INDISCERNIBLE SEQUENCES IN BANACH SPACE GEOMETRY

by

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1. Introduction

A close analysis of the concepts and techniques that have played an important role in the development Banach space theory in the last thirty years reveals that a number of these are closely related to concepts that are studied in model theory. Examples are:

1. Indiscernible sequences (called 1-*subsymmetric* sequences in Banach space theory);
2. Ordinal ranks;
3. Ehrenfeucht-Mostowski models (called *spreading models* in Banach space theory);
4. Spaces of types;
5. Stability;
6. Ultrapowers;

In some cases, these concepts have been introduced by adapting directly a construction from model theory to the context of Banach space theory (prominently, the case of Banach space ultrapowers, introduced by D. Dacunha-Castelle and J.-L. Krivine in [12]), in other cases, by analogy (*e.g.*, the case of Banach space stability, introduced by J.-L. Krivine and B. Maurey in [46]), and yet in other cases, concepts which are studied in model theory, as well as their connections with others, have been discovered independently by analysts (as in the case of indiscernible sequences — and their construction using Ramsey's Theorem — which were introduced by A. Brunel and L. Sucheston in the study of ergodic properties of Banach spaces; see [5]).

In addition, some concepts that play a central role in Banach space theory (*e.g.*, that of finite representability) can be seen naturally as model theoretical phenomena (loosely speaking a Banach space X is finitely represented in a Banach space Y if and only if Y is a model of the existential theory of X). There are even similarities between classification programs in both fields. For example, the dichotomy reflexive/unreflexive in Banach space theory is equivalent, in a categorical sense, to the dichotomy stable/unstable in model theory. (See [37].) Also, in both fields, the role played by partition theorems is regarded as fundamental.

These phenomena suggest that the relation between these two fields is rather deep. Given the remarkable technical complexity that both fields have attained in the last thirty years, it is natural to suggest that it would be desirable to have clearly understood channels of communication between them so that techniques from one field might become useful in the other. Some considerations are in order, however.

1. First order logic is not the natural logic to analyze Banach spaces as models. Banach space theory is carried out in higher order logics, as is functional analysis in general. Furthermore, a result of S. Shelah and J. Stern shows that the first order theory of Banach spaces is equivalent to a second order logic. (See [64].)
2. The concepts from Banach space theory listed above are not the literal translations of their first order analogs. For instance, a Banach space ultrapower of a Banach space X is not an ultrapower of X in the sense customarily considered in model theory, and is not an elementary extension of X in the sense of first order logic. However, there is a strong analogy between the role

played by Banach space ultrapowers in Banach space theory and that played by elementary extensions in model theory.

Let us illustrate this point with a second example. What is regarded in Banach space theory as the "space of types" is not what is understood as the space of types in the first order sense. Let us recall definition given in [46]:

> Let X be a fixed separable Banach space. A *type* is a function $\tau(x)\colon X \to \mathbb{R}$ such that there exists a sequence (x_n) in X satisfying
>
> $$\tau(x) = \lim_{n\to\infty} \|x + x_n\|.$$
>
> The set of types is regarded as a topological space with the topology of pointwise convergence.

This notion of space of types is motivated by the corresponding notion from first order logic. A priori, it is not entirely clear what the analogy is. However, as we shall see, both notions are connected by a natural translation.

A formal framework for a model theoretical analysis of Banach spaces was introduced by C. W. Henson in [31]. Although this framework was originally introduced for Banach spaces, it generalizes naturally to include rich classes of structures from functional analysis. The unique feature of this logical approach to analysis is that, although it is appropriate for structures from functional analysis, it preserves many of the desirable characteristics of first order model theory, *e.g.*, the compactness theorem, Löwenheim-Skolem theorems, and omitting types theorem. (In fact, it provides a natural setting for the classification theory, in the sense of [63], of structures from infinite dimensional analysis.) Furthermore, it provides a uniform foundation for the contributions mentioned above. For example, the role played by analytic ultrapowers in this framework mirrors that played by algebraic ultrapowers in first order model theory; also, types in the sense of [46] described above correspond exactly to quantifier-free types in this context, indiscernibles in the sense of [5] are quantifier-free indiscernibles, and the kind of Banach space stability introduced in [46] corresponds exactly to quantifier-free stability of the structure.

The problem of how the classical sequence spaces ℓ_p $(1 \le p < \infty)$ and c_0 occur inside every Banach space has played a central role in Banach space geometry for more than half a century. The first example of a Banach space not containing ℓ_p or c_0 was constructed by B. S. Tsirel'son [67]. Shortly after Tsirel'son's example appeared in print, J.-L. Krivine [45] published a celebrated result (now known as Krivine's Theorem) which states that for every Banach space X there exists p with $1 \le p \le \infty$ such that ℓ_p is block finitely represented in X. The spectacular breakthroughs that have taken place in Banach space theory in the 1990's (see the historical notes at the end of the paper) confirm the long held belief that Krivine's Theorem in fact states the ultimate way in which the classical spaces ℓ_p and c_0 occur as subspaces of every Banach space.

A question that still remains open is what conditions on the norm of a Banach space guarantee that the space contains ℓ_p or c_0 almost isometrically. The most elegant partial answer to this question known so far is the theorem proved by J.-L. Krivine and B. Maurey in [46] which states that every stable Banach space contains some ℓ_p almost isometrically.

In this paper we use the model theoretical framework introduced by Henson to prove these two famous results. In the original proofs, various concepts motivated by analogies with model theory played a fundamental role (prominently, that of Banach space ultrapowers). However, these connections are in the background of the proofs and not easily visible. Here, we bring the model theoretical ideas to the foreground. We prove a general principle about block representability of ℓ_p in arbitrary indiscernible sequences (Theorem 11.1) from which both Krivine's Theorem and the Krivine-Maurey theorem about ℓ_p subspaces of stable spaces follow easily.

The exposition is entirely self contained. A basic course in model theory (for example, the first three chapters of [10]) will more than suffice for the prerequisites in logic. The prerequisites in Banach space theory are minimal. We assume that the reader is familiar with the definition of the ℓ_p sequence spaces and with the definition of Banach space operator.

The historical notes at the end of the paper should be regarded as an integral part of the exposition. We suggest that the reader consult the notes corresponding to each section before and after studying the section. By no means have we tried to be exhaustive. We have mentioned only the writings that have shaped the author's view of the subject.

A word about notation. Model theorists use the letters p, q, etc. to denote types. However, in Banach space theory, these letters are reserved to denote certain parameters, specifically, the parameter p in the $L_p(\mu)$ spaces. For this reason, we have denoted types by the letters t, t', etc. We have also avoided using the letter T to denote theories, as in Banach space theory it is customarily used to denote operators.

2. Preliminaries: Banach Space Models

2.1. **Banach Space Ultrapowers.** A Banach space is finite dimensional if and only if the unit ball is compact, *i.e.*, if and only if for every bounded family $(x_i)_{i\in I}$ and every ultrafilter $\mathcal{U}$ on the set I, the $\mathcal{U}$-limit

$$\lim_{i,\mathcal{U}} x_i$$

exists. If X is an infinite dimensional Banach space and $\mathcal{U}$ is an ultrafilter on a set I, there is a canonical way of expanding X to a larger Banach space $\hat{X}$ by adding for every bounded family $(x_i)_{i\in I}$ in X an element $\hat{x} \in \hat{X}$ such that $\|\hat{x}\| = \lim_{i,\mathcal{U}} \|x_i\|$. This is the construction of *Banach space ultrapower* introduced by D. Dacunha-Castelle and J.-L. Krivine in [12].

Let $(X_i)_{i\in I}$ be a family of Banach spaces. Define

$$\ell_\infty(\prod_{i\in I} X_i) = \{\ (x_i) \in \prod_{i\in I} X_i \ \mid\ \sup_{i\in I} \|x_i\| < \infty\ \}$$

$\ell_\infty(\prod_{i\in I} X_i)$ is naturally a vector space. An ultrafilter $\mathcal{U}$ on I induces a seminorm on $\ell_\infty(\prod_{i\in I} X_i)$ by defining

$$\|(x_i)\| = \lim_{i,\mathcal{U}} \|x_i\|.$$

The set $N_\mathcal{U}$ of families (x_i) in $\ell_\infty(\prod_{i\in I} X_i)$ such that $\|(x_i)\| = 0$ is obviously a closed subspace of $\ell_\infty(\prod_{i\in I} X_i)$. We define

$$\prod_{i\in I} X_i/\mathcal{U} = \ell_\infty(\prod_{i\in I} X_i)/N_\mathcal{U}.$$

The space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultraproduct of* $(X_i)_{i\in I}$. If $X_i = X$ for every $i \in I$, the space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultrapower of* X and is denoted $X^I/\mathcal{U}$.[1] If $X^I/\mathcal{U}$ is an ultrapower of a Banach space X, the map $x \mapsto (x_i)$, where $x_i = x$ for every $i \in I$, is an isometric embedding of X into $X^I/\mathcal{U}$. Hence, we may regard X as a subspace of $X^I/\mathcal{U}$. This embedding is not surjective, except in the trivial cases when $\mathcal{U}$ is a principal ultrafilter or the space X is finite dimensional.

An operator T of X can be extended naturally to an operator $T^I/\mathcal{U}$ on by defining, for $(x_i) + N_{\mathcal{U}}$ in $X^I/\mathcal{U}$,

$$T^I\big((x_i) + N_{\mathcal{U}}\big) = (\,T(x_i)\,) + N_{\mathcal{U}}.$$

Clearly, $\|T^I\| = \|T\|$.

If $\{\,T_i\,\}_{i\in I}$ is a family of operators on X and C is a subset of X, we will refer to the structure

$$\mathbf{X} = (\ X,\ T_j,\ c \ \mid\ j \in J,\ c \in C\ )$$

as a *Banach space structure*, and to the structure

$$(\ X^I/\mathcal{U},\ T_j^I/\mathcal{U},\ c \ \mid\ j \in J,\ c \in C\ )$$

as the $\mathcal{U}$-ultrapower of $\mathbf{X}$.

Let $\{\,\mathbf{X}\,\}_{i\in I}$ be a family of Banach space structures such that

1. There exist sets J, K such that for each $i \in I$
$$\mathbf{X}_i = (\ X,\ T_{i,j},\ c_{i,k} \ \mid\ j \in J,\ k \in K\ );$$
2. $\sup_{i\in I} \|T_{i,j}\| < \infty$ for every $j \in J$;
3. $\sup_{i\in I} \|c_{i,k}\| < \infty$ for every $k \in K$.

Then it is natural to define for each $j \in J$ an operator $\prod_{i\in I} T_{i,j}/\mathcal{U}$ on $\prod_{i\in I} X_i/\mathcal{U}$ by letting

$$\prod_{i\in I} T_{i,j}/\mathcal{U}\,\big((x_i) + N_{\mathcal{U}}\big)_{i\in I} = (\,T_{i,j}(x_i)\,) + N_{\mathcal{U}}.$$

For every $j \in J$ and $k \in K$, we have

$$\|\prod_{i\in I} T_{i,j}/\mathcal{U}\| = \lim_{i,\mathcal{U}} \|T_{i,j}\|, \qquad \|(c_{i,k})_{i\in I} + N_{\mathcal{U}}\| = \lim_{i,\mathcal{U}} \|c_{i,k}\|.$$

What is the relation between a Banach space structure and its ultrapowers? In order to answer this question we need to discuss the logic of *positive bounded formulas* and *approximate satisfaction* introduced by C. W. Henson in [30] and [31].

2.2. **Positive Bounded Formulas.** The fundamental distinction between the concept of language in Banach space model theory and the usual concept first-order language is that a Banach space language is required to come equipped with norm bounds for the constants and operators.

Suppose that X is a Banach space, C is a subset of X, and $\{\,T_i\,\}_{i\in I}$ is a family of operators on X. Let

$$\mathbf{X} = (\ X,\ T_j,\ c \ \mid\ j \in J,\ c \in C\ )$$

[1]From a model theorist's point of view, a Banach space ultraproduct is the result of eliminating the elements of infinite norm from an ordinary ultraproduct and dividing by infinitesimals. Instead of algebraic ultrapowers, one can deal with arbitrary models of a certain theory (as in [32]). However, we have chosen to use Banach space ultrapowers as, for our purposes, they provide the most straightforward approach.

A *language* L for $\mathbf{X}$ consists of the following items.

- A binary function symbol $+$ for the vector space addition of X;
- For each rational number r, a monadic function symbol (which we denote also by r) for scalar multiplication by r;
- For each rational number $M > 0$, monadic predicates for the sets

$$\{ x \in X \mid \|x\| \leq M \} \quad \text{and} \quad \{ x \in X \mid \|x\| \geq M \};$$

- A monadic function symbol (an *operator symbol*) for each operator T_i;
- A constant symbol for each element of C;
- Upper norm bounds for each element of C and each operator T_i.

We say that $\mathbf{X}$ is a Banach space L-structure, or simply, an L-structure. We have discussed the fact that class of L-structures is naturally closed under ultrapowers.

The terms and formulas of L are defined as usual. The class of *positive bounded formulas* of L (or *positive bounded L-formulas*) is the class of formulas built up from the atomic formulas

$$\|t\| \leq M, \qquad \|t\| \geq M$$

(where t is a term of L and $M > 0$) by using the *positive connectives* $\wedge, \vee$ and the *bounded quantifiers*

$$\exists x(\|x\| \leq M \wedge \ \ldots\ ) \quad \text{and} \quad \forall x(\|x\| \leq M \rightarrow \ \ldots\ )$$

(where $M > 0$).

If φ is a positive bounded formula, an *approximation* of φ is a positive bounded formula that results from "relaxing" all the norm estimates in φ, as indicated by the following table.

In φ	In approximations of φ
$\|t\| \leq M$	$\|t\| \leq N \quad (N > M)$
$\|t\| \geq M$	$\|t\| \geq N \quad (N < M)$
$\exists x(\|x\| \leq M \wedge \ \ldots\ )$	$\exists x(\|x\| \leq N \wedge \ \ldots\ ) \quad (N > M)$
$\forall x(\|x\| \leq M \rightarrow \ \ldots\ )$	$\forall x(\|x\| \leq N \rightarrow \ \ldots\ ) \quad (N < M)$

2.1. Notation.

1. If φ, ψ are positive bounded formulas, we write $\varphi < \varphi'$ to denote the fact that φ' is an approximation of φ.
2. If Γ is a set of positive bounded formulas, we denote by Γ_+ the set of approximations of formulas in Γ.

The negation connective is not allowed in positive bounded formulas, nor is the implication connective, except when it occurs as part of the bounded universal quantifiers. However, for every positive bounded formula φ there is a positive bounded formula $\operatorname{neg}(\varphi)$ which in Banach space model theory plays a role analogous to that of the negation of φ. We define the formula $\operatorname{neg}(\varphi)$ by means of the following table.

In φ	In $\text{neg}(\varphi)$
$\|t\| \leq M$	$\|t\| \geq M$
$\|t\| \geq M$	$\|t\| \leq M$
$\wedge$	$\vee$
$\vee$	$\wedge$
$\exists x(\|x\| \leq M \wedge \ \dots\ )$	$\forall x(\|x\| \leq M \rightarrow \ \dots\ )$
$\exists x(\|x\| \leq M \wedge \ \dots\ )$	$\forall x(\|x\| \leq M \rightarrow \ \dots\ )$

2.2. Remarks.

1. If φ, φ' are positive bounded formulas, then $\varphi < \varphi'$ if and only if $\text{neg}(\varphi') < \text{neg}(\varphi)$.
2. If $\mathbf{X}$ is a Banach space L-structure and φ is a positive bounded L-sentence, then $\mathbf{X} \not\models \varphi$ if and only if there exists $\varphi' > \varphi$ such that $\mathbf{X} \models \text{neg}(\varphi')$.

2.3. Proposition (Perturbation Lemma). *For every positive bounded L-formula $\varphi(x_1, \dots, x_n)$, every $\varphi' > \varphi$, and every $M > 0$ there exists $\delta > 0$ such that for every Banach space L-structure X,*

$$X \models \bigwedge_{1 \leq i \leq n} \|a_i\| \leq M \ \wedge \ \bigwedge_{1 \leq i \leq n} \|a_i - b_i\| \leq \delta \ \wedge \ \varphi(a_1, \dots, a_n)$$

implies

$$X \models \varphi'(b_1, \dots, b_n).$$

Proof. By induction on the complexity of φ, using the fact that both the norm and the operator symbols of X are uniformly continuous on every bounded subset of X (and the moduli of uniform continuity are given by the language L, so do not depend on the structure X). ⊣

2.3. **Approximate Satisfaction.** In order to simplify the notation, from this point on we will identify a Banach space structure with its universe.

If X is a Banach space L-structure and φ is a positive bounded L-sentence, we say that X *approximately satisfies* φ, and write

$$X \models_{\mathcal{A}} \varphi,$$

if $X \models \varphi'$ for every approximation φ' of φ.

If Γ is a set of positive bounded sentences, we say that X *approximately satisfies* Γ or that X *is a model of* Γ, and write $X \models_{\mathcal{A}} \Gamma$, if X approximately satisfies every sentence in Γ. In the notation introduced in 2.1, $X \models_{\mathcal{A}} \Gamma$ if and only if $X \models \Gamma_+$.

The notion of approximate satisfaction, rather than the usual notion of satisfaction, provides the appropriate semantics for a model theoretical analysis of Banach space structures.

2.4. Proposition. *If X is a Banach space L-structure and φ is a positive bounded L-sentence, then $X \not\models_{\mathcal{A}} \varphi$ if and only there exists $\varphi' > \varphi$ such that $X \models_{\mathcal{A}} \text{neg}(\varphi')$.*

Proof. If $X \not\models_{\mathcal{A}} \varphi$, there exists $\varphi' > \psi$ such that $X \not\models \varphi'$. Then $X \models \text{neg}(\varphi')$ and hence $X \models_{\mathcal{A}} \text{neg}(\varphi')$. Assume, conversely, that there exists $\varphi' > \varphi$ such that $X \models_{\mathcal{A}} \text{neg}(\varphi')$ and take sentences ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Then $X \models \text{neg}(\psi')$ (by Remark 2.2) and hence $X \not\models \psi$, so $X \not\models_{\mathcal{A}} \varphi$. ⊣

2.5. Theorem (Compactness). *Let Γ be a set of positive bounded L-sentences such that every finite subset of Γ is approximately satisfied by some Banach space L-structure. Then there exists a Banach space L-structure which approximately satisfies every sentence in Γ.*

Proof. Let I be the set of finite subsets of Γ_+, and for each $i \in I$ let X_i be a Banach space L-structure satisfying every sentence in i. For every finite subset Δ of Γ_+ let F_Δ be the set of all $i \in I$ such that $X_i \models \Delta$. The family $\mathcal{F}$ of sets of the form F_Δ is closed under finite intersections. If $\mathcal{U}$ is an ultrafilter on I extending $\mathcal{F}$, then

$$\prod_{i\in I} X_\varphi/\mathcal{U} \models_{\mathcal{A}} \Gamma.$$

⊣

A *positive bounded theory* is a set of positive bounded sentences. If X is a Banach space structure, we denote by $\mathrm{Th}_{\mathcal{A}}(X)$ the set of sentences which are approximately satisfied by X.

2.6. Corollary. *The following conditions are equivalent for a positive bounded theory Γ in a language L.*

1. *There exists a Banach space L-structure X such that $\Gamma = \mathrm{Th}_{\mathcal{A}}(X)$;*
2. *(a) Every finite subset of Γ is approximately satisfied in some Banach space L-structure;*
   (b) For every positive bounded L-sentence φ, either $\varphi \in \Gamma$ or there exists $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma$.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from Proposition 2.4. To prove (2) $\Rightarrow$ (1), use Theorem 2.5 to fix a Banach space L-structure X such that $X \models_{\mathcal{A}} \Gamma$. Then $\mathrm{Th}_{\mathcal{A}}(X) \subseteq \Gamma$, for if φ were in $\mathrm{Th}_{\mathcal{A}}(X) \setminus \Gamma$, there would exist $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma \subseteq \mathrm{Th}_{\mathcal{A}}(X)$, which is impossible. Hence $\Gamma = \mathrm{Th}_{\mathcal{A}}(X)$. ⊣

If X and Y are Banach space L-structures, we say that X and Y are *approximately elementarily equivalent*, and write

$$X \equiv_{\mathcal{A}} Y,$$

if X and Y approximately satisfy the same positive bounded L-sentences. If X is a substructure of Y, we say that X is an *approximately elementary substructure of* Y, and write

$$X \prec_{\mathcal{A}} Y,$$

if $(X, a \mid a \in X) \equiv_{\mathcal{A}} (Y, a \mid a \in X)$.

2.7. Proposition. *Let X and Y be L-structures.*

1. *If A is a common subset of X and Y and A_0 is a dense subset of A, then*

$$(X, a \mid a \in A_0) \equiv_{\mathcal{A}} (Y, a \mid a \in A_0) \quad \textit{implies} \quad (X, a \mid a \in A) \equiv_{\mathcal{A}} (Y, a \mid a \in A).$$

2. (Tarski-Vaught Test.) *If X is an L-substructure of Y, then $X \prec_{\mathcal{A}} Y$ if and only if for every positive bounded sentence φ in a language for $(Y, a \mid a \in X)$ of the form $\exists x(\psi(x))$ such that $Y \models_{\mathcal{A}} \varphi$ and every approximation ψ' of ψ there exists $a \in X$ such that $Y \models_{\mathcal{A}} \psi'(a)$;*

Proof. Part (1) follows from the Perturbation Lemma (Proposition 2.3). Part (2) is a straightforward induction. ⊣

2.8. Proposition. *Let X be a Banach space structure.*

1. *If $\hat{X}$ is an ultrapower of X, then $X \prec_{\mathcal{A}} \hat{X}$;*
2. *If Y is a Banach space structure, then $Y \equiv_{\mathcal{A}} X$, if and only if there exists a Banach space structure $\hat{X} \succ_{\mathcal{A}} X$ and an embedding $f : Y \to \hat{X}$ such that $f(Y) \prec_{\mathcal{A}} X$.*

Proof. Exercise. ⊣

Recall that the *density character* of a topological space is the smallest cardinality of a dense subset of the space. For example, a space is separable if and only if its density character is $\aleph_0$.

2.9. Proposition. *Let X be a Banach space structure in a countable language.*

1. (Downward Löwenheim-Skolem Theorem.) *For every set $A \subseteq X$ there exists a substructure Y of X such that $A \subseteq Y \prec_{\mathcal{A}} X$ and*

$$\operatorname{density}(Y) \leq \operatorname{density}(A).$$

2. (Upward Löwenheim-Skolem Theorem.) *If X is infinite-dimensional, then for every cardinal κ with $\kappa \geq \operatorname{density}(X)$ there exists an approximately elementary extension of X of density character κ.*

Proof. To prove (1), let A_0 be a dense subset of A and expand the language with constant symbols and norm bounds for the elements of A_0. Now apply Proposition 2.7 to the structure $(X, a \mid a \in A_0)$.

To prove (2), let X_0 be a dense subset of X and expand the language with constants symbols and norm bounds for the elements of X_0. Expand the language further with new constants symbols $\{ c_i \}_{i<\kappa}$ and norm bounds $\|c_i\| = 1$ for $i < \kappa$. Every finite subset of the theory

$$\operatorname{Th}_{\mathcal{A}}(X, a \mid a \in X_0) \cup \{ \|c_i - c_j\| = 1 \mid i < j < \kappa \}.$$

is approximately satisfied in X, so the conclusion now follows from (1). ⊣

2.4. **$(1+\epsilon)$-Isomorphism and $(1+\epsilon)$-Equivalence of Structures.** When do two Banach spaces have isometric approximately elementary extensions? By Proposition 2.8, this happens if and only if the two Banach spaces are approximately elementary equivalent. Furthermore, two structures

$$(X, c_i \mid i \in I)$$

and

$$(Y, d_i \mid i \in I)$$

are approximately elementary equivalent if and only if there are approximately elementary extensions $\hat{X} \succ_{\mathcal{A}} X$ and $\hat{Y} \succ_{\mathcal{A}} Y$ and an isometry from $f \colon \hat{X} \to \hat{Y}$ such that $f(c_i) = d_i$ for every $i \in I$.

We now address the question of when two Banach spaces have isomorphic (as opposed to isometric) approximately elementary extensions.

In the following discussion, L will denote a language that contains no operator symbols.

For every formula φ of L and every rational $\epsilon > 0$ we define an approximation $\varphi_{1+\epsilon}$ as follows.

In φ	In $\varphi_{1+\epsilon}$
$\|t\| \leq M$	$\|t\| \leq M(1+\epsilon)$
$\|t\| \geq M$	$\|t\| \geq \frac{M}{1+\epsilon}$
$\exists x(\|x\| \leq M \wedge \ \dots\ )$	$\exists x(\|x\| \leq M(1+\epsilon) \wedge \ \dots\ )$
$\forall x(\|x\| \leq M \rightarrow \ \dots\ )$	$\forall x(\|x\| \leq \frac{M}{1+\epsilon} \rightarrow \ \dots\ )$

If Γ is a set of formulas of L, we denote by $\Gamma_{1+\epsilon}$ the set of $(1+\epsilon)$-approximations of formulas in Γ.

We say that two Banach space L-structures X and Y are $(1+\epsilon)$-*equivalent*, and write

$$X \equiv_{1+\epsilon} Y,$$

if

$$X \models_{\mathcal{A}} \varphi \qquad \text{implies} \qquad Y \models_{\mathcal{A}} \varphi_{1+\epsilon}.$$

Let us prove that $\equiv_{1+\epsilon}$ is a symmetric relation. Suppose

$$(\operatorname{Th}_{\mathcal{A}}(X))_{1+\epsilon} \subseteq \operatorname{Th}_{\mathcal{A}}(Y)$$

and let φ be a positive bounded sentence such that $Y \models_{\mathcal{A}} \varphi$. Fix $\varphi' > \varphi$. If $X \not\models \varphi'_{1+\epsilon}$, then $X \models \operatorname{neg}(\varphi'_{1+\epsilon})$. By assumption, $Y \models_{\mathcal{A}} (\operatorname{neg}(\varphi'_{1+\epsilon}))_{1+\epsilon}$. But $(\operatorname{neg}(\varphi'_{1+\epsilon}))_{1+\epsilon}$ is equivalent to $\operatorname{neg}(\varphi')$, so $Y \models_{\mathcal{A}} \operatorname{neg}(\varphi')$. This contradicts the choice of φ, by Proposition 2.4.

If $\epsilon > 0$, two structures

$$(\ X,\ c_i \ \mid\ i \in I\ )$$

and

$$(\ Y,\ d_i \ \mid\ i \in I\ )$$

are said to be $(1+\epsilon)$-*isomorphic* if there exists a linear isomorphism $f\colon X \to Y$ such that $f(c_i) = d_i$ for every $i \in I$ and $\|f\|, \|f^{-1}\| \leq 1+\epsilon$, *i.e.*,

$$(1+\epsilon)^{-1}\|x\| \leq \|T(x)\| \leq (1+\epsilon)\|x\|$$

for every $x \in X$. The function f is called a $(1+\epsilon)$-*isomorphism*.

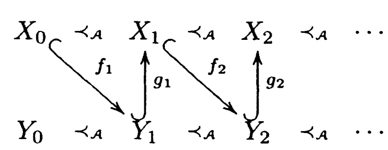
It is easy to see that two $(1+\epsilon)$-isomorphic structures are $(1+\epsilon)$-equivalent. The following is a converse of this observation.

2.10. Theorem. *Two Banach space L-structures are $(1+\epsilon)$-equivalent if and only if they have $(1+\epsilon)$-isomorphic approximately elementary extensions.*

Proof. We prove the nontrivial implication. Assume $X \equiv_{1+\epsilon} Y$. By compactness (Theorem 2.5), we construct chains of extensions extensions

$$X = X_0 \prec_{\mathcal{A}} X_1 \prec_{\mathcal{A}} X_2 \prec_{\mathcal{A}} \cdots$$
$$Y = Y_0 \prec_{\mathcal{A}} Y_1 \prec_{\mathcal{A}} Y_2 \prec_{\mathcal{A}} \cdots$$

and embeddings



such that

$$f_n \subseteq g_n^{-1} \subseteq f_{n+1}, \qquad \text{for } n = 1, 2, \ldots$$

and for every quantifier-free formula $\varphi(\bar{x})$,

$$X_n \models \varphi(\bar{a}) \qquad \text{implies} \qquad Y_{n+1} \models \varphi_{1+\epsilon}(\, f_{n+1}(\bar{a})\,)$$

and

$$Y_n \models \varphi(\bar{a}) \qquad \text{implies} \qquad X_n \models \varphi_{1+\epsilon}(\, g_n(\bar{a})\,).$$

Let $\hat{X}$ be the completion of $\bigcup X_n$ and $\hat{Y}$ the completion of $\bigcup Y_n$. Then $\hat{X} \succ_{\mathcal{A}} X$, $\hat{Y} \succ_{\mathcal{A}} Y$, and $\bigcup_{n>0} f_n$ extends to a $(1+\epsilon)$-isomorphism between $\hat{X}$ and $\hat{Y}$. ⊣

2.5. **Finite Representability.** The notion of finite representability is the central notion in local Banach space geometry.

A Banach space X is *finitely represented* in a Banach space Y if for every finite dimensional subspace E of X and for every $\epsilon > 0$ there exists a finite dimensional subspace F of Y such that E and F are $(1+\epsilon)$-isomorphic.

If X is a Banach space structure, the *existential theory of* X, denoted $\exists\,\mathrm{Th}_{\mathcal{A}}(X)$ is the set of existential positive bounded sentences which are approximately satisfied by X.

2.11. Proposition. *Let X and Y be Banach spaces. The following conditions are equivalent.*

1. *X is finitely represented in Y;*
2. *$\exists\,\mathrm{Th}_{\mathcal{A}}(X) \subseteq \mathrm{Th}_{\mathcal{A}}(Y)$;*
3. *There exists an ultrapower of Y which contains an isometric copy of X.*

Proof. The implication (3) ⇒ (1) is immediate, since an ultrapower of Y is always finitely represented in Y. The implication (1) ⇒ (2) follows from the fact that the unit ball of a finite dimensional space is compact. To prove (2) ⇒ (3), assume that X is finitely represented in Y and let Γ be set of quantifier-free diagram of X. By compactness (Theorem 2.5), there is an ultrapower $\hat{Y}$ of Y such that $\hat{Y} \models_{\mathcal{A}} \Gamma$. Since $\models_{\mathcal{A}}$ and $\models$ coincide for quantifier-free formulas, we have $\hat{Y} \models \Gamma$, so $\hat{Y}$ contains an isometric copy of X. ⊣

2.6. **Types.** Suppose that X is a Banach a space structure and A is a subset of X. If $\bar{c} \in X$, the *type of* $\bar{c}$ *over* A is the set

$$\mathrm{tp}(\bar{c}/A) = \{\ \varphi(\bar{x}, \bar{a})\ \mid \bar{a} \in A,\ (\, X, a \mid a \in A\,) \models_{\mathcal{A}} \varphi(\bar{c}, \bar{a})\ \}.$$

2.12. Proposition. *Let X be a Banach space structure, let A be a subset of X, and let L be a language for the structure $(\, X, a \mid a \in A\,)$. The following conditions are equivalent for a set of positive bounded L-formulas $t(\bar{x}) = t(x_1, \ldots, x_n)$.*

1. *There exists a Banach space structure $Y \succ_{\mathcal{A}} X$ and $\bar{c} \in Y$ such that $t(\bar{x}) = \mathrm{tp}(\bar{c}/A)$.*
2. *(a) There exists $M > 0$ such that the formula*

$$\bigwedge_{1 \le i \le n} \|x_i\| \le M$$

is in t;

(b) Every L-formula of t_+ is satisfied in $(\, X, a \mid a \in A\,)$;

(c) *For every L-formula $\varphi(\bar{x})$, either $\varphi \in t$, or there exists $\varphi' > \varphi$ such that* $\operatorname{neg}(\varphi') \in t$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate from Proposition 2.4. The implication (2) $\Rightarrow$ (1) follows from compactness (Theorem 2.5) and, again, Proposition 2.4. $\dashv$

If X is a Banach space structure, A is a subset of X, and $t(\bar{x})$ is a set of positive formulas satisfying the equivalent conditions of Proposition 2.12, we say that t is a *type over* A and $\bar{c}$ *realizes* t in Y. If $\bar{x} = x_1, \dots, x_n$, we call t an *n-type.*

Fix a Banach space structure X, a subset A of X, and a language L for $(X, a \mid a \in A)$. For a positive bounded L-formula φ, let $[\varphi]$ denote the set of types over A which contain φ. The *logical topology* is the topology on the set of types over A where the basic open neighborhoods of a type t are the sets of the form $[\varphi]$, with $\varphi \in t_+$.

If $t(x_1, \dots, x_n)$ is a type and $(c_1, \dots, c_n)$ is a realization of t, the *norm* of t is $\max_{1 \le i \le n} \|c_i\|$.

2.13. Proposition. *For any $M > 0$, the set of types of norm less than or equal to M is compact with respect to the logical topology.*

Proof. Fix a Banach space structure X and a subset A of X. Let $\{ t_i \}_{i \in I}$ be a family of types over A and let $\mathcal{U}$ be an ultrafilter on I. By compactness (Theorem 2.5), we may take $Y \succ_{\mathcal{A}} X$ such that every t_i is realized in Y. For each $i \in I$ let $\bar{c}_i$ be a realization of t_i in Y. It is now easy to see that the type over A of the element of $\prod_{i \in I} Y_i/\mathcal{U}$ represented by $(\bar{c}_i)_{i \in I}$ is $\lim_{i,\mathcal{U}} t_i$. $\dashv$

2.14. Remark. It is not true that the set of types over A is compact with respect to the logical topology. Indeed, for each $n > 0$, the set $[\, \|x\| \ge n\,]$ is closed in the logical topology. However,

$$\bigcap_{n>0} [\, \|x\| \ge n\,] = \emptyset.$$

2.7. **Saturated and Homogeneous Structures.** Let κ be an infinite cardinal. A Banach space structure X is called *κ-saturated* if every type over every subset of X of cardinality less than κ is realized in X.

The proof that every Banach space structure X has a κ-saturated approximately elementary extensions is completely analogous to the proof of the corresponding fact in first order model theory; specifically, one constructs a chain of approximately elementary extensions

$$X = X_0 \prec_{\mathcal{A}} X_1 \prec_{\mathcal{A}} \cdots \prec_{\mathcal{A}} X_i \prec_{\mathcal{A}} \cdots \qquad (i < \kappa^+) \tag{1}$$

such that whenever $i < j < \kappa^+$, every type over every subset of X_i of cardinality less than κ is realized in X_j. Then, the completion of $\bigcup_{i<\kappa^+} X_i$ is a κ-saturated approximately elementary extension of X.

Now suppose that we have a chain of structures as in (1) above such that whenever $i < j < \kappa^+$, the structure X_j is $|X_i|^+$-saturated, and let $\hat{X} = \bigcup i < \kappa^+ X_i$. We say that the structure $\hat{X}$ is *κ-special.* Arguing as in Theorem 2.10, one proves that a κ-special structure $\hat{X}$ has the following property: if A is a subset of X of cardinality less than κ and $f\colon A \to X$ is such that

$$(\hat{X},\, a \mid a \in A) \equiv_{\mathcal{A}} (\hat{X},\, f(a) \mid a \in A),$$

there exists a bijection $F\colon X \to X$ extending f such that

$$(\hat{X},\, a \mid a \in X) \equiv_{\mathcal{A}} (\hat{X},\, F(a) \mid a \in X).$$

We express this fact by saying that $\hat{X}$ is *strongly κ-homogeneous.* The argument of Theorem 2.10 also shows that if the language contains no operator symbols, then every $(1+\epsilon)$-isomorphism between two approximately elementary substructures of $\hat{X}$ of density character less than κ can be extended to a $(1+\epsilon)$-automorphism of $\hat{X}$.

2.8. **The Monster Model.** In what follows, X will denote a Banach space structure and we will regard X as being embedded as an approximately elementary substructure in a single κ-saturated, κ-special structure, where κ is a cardinal larger than any cardinal mentioned in the proofs.[2] Following the tradition (started by Shelah), we will refer to this structure as the "monster model", and denote it $\mathfrak{C}$. Our assumption on the monster model allows us to regard all the structures approximately elementary equivalent to X as substructures of $\mathfrak{C}$, and all the realizations of types over subsets of them as living inside $\mathfrak{C}$.

If $\varphi(x_1, \dots, x_n)$ is a positive bounded formula, we denote by $\varphi(\mathfrak{C})$ the subset of $\mathfrak{C}^n$ defined by φ.

Notice that, by the $\aleph_1$-saturation of the monster model implies that satisfaction and approximate satisfaction are equivalent on it.

The terms "structure", formula, "type", and "consistent" stand, respectively, for "Banach space structure", " positive bounded formula", "positive bounded type", and "satisfied in the monster model".

3. Semidefinability of Types

3.1. Definition. Suppose $A \subseteq B$ and let $t(\bar{x})$ be a type over B. We say that t *splits over* A if there exist tuples $\bar{b}, \bar{c}$ with $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$, a formula $\varphi(\bar{x}, \bar{y})$, and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\mathrm{neg}(\varphi'(\bar{x}, \bar{c})) \in t(\bar{x})$.

3.2. Proposition. *Suppose that $(a_i \mid i < \gamma)$ is a sequence such that*

(i) $\mathrm{tp}(a_\alpha/A \cup \{a_i \mid i < \beta\}) \subseteq \mathrm{tp}(a_\beta/A \cup \{a_i \mid i < \alpha\})$ *for $\alpha < \beta < \gamma$;*

(ii) $\mathrm{tp}(a_\alpha/A \cup \{a_i \mid i < \alpha\})$ *does not split over A for $\alpha < \gamma$.*

Then the sequence $(a_i \mid i < \gamma)$ is indiscernible.

Proof. We prove by induction on n that

$$\mathrm{tp}(a_{i(0)}, \dots, a_{i(n-1)}/A) = \mathrm{tp}(a_0, \dots, a_{n-1}/A), \qquad \text{for } i(0) < \cdots < i(n-1) < \gamma.$$

For $n = 1$, this is given by (i). Assume that the result is true for n and take $i(0) < \cdots < i(n) < \gamma$. By the induction hypothesis (ii) and the fact that

$$\mathrm{tp}(a_{i(n)}/A \cup \{a_i \mid i < i(n)\})$$

does not split over A,

$$\mathrm{tp}(a_{i(n)}/\{a_{i(0)}, \dots, a_{i(n-1)}\} \cup A) = \mathrm{tp}(a_{i(n)}/\{a_0, \dots, a_{n-1}\} \cup A),$$

and by (i)

$$\mathrm{tp}(a_{i(n)}/\{a_0, \dots, a_{n-1}\} \cup A) = \mathrm{tp}(a_n/\{a_0, \dots, a_{n-1}\} \cup A).$$

[2]Given that we are mostly interested in separable spaces, $\kappa = (2^{\aleph_0})^+$ will typically suffice.

Putting together these two equalities, we get

$$\operatorname{tp}(\, a_{i(n)} \,/\, \{\, a_{i(0)}, \ldots, a_{i(n-1)} \,\} \cup A \,) = \operatorname{tp}(\, a_n \,/\, \{\, a_0, \ldots, a_{n-1} \,\} \cup A \,).$$

⊣

3.3. Definition. Suppose $A \subseteq B$. A type t over B is called *semidefinable over A* if every approximation of every finite subset of t is realized in A.

3.4. Proposition. *Suppose that $A \subseteq B$. A type t over B which is semidefinable over A does not split over A.*

Proof. Suppose that $t(\bar{x})$ splits over A. Take $\bar{b}, \bar{c} \in B$ with $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$, a formula $\varphi(\bar{x}, \bar{y})$, and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\operatorname{neg}(\, \varphi'(\bar{x}, \bar{c}) \,) \in t(\bar{x})$. Take formulas ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Since t is semidefinable over A, there exists $\bar{a} \in A$ such that $\models \psi(\bar{a}, \bar{b}) \wedge \operatorname{neg}(\, \psi'(\bar{a}, \bar{c}) \,)$. But this contradicts the fact that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$. ⊣

3.5. Proposition. *Suppose that $A \subseteq B \subseteq C$ and let $t(\bar{x})$ be a type over B which is semidefinable over A.*

1. *t has an extension $t'(\bar{x})$ over C which semidefinable over A;*
2. *If for every $n > \omega$ every n-type over A is realized in B, then t has a unique extension $t'(\bar{x})$ over C which semidefinable over A;*

Proof. (1): Let

$$\Gamma(\bar{x}) = \{\ \operatorname{neg}(\, \varphi(\bar{x}, \bar{c}) \,) \ \mid\ \bar{c} \in C,\ A \cap \varphi(\mathfrak{C}, \bar{c}) = \emptyset\ \}.$$

Let us show that $t \cup \Gamma$ is consistent. If $t \cup \Gamma$ were inconsistent, there would exist formulas $\psi(\bar{x}) \in t$ and $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$ such that $A \cap \varphi(\mathfrak{C}, \bar{c}) = \emptyset$ and $\psi(\mathfrak{C}) \cap \operatorname{neg}(\, \varphi(\mathfrak{C}, \bar{c}) \,) = \emptyset$. But then

$$A \cap \psi(\mathfrak{C}, \bar{c}) \subseteq A \cap \varphi(\mathfrak{C}, \bar{c}) = \emptyset,$$

which is a contradiction.

It is easy to see that any type type $t'(\bar{x})$ over C which extends $t \cup \Gamma$ must be semidefinable over A.

(2): Suppose that $t_1(\bar{x})$ and $t_2(\bar{x})$ are distinct extensions of t over C which are semidefinable over A. Then there exist a formula $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$ and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{c}) \in t_1$ and $\operatorname{neg}(\, \varphi'(\bar{x}, \bar{c}) \,) \in t_2$. Take $\bar{b} \in B$ such that $t(\bar{b}, A) = \operatorname{tp}(\bar{c}, A)$. Take also formulas ψ, ψ' such such that $\varphi < \psi < \psi' < \varphi'$. By Proposition 3.4 t_1 does not split over A, so $\psi(\bar{x}, \bar{b}) \in t_1 \restriction B = t$; similarly, t_2 does not split over A, so $\operatorname{neg}(\, \psi'(\bar{x}, \bar{c}) \,) \in t_2 \restriction B = t$. This is, of course, a contradiction. ⊣

4. Ramsey's Theorem for Analysis

In this section we discuss a form of Ramsey's Theorem which was used by A. Brunel and L. Sucheston in [5] to produce 1-subsymmetric sequences (*i.e.*, quantifier-free indiscernible sequences). The method of Brunel and Sucheston has now become standard in Banach space geometry, and in [61] H. P. Rosenthal labelled it: *The Ramsey principle for analysts.*

4.1. Proposition. *Let $(a_{m,n})_{m,n<\omega}$ be a matrix of real numbers such that $\lim_n a_{m,n}$ exists for every m, and*

$$\lim_m \lim_n a_{m,n} = \alpha.$$

Then there exist $k(0) < k(1) < \dots$ *such that*

$$\lim_{i<j} a_{k(i),k(j)} = \alpha.$$

Proof. By definition, for every $\epsilon > 0$ there exists a positive integer M_ϵ such that

$$m \geq M_\epsilon \qquad \text{implies} \qquad |\lim_n a_{m,n} - \alpha| \leq \epsilon.$$

Also, for every $\epsilon > 0$ and every fixed integer $\hat{m}$ there exists $N_\epsilon^{\hat{m}}$ such that

$$n \geq N_\epsilon^{\hat{m}} \qquad \text{implies} \qquad |a_{\hat{m},n} - \lim_n a_{\hat{m},n}| \leq \epsilon.$$

Take $k(0) < k(1) < \dots$ such that

$$k(0) \geq M_1$$
$$k(l+1) \geq \max\{ M_{2^{-l}}, N_{2^{-l}}^{k(0)}, \dots, N_{2^{-l}}^{k(l)} \}.$$

It is easy to see that

$$i < j \qquad \text{implies} \qquad |a_{k(i),k(j)} - \alpha| \leq 1/2^{i-1}.$$

⊣

We need the multidimensional version of Proposition 4.1. The proof is similar. (It can also be easily derived from Proposition 4.1 by induction and diagonalization.)

4.2. Proposition. *Let*

$$(a_{m_1,m_2,\dots,m_d} \mid (m_1, m_2, \dots, m_d) \in \omega^d)$$

be a family of real numbers such the iterated limits

$$\lim_{m_1} \dots \lim_{m_d} a_{m_1,m_2,\dots,m_d}$$

exist. Then there exist $k(0) < k(1) < \dots$ *such that*

$$\lim_{i_1<i_2<\dots<i_d} a_{k(i_1),k(i_2),\dots,k(i_d)} = \lim_{m_1} \dots \lim_{m_d} a_{m_1,m_2,\dots,m_d}.$$

5. Quantifier-Free Types over Normed Spaces

At this point and for the rest of the paper, we concentrate our attention on quantifier-free types. Thus, hereafter, the word "type" will mean "quantifier-free type". If $\bar{a}$ is a finite tuple and C is a subset of the monster model, $\text{tp}(\bar{a}/C)$ denotes the quantifier-free type of $\bar{a}$ over C.

We now argue that it is sufficient to focus on the case when $\bar{a}$ is an element (rather than a tuple of elements) and C is a Banach space.

We study only 1-types: The type of a tuple $a_0, \dots, a_n$ over a set C is completely determined by the types of the elements of the linear span of $a_0, \dots, a_n$. This allows us to focus on types of elements of the monster model, rather than tuples.

We consider only types over Banach spaces: If a be an element and C is a subset of the monster model, the quantifier-free type of a over C is completely determined by the formulas of the form

$$\|a + c\|,$$

where c is an element of the linear span of C. Since the function $c \mapsto \|a + c\|$ is uniformly continuous, it has a unique extension to the closed span of C. Thus, we can assume without loss of generality that C is a Banach space.

Recall that the *norm* of a 1-type is the norm of an element realizing the type.

5.1. Remark. If $M > 0$, the set of types of norm less than or equal to M is compact, because if $(\operatorname{tp}(a_i/X))_{i\in I}$ is a family of types with $\|a_i\| \leq M$ and $\mathcal{U}$ is an ultrafilter of I, then $\lim_{i,\mathcal{U}} \operatorname{tp}(a_i/X)$ is exactly the type over X realized in the $\mathcal{U}$-ultrapower of $\overline{\operatorname{span}}\{X \cup \{a_i \mid i \in I\}\}$ by the element represented by the family $(a_i)_{i\in I}$.

The type of a over a Banach space X can be identified with the real-valued function

$$x \mapsto \|x + a\| \qquad (x \in X).$$

Furthermore, it is easy to see that in this identification the logical topology corresponds exactly to the product topology inherited from $\mathbb{R}^X$. Proposition 5.2 shows that the space of types over X corresponds the closure of the set of realized types (*i.e.*, the types of the form $\operatorname{tp}(a/X)$, where $a \in X$). Thus, in particular, the space of types is separable if X is separable. [3]

5.2. Proposition. *Let X be a separable normed space and let τ be a real-valued function on X. Then the following conditions are equivalent.*

1. *τ is the function corresponding to a type over X;*
2. *There exists a sequence (x_n) in X such that*

$$\tau(x) = \lim_{n\to\infty} \|x_n + x\|, \qquad \textit{for every } x \in X.$$

Proof. Notice that if (x_n) is as in (2), then (x_n) is bounded. Hence, (2) $\Rightarrow$ (1) follows from Remark 5.1. To prove (1) $\Rightarrow$ (2), suppose that τ corresponds to $\operatorname{tp}(c/X)$. Then let $\{d_n \mid n \in \omega\}$ be a dense subset of X. Since every approximation formula of every formula in $\operatorname{tp}(c/X)$ is satisfied X, we can find a sequence (x_n) in X such that

$$\big|\, \|x_n + d_k\| - \|c + d_k\| \,\big| < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Then we have $\lim_{n\to\infty} \|x_n + x\| = \|c + x\| = \tau(x)$ for every $x \in X$. ⊣

5.3. Definition. Let $t(x)$ be a type over a normed space Y and let X be a subspace of Y. A sequence (x_n) in X is called *approximating* for t if

$$\lim_{n\to\infty} \operatorname{tp}(x_n/X) = t(x).$$

We say that (x_n) *approximates* t.

5.4. Proposition. *Every bounded sequence in a separable Banach space X has a subsequence which approximates some type over X.*

Proof. By Remark 5.1 and Proposition 5.2. ⊣

[3] This argument shows that in general, the density character of the space of types over X equals the density character of X. However, for the kinds of results that we wish to prove in this paper, we do not lose generality by restricting our attention to separable spaces.

5.5. Proposition. *Suppose that t is a type over a separable space Y and X is subspace of Y. Then every approximating sequence for $t \upharpoonright X$ has a subsequence which is approximating for t.*

Proof. Let (x_n) be an approximating sequence for $t \upharpoonright X$ and let a be a realization of t. Then,

$$\lim_n \|x_n + x\| = \|a + x\|, \qquad \text{for every } x \in X.$$

Let D be a countable dense subset of $Y \setminus X$. By a simple diagonalization argument, we can find a subsequence (x'_n) of (x_n) such that

$$\lim_n \|x'_n + d\| = \|a + d\|, \qquad \text{for every } x \in D.$$

The sequence (x'_n) approximates t. ⊣

5.6. Proposition. *Let Y be a normed space and let X be a separable subspace of Y. Then the following conditions are equivalent.*

1. $\mathrm{tp}(a/Y)$ *is semi-definable over X;*
2. *There exists a sequence in X which approximates* $\mathrm{tp}(a/X)$

Proof. (2) ⇒ (1) is clear. We prove (1) ⇒ (2). Let $\{ d_n \mid n \in \omega \}$ be a dense subset of X. Since $\mathrm{tp}(a/Y)$ is semidefinable over X, we can find a sequence (x_n) in X such that

$$|\, \|x_n + d_k\| - \|a + d_k\| \,| < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Clearly, $\lim_{n\to\infty} \mathrm{tp}(x_n/X) = \mathrm{tp}(a/X)$. ⊣

6. Fundamental Sequences

6.1. Definition. Let X be a normed space and let $t(x)$ be a type over X. We will say that a sequence (a_n) is a *fundamental sequence for t* if

1. For every $n < \omega$, $\mathrm{tp}(a_n/X) = t$;
2. (a_n) is indiscernible over X;
3. For every $n < \omega$, the type
$$\mathrm{tp}(\, a_n \,/\, X \cup \{ a_i \mid i < n \} \,)$$
is semidefinable over X.

We say that (a_n) is a *fundamental sequence* if there exists a type t such that (a_n) is fundamental for t.

6.2. Proposition. *Every type has a fundamental sequence for it.*

Proof. By the results in Section 3. ⊣

6.3. Remark. When the space X is stable with respect to quantifier-free types, then a type is semidefinable over A if and only if it is nonforking over A, so the concept of fundamental sequence coincides with that of *Morley sequence.*

6.4. Proposition. *Let X be a separable Banach space. Then the following conditions are equivalent for a bounded sequence (a_n) in X.*

1. *(a_n) is a fundamental sequence for a type over X;*

2. *There exists a bounded sequence* (x_n) *in* X *such that whenever* $r_0, \dots, r_k \in \mathbb{R}$ *and* $x \in X$,

$$\lim_{n_k} \dots \lim_{n_0} \|r_0 x_{n_0} + \dots + r_k x_{n_k} + x\| = \|r_0 a_0 + \dots + r_k a_k + x\|;$$

3. *There exists a bounded sequence* (x_n) *in* X *such that whenever* $r_0, \dots, r_k \in \mathbb{R}$ *and* $x \in X$,

$$\lim_{n_0 < \dots < n_k} \|r_0 x_{n_0} + \dots + r_k x_{n_k} + x\| = \|r_0 a_0 + \dots + r_k a_k + x\|;$$

Proof. The equivalence between (2) and (3) follows from Ramsey's Theorem (Proposition 4.2). Now, (1) follows from these two equivalent conditions because (3) trivially implies that (a_n) is indiscernible, and (2) implies that, for every $n < \omega$, $\mathrm{tp}(a_n / X \cup \{ a_i \mid i < n \})$ is semidefinable over X.

(1) $\Rightarrow$ (2): Let

$$t = \bigcup_{n<\omega} \mathrm{tp}(a_n / X \cup \{ a_i \mid i < n \}).$$

Then t is semidefinable over X. Let (x_n) be a sequence in X which approximates t. Then (x_n) approximates $\mathrm{tp}(a_n / X \cup \{ a_i \mid i < n \})$ for every $n < \omega$. Hence, if $r_0, \dots, r_k \in \mathbb{R}$ and $x \in X$,

$$\|r_0 a_0 + \dots + r_k a_{n_k} + x\| = \lim_{n_k} \dots \lim_{n_0} \|r_0 x_{n_0} + \dots + r_k x_{n_k} + x\|.$$

⊣

6.5. Proposition. *Let* X *be a separable Banach space. Then for every bounded sequence* (x_n) *in* X *there exist:*

- *A subsequence* (x'_n) *of* (x_n);
- *A type* t *over* X *such that* (x'_n) *approximates* t *over* X,
- *A sequence* (a_n) *fundamental for* t *such that for* $r_0, \dots, r_k \in \mathbb{R}$ *and* $x \in X$;

$$\lim_{n'_0 < \dots < n'_k} \|r_0 x'_{n_0} + \dots + r_k x'_{n_k} + x\| = \|r_0 a_0 + \dots + r_k a_k + x\|.$$

Proof. By extracting a subsequence, we can assume that $\mathrm{tp}(x_n/X)$ converges to a type t over X. Let (a_n) be a fundamental sequence for t (which exists by Proposition 6.2) and define

$$\hat{t} = \bigcup_{n<\omega} \mathrm{tp}(a_n / X \cup \{ a_k \mid k < n \}).$$

Then $\hat{t}$ is semidefinable over X. By Proposition 5.5 we can assume that (x_n) approximates $\hat{t}$. Then, for $r_0, \dots, r_k \in \mathbb{R}$ and $x \in X$,

$$\lim_{n_k} \dots \lim_{n_0} \|r_0 x_{n_0} + \dots + r_k x_{n_k} + x\| = \|r_0 a_0 + \dots + r_k a_k + x\|;$$

By Ramsey's Theorem (Proposition 4.2), we can now take a subsequence (x'_n) of (x_n) such that the conclusion of the proposition holds. ⊣

6.6. Definition. If (a_n) is a fundamental sequence for a type over a normed space X, the space generated by X and the sequence (a_n) is called the *spreading model of the fundamental sequence* (a_n) *over* X.

7. Symmetric Types

We can define a scalar multiplication of types naturally as follows.

7.1. Definition. Suppose that t is $\mathrm{tp}(a/X)$. If r is a scalar, we denote by rt the type $\mathrm{tp}(ra/X)$.

7.2. Definition. A type t is *symmetric* if $t = -t$.

7.3. Proposition. *For every space X there exists a nonzero symmetric type over X.*

To prove Proposition 7.3, we will invoke a famous result from finite-dimensional topology. For an integer $k \geq 1$, let S_k denote the k-dimensional unit sphere, *i.e.*,

$$S_k = \{ x \in \mathbb{R}^{k+1} \mid \|x\| = 1 \},$$

where $\| \ \|$ denotes the usual Euclidean norm. A map $f \colon S_k \to \mathbb{R}^l$ is called *antipodal* if $f(-x) = -f(x)$.

7.4. Theorem (Borsuk-Ulam Antipodal Map Theorem). *Let $k \geq 1$ and let $f \colon S_k \to \mathbb{R}^k$ be a continuous antipodal map. Then there exists $s \in S_k$ such that $f(s) = 0$.*

7.5. Remark. An analysis of the proof of the Borsuk-Ulam Theorem reveals that the Euclidean norm can be replaced with any norm. Hence, $\mathbb{R}^k$ can be replaced by any finite dimensional Banach space E, and S_k by the unit sphere of E.

Proof of Proposition 7.3. By compactness, we just have to prove the assertion in the case when X is finite dimensional. Assume, then, that X is generated by $x_0, \dots, x_n$, and define a map f on the unit sphere of X by letting

$$f(x) = (\|x + x_0\| - \|x - x_0\|, \dots , \|x + x_n\| - \|x - x_n\|).$$

Then f is continuous and antipodal. By the Borsuk-Ulam Antipodal Map Theorem, there exists s in the unit sphere of X such that $f(s) = 0$. Then, $\mathrm{tp}(s/X)$ is symmetric. ⊣

Two sequences (a_n) and (b_n) are called *equivalent* if the map $a_n \mapsto b_n$ determines an isometry between the span of $\{ a_n \mid n < \omega \}$ and the span of $\{ a_n \mid n < \omega \}$.

7.6. Definition. A sequence (a_n) in a Banach space is called 1-*unconditional* if whenever (ϵ_n) is a sequence such that $\epsilon_n = \pm 1$, the sequence $(\epsilon_n a_n)$ is equivalent to (a_n).

7.7. Proposition. *Every sequence which is fundamental for a symmetric types is indiscernible and* 1*-unconditional.*

Proof. Immediate from Proposition 6.4. ⊣

8. ℓ_p- and c_0-Types

In Definition 7.1 we defined an operation on types, namely, scalar multiplication. In this section we introduce another operation on types, that of *convolution.* The convolution is a binary operation defined on scalar multiples of a given type.

8.1. Definition. Let t be a type over X, and let (a_n) be a fixed fundamental sequence for t. If $r_0, \dots, r_n$ are scalars, we define $r_0 t * \cdots * r_n t$, the *convolution* of $r_0 t, \dots, r_n t$, as the type $\mathrm{tp}(r_0 a_0 + \cdots + r_n a_n \,/\, X)$. We denote by $[t]$ the set of types of the form $r_0 t * \cdots * r_n t$, where $r_0, \dots, r_n$ are scalars and $n < \omega$.

8.2. Remark. The *convolution* of scalar multiples of a type t depends on a given fundamental sequence for t. Thus, when we refer to convolutions of scalar multiples t, it should be understood that a fundamental sequence for t is fixed in the background.

8.3. Definition. Let p be a real number satisfying $1 \leq p < \infty$. A type t over X is called an *ℓ_p-type* if

- t is symmetric;
- If $r, s \geq 0$, then $r^p t * s^p t = (r^p + s^p)^{1/p} t$.

The type t is called a *c_0-type* if

- t is symmetric;
- If $r, s \geq 0$, then $rt * st = \max(r, s)t$.

8.4. Definition. Let p be a real number satisfying $1 \leq p < \infty$. A sequence (a_n) is said to be *ℓ_p over X* if whenever $x \in X$ and $r_0, \ldots, r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{n} r_i a_i \right\| = \left\| x + \Big(\sum_{i=0}^{n} |r_i|^p \Big)^{1/p} a_0 \right\|.$$

The sequence (a_n) is said to be *c_0 over X* if whenever $x \in X$ and $r_0, \ldots, r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{n} r_i a_i \right\| = \left\| x + \big(\max_i |r_i| \big) a_0 \right\|.$$

8.5. Proposition. *Let t be a symmetric type over X and let (a_n) be a fundamental sequence inducing a convolution on multiples of t. Then the following conditions are equivalent for $1 \leq p < \infty$.*

1. *t is an ℓ_p-type;*
2. *(a_n) is ℓ_p over X;*
3. *For every $x \in X$ and every natural number k,*

$$\left\| x + \sum_{i=0}^{m-1} r_i a_i + k^{1/p} a_m + \sum_{i=m+1}^{n} r_i a_i \right\| = \left\| x + \sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k} \right\|.$$

Proof. (1) $\Rightarrow$ (2): We prove by induction on n that the first equality in Definition 8.4 holds. If $n \leq 1$, the equality is immediate. Assume $n \geq 1$. Let (x_k) be an approximating sequence for t in X. Then,

$$\begin{aligned}\Big\|x+\sum_{i=0}^{n} r_i a_i\Big\| &= \lim_{k_{n-2}} \dots \lim_{k_0} \Big\|x+\sum_{i=0}^{n-2} r_i x_{k_i} + r_{n-1}a_{n-1} + r_n a_n\Big\| \\ &= \lim_{k_{n-2}} \dots \lim_{k_0} \Big\|x+\sum_{i=0}^{n-2} r_i x_{k_i} + (|r_{n-1}|^p + |r_n|^p)^{1/p} a_n\Big\| \\ &= \Big\|x+\sum_{i=0}^{n-2} r_i a_i + (|r_{n-1}|^p + |r_n|^p)^{1/p} a_n\Big\| \\ &= \Big\|x+\big(\sum_{i=0}^{n-2} |r_i|^p\big)^{1/p} a_0 + (|r_{n-1}|^p + |r_n|^p)^{1/p} a_n\Big\| \\ &= \Big\|x+\big(\sum_{i=0}^{n} |r_i|^p\big)^{1/p} a_0\Big\|.\end{aligned}$$

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are immediate. We prove (3) $\Rightarrow$ (2).

Fix scalars $r_0, \dots, r_n$. Since t is symmetric, we can also assume $r_0, \dots, r_n \geq 0$. Furthermore, by the uniform continuity of the norm, we can assume that r_i^p is rational, for $i = 0, \dots, n$. Therefore, we can take positive integer such that Mr_i^p is an integer, for $i = 0, \dots, n$. By the indiscernibility of (a_n) over X, for every $x \in X$ we have

$$\begin{aligned}\Big\|M^{1/p}x + \big(\sum_{i=0}^{n} Mr_i^p\big)^{1/p} a_0\Big\| &= \Big\|M^{1/p}x + \sum_{i=0}^{n}\sum_{j=0}^{Mr_i^p} a_{i+j}\Big\| \\ &= \Big\|M^{1/p}x + \big(\sum_{i=0}^{n} Mr_i^p\big)^{1/p} a_i\Big\|.\end{aligned}$$

Dividing by $M^{1/p}$, we obtain the desired result. ⊣

8.6. Proposition. *Let t be a symmetric type over X and let (a_n) be a fundamental sequence inducing a convolution on multiples of t. Then the following conditions are equivalent.*

1. *t is a c_0-type.*
2. *(a_n) is c_0 over X.*
3. *For every $x \in X$ and every natural number k,*

$$\Big\|x+\sum_{i=0}^{m-1} r_i a_i + a_m + \sum_{i=m+1}^{n} r_i a_i\Big\| = \Big\|x+\sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k}\Big\|.$$

Proof. Similar to the proof of Proposition 8.5 ⊣

8.7. Exercise. Prove that if t is an ℓ_p-type (or c_0-type) and (x_n) is an approximating sequence for t, then or every $\epsilon > 0$ the sequence (x_n) contains a subsequence which is $(1+\epsilon)$-isomorphic to the standard basis of ℓ_p (respectively, c_0). [*Hint*: Fix $\epsilon_0, \epsilon_1 \dots$ such that $0 < \epsilon_0 < \epsilon_1 < \dots < \epsilon$. Then, by induction, find $n_0 < n_1 < \dots$ such that $(x_{n_i})_{i \leq k}$ is $(1+\epsilon_k)$-isomorphic to the standard basis of $\ell_p(n)$.]

9. Where Does the Number p Come From?

Our goal is to find ℓ_p-like sequences inside Banach spaces. A common question is: how does the p arise? Generally, p is obtained through a combination of Proposition 10.3 and the proposition in this section, as follows. One starts with a type, and obtains an indiscernible sequence by taking fundamental sequences. An order-homomorphism of the sequence induces an operator on the Banach space X spanned by the sequence. In general, one will have countably many such operators, say, we have a family $\{ T_n \}$ of operators on X. Proposition 10.3 then provides an extension $\hat{X}$ of X and a subspace Y of X such that $T_n = \lambda_n I$ on Y. By choosing the operators appropriately, we will be able to assure that the sequence $\{ \lambda_n \}$ is nondecreasing. Then, the following proposition will provide the desired number p.

9.1. Proposition. *Let $\{ \lambda_n \}_{n\geq 1}$ be a sequence of real numbers such that*

1. $1 = \lambda_1 \leq \lambda_2 \leq \dots$
2. $\lambda_m \lambda_n = \lambda_{mn}$.

Then, either $\lambda_n = 1$ for every n, or there exists a number $p \geq 1$ such that $\lambda_n = n^{1/p}$ for every n.

Proof. Suppose $\lambda_2 > 1$ and let $p = \frac{\log 2}{a_2}$. Fix integers $m, n \geq 2$. For every integer k there exists an integer $h = h(k)$ such that $m^{h(k)} \leq n^k < m^{h(k)+1}$. By (1) and (2), we have $\lambda_m^{h(k)} \leq \lambda_n^k \leq \lambda_m^{h(k)+1}$. Hence,

$$\Big| \Big\lfloor k\frac{\log n}{\log m} \Big\rfloor - \Big\lfloor k\frac{\log \lambda_n}{\log \lambda_m} \Big\rfloor \Big| \leq 1$$

(where $\lfloor r \rfloor$ denotes the largest integer less than or equal to r). By letting $k \to \infty$, we conclude $\frac{\log \lambda_n}{\log n} = \frac{\log \lambda_m}{\log m}$. Hence, $\lambda_n = n^{1/p}$. ⊣

10. A Little Operator Theory

In this section we include a few basic facts from operator theory. We have included the proofs for completeness. Proposition 10.3 will be used in Section 11 to transform indiscernible sequences.

Recall that the set of operators on a Banach space is a Banach space, with the norm of an operator T defined by $\sup_{\|x\|\leq 1} \|T(x)\|$. The identity operator is denoted I. Note that if T, W are operators on X, then $\|TW\| \leq \|T\|\,\|W\|$.

10.1. Proposition. *Let X be a Banach space.*

1. *If T is an operator on X with $\|T\| < 1$, then $I - T$ is invertible.*
2. *The set of invertible operators on X is open in the norm topology.*

Proof. (1): Let $W = \sum_n T^n$. It is easy to see that W is an operator on X and $(I-T)W = W(I-T) = I$.

(2): Suppose that W is an invertible operator on X. If T is any other operator, $\|I - TW^{-1}\| \leq \|W - T\|\,\|W^{-1}\|$. Thus, if $\|W - T\| < \|W^{-1}\|^{-1}$, then TW^{-1} is invertible by (1), and hence so is T. ⊣

The *spectrum* of an operator T is

$$\{ \lambda \in \mathbb{R} \mid T - \lambda I \text{ is not invertible} \}.$$

It follows from Proposition 10.1 that the spectrum of an operator is a closed subset of $\mathbb{R}$.

10.2. Proposition. *Let T be an operator on a Banach space X, and λ be an element of the boundary of the spectrum of T. Then there exists an ultrapower $(\hat{X},\hat{T})$ of (X,T) and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}(e) = \lambda e$.*

Proof. By replacing T with $T - \lambda I$, we can assume that $\lambda = 0$.

Suppose that the conclusion of the proposition is false. Then there exists $\delta > 0$ such that $\inf_{\|x\|=1} \|T(x)\| \geq \delta$. Also, since 0 is in the closure of the spectrum of T, we can find arbitrarily small real numbers μ such that $T - \mu I$ is invertible. Fix such μ with $|\mu| < \frac{\delta}{2}$. Then, by Proposition 10.1, the operator $1 + \mu(T - \mu I)^{-1}$ is invertible. But then so is

$$(T - \mu I)(1 + \mu(T - \mu I)^{-1}) = T,$$

which contradicts the fact that 0 is in the spectrum of T (since it is in the boundary and the spectrum is closed). ⊣

10.3. Proposition. *Let $\{\, T_i \,\}_{i\in I}$ be a family of operators on a Banach space X such that $T_iT_j = T_jT_i$ for $i, j \in I$. For each $i \in I$, let λ_i be an element of the boundary of the spectrum of T_i. Then there exists an ultrapower $(\hat{X},\hat{T})$ of (X,T) and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for every $i \in I$.*

Proof. By compactness, it suffices to consider the case when I is finite. We prove the proposition by induction on the number of elements of I. If I is a singleton, our proposition is just Proposition 10.2. Assume, then, that $I = \{\, 1, \dots, n \,\}$.

By induction hypothesis, there exists an ultrapower $(\, \hat{X}, \hat{T}_i \mid i \leq n \,)$ of $(\, X, T_i \mid i \leq n \,)$ and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for $i < n$. Let

$$Y = \{\; x \in \hat{X} \;\mid\; \hat{T}_i(x) = \lambda_i x \text{ for } i < n \;\}.$$

Since $\hat{T}_n$ commutes with $\hat{T}_i$ for $i < n$, we have $\hat{T}_n(Y) \subseteq Y$. By Proposition 10.2 and compactness there exist an ultrapower $\hat{Y}$ of Y and an ultrapower $(\, \hat{\hat{X}}, \hat{\hat{T}}_i \mid i \leq n \,)$ of $(\, \hat{X}, \hat{T}_i \mid i \leq n \,)$ such that $\hat{\hat{X}}$ contains $\hat{Y}$ and there exists $f \in \hat{Y}$ with $\|f\| = 1$ satisfying $\hat{\hat{T}}_i(f) = \lambda_i f$ for $i = 1, \dots, n$. ⊣

11. Block Representability of ℓ_p in Types

11.1. Theorem. *Let t be a symmetric type over X and let $*$ be a convolution on the scalar multiples of t. Then there exists a sequence (e_n) such that*

1. *(e_n) is c_0 or ℓ_p over X, for some p with $1 \leq p < \infty$.*
2. *There exists a sequence of types (u_l) in $[t]$ such that:*
   (a) (e_n) is fundamental for $\lim_l u_l$;
   (b) Whenever $r_0, \dots, r_k$ are scalars,

$$\operatorname{tp}(\, r_0e_0 + \cdots + r_ke_k \,/\, X \,) = \lim_l \,(r_0u_l * \cdots * r_ku_l).$$

Proof. Let $(a_q)_{q\in\mathbb{Q}\cap[0,1)}$ be an indiscernible family such that, for any scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\, r_0a_{q_0} + \cdots + r_ka_{q_k} \,/\, X \,) = r_0t * \cdots * r_kt, \qquad \text{if } q_1 < \cdots < q_k.$$

Let Y be the subspace spanned by X and (a_q). For each $n \geq 1$ define an operator $T_n : Y \to Y$ as follows. If $x \in X$ and $q_0 < \cdots < q_k$ are in $\mathbb{Q} \cap [0,1)$,

$$T_n\Big(x + \sum_{i=0}^{k} r_i a_{q_i}\Big) = x + \sum_{j=0}^{n-1} \sum_{i=0}^{k} r_i \, a_{\frac{q_i}{n} + \frac{j}{n}}\,.$$

Then, for every m, n,

(i) $T_m \circ T_n = T_{mn}$;
(ii) $1 \leq \|T_n\| \leq n$.

(The second inequality in (ii) is immediate from the indiscernibility of (a_q); the first one is an easy exercise; you'll need the fact that t is symmetric.) Now we apply Proposition 10.3 to find an extension $(\hat{Y}, \hat{T}_n \mid n \geq 1,)$ of $(Y, T_n \mid n \geq 1)$ and a nonzero element $e \in \hat{Y}$ such that $\hat{T}_n(e) = \lambda_n e$ for some real number λ_n. By Proposition 9.1 and (i)–(ii) above, we conclude that either $\lambda_n = 1$ for every n, or there exists a real number $1 \leq p < \infty$ such that $\lambda_n = n^{1/p}$.

Let (y_l) be a sequence in the span of (a_q) such that $\lim_l \mathrm{tp}(y_l/X) = \mathrm{tp}(e/X)$, and let $u_l \in [t]$ be such that $\mathrm{tp}(y_l/X) = u_l$.

Let $\{ c_n \mid n < \omega \}$ be a set of new constants and let $\Gamma(c_n)_{n<\omega}$ be a set of sentences expressing the following facts:

(iv) (c_n) is indiscernible over X and fundamental for $\mathrm{tp}(e/X)$;
(v) $\mathrm{tp}(r_0 c_0 + \cdots + r_k c_k \,/\, X) = \lim_l (r_0 u_l * \cdots * r_k u_l)$ for any scalars $r_0, \ldots, r_k$;
(vi) If $x \in X$, and $r_0, \ldots r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{m-1} r_i c_i + \lambda_m c_m + \sum_{i=m+1}^{n} r_i c_i \right\| = \left\| x + \sum_{i=0}^{m-1} r_i c_i + \sum_{i=m}^{m+k} c_i + \sum_{i=m+1}^{n} r_i c_{i+k} \right\|.$$

Every finite finite subset of $\Gamma(c_n)_{n<\omega}$ is realized in Y by interpreting the constants with the blocks of $T_n(y_l)$ for sufficiently large n and l.

Let $(e_n)_{n<\omega}$ realize $\Gamma(c_n)_{n<\omega}$. We prove first that (e_n) is fundamental for $\mathrm{tp}(e_0/X)$. By Proposition 6.4, for every $l < \omega$, there exists a sequence (z_j^l) in X such that

- For any $k < \omega$ and scalars $r_0, \ldots, r_k$,
$$r_0 u_l * \cdots * r_k u_l = \lim_{j_0 < \cdots < j_k} \|r_0 z_{j_0}^l + \cdots + r_k z_{j_k}^l + x\|;$$
- If $l < l'$, the sequence $(z_j^{l'})$ is a subsequence of (z_j^l).

By diagonalization, we obtain a sequence (z_j) such that whenever $k < \omega$ and $r_0, \ldots, r_k$ are scalars,

$$\|r_0 e_0 + \cdots + r_k e_k + x\| = \lim_l (r_0 u_l * \cdots * r_k u_l) = \lim_{j_0 < \cdots < j_k} \|r_0 z_{j_0}^l + \cdots + r_k z_{j_k}^l\|.$$

Then the sequence (e_n) is fundamental by Proposition 6.4, and condition (2) of the theorem is satisfied.

If $\lambda_n = n^{1/p}$, then (e_n) is ℓ_p over X by (vi) and Proposition 8.5. Otherwise $\lambda_n = 1$ for every n, and (e_n) is c_0 over X by (vi) and Proposition 8.6. ⊣

12. Krivine's Theorem

If $(a_0, \ldots, a_k)$ and $(b_0, \ldots, b_k)$ are finite sequences, X is a Banach space, and $\epsilon > 0$, we write

$$\mathrm{tp}(\, a_0, \ldots, a_k \,/\, X \,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_n, \ldots, b_k \,/\, X \,)$$

and say that the types $\mathrm{tp}(\, a_0, \ldots, a_k \,/\, X\,)$ and $\mathrm{tp}(\, b_0, \ldots, b_k \,/\, X\,)$ $(1+\epsilon)$-*equivalent over* X if there is a $(1+\epsilon)$-isomorphism f from $\overline{\mathrm{span}}\{ \{ a_i \mid i \leq k \} \cup X \}$ onto $\overline{\mathrm{span}}\{ \{ b_i \mid i \leq k \} \cup X \}$ such that $f(a_i) = b_i$ for $i = 1, \ldots, k$ and f fixes X pointwise.

Let (a_n) be a sequence in a Banach space. We say that $b_0, \dots, b_k$ are *blocks* of (a_n) if there exist finite sets $F_0, \dots, F_k \subseteq \omega$ such that $\max F_i < \min F_{i+1}$ and $b_i \in \operatorname{span}\{ a_n \mid n \in F_i \}$ for $n = 1, \dots, k$.

12.1. Proposition. *Suppose (a_n) is a fundamental sequence for a symmetric type over a Banach space X. Then there exists a sequence (e_n) such that*

1. *(e_n) is c_0 or ℓ_p over X, for some p with $1 \leq p < \infty$;*
2. *For every $\epsilon > 0$ and every $k \in \omega$ there exist blocks $b_0, \dots, b_k$ of (a_n) satisfying*

$$\operatorname{tp}(e_0, \dots, e_k / X) \overset{1+\epsilon}{\sim} \operatorname{tp}(b_0, \dots, b_k / X).$$

Proof. Suppose that (a_n) is fundamental for a nontrivial symmetric type t over a Banach space X and let $*$ be a convolution on the scalar multiples of t. By Theorem11.1 there exists a sequence (e_n) such that

1. (e_n) is c_0 or ℓ_p over X, for some p with $1 \leq p < \infty$.
2. There exists a sequence of types (u_l) in $[t]$ such that:
   (a) (e_n) is fundamental for $\lim_l u_l$;
   (b) Whenever $r_0, \dots, r_k$ are scalars,

$$\operatorname{tp}(r_0 e_0 + \cdots + r_k e_k / X) = \lim_l (r_0 u_l * \cdots * r_k u_l).$$

Fix $\epsilon > 0$ and $k \in \omega$. By (2-b) above and the fact that the unit ball of $(\mathbb{R}^k, \| \|_\infty)$ is compact, we find blocks $b_0, \dots, b_k$ of (a_n) such that whenever $r_0, \dots, r_k$ are scalars,

$$\operatorname{tp}(r_0 e_0 + \cdots + r_k e_k / X) \overset{1+\epsilon}{\sim} \operatorname{tp}(r_0 b_0 + \cdots + r_k b_k / X).$$

The conclusion of the proposition now follows. ⊣

A sequence (e_n) is *block finitely represented* in a sequence (a_n) if for every $\epsilon > 0$ and every $k < \omega$ there exist blocks $e_0, \dots, e_k$ of (a_n) such that

$$\operatorname{tp}(e_0, \dots, e_k / \emptyset) \overset{1+\epsilon}{\sim} \operatorname{tp}(b_0, \dots, b_k / \emptyset).$$

12.2. Theorem (Krivine's Theorem). *Given any bounded sequence (x_n) in a Banach space, either there exists p with $1 \leq p < \infty$ such that ℓ_p is block finitely represented in (x_n), or c_0 is block finitely represented in (x_n).*

Proof. Let (x_n) be a sequence a separable Banach space X. By extracting a subsequence, we may assume that (x_n) approximates a type t over X. Let (a_n) be a fundamental sequence for t. By Proposition6.5, we may refine (x_n) so that whenever $r_0, \dots, r_k$ are scalars,

$$(\dagger) \qquad \lim_{n_0 < \cdots < n_k} \operatorname{tp}(r_0 x_{n_0} + \cdots + r_k x_{n_k} / X) = \operatorname{tp}(r_0 a_0 + \cdots + r_k a_k / X)$$

Let now $X' = \overline{\operatorname{span}}\{ a_n \mid n < \omega \}$ and let (a'_n) be a fundamental sequence for a nonzero symmetric type over X' (which exists by Proposition 7.3).

Fix $\epsilon > 0$ and $k < \omega$. By Proposition 12.1, there exists a sequence (e_n) such that

1. (e_n) is c_0 or ℓ_p over X, for some p with $1 \leq p < \infty$;
2. There exist blocks $b_0, \dots, b_k$ of (a'_n) with

$$(\ddagger) \qquad \operatorname{tp}(e_0, \dots, e_k / X) \overset{1+\epsilon}{\sim} \operatorname{tp}(b_0, \dots, b_k / X).$$

Using (†) and the fact that the unit ball of $(\mathbb{R}^k, \|\ \|_\infty)$ is compact, we find blocks $y_0, \dots, y_k$ of (x_n) such that

$$\mathrm{tp}(\, y_{n_0}, \dots, y_{n_k} \,/\, X\,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, X\,).$$

Putting this together with (‡), we obtain

$$\mathrm{tp}(\, e_0, \dots, e_k \,/\, X\,) \overset{(1+\epsilon)^2}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, X\,).$$

Krivine's Theorem now follows, since ϵ is arbitrary. ⊣

13. Stable Banach Spaces

A separable Banach space X is *stable* if whenever (x_m) and (y_n) are bounded sequences in X and $\mathcal{U}, \mathcal{V}$ are ultrafilters on $\mathbb{N}$,

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\|.$$

Let $\varphi(\bar{x}, \bar{y})$ be a positive bounded formula and let $\varphi'(\bar{x}, \bar{y})$ be an approximation of φ (see Section 2). We will say that the pair φ, φ' *has the order property* in the space X if there exist bounded sequences $(\bar{x}_m)$ and $(\bar{y}_n)$ in X such that

$$\begin{aligned} X &\models \varphi(\bar{x}_m, \bar{y}_n), && \text{if } m \le n; \\ X &\models \mathrm{neg}(\,\varphi(\bar{x}_m, \bar{y}_n)\,), && \text{if } m > n. \end{aligned}$$

13.1. Proposition. *A separable Banach space X is stable if an only if no pair of quantifier-free positive bounded formulas has the order property in X.*

Proof. Every quantifier-free positive formula $\varphi(\bar{x}, \bar{y})$ is equivalent to a conjunction of disjunctions of formulas of the form

$$\|\Lambda(\bar{x}, \bar{y})\| \le r \qquad \text{or} \qquad \|\Lambda(\bar{x}, \bar{y})\| \ge r,$$

where r is a scalar and $\Lambda(\bar{x}, \bar{y})$ is a linear combination of $\bar{x}$ and $\bar{y}$. Hence, by the pigeonhole principle, a pair of quantifier-free formulas has the order property in X if and only if there exist bounded sequences (x_m) and (y_n) in X such that

$$\sup_{m<n} \big(\|x_m + y_n\|\big) \ne \inf_{m>n} \big(\|x_m + y_n\|\big).$$

But, by Ramsey's Theorem (Proposition 4.1), this is equivalent to unstability of X. ⊣

Suppose that (x_m) and (x'_m) are bounded sequences in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{m,\mathcal{U}} \mathrm{tp}(x_m/X) = \lim_{m,\mathcal{U}} \mathrm{tp}(x'_m/X).$$

Then, if (y_m) is a bounded sequence in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$,

$$\lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x'_m + y_n\|.$$

Similarly, if (y_n) and (y'_n) are bounded sequences in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{n,\mathcal{V}} \mathrm{tp}(y_n/X) = \lim_{n,\mathcal{V}} \mathrm{tp}(y'_n/X),$$

then, whenever (x_m) is a bounded sequence in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, we have

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y'_n\|.$$

Thus, if X is stable, we can define a binary operation $*$ on the space of types over X as follows. Let t, t' be types over X and let (x_m) and (y_n) be sequences in X such that $t = \lim_{m,\mathcal{U}} \mathrm{tp}(x_m/X)$ and $t' = \lim_{n,\mathcal{V}} \mathrm{tp}(y_n/X)$. We define

$$t * t' = \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\|$$

The preceding remarks prove that this operation is well defined. This operation is called the *convolution* on the space of types of X

13.2. Proposition. *The convolution on the space of types of a stable Banach space is commutative and separately continuous.*

Proof. Immediate from the definitions. ⊣

13.3. Remark. A space X is stable if and only if there exists a separately continuous binary operation $*$ on the space of types over X which extends the addition of X in the sense that if $x, y \in X$,

$$\mathrm{tp}(x/X) * \mathrm{tp}(y/X) = x + y.$$

13.4. Remark. In Definition 8.1 we defined a convolution on the scalar multiples of a type, by fixing for every type t a fundamental sequence (a_n) for it and for scalars $r_0, \dots, r_k$ letting

$$r_0 t * \cdots * r_k t = \|r_0 a_0 + \cdots + r_k a_k\|. \tag{2}$$

There is no conflict between this notion of convolution and that defined in this section. If X is stable, then (2) holds for any type t over X and any fundamental sequence (a_n) for t.

Examples of stable Banach spaces include the ℓ_p and L_p spaces. For a proof that these spaces are stable, we refer the reader to [46]. For further examples of stable spaces, see [16, 55, 56].

The space c_0 is not stable. For each $n < \omega$ let x_n be the nth vector of the standard basis of c_0, and let $y_n = x_0 + \cdots + x_n$. Then

$$\|x_n + y_m\| = \begin{cases} 1, & \text{if } m > n \\ 2, & \text{if } m \leq n. \end{cases}$$

Since the property of being stable is closed under subspaces, no stable space can contain c_0.

14. Block Representability of ℓ_p in Types Over Stable Spaces

14.1. Definition. Let t be a symmetric type over X and let $1 \leq p < \infty$. We will say that ℓ_p (or ℓ_∞) is *block represented in* $[t]$ if there exists a sequence (e_n) such that

1. (e_n) is ℓ_p (respectively, c_0) over X;
2. There exists a sequence of types (u_l) in $[t]$ such that:
   (a) (e_n) is fundamental for $\lim_l u_l$;

(b) Whenever $r_0, \dots, r_k$ are scalars,

$$\mathrm{tp}(\, r_0 e_0 + \cdots + r_n e_k \,/\, X\,) = \lim_l (r_0 u_l * \cdots * r_k u_l).$$

For a symmetric type t over X, we define

$$\mathfrak{p}(t) = \{\; p \in [1,\infty] \;\mid\; \ell_p \text{ is block represented in } [t] \;\}$$

Theorem 11.1 says exactly that for every Banach space X and every symmetric type t over X, the set $\mathfrak{p}(t)$ is nonempty.

14.2. Proposition. *Suppose that X is stable. If t, t' are symmetric types over X such that $t \in \overline{[t']}$, then $\mathfrak{p}(t) \subseteq \mathfrak{p}[t']$.*

Proof. Suppose that $p \in \mathfrak{p}(t)$ and take (e_n), and (u_l) corresponding to p and $[t]$ as in Theorem 11.1. Since $u_l \in [t]$, we can write

$$u_l = s_0^l t * \cdots * s_{j(l)}^l t,$$

where $s_0^l, \dots, s_{j(l)}^l$ are scalars. Also, since $t \in \overline{[t]}$, there exists a sequence (w_m) in $[t']$ such that $t = \lim_m w_m$. Then for any scalars $r_1, \dots, r_k$ we have the following equalities. The last one follow from the separate continuity of the convolution and Ramsey's Theorem (Proposition 4.2).

$$\mathrm{tp}(\, r_0 e_0 + \cdots + r_n e_k \,/\, X\,) =$$

$$\lim_l \;\big[\, r_0 (s_0^l t * \cdots * s_{j(l)}^l t) * \cdots * r_k (s_0^l t * \cdots * s_{j(l)}^l t) \,\big] =$$

$$\lim_l \big[\, r_0 \big(s_0^l \lim_m w_m * \cdots * s_{j(l)}^l \lim_m w_m\big) * \cdots * r_k \big(s_0^l \lim_m w_m * \cdots * s_{j(l)}^l \lim_m w_m\big) \,\big] =$$

$$\lim_{m_0^0 < \cdots < m_{j(l)}^0 < \cdots < m_0^k < \cdots < m_{j(l)k}^k < l} \big[\, r_0 (s_0^l w_{m_0^0} * \cdots * s_{j(l)}^l w_{m_{j(l)}^0}) * \cdots * r_k (s_0^l w_{m_0^k} * \cdots * s_{j(l)}^l w_{m_{j(l)}^k}) \,\big].$$

We conclude that $p \in \mathfrak{p}(t')$. ⊣

14.3. Proposition. *Suppose that X is stable. Then there exists a type t over X such that*

1. *$\|t\|$ is symmetric;*
2. *$\|t\| = 1$;*
3. *$\mathfrak{p}(t') = \mathfrak{p}(t)$ for every type $t' \in \overline{[t]}$ of norm 1.*

Proof. Suppose that the conclusion of the proposition is false. We construct, inductively, a sequence $(\, t_i \,)_{i < (2^{\aleph_0})^+}$ of types over X such that

1. $\|t_i\|$ is symmetric;
2. $\|t_i\| = 1$;
3. $t_i \in \overline{[t_j]}$ for $i > j$;
4. $\mathfrak{p}(t_i) \subsetneq \mathfrak{p}(t_j)$ for $i > j$.

This is clearly impossible.

We construct t_i by induction on i. The case when i is a successor ordinal is given by assumption. Suppose that i is a limit ordinal. Fix an ultrafilter $\mathcal{U}$ on i. By compactness, there exists a type t' over X such that $\lim_{j<i,\mathcal{U}} t_j = t'$. Conditions (1)-(3) are satisfied by letting $t_i = t'$.

⊣

15. ℓ_p-Subspaces of Stable Banach Spaces

Let $(\Sigma, \leq)$ be a partially ordered set. For an ordinal α we define the set Σ^α as follows.

- $\Sigma^0 = \Sigma$;
- If $\alpha = \beta + 1$,

$$\Sigma^{\alpha+1} = \{ \ \xi \in \Sigma^\alpha \ \mid \ \text{There exists } \eta \in \Sigma^\alpha \text{ with } \eta > \xi \ \}$$

- If α is a limit ordinal,

$$\Sigma^\alpha = \bigcap_{\beta<\alpha} \Sigma^\beta.$$

The *rank* of Σ, denoted $\operatorname{rank}(\Sigma)$, is the smallest ordinal α such that $\Sigma^{\alpha+1} = \emptyset$. If such an ordinal does not exist, we say that Σ has *unbounded rank* and write $\operatorname{rank}(\Sigma) = \infty$.

15.1. Proposition. *Suppose that* $\operatorname{rank}(\Sigma) = \infty$. *Then there exists a sequence* (ξ_n) *in* Σ *such that* $\xi_0 < \xi_1 < \dots$

Proof. Fix an ordinal α such that $\Sigma^\alpha = \Sigma^\beta$ for every $\beta > \alpha$. Take $\xi_0 \in \Sigma^\alpha$. Then $\xi \in \Sigma^{\alpha+1}$, so there exists $\xi_1 \in \Sigma^\alpha$ with $\xi_1 > \xi_0$. Now, $\xi_1 \in \Sigma^{\alpha+1}$, so there exists $\xi_2 \in \Sigma^\alpha$ with $\xi_3 > \xi_2$. Continuing in this fashion, we find (ξ_n) as desired. ⊣

Let $X^{<\omega}$ denote the set of finite sequences of X. If $\xi, \eta \in X^{<\omega}$, we write $\xi < \eta$ if η extends ξ.

15.2. Proposition. *Suppose that X is stable. Then there exists* $p \in [1, \infty]$ *such that for every* $\epsilon > 0$, *the set*

$$\{ \ \xi \in X^{<\omega} \ \mid \ \xi \overset{1+\epsilon}{\sim} \ell_p(n) \text{ for some } n < \omega \ \}$$

has unbounded rank.

Before proving the proposition, let us invoke it to prove the following famous result.

15.3. Theorem (Krivine-Maurey, 1980). *For every stable Banach space X there exists a number* $p \in [1, \infty)$ *such that for every* $\epsilon > 0$ *there exists a sequence in X which is* $(1 + \epsilon)$*-equivalent to the standard basis of* ℓ_p.

Proof. By Proposition 15.2, there exists $p \in [1, \infty]$ such that for every $\epsilon > 0$ there exists a sequence in X which is $(1 + \epsilon)$-equivalent to the standard basis of ℓ_p. But the stability of X rules out the case $p = \infty$, so the theorem follows. ⊣

Proof of Proposition 15.2. Use Proposition 14.3 to fix a symmetric type t_0 over X of norm 1 and such that $\mathfrak{p}(t) = \mathfrak{p}(t)$ for every type $t \in \overline{[t_0]}$ of norm 1 . Fix $p \in \mathfrak{p}(t)$.

We construct for every ordinal α a type t_α over X such that

1. $\|t_\alpha\| = 1$;
2. t_α is symmetric;
3. $t_\alpha \in \overline{[t_\beta]}$ for every $\beta < \alpha$;

4. For every $\epsilon > 0$, every finite dimensional subspace E of X, and every element c with $\text{tp}(c/X) \in [t_\alpha]$, the set

$$\Sigma[\epsilon, E, c] = \Big\{ (x_0, \dots, x_n) \in X^{<\omega} \;\Big|\; \text{tp}\,(\sum_{i=0}^{n} \lambda_i x_i \,/\, E) \overset{1+\epsilon}{\sim} (\sum_{i=0}^{n} |\lambda_i|^p)^{1/p}\, \text{tp}(c/E) \text{ whenever } \lambda_0, \dots, \lambda_n \text{ are scalars} \Big\}$$

has rank $\geq \alpha$.

Notice that if $(x_0, \dots, x_n) \in \Sigma[\epsilon, E, c]$ and $c \neq 0$, then $(\frac{x_0}{\|c\|}, \dots, \frac{x_n}{\|c\|}) \in \Sigma[\epsilon, E, c]$ Hence, condition (4) ensures that $\text{rank}(\Sigma[p, \epsilon]) = \infty$. The other conditions are set so that the inductive construction goes through.

Note that (3) implies that $p \in \mathfrak{p}(t_\alpha)$ for every ordinal α.

The type t_0 defined above, satisfies (1)-(3). Condition (4) is immediate from the symmetry of t and the fact that every approximation of a type over X is realized in any finite dimensional subspace of X.

Suppose that t_α has been defined, in order to define $t_{\alpha+1}$. Fix $\epsilon > 0$ and a finite dimensional subspace E of X. Take real numbers δ_1, δ_2 such that $0 < \delta_1 < \delta_2 < \epsilon$ and $(1 + \delta_2)^2 < \epsilon$.

Let (u_l) be a sequence of types of norm 1 in $[t_\alpha]$ which witness the fact that $p \in \mathfrak{p}(t_\alpha)$. Let $t_{\alpha+1} = \lim u_l$. Conditions (1)–(3) are clearly satisfied. We prove (4).

By (2), if $r_0, \dots, r_n$ are scalars,

$$(\dagger) \qquad (\sum_{i=0}^{n} |r_i|^p)^{1/p} t_\alpha = \lim_{l} (r_0 u_l * \cdots * r_n u_l).$$

Fix an element c such that $\text{tp}(c/X) \in [t_{\alpha+1}]$. Each u_l is in $[t_\alpha]$, so using (†) and the fact that the convolution is commutative and separately continuous, we find types $w_0, \dots, w_n \in [t_\alpha]$ such that

$$(\sum_{i=0}^{n} |r_i|^p)^{1/p} \text{tp}(c/X) \overset{1+\delta_1}{\sim} r_0 w_0 * \cdots * r_n w_n$$

whenever $r_0, \dots, r_n$ are scalars. Let d be a realization of w_0. Since E is finite dimensional, there exist $a_1, \dots, a_n \in E$ such that

$$(\ddagger) \qquad (\sum_{i=0}^{n} |r_i|^p)^{1/p} \text{tp}(c/E) \overset{1+\delta_2}{\sim} \text{tp}\,(r_0 d + \sum_{i=1}^{n} r_i a_i \,/\, E)$$

whenever $r_0, \dots, r_n$ are scalars. Fix $(x_0, \dots, x_n) \in \Sigma[\epsilon, E, c]$. We now prove that $(x_0, \dots, x_n, a_1, \dots, a_n) \in \Sigma[\epsilon, E, c]$; This will conclude the proof of (4). Fix scalars $\lambda_0, \dots, \lambda_n, \mu_1, \dots, \mu_n$. Since $\text{tp}(d/X) = w_0 \in [t_\alpha]$, by induction hypothesis we have

$$\text{tp}\,(\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i a_i \,/\, E) \overset{1+\delta_2}{\sim} \text{tp}\,((\sum_{i=0}^{n} |\lambda_i|^p)^{1/p} d + \sum_{i=1}^{n} \mu_i a_i \,/\, E)$$

Hence, by (‡),

$$\text{tp}\,(\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i a_i \,/\, E) \overset{(1+\delta_2)^2}{\sim} (\sum_{i=0}^{n} |\lambda_i|^p + \sum_{i=1}^{n} |\mu_i|^p)^{1/p} \text{tp}(c/E)$$

Since $(1+\delta)^2 < \epsilon$, it follows that $(x_0, \dots, x_n, a_1, \dots, a_n) \in \Sigma[\epsilon, E, c]$. Hence, $\operatorname{rank} \Sigma[\epsilon, E, c] \geq \alpha + 1$.

If α is a limit ordinal, we take an ultrafilter $\mathcal{U}$ on α and define $t_\alpha = \lim_{\beta<\alpha,\mathcal{U}} t_\beta$. ⊣

16. Historical Remarks

Section 2: The general construction of Banach space ultrapower was introduced by D. Dacunha-Castelle and J. L. Krivine in [12] (although ultrapowers had been used by Krivine and others in earlier publications; see [11]). The classical reference for Banach space ultrapowers is [25]. A somewhat more recent survey is [65].

The ultrapower construction is a particular case of the *nonstandard hull* construction introduced by W. A. J. Luxemburg in [48]. For a survey on applications of nonstandard hulls to Banach space theory, see [34].

The logic of positive bounded formulas and approximate satisfaction was introduced by C. W. Henson in [31]. The precursor was [30]. (See also [26, 27, 28, 29, 33].) In the general framework of Banach space model theory, one considers structures of the form

$$(\ X,\ R_i,\ f_j,\ c_k\ \mid\ i \in I,\ j \in J,\ k \in K\ ),$$

where the c_k's are constants, the f_j's are functions from X^n into X, (for some n depending on j), and the R_i's are *real-valued relations, i.e.*, functions from X^n (for some n) into the extended real numbers. The functions and real-valued relations are required to be uniformly continuous on every bounded subset of X, and the language is required to come equipped with norm bounds for the constants and moduli of uniform continuity for the functions and real-valued relations on each bounded subset of X. One does not generally deal with ultrapowers, but rather with general models.

The notion of $(1+\epsilon)$-approximation and Theorem 2.10 are due to S. Heinrich and C. W. Henson [26].

In this section we have discussed only the most basic aspects of Banach space model theory. For more advanced aspects of the theory, *e.g.*, forking and stability, see [36, 35, 40, 41].

Related, but less general approaches to Banach spaces as models were proposed by J.-L. Krivine [43, 44] and J. Stern [66].

Section 3: The notions of splitting and semidefinability in model theory are due to S. Shelah, and the results in this section are straightforward adaptations of results in [63].

Section 4: For a survey on applications of Ramsey's Theorem to Banach space geometry, see [50].

Powerful strengthenings of Ramsey's Theorem due to W. T. Gowers and B. Maurey have led to the construction of Hereditarily Indecomposable spaces and to a chain of some of the most spectacular breakthroughs in the history of Banach space theory. For a nontechnical exposition, see [19] and [53]. The paper [20] contains a more recent although more technical survey. Further remarks on these important developments are at the end of this paper.

Section 5: The definition of "type" in analysis was introduced in [46]. The definition in [46] is as follows. A separable Banach space X is fixed. If $a \in X$, the function $\tau_a \colon X \to \mathbb{R}$ is defined by $\tau_a(x) = \|a + x\|$. The *space of types* is the closure of the set $\{ \tau_a \mid a \in X \}$ in the product space $\mathbb{R}^X$. Proposition 5.2 shows that the space of types in this sense is exactly the space of quantifier-free types over X.

For further applications of the concept of type to Banach space geometry, see, for example [8, 14, 24, 23, 49, 58, 60, 61].

The definition of "approximating sequence" is also given in [16]; it appears there, however, without the clause "over X", since there, the space X is regarded as fixed throughout.

Section 6: Spreading models were introduced in analysis by A. Brunel and L. Sucheston [4, 5] in the study of sumability of sequences in Banach spaces. The authors proved in [5] that whenever (x_n) is a bounded sequence in a Banach space X, there exists a subsequence (x'_n) of (x_n) such that the limit

$$\lim_{n'_0 < \cdots < n'_k} \| r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x \|$$

exists for every $r_0, \ldots, r_k \in \mathbb{R}$. The sequence (x'_n) is called a *good subsequence* of (x_n). We outline the argument of Brunel and Sucheston. A good subsequence (x'_n) induces a seminorm on $\mathbb{R}^\omega$ (or $\mathbb{C}^\omega$ if the space X is complex) as follows. If (e_n) is the standard basis of unit vectors in $\mathbb{R}^\omega$,

$$\| \sum_i r_i e_i \| = \lim_{n'_0 < \cdots < n'_k} \| r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x \|.$$

This seminorm is a norm if (and only if) the sequence (x'_n) is nonconvergent. The resulting Banach space is called the *spreading model defined by the sequence* (x_n). The clause "over X" is not used by analysts, since the space X is normally regarded as fixed. The sequence (a_n) in Proposition 6.5 is called the *fundamental sequence* of the model. It should be remarked that, despite this terminology, neither the good sequence (x'_n) nor the sequence (a_n) are uniquely determined by (x_n).

The indiscernibility of the fundamental sequence is expressed by analysts by saying that the fundamental sequence of a spreading models is 1-*subsymmetric*.

J.-L. Krivine constructed spreading models in [45] using iterated Banach space ultrapowers. Both constructions are presented in detail in [2].

Section 7: Symmetric types (and types in Banach space theory in general) were explicitly introduced in [46] in the context of stable Banach spaces. Under the presence of stability, the existence of a symmetric type is immediate, for if t is a nonzero type, then $t * (-t)$ is symmetric, since $*$ is commutative (see Proposition 13.2).

We obtained our proof of existence of symmetric types using the Borsuk-Ulam Theorem from [61].

Section 8: ℓ_p- and c_0-types were introduced in [46].

Section 10: The simplification of the proof of Krivine's Theorem through the use of eigenvectors of operators (Proposition 10.2) is due to H. Lemberg [47]. See the comments on Sections 11 and 12 for further remarks on Lemberg's proof.

Section 11: Our proof of Theorem 11.1 is based on H. Lemberg's proof of Krivine's Theorem [47]. We have tried to highlight the fact that, from a model theoretical perspective, the main idea is in fact simple.

Section 12: The original statement of Krivine's Theorem in [45] was that given any bounded sequence (x_n) in a Banach space, either there exists p with

$1 \leq p < \infty$ such that ℓ_p is block finitely represented in (x_n), or there exists a permutation of (x_n) such that c_0 is block finitely represented in (x_n). In [59], H. P. Rosenthal expounded Krivine's Theorem and showed that the permutation of (x_n) in the c_0 case was unnecessary. In [47], H. Lemberg extracted the essential aspects of Rosenthal's proof, and simplified the argument further by using Proposition 10.2.

For a long time, it was an open problem whether every Banach space has a spreading model containing ℓ_p $(1 \leq p < \infty)$ or c_0. The question was answered negatively by E. Odell and Th. Schlumprecht in [54]. In the same paper, the authors also provided an example of a space with an unconditional basis for which ℓ_p and c_0 are block-finitely represented in all block bases. Proposition 12.1 shows that every spreading model has in turn a spreading model with a fundamental sequence (e_n) which is equivalent to the standard basis of c_0 or ℓ_p, for some p with $1 \leq p < \infty$.

Section 14: Proposition 14.3 is from [6], and it plays a role analogous to that played by *minimal cones* in [46].

Section 15: The question of what Banach spaces contain ℓ_p or c_0 almost isometrically has played a central in the history of Banach space geometry. The first example of a Banach space not containing ℓ_p or c_0 (not even isomorphically) was constructed by B. S. Tsirel'son [67]. This phenomenon was even more dramatic for the dual of the original Tsirel'son space [15], which later became also known as the Tsirel'son space and has been used as cornerstone for further variations of the original. *Tsirel'son spaces* became an object of rather intense study. (See [7].)

In 1981, using probabilistic methods, D. Aldous proved [1] that every subspace of L_1 contains c_0 or some ℓ_p $(1 \leq p < \infty)$ almost isometrically. Almost immediately, J.-L. Krivine and B. Maurey generalized the methods of Aldous to a wider class of spaces: the class of stable Banach spaces. The role played by types in [46] (regarded as real-valued functions, see the notes on Section 5 above) is analogous to that played by random measures in Aldous' proof.

A wealth of examples of stable Banach spaces is exhibited in [46]. Furthermore, the authors provide methods to construct new stable Banach spaces from old ones; specifically, it is proved that if X is stable, then the space $L_p(X)$ is stable, for $1 \leq p < \infty$. Further examples are given in [16] and [55].

The general theory of model theoretical stability for Banach space structures (*e.g.*, forking, stability spectrum, etc.) was developed in [39]. See [36, 35, 40].

Our proof of Theorem 15.3 is based on a proof by S. Q Bu [6]. In [6], Bu invokes a principle from descriptive set theory that C. Dellacherie in [13] labelled *the Kunen-Martin Theorem.* Bu proves Theorem 15.3 by showing that there are types of arbitrarily high countable rank. Our argument shows that one need not invoke the Kunen-Martin Theorem if one considers values on all ordinals, rather than countable ones.

For an important application of ordinal ranks in Banach space theory, we refer the reader to [3].

F. Chaatit [9] showed that a Banach space is stable if and only if it can be embedded in the group of isometries of a reflexive Banach space.

It was noticed by Krivine and Maurey that if X is a stable Banach space, then the space of types over X is *strongly separable*, *i.e.*, separable with respect to the topology of uniform convergence on bounded subsets of X (recall that for Banach space theorists stable spaces are by definition separable, and types are real-valued functions; see the notes on Section 5 above). E. Odell proved (see [50] or [57]) that strong separability of the space of types does not imply stability by showing that the space of types over the Tsirel'son space of [15] is strongly separable. Later, in [24], R. Haydon and B. Maurey proved that every space with a strongly separable space of types contains either a reflexive subspace or a copy of ℓ_1. In [38], the author identified topological conditions on the space of types of a Banach space that characterize stability of the space.

We conclude by remarking that the decade of the 1990's has been a time of historical developments in Banach space geometry. A remarkable number of problems that had remained open since Banach's time and were regarded as intractable has been solved. The key lay in a deeper understanding of Tsirel'son's space. A central protagonist in these events has been W. T. Gowers.

Based on a construction of Th. Schlumprecht [62], Gowers and Maurey [22] constructed a *Hereditarily Indecomposable space i.e.*, a Banach space such that no subspace X of it is isomorphic to a sum of two infinite dimensional subspaces of X. The authors proved that a Hereditarily Indecomposable space does not contain an unconditional basic sequence (*i.e.* no sequence (x_n) satisfying $\|\sum \theta_n r_n x_n\| \leq K\|\sum r_n x_n\|$ for some $K > 0$, and all scalars r_n for which $\sum r_n x_n$ converges, and all θ_n with $|\theta_n| = 1$), thus solving the Unconditional Base Problem. The authors also proved that a Hereditarily Indecomposable space cannot be isomorphic to any of its subspaces, and therefore it cannot be isomorphic (let alone isometric) to any of its hyperplanes. This solves Banach's Hyperplane Problem. (Gowers had just presented a solution to the Hyperplane Problem in [17].)

Later, Gowers [18] refined the techniques of [22] to exhibit a space that contains no isomorphic copy of c_0, ℓ_1, or an infinite dimensional reflexive space, answering a long standing question.

More recently [21], Gowers solved negatively the Schroeder-Bernstein Problem by exhibiting two nonisomorphic Banach spaces that are isomorphic to complemented subspaces of each other. The construction is based on the space with no unconditional basic sequence provided in [22].

In [20], using topological games and sophisticated forms of Ramsey's Theorem, Gowers provided the final positive solution to Mazur's Homogeneous Space Problem. A space is homogeneous if it is isomorphic to all of its infinite dimensional subspaces. Gowers shows in [20] that any Banach space either has a subspace with an unconditional basis, or contains a Hereditarily Indecomposable subspace. Hence a homogeneous space must have an unconditional basis, and by a result of R. Komorowski and N. Tomczak-Jaegermann [42], it must be isomorphic to ℓ_2.

A Banach space $(X, \| \ \|)$ is said to be *distortable* if there exist an equivalent norm $| \ |$ on X and a $\epsilon > 0$ such for every infinite dimensional Y of X we have

$$\sup\{ \ |y|/|x| \ \mid \ x, y \in Y, \ \|x\| = \|y\| = 1 \ \} > 1 + \epsilon.$$

The Distortion Problem is whether every Hilbert space is distortable. In [52], E. Odell and Th. Schlumprecht solved affirmatively the Distortion Problem. (The solution had been announced earlier in [51]. See also [53].) Furthermore, the authors proved that any space not containing an isomorphic copy of ℓ_1 or c_0 contains a distortable subspace.

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