, A, M)$. Hence, if $\lambda^+ > |A|$, then there exists $\bar{a}_\alpha \in J \subseteq p(M)$ realizing $\mathrm{Av}(J, A, M)$ (as $\lambda \geq \mu + |L(M)|$). This shows that $\mathrm{Av}(J, A, M) \in S_p(A, M)$.

Let $\bar{c} \in p(M)$ and $\phi(\bar{x}, \bar{y}) \in L(M)$ be given. Since $\bar{c} \in p(M)$ then $\mathrm{tp}(\bar{c}/A_\alpha, M) \in S_p(A_\alpha, M)$. Hence, by (5), we can find $\{\bar{c}_\alpha \mid \alpha \in S\} \subseteq p(M)$ satisfying

(9) $\bar{c}_\alpha \in A_{\alpha+2}$.
(10) $\bar{c}_\alpha$ realizes $\mathrm{tp}(\bar{c}/A_{\alpha+1}, M)$.

We will prove (*) by finding a set of ordinals E of cardinality μ such that either $\{\alpha \in S : M \models \phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$ or $\{\alpha \in S : M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$.

We construct the set E, as well as a set $C \subseteq A_{\lambda^+}$ with the following properties:

(11) $|E| \leq \mu$ and $|C| \leq \mu$.
(12) $\lambda^+ \in E$.
(13) If $\alpha + 1 \in E$ then $\alpha \in E$ and if $\delta \in E$ and $\mathrm{cf}(\delta) \leq \mu$ then $\sup(E \cap \delta) = \delta$.
(14) If $\delta \in E$ and $\mathrm{cf}(\delta) \geq \mu^+$, then $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C \cap A_\delta$. Moreover, $C \cap A_\delta \subseteq A_{\sup(E\cap\delta)}$.

This is possible: Construct E_n and C_n of cardinality at most μ by induction on $n < \omega$. Let $E_0 = \{\lambda^+\}$ and $C_0 = \emptyset$. Then, by (6) and Lemma 7 we can find C_{n+1} of cardinality μ such that $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C_{n+1} \cap A_\delta$ for each $\delta \in E_n$ with $\mathrm{cf}(\delta) \geq \mu^+$. Furthermore, we can add at most μ many ordinals to E_{n+1} to ensure that $C_{n+1} \subseteq A_{\sup(E_{n+1}\cap\delta)}$. Thus, $E = \bigcup_{n<\omega} E_n$ and $C = \bigcup_{n<\omega} C_n$ are as desired.

This is enough to prove (*). In fact, to show that $\{\alpha \in S : M \models \phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$ or $\{\alpha \in S : M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$, it clearly suffices to show

(**) $$M \models \phi[\bar{a}_\alpha, \bar{c}] \leftrightarrow \phi[\bar{a}_\beta, \bar{c}], \quad \text{for every } \alpha, \beta \in S \setminus E.$$

Notice that by construction (11)–(14) the set $S \setminus E$ is partitioned into at most μ *intervals* of the form $\{\alpha \in S \mid \sup(E \cap \delta) \leq \alpha < \delta\}$, where $\delta \in E$ with $\mathrm{cf}(\delta) \geq \mu^+$. If such an interval is nonempty, then it must have size at least μ^+. We will make use of this and prove (**) in two stages. In the first part, we will show that (**) holds, provided α and β belong to the same interval, and then in the second part, that (**) holds also when α and β belong to different intervals.

Let $\delta \in E$ be such that $\mathrm{cf}(\delta) \geq \mu^+$. Denote by $\delta_0 = \sup(E \cap \delta)$. Now let $\alpha, \beta \in S$ such that $\delta_0 \leq \alpha < \beta < \delta$. Without loss of generality, assume that $M \models \phi[\bar{a}_\alpha, \bar{c}]$. Then $\phi(\bar{a}_\alpha, \bar{y}) \in \mathrm{tp}(\bar{c}/A_\delta, M)$. By (14) the type $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C \cap A_\delta \subseteq A_{\delta_0}$. But, by (8), we have $\mathrm{tp}(\bar{a}_\alpha/A_{\delta_0}, M) = \mathrm{tp}(\bar{a}_\beta/A_{\delta_0}, M)$. Hence, by nonsplitting $\phi(\bar{a}_\beta, \bar{y}) \in \mathrm{tp}(\bar{c}/A_\delta, M)$ and so $M \models \phi[\bar{a}_\beta, \bar{c}]$.

To prove the second part, we first claim that

(†) $$M \models \phi[\bar{a}_{\alpha_1}, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_{\alpha_2}, \bar{c}_{\beta_2}], \quad \text{for every } \alpha_1 > \beta_1 \text{ and } \alpha_2 > \beta_2 \text{ in } S.$$

To see this, let $\gamma = \max(\alpha_1, \alpha_2)$. Then by (8) and (9) (recall that ordinals in S are limit), we have $M \models \phi[\bar{a}_{\alpha_1}, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_1}]$ and also $M \models \phi[\bar{a}_{\alpha_2}, \bar{c}_{\beta_2}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_2}]$. Now by (10) we have that $\mathrm{tp}(\bar{c}_{\beta_1}/A_{\alpha(*)}, M) = \mathrm{tp}(\bar{c}_\gamma/A_{\alpha(*)}, M)$, and by (9), both $\bar{c}_{\beta_1}, \bar{c}_{\beta_2} \in A_\gamma$. But by (7) the type $\mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$ does not split over $A_{\alpha(*)}$. Hence, $\phi(\bar{x}, \bar{c}_{\beta_1}) \in \mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$ if and only if $\phi(\bar{x}, \bar{c}_{\beta_2}) \in \mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$, and therefore, $M \models \phi[\bar{a}_\gamma, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_2}]$. This proves (†).

Now for the second part, let $\delta, \xi \in E$ with $\mathrm{cf}(\delta) \geq \mu^+$ and $\mathrm{cf}(\xi) \geq \mu^+$. Denote by $\delta_0 = \sup(E \cap \delta)$ and $\xi_0 = \sup(E \cap \xi)$. Assume that $\delta_0 < \xi_0$ and let $i \in S$ with $\delta_0 \leq i < \delta$ and $j \in S$ with $\xi_0 \leq j < \xi$. To show: $M \models \phi[\bar{a}_i, \bar{c}] \leftrightarrow \phi[\bar{a}_j, \bar{c}]$. Suppose $M \models \neg(\phi[\bar{a}_i, \bar{c}] \leftrightarrow \phi[\bar{a}_j, \bar{c}])$. We will derive a contradiction by showing that M has the (μ^+, p)-order property.

Assume, without loss of generality, that $M \models \phi[\bar{a}_i, \bar{c}]$ and $M \models \neg\phi[\bar{a}_j, \bar{c}]$. We distinguish two cases.

Case 1: Suppose $M \models \phi[\bar{a}_j, \bar{c}_i]$ (recall $j > i$). Then, by (†), we have that $M \models \phi[\bar{a}_\alpha, \bar{c}_\beta]$, for every $\alpha, \beta \in S$ with $\alpha > \beta$. On the other hand since $M \models \neg\phi[\bar{a}_j, \bar{c}]$, the first part of this argument shows that $M \models \neg\phi[\bar{a}_\alpha, \bar{c}]$, for each $\alpha \in S$ with $\xi_0 \leq \alpha < \xi$. Hence, by (10), for each $\beta \in S$ with $\alpha \leq \beta$ we have that $M \models \neg\phi[\bar{a}_\alpha, \bar{c}_\beta]$. Thus, for $\alpha, \beta \in S \cap [\xi_0, \xi)$, we have

$$M \models \neg\phi[\bar{a}_\alpha, \bar{c}_\beta] \quad \text{if and only if} \quad \alpha \leq \beta.$$

This implies easily that M has the (μ^+, p)-order property.

Case 2: Suppose $M \models \neg\phi[\bar{a}_j, \bar{c}_i]$. Similarly to Case 1, we obtain the (μ^+, p)-order property by using the interval $S \cap [\delta_0, \delta)$ and the fact that $M \models \phi[\bar{a}_i, \bar{c}]$.

□

3. Local order and stability for nonelementary classes

In this short section, we will examine the stability of p with respect to all the models of a given class of models $\mathcal{K}$. Let us fix the concepts. We will work inside the class $\mathcal{K} = \mathrm{PC}(T_1, T, \Gamma)$. Recall that for $T \subseteq T_1$ and Γ a set of T_1-types over the empty set, we let

$$\mathrm{PC}(T_1, T, \Gamma) = \{M \restriction L(T) \ : \ M \models T_1 \text{ and } M \text{ omits every type in } \Gamma\}$$

We will denote by $\mu(\mathcal{K}) = \mu(|T_1|, |\Gamma|)$, the Hanf-Morley number for $\mathcal{K}$. Recall that $\mu(\lambda, \kappa)$ is the least cardinal μ with the property that for every $\mathrm{PC}(T_1, T, \Gamma)$ with $|T_1| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{PC}(T_1, T, \Gamma)$ contains a model of cardinality μ, then it contains models of arbitrarily large cardinality. It is known for example that when $\kappa = |\Gamma| = 0$, then $\mu(\mathcal{K}) = \aleph_0$. For $|\Gamma| \geq 1$, then $\mu(\mathcal{K}) = \beth_{\delta(|T_1|, |\Gamma|)}$. Recall that $\delta(\lambda, \kappa)$ is the least ordinal δ with the property that for every $\mathrm{PC}(T_1, T, \Gamma)$ with $|T_1| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{PC}(T_1, T, \Gamma)$ contains a model with a predicate whose order type is δ, then it contains a model where this predicate is not well-ordered. Much is known about such numbers. Here are some of the known facts. First $\delta(\lambda, 0) = \omega$ and $\delta(\lambda, \kappa)$ is always a limit ordinal. We have monotonicity properties: if $\lambda_1 \leq \lambda_2$ and $\kappa_1 \leq \kappa_2$, then $\delta(\lambda_1, \kappa_1) \leq \delta(\lambda_2, \kappa_2)$. Also, if $1 \leq \kappa \leq \lambda$ then $\delta(\lambda, \kappa) = \delta(\lambda, 1)$. In general $\delta(\lambda, \kappa) \leq (2^\lambda)^+$. Finally, suppose $\kappa \leq \lambda$ and λ is a strong limit cardinal of cofinality $\aleph_0$, then $\delta(\lambda, \kappa) = \lambda^+$. See Lemma VII.5.1 and Theorem VII.5.5 of [Sh a] or [Gr b]

Choosing to carry out the theorems of this section in a PC-class is arbitrary. We could have chosen to study any sufficiently general class of models extending the first order case in which the compactness theorem fails. For example, the class of models of an infinitary sentence $\psi \in L_{\omega_1\omega}$ or $L_{\lambda^+\omega}$. All the results of this section hold for such classes and the proofs can usually be used verbatim.

As in the previous section, we will fix p a set of $L(T)$-formulas (with parameters).

We expand the definitions we made in the first section for the class $\mathcal{K}$.

Definition 13. (1) Let λ be a cardinal. We say that p is *stable in* λ, if for every $M \in \mathcal{K}$, M is (λ, p)-stable.
(2) We say that p is *stable* if there exists a cardinal λ such that p is stable in λ.

Definition 14. (1) We say that p has the λ-*order property* if there exists $M \in \mathcal{K}$ such that M has the (λ, p)-order property.
(2) We say that p has the *order property* if p has the λ-order property for every λ.

Using the tools of [Sh12] (finer results are in [GrSh1] and [GrSh2]) we observe:

Fact 15. *The following conditions are equivalent.*

(1) *p has the order property;*
(2) *p has the λ-order property for every $\lambda < \mu(\mathcal{K})$;*
(3) *p has the $\mu(\mathcal{K})$-order property;*
(4) *There exists a model $M \in \mathcal{K}$, a formula $\phi(\bar{x}, \bar{y})$, and an indiscernible sequence $\{\bar{a}_i \mid i < \mu(\mathcal{K})\} \subseteq p(M)$, such that*

$$M \models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \mu(\mathcal{K}).$$

We now prove a version of the stability spectrum and the equivalence between local instability and local order. Nonlocal theorems of this vein appear in [Sh12].

Theorem 16. *The following conditions are equivalent.*

(1) *p is stable;*
(2) *There exists a cardinal $\kappa(\mathcal{K}) < \mu(\mathcal{K}) + |L(T)|^+$ such that p is stable in every $\lambda \geq \mu(\mathcal{K})$ satisfying $\lambda^{\kappa(\mathcal{K})} = \lambda$.*
(3) *p does not have the order property.*

Proof. (2) $\Rightarrow$ (1) trivially.

(3) $\Rightarrow$ (2): Since p does not have the order property, by Fact 15 there exists a cardinal $\kappa < \mu(\mathcal{K})$ such that no model of $\mathcal{K}$ has the (κ^+, p)-order property. Let

$\lambda \geq \mu(\mathcal{K})$. Then, automatically, since $\kappa < \mu(\mathcal{K})$ and $\mu(\mathcal{K})$ is either $\aleph_0$ or a strong limit, we have $\lambda \geq 2^{2^\kappa}$. Let $\kappa(\mathcal{K}) = \kappa + |L(T)|$. Hence, if $\lambda \geq \mu(\mathcal{K})$ satisfies $\lambda^{\kappa(\mathcal{K})} = \lambda$, and $M \in \mathcal{K}$, then Theorem 5 implies that M is (λ, p)-stable. Thus, p is stable in λ.

(1) $\Rightarrow$ (3): This is again a standard application of Hanf number techniques. We give just a sketch. Suppose p is stable in λ. Let T^* be an expansion of T_1 with Skolem functions, such that $|T^*| = |T_1|$. Let κ be smallest such that $2^\kappa > \lambda$. Using the order property and the methods of Morley, we can find $M^* \models T^*$ such that $M = M^* \restriction L(T) \in \mathcal{K}$, with $\phi(\bar{x}, \bar{y})$, and $\{\bar{a}_i \mid i < \omega\} \subseteq p(M)$ demonstrating the p-order property. Furthermore $\{\bar{a}_i \mid i < \omega\} \subseteq p(M)$ is T^*-indiscernible. Hence, by compactness, we can find a model $N^* \models T^*$ and a set $\{\bar{a}_\eta \mid \eta \in {}^{\kappa \geq}2\} \subseteq p(N^*)$ demonstrating the p-order property with respect to the lexicographic order. Furthermore, for every $n < \omega$

$$\text{tp}(\bar{a}_{\nu_0}, \dots, \bar{a}_{\nu_n}/\emptyset, N^*) = \text{tp}(\bar{a}_0, \dots, \bar{a}_n/\emptyset, M^*), \text{ for every } \nu_0 < \dots < \nu_n.$$

We may assume that N^* is the Ehrenfeucht-Mostowski closure of $\{\bar{a}_\eta \mid \eta \in {}^{\kappa \geq}2\}$, since T^* has Skolem functions. Let $N = N^* \restriction L(T)$. Then $N \in \mathcal{K}$. Consider $A = \bigcup_{\eta \in {}^{\kappa >}2} \bar{a}_\eta \subseteq p(N)$. Then $|A| \leq 2^{<\kappa} \leq \lambda$ and $|S_p(A, N)| = 2^\kappa > \lambda$. Thus, N is not (λ, p)-stable, a contradiction. □

Remark 17. In the first order case, $\mu(\mathcal{K}) = \aleph_0$ and so p is stable if and only if p is stable in every λ such that $\lambda^{|L(T)|} = \lambda$. In the first order case, most authors define *stable* types using (3) with $\mu(\mathcal{K}) = \aleph_0$.

4. Local order, independence, and strict order in the first order case

In this section, we will fix a complete, first order theory T and obtain results for the class of models of T. As usual, we work inside the *monster model* $\mathfrak{C}$, a model which is $\bar{\kappa}$-saturated, for a cardinal $\bar{\kappa}$ larger than any cardinality mentioned in this paper. Hence, all sets will be assumed to be inside $\mathfrak{C}$ and satisfaction is defined with respect to $\mathfrak{C}$. We will write $S_p(A)$ for $S_p(A, \mathfrak{C})$ and $\text{Av}(I, A)$ for $\text{Av}(I, A, \mathfrak{C})$ as is customary. As before, we fix a (nonalgebraic) T-type p. Denote by $\text{dom}(p)$ the set of parameters of p.

All the results we have obtained so far hold with $\mu(\mathcal{K}) = \aleph_0$.

We first give local versions of Saharon Shelah's first order notion of independence and strict order property (see [Sh a]).

For a statement $\mathbf{t}$ and a formula ϕ, we use the following notation: $\phi^{\mathbf{t}} = \neg\phi$ if the statement $\mathbf{t}$ is false and $\phi^{\mathbf{t}} = \phi$, if the statement $\mathbf{t}$ is true. We will use the same notation when $\mathbf{t} \in \{0, 1\}$, where 0 stands for false and 1 stands for truth.

Definition 18. (1) We say that $\phi(\bar{x},\bar{y})$ has the *p-independence property* if for every $n<\omega$ there exists $\{\bar{a}_i \mid i<n\} \subseteq p(\mathfrak{C})$ such that

$$p(\bar{x}) \cup \{\phi(\bar{x},\bar{a}_i)^{i\in w} \mid i<n\} \text{ is consistent, } \quad \text{for every } w \subseteq n.$$

We say that p has the *independence property* if there exists a formula $\phi(\bar{x},\bar{y})$ with the p-independence property;

(2) A formula $\phi(\bar{x},\bar{y})$ is said to have the *p-strict order property* if for every $n<\omega$ there exists $\{\bar{a}_i \mid i<n\} \subseteq p(\mathfrak{C})$ such that

$$\models \exists\bar{x}(\neg\phi(\bar{x},\bar{a}_i) \wedge \phi(\bar{x},\bar{a}_j)) \quad \text{if and only if} \quad i<j<n.$$

We say that p has the *strict order property* if there exists a formula $\phi(\bar{x},\bar{y})$ with the p-strict order property.

Proposition 19. *If p has the independence property or the strict order property, then p has the order property.*

Proof. Suppose first that p has the independence property. Then, some $\phi(\bar{x},\bar{y})$ has the p-independence property. Hence, by compactness there exist $I=\{\bar{a}_i \mid i<\omega\} \subseteq p(M)$ such that for every $n<\omega$ and $w \subseteq n$ there exists $\bar{c} \in p(\mathfrak{C})$ realizing the formula $\bigwedge_{i<n}\phi(\bar{x},\bar{a}_i)^{i\in w}$. We show that ϕ has the p-order property. For each $k<n$, let $\bar{c}_k \in p(\mathfrak{C})$ realize $\{\phi(\bar{x},\bar{a}_i) \mid i<k\} \cup \{\neg\phi(\bar{x},\bar{a}_i) \mid i \geq k, i<n\}$. Then, we use $\{\bar{c}_i\hat{\ }\bar{a}_i \mid i<n\}$ and the compactness theorem to show that the formula $\psi(\bar{x}_0,\bar{x}_1;\bar{y}_0,\bar{y}_1) := \phi(\bar{x}_0,\bar{y}_1)$ has the p-order property.

Suppose that p has the strict order property. Let $\phi(\bar{x},\bar{y})$ have the p-strict order property. Then, the formula $\psi(\bar{y}_1,\bar{y}_2) := \exists\bar{x}(\neg\phi(\bar{x},\bar{y}_1) \wedge \phi(\bar{x},\bar{y}_2))$ has the p-order property. □

The next two results depend explicitly on the parameters of p.

Theorem 20. *Let $\phi(\bar{x},\bar{y})$ be a formula with the p-order property. Then, either $\phi(\bar{x},\bar{y})$ has the p-independence property, or there exist $\chi(\bar{x})$, the conjunction of finitely many formulas of p, an integer $n<\omega$ and a sequence $\eta \in {}^{n}2$ such that the formula $\chi(\bar{x}) \wedge \bigwedge_{l<n}\phi(\bar{x},\bar{y}_l)^{\eta[l]}$ has the p-strict order property (maybe with parameters from* $\mathrm{dom}(p)$*).*

Proof. By Fact 15 (4) there exists an indiscernible sequence $\{\bar{a}_i \mid i<\omega\} \subseteq p(\mathfrak{C})$ such that

$$\models \phi[\bar{a}_i,\bar{a}_j] \quad \text{if and only if} \quad i<j<\omega.$$

Further, by a standard compactness argument using Ramsey's Theorem, we may assume that $\{\bar{a}_i \mid i<\omega\}$ is indiscernible over $\mathrm{dom}(p)$, the set of parameters of p.

If $\phi(\bar{x},\bar{y})$ does not have the p-independence property, then there exists $n<\omega$ and $w \subseteq n$ such that

(*) $$p(\bar{x}) \cup \{\phi(\bar{x},\bar{a}_l)^{l\in w} \mid l<n\} \text{ is not consistent.}$$

Let $w^* = \{n-|w|, n-|w|+1, n-|w|+2, \dots, n-1\}$. Since ϕ has the p-order property, we have that $\models \phi[\bar{a}_{n-|w|-1}, \bar{a}_l]$ if and only if $n-|w| \leq l$. Therefore, by definition of w^*, the tuple $\bar{a}_{n-|w|-1}$ realizes $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w^*} \mid l < n\}$, and so

(**) $$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w^*} \mid l < n\} \text{ is consistent.}$$

Now, construct a sequence $\langle w_i \mid i \leq i^* \rangle$ of subsets of n of cardinality $|w|$ such that $w_0 = w$, $w_{i^*} = w^*$, and for each $i < i^*$, there exists $k \in w_i$ such that $w_{i+1} = w_i \cup \{k+1\} \setminus \{k\}$. Notice that because of (*) and (**) and the definition of $\langle w_i \mid i \leq i^* \rangle$, we can find $i < i^*$ such that $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_{i+1}} \mid l < n\}$ is consistent, while $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i} \mid l < n\}$ is not.

Let $k \in w_i$ such that $w_{i+1} = w_i \setminus \{k\} \cup \{k+1\}$ (note that $k+1 \notin w_i$). We then have,

(†)

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i}, \neg\phi(\bar{x}, \bar{a}_k), \phi(\bar{x}, \bar{a}_{k+1}) \mid l < n, l \neq k, k+1\} \text{ is consistent}$$

and

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i}, \neg\phi(\bar{x}, \bar{a}_{k+1}), \phi(\bar{x}, \bar{a}_k) \mid l < n, l \neq k, k+1\} \text{ is inconsistent.}$$

Hence, by the finite character of consistency, we can find $\chi(\bar{x})$, the conjunction of finitely many formulas of p, such that

(‡) $$\models \neg\exists\bar{x}[\chi(\bar{x}) \wedge (\bigwedge_{l<n, l\neq k, l\neq k+1} \phi(\bar{x}, \bar{a}_l)^{l \in w_i}) \wedge \neg\phi(\bar{x}, \bar{a}_{k+1}) \wedge \phi(\bar{x}, \bar{a}_k)].$$

Define the formula $\psi(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = \bar{z}_0, \dots, \bar{z}_{k-1}, \bar{z}_{k+2}, \dots, \bar{z}_{n-1}$ by

$$\chi(\bar{x}) \wedge (\bigwedge_{l<n, l\neq k, l\neq k+1} \phi(\bar{x}, \bar{z}_l)^{l \in w_i}) \wedge \phi(\bar{x}, \bar{y}).$$

To conclude the proof we show that $\psi(\bar{x}, \bar{y}, \bar{z})$ has the p-strict order property:

Let $m < \omega$ be given. For every $j < m$ we let

$$\bar{c}^j = \bar{a}_{k+j}\hat{\ }\bar{a}_0\hat{\ }\bar{a}_1\hat{\ }\dots\hat{\ }\bar{a}_{k-1}\hat{\ }\bar{a}_{m+k+2}\hat{\ }\dots\hat{\ }\bar{a}_{m+n-1},$$

where $\bar{a}_{k+j}$ is to be substituted for the $\bar{y}$-variable, and $\bar{a}_0\hat{\ }\dots\hat{\ }\bar{a}_{m+n-1}$ is to be substituted for the variable $\bar{z}_0\hat{\ }\dots\hat{\ }\bar{z}_{k-1}\hat{\ }\bar{z}_{k+2}\hat{\ }\dots\hat{\ }\bar{z}_{n-1}$.

It is enough to check that

$$\models \exists\bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2})) \quad \text{if and only if} \quad j_1 < j_2.$$

For convenience, denote by $\bar{c}$ the following sequence $\bar{a}_0 \dots \bar{a}_{k-1}\bar{a}_{k+2} \dots \bar{a}_{n-1}$. By indiscernibility of $\{\bar{a}_i \mid i < \omega\}$, we have the following equalities

(***) $$\begin{aligned} \operatorname{tp}(\bar{c}^{j_1}, \bar{c}^{j_2}/\operatorname{dom}(p)) &= \operatorname{tp}(\bar{a}_k\bar{c}, \bar{a}_{k+1}\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 < j_2, \\ &= \operatorname{tp}(\bar{a}_k\bar{c}, \bar{a}_k\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 = j_2, \\ &= \operatorname{tp}(\bar{a}_{k+1}\bar{c}, \bar{a}_k\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 > j_2. \end{aligned}$$

We distinguish three cases.

If $j_1 < j_2$. By the first equality of (***), it is enough to check that $\models \exists \bar{x}(\neg\psi(\bar{x}, \bar{a}_k, \bar{c}) \wedge \psi(\bar{x}, \bar{a}_{k+1}, \bar{c})))$. By definition of ψ, it is enough to show that $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i}, \neg\phi(\bar{x}, \bar{a}_k), \phi(\bar{x}, \bar{a}_{k+1})) \mid l < n, l \neq k, l \neq k+1\}$ is consistent, which is true by (†).

If $j_1 = j_2$, then by the second equality of (***) $\models \exists \bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))$ if and only if $\models \exists \bar{x}(\neg\psi(\bar{x}, \bar{a}_k, \bar{c}) \wedge \psi(\bar{x}, \bar{a}_k, \bar{c}))$. Therefore, we have $\models \neg[\exists \bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))]$.

If $j_1 > j_2$, then use the third equality of (***), and (‡) to conclude that $\models \neg[\exists \bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))]$. □

The next corollary is the local version of Shelah's Trichotomy Theorem (see Theorem II 4.7 of [Sh a]).

Corollary 21. *Assume that p has no parameters. The type p has the order property if and only if p has the independence property or p has the strict order property.*

Proof. Suppose p has the order property. Then some formula ϕ has the p-order property. Thus, by Theorem 20 p has the independence property or the strict order property (without parameters, since $\mathrm{dom}(p) = \emptyset$).

The converse is Proposition 19. □

The following is an improvement of Theorem II.2.20 of [Sh a].

Lemma 22. *The following conditions are equivalent*

(1) *p does not have the independence property;*
(2) *For every infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and for every formula $\phi(\bar{x}, \bar{y}) \in L(T)$ there exists an integer $n_\phi < \omega$ such that for every $\bar{c} \in p(M)$ either*

$$|\{\bar{a} \in I : \models \phi[\bar{a}, \bar{c}]\}| \leq n_\phi \quad \textit{or} \quad |\{\bar{a} \in I : \models \neg\phi[\bar{a}, \bar{c}]\}| \leq n_\phi.$$

Proof. (1) ⇒ (2) Let $\phi(\bar{x}, \bar{y})$ and I be given. Suppose (2) fails. Then, by compactness, we can find $\bar{c} \in p(\mathfrak{C})$ and a sequence $\{\bar{a}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ indiscernible over $\mathrm{dom}(p)$ such that

(*) $\quad |\{i < \omega : \models \phi[\bar{a}_i, \bar{c}]\}| = \aleph_0 \quad$ and $\quad |\{i < \omega : \models \neg\phi[\bar{a}_i, \bar{c}]\}| = \aleph_0.$

We are going to show that $\phi(\bar{x}, \bar{y})$ has the p-independence property. Let $n < \omega$ and $w \subseteq n$. It is enough to show that

(**) $\quad p(\bar{y}) \cup \{\phi(\bar{a}_i, \bar{y})^{i \in w} \mid i < n\}$ is consistent.

To see this, construct a strictly increasing sequence of integers $\langle i_m \mid m < n\rangle$ such that $\mathfrak{C} \models \phi[\bar{a}_{i_m}, \bar{c}]$ if and only if $m \in w$. This is easily done by induction using

(*). By indiscernibility of $\{\bar{a}_i \mid i < \omega\}$, (**) holds if and only the set of formulas $p(\bar{y}) \cup \{\phi(\bar{a}_{i_m}, \bar{y})^{m \in w} \mid m < n\}$ is consistent, which is the case, since it is realized by $\bar{c}$.

(2) $\Rightarrow$ (1) Suppose that $\phi(\bar{x}, \bar{y})$ has the p-independence property and $I = \{\bar{a}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ demonstrate this. Then, for each $n < \omega$, and for each $w \subseteq n$ we have

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_i)^{i \in w} \mid i < n\} \text{ is consistent.}$$

Hence, by compactness, we can find an indiscernible sequence $J = \{\bar{b}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ and $\bar{c} \in p(\mathfrak{C})$ such that both $\{i < \omega \ : \models \phi[\bar{c}, \bar{b}_i]\}$ and $\{i < \omega \ : \models \neg\phi[\bar{c}, \bar{b}_i]\}$ are infinite. Hence both $\phi(\bar{c}, \bar{y})$ and $\neg\phi(\bar{c}, \bar{y})$ belong to $\mathrm{Av}(J, \bar{c})$. Thus $\mathrm{Av}(J, \bar{c})$ is not consistent, which contradicts (2). □

We can now answer the question of when averages are well-defined and characterize types without the independence property.

Theorem 23. *The following conditions are equivalent:*

1. *p does not have the independence property;*
2. *For every infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and every subset $A \subseteq p(\mathfrak{C})$ the average $\mathrm{Av}(I, A)$ is a complete type. Furthermore, $\mathrm{Av}(I, A) \in S_p(A)$.*

Proof. (1) $\Rightarrow$ (2): Let $I, A \subseteq p(\mathfrak{C})$ and I be an infinite indiscernible sequence. By Lemma 22 (1) $\Rightarrow$ (2), we have that $\mathrm{Av}(I, A) \in S(A)$. Furthermore, since $I \subseteq p(\mathfrak{C})$, we have $\mathrm{Av}(I, A) \in S_p(A)$.

(2) $\Rightarrow$ (1): We prove the contrapositive. Suppose that p has the independence property. Then, by Lemma 22 (2) $\Rightarrow$ (1), there exists an infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and $\bar{a} \in p(\mathfrak{C})$ such that both $\phi(\bar{x}, \bar{a})$ and $\neg\phi(\bar{x}, \bar{a})$ belong to $\mathrm{Av}(I, \bar{a})$. This contradicts (2). □

We now give an easy characterization of stable types in simple theories. The following fact is due to Shelah and appears in [Sh93].

Fact 24. *If T is simple then T does not have the strict order property.*

We make use of the following observation.

Proposition 25. *If the formula $\phi(\bar{x}, \bar{y}, \bar{b})$ with parameter $\bar{b} \in \mathfrak{C}$ has the p-strict order property, then T has the strict order property.*

Proof. We show that T has the strict order property, by showing that $\phi(\bar{x}, \bar{y}, \bar{z})$ has the strict order property. But, for each $n < \omega$, there exists $\{\bar{a}_i \mid i < n\} \subseteq p(\mathfrak{C})$ such that

$$\models \exists \bar{x}(\neg\phi(\bar{x}, \bar{a}_i, \bar{b}) \wedge \phi(\bar{x}, \bar{a}_j, \bar{b})) \quad \text{if and only if} \quad i < j < n.$$

Thus, for each $n < \omega$, the set $\{\bar{a}_i\hat{}\bar{b} \mid i < n\}$ shows that $\phi(\bar{x}, \bar{y}, \bar{z})$ has the strict order property. □

Corollary 26. *Let T be simple. The following conditions are equivalent:*

(1) *p is stable;*
(2) *For every infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and for every $A \subseteq p(\mathfrak{C})$, we have $\mathrm{Av}(I, A) \in S_p(A)$.*

Proof. (1) $\Rightarrow$ (2): Let p be stable, then p does not have the order property by Theorem 16. Hence p does not have the independence property by Proposition 19. Hence, (2) follows from Theorem 23.

(2) $\Rightarrow$ (1): Suppose p is not stable. Then p has the order property by Theorem 16. Thus, p has either the independence property or the strict order property (maybe with parameters) by Theorem 20. Since T is simple, by Fact 24, we have that T does not have the strict order property. But, if p has the strict order property with parameters, then T has the strict order property by Proposition 25. Therefore, p has the independence property, and so (2) fails by Lemma 23. □

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