CHARACTERIZATION OF HOMOGENEOUS GRADIENT YOUNG MEASURES IN THE CASE OF ARBITRARY INTEGRANDS

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Characterization of homogeneous gradient Young measures in the case of arbitrary integrands*

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Abstract

In the case of a continuous integrand $L : \mathbf{R}^{nm} \to \mathbf{R} \cup \{\infty\}$ and a probability measure ν supported in $\mathbf{R}^{nm}$ we indicate conditions both necessary and sufficient for this measure to be generated as a homogeneous Young measure by gradients of piece-wise affine functions $u_k \in l_A + W_0^{1,\infty}(\Omega)$ with the property $L(Du_k) \rightharpoonup \langle L; \nu\rangle$ in $L^1(\Omega)$. Here A is the center of mass of ν and l_A is a linear function with gradient equal to A everywhere. We show also that in the scalar case $m = 1$ any probability measure with finite action on L has this property. We provide elementary proofs of these results.

1 Introduction

Recent results in the area of Young measure theory, see [Ba], [B1], [KP1], [KP2], [Kr], [P], [S1]-[S3], showed that this theory presents a powerful tool

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for studying classical problems of the Calculus of Variations related to behavior of integral functionals on weakly convergent sequences. In fact, the relaxation theorem was proved under optimal assumptions on integrands satisfying standard growth conditions using this theory in [S2]. Moreover the proof is based on quite different technique comparing with the previous ones proposed for relaxation results, see [AF], [Bu], [D], etc. For other results obtained by the same methods see [S1]-[S3].

The basic idea of this technique is to work directly with Young measures instead of sequences generating them, provided the action of a measure on an integrand is equal to the limit of values assumed by the integral functional at the sequence. In the case of integrands $L = L(Du)$ with p-growth

$$A_1|Du|^p + B_1 \leq L(Du) \leq A_2|Du|^p + B_2,\ p > 1,\ A_2 \geq A_1 > 0$$

the class of Young measures generated by gradients and having the above property was characterized by Kinderlehrer & Pedregal in [KP2]. These measures were named homogeneous gradient p-Young measures.

In order to move analysis further towards the realistic problems in Elasticity (cf. [B2], [B3], [BM], [C,Ch.4]) we have to obtain a characterization of Young measures arising in the same way in the case of arbitrary integrands. Note that even the case of realistic homogeneous isotropic materials demands to deal with integrands $L = L(Du) : \mathbf{R}^{3\times 3} \to \mathbf{R} \cup \{\infty\}$ meeting the requirement

$$L(Du) \to 0 \text{ as } \det Du \to +0.$$

Therefore, the basic assumption on L in this paper will be

(H1)

$$L : \mathbf{R}^{nm} \to \bar{\mathbf{R}} := \mathbf{R} \cup \{+\infty\} \text{ is continuous and } L(v) \to \infty, v \to \infty$$

We adopt the following conventions: for a subset A of $\mathbf{R}^n$ the sets int$A$, reint$A$, co$A$, and extr$A$ are respectively the interior, the relative interior, the convex hull, and the set of extremum points of A. $B(a,\epsilon)$ denotes the ball of radius ϵ centered at the point $a \in \mathbf{R}^n$; l_a is a linear function with gradient equal to a everywhere. Weak and strong convergences of sequences are denoted by $\rightharpoonup$ and $\to$ respectively. We will assume that Ω is a bounded open subset of R^n with meas$(\partial\Omega) = 0$.

We will use notation $\tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ for the set of piece-wise affine functions vanishing at the boundary: $u \in \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ if $u \in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ and there exists at most countable decomposition of Ω in Lipschitz domains such that the restriction of the function u to the closure of each of these domains is an affine function. $C_0(\mathbf{R}^l)$ will denote the class of continuous functions $\Phi : \mathbf{R}^l \to \mathbf{R}$ vanishing at infinity. We use notation $\langle \cdot ; \cdot \rangle$ to denote the action of a measure on a function.

We will use the following

Definition 1.1 *For an integrand L, which satisfies the condition (H1), and a probability measure ν, which has finite action on L and is centered at a point $A \in \mathbf{R}^{nm}$, we call this measure a homogeneous gradient L-Young measure provided there exists a sequence $u_k \in l_A + \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ such that Du_k generates ν as a Young measure:*

$$\Phi(Du_k) \stackrel{*}{\rightharpoonup} \langle \Phi;\nu\rangle \text{ in } L^\infty(\Omega) \text{ for all } \Phi \in C_0(\mathbf{R}^{nm}),$$

and $L(Du_k) \rightharpoonup \langle L;\nu\rangle$ in L_1 as $k \to \infty$.

Remark 1 We do not associate ν with the set Ω since validity of this definition for Ω implies its validity for all bounded open sets, cf. Lemma 2.2.

The first result of this paper is

Theorem 1.2 *Let L satisfy (H1), and let ν be a probability measure, which is supported in $\mathbf{R}^{nm}$ and is centered at $A \in \mathbf{R}^{nm}$, with finite action on L. Then ν is a gradient L-Young measure if and only if for each $\Phi \in C_0(\mathbf{R}^{nm})$ the inequality*

$$\inf_{\psi\in\tilde{C}_0^\infty(\Omega;\mathbf{R}^m)} \frac{1}{\text{meas }\Omega}\int_\Omega \{L(A+D\psi(x))+\Phi(A+D\psi(x))\}dx \le \langle L+\Phi;\nu\rangle \quad (1.1)$$

holds.

Remark 2 As it will follow from the proof, the analogous result holds if in the definition of gradient L-Young measures the class $\tilde{C}_0^\infty(\Omega;\mathbf{R}^n)$ is replaced by the Sobolev class $W_0^{1,p}(\Omega;\mathbf{R}^m)$, $p \in [1,\infty]$. In this case ψ in (1.1) should be taken in the same class.

Remark 3 Note that in (1.1) the function L can be replaced by an equivalent integrand $\tilde{L}$, i.e.

$$C_1 L + B_1 \leq \tilde{L} \leq C_2 L + B_2, C_2 \geq C_1 > 0, B_2, B_1 \in \mathbf{R}.$$

In the case of an integrand L with p-growth this means that ν is a gradient L-Young measure if and only if

$$\inf_{\psi \in C_0^\infty(\Omega;\mathbf{R}^m)} \int_\Omega \{|A + D\psi(x)|^p + \Phi(A + D\psi(x))\} dx \leq \langle |\cdot|^p + \Phi(\cdot); \nu \rangle$$

for each $\Phi \in C_0(\mathbf{R}^{nm})$ (here A is the center of mass of ν).

The original result of [KP2] says that ν is a gradient p-Young measure if and only if for each quasiconvex L with p-growth the inequality $\langle L; \nu \rangle \geq L(A)$ holds. Our result shows that one can avoid checking this inequality with quasiconvex functions. Instead it is enough to verify the inequality for finite perturbations of the original integrand. Moreover, the arguments we introduce here let us prove an analogous result for arbitrary integrands satisfying the condition (H1).

In the case $m = 1$ we can prove that an arbitrary probability measure ν with finite action on L is a gradient L-Young measure.

Theorem 1.3 *Let L satisfy the condition (H1) with $m = 1$ and let L have superlinear growth:*

$$L(v) \geq \theta(v), \text{ where } \theta(v)/|v| \to \infty \text{ as } |v| \to \infty.$$

Let ν be a probability measure supported in $\mathbf{R}^m$ and having finite action on L. Then ν is a gradient L-Young measure.

This fact follows from a possibility to generate any convex combination of Dirac masses by gradients of a sequence of piece-wise affine functions, which is bounded in $W^{1,\infty}(\Omega)$, see Lemma 4.2. In the case $m > 1$ far not each probability measure is a gradient L-Young measure. For different types of nontrivial restrictions these measures have to satisfy see [Sv1], [Sv2].

Note that we do not make an attempt to characterize nonhomogeneous gradient Young measures $(\nu_x)_{x\in\Omega}$ since difficulties associated with this case are more or less equivalent to the problem of approximation of admissible Sobolev functions (admissibility means that the value assumed by the integral functional at the function is finite) by piece-wise affine admissible ones. One of the aims of our work with Young measures is just to avoid dealing with the latter problem - the problem which completely stopped research in the area of realistic problems in Elasticity - through proving the Localization principle (for a.e. $x \in \Omega$ the measure ν_x is a homogeneous gradient L-Young measure) for such problems.

We will prove some auxiliary results about Young measures in §2. In §3 we will prove the main result - Theorem 1.2. Theorem 1.3 will be proved in §4.

2 Some facts from Young measure theory

In this section we recall some facts from Young measure theory which will be involved in the proof of the main result.

Recall that a sequence $\xi_k : \Omega \to \mathbf{R}^l$ *generates a homogeneous Young measure* ν if ν is a probability measure and for each $\Phi \in C_0(\mathbf{R}^l)$ the convergence

$$\Phi(\xi_k) \stackrel{*}{\rightharpoonup} \langle \Phi; \nu \rangle \text{ in } L^\infty(\Omega) \tag{2.1}$$

holds .

Let $\mathrm{Av}(\xi_k)_\Omega$ (k is fixed) be a measure defined as

$$\langle \Phi; \mathrm{Av}(\xi_k)_\Omega \rangle := \frac{1}{\text{meas } \Omega} \int_\Omega \Phi(\xi_k(x))dx, \ \forall \Phi \in C_0(\mathbf{R}^l).$$

It is easy to prove that $\mathrm{Av}(\xi_k)_\Omega$ is a homogeneous Young measure generated by scaled copies of the function ξ_k. Therefore the convergence in (2.1) implies the convergence $\mathrm{Av}(\xi_k)_\Omega \stackrel{*}{\rightharpoonup} \nu$, i.e.

$$\langle \Phi; \mathrm{Av}(\xi_k)_\Omega \rangle \to \langle \Phi; \nu \rangle, \ k \to \infty, \forall \Phi \in C_0(\mathbf{R}^l).$$

In the proof of Theorem 1.2 we will use a similar construction showing that for a special sequence $\phi_k \in l_A + \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$ the convergence $\langle L; \mathrm{Av}(D\phi_k)_\Omega \rangle \to \langle L; \nu \rangle$ holds.

To make the proof complete we will need

1) to show that $\text{Av}(D\phi_k)_\Omega$ are gradient L-Young measures provided $L(D\phi_k) \in L_1(\Omega)$;

2) to show that the convergences $\text{Av}(D\phi_k)_\Omega \stackrel{*}{\rightharpoonup} \nu$, $\langle L; \text{Av}(D\phi_k)_\Omega\rangle \to \langle L; \nu\rangle$ imply that ν is a gradient L-Young measure;

3) to establish a connection of these convergences with the inequality in the statement of Theorem 1.2.

To answer the third question we need

Lemma 2.1 *Let ν_k, $k = 0, 1, \ldots$, be a sequence of probability measures supported in $\mathbf{R}^l$. Then $\nu_k \stackrel{*}{\rightharpoonup} \nu_0$ if and only if $\rho(\nu_k, \nu_0) \to 0$, where*

$$\rho(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i ||\Phi_i||_C} |\langle \Phi_i; \mu\rangle - \langle \Phi_i; \nu\rangle|$$

and $\{\Phi_i\} \subset C_0(\mathbf{R}^l)$ is dense in $C_0(\mathbf{R}^l)$.

Proof is straightforward since the convergence $\nu_k \stackrel{*}{\rightharpoonup} \nu_0$ means convergence $\langle \Phi; \nu_k\rangle \to \langle \Phi; \nu_0\rangle$, $k \to \infty$, for all $\Phi \in C_0(\mathbf{R}^l)$.

Lemma 2.2 *Let $\phi \in l_A + \tilde{C}_0^\infty(\Omega; R^m)$ and let $\tilde{\Omega}$ be an open bounded subset of R^n. Then, there exists a sequence $\phi_k \in l_A + \tilde{C}_0^\infty(\tilde{\Omega}; R^m)$ such that $D\phi_k$ generates $\text{Av}(D\phi)_\Omega$ as a homogeneous Young measure in $\tilde{\Omega}$ and for each $k \in \mathbf{N}$ the function $D\phi_k$ has the same distribution in $\tilde{\Omega}$ as $D\phi$ in Ω, that is $\text{Av}(D\phi_k)_{\tilde{\Omega}} = \text{Av}(D\phi)_\Omega$.*

Moreover, if L satisfy (H1) and $\int_\Omega L(D\phi(x))dx < \infty$ then $D\phi_k$ generates $\text{Av}(D\phi)_\Omega$ as a homogeneous gradient L-Young measure.

In the proof we will use the following standard result: a family F of closed subsets of $\mathbf{R}^n$ is said to be a *Vitaly cover* of a bounded measurable set A if for any $x \in A$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x, \epsilon_k)$ with $\epsilon_k \to 0$, and a sequence $C_k \in F$ such that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and $(\text{meas}\, C_k / \text{meas}\, B(x, \epsilon_k)) > r(x)$ for all $k \in \mathbf{N}$.

The version of Vitaly covering theorem from [Sa,p.109] says that each Vitaly cover of A contains at most countable subfamily of disjoint sets C_k such that $\text{meas}\,(A \setminus \cup_k C_k) = 0$.

Proof of Lemma 2.2

Let $\tilde{\Omega}$ be a bounded open subset of R^n.

By the above version of Vitaly covering theorem for each $k \in \mathbf{N}$ we can find decomposition of $\tilde{\Omega}$ in sets $\Omega_i^k := x_i^k + \epsilon_i^k \bar{\Omega} \subset \tilde{\Omega}$, where $\epsilon_i^k \leq 1/k$ for all $i \in \mathbf{N}$, and a set N_k of zero measure. We can assume also that for $k' \geq k$, $i, i' \in \mathbf{N}$ either $\Omega_{i'}^{k'} \subset \Omega_i^k$ or $\Omega_{i'}^{k'} \cap \Omega_i^k = \emptyset$.

Define ϕ_k as follows:

$$\phi_k(x) = \epsilon_i^k \phi(\frac{x - x_i^k}{\epsilon_i^k}) \text{ if } x \in \Omega_i^k,\ i \in \mathbf{N};\ \phi_k(x) = l_A(x) - \text{ otherwise.}$$

Then $\phi_k \in l_A + \tilde{C}_0^\infty(\tilde{\Omega}; \mathbf{R}^m)$. Notice that for each Ω_i^j and all $k \geq j$ the identity

$$\int_{\Omega_i^j} \Phi(D\phi_k)dx = \langle \Phi; \mathrm{Av}(D\phi)_\Omega \rangle \text{ meas } \Omega_i^j \tag{2.2}$$

holds. To show that for each $\Phi \in C_0(\mathbf{R}^{nm})$

$$\Phi(D\phi_k) \stackrel{*}{\rightharpoonup} \langle \Phi; \mathrm{Av}(D\phi)_\Omega \rangle \text{ in } L^\infty(\tilde{\Omega})$$

we have to establish convergence

$$\frac{1}{\text{meas } K} \int_K \Phi(D\phi_k(x))dx \to \langle \Phi; \mathrm{Av}(D\phi)_\Omega \rangle$$

for all compact subsets K of $\tilde{\Omega}$. In fact, we can replace K by $K' := K \cap \Omega'$, where $\Omega' := \cap_{j=1}^\infty \cup_{i=1}^\infty (\mathrm{int}\Omega_i^j)$, since Ω' has full measure.

Note that the sets Ω_j^i, $i, j \in \mathbf{N}$, form a Vitaly cover of K'. Hence for each $\epsilon > 0$ there exists a sequence of disjoint closed sets $\bar{\Omega}_{j(l)}^{i(l)}$, $l \in \mathbf{N}$, such that

$$\text{meas}\,\{(K \setminus \cup_l \Omega_{j(l)}^{i(l)}) \cup (\cup_l \Omega_{j(l)}^{i(l)} \setminus K)\} \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary and (2.2) holds for each pair $i, j \in \mathbf{N}$ with $k \in \mathbf{N}$ sufficiently large, we infer

$$\Phi(D\phi_k\big|_K) \to \langle \Phi; \mathrm{Av}(D\phi)_\Omega \rangle \text{ meas } K,\ \forall \Phi \in C_0(\mathbf{R}^{nm}).$$

Since K is an arbitrary compact subset of $\tilde{\Omega}$ we obtain

$$\Phi(D\phi_k) \stackrel{*}{\rightharpoonup} \langle \Phi; \mathrm{Av}(D\phi)_\Omega \rangle \text{ in } L^\infty(\tilde{\Omega}), \forall \Phi \in C_0(\mathbf{R}^{nm}). \tag{2.3}$$

By construction $D\phi_k$, $k \in \mathbf{N}$, has the same distribution in $\tilde{\Omega}$ as $D\phi$ in Ω, therefore $\mathrm{Av}(D\phi_k)_{\tilde{\Omega}} = \mathrm{Av}(D\phi)_\Omega$. This proves the first assertion of the lemma.

To prove the second one notice that the convergence $L(D\phi_k) \rightharpoonup \langle L; \mathrm{Av}(D\phi)_\Omega\rangle$ in $L^1(\tilde{\Omega})$ can be proved by the same arguments as (2.3) since the sequence $L(D\phi_k)$ is equi-integrable - it is easy to see that it has modulus of equi-integrability of the function $|L(D\phi)|$ multiplied by the factor (meas $\tilde{\Omega}/$ meas Ω).

QED

Lemma 2.3 *Let $\nu_k, k \in \mathbf{N}$, be a sequence of homogeneous Young measures generated by gradients of functions $\phi_i^k \in l_A + \tilde{C}_0^\infty(\Omega; R^m)$ respectively, and let $\nu_k \overset{*}{\rightharpoonup} \nu$.*

Then ν is generated as a homogeneous Young measure by gradients of a sequence $\phi_{i(k)}^k$, $k \in \mathbf{N}$. Moreover, if ν_k are gradient L-Young measures then ν is a gradient L-Young measure provided $\langle L; \nu_k\rangle \to \langle L; \nu\rangle$.

Proof

For each fixed k there exists a sequence $\phi_i^k \in \tilde{C}_0^\infty(\Omega; \mathbf{R}^m)$, $i \in \mathbf{N}$, such that ϕ_i^k generate ν_k as a homogeneous L-Young measure:

$$\Phi(D\phi_i^k) \overset{*}{\rightharpoonup} \langle \Phi; \nu_k\rangle \text{ in } L^\infty(\Omega), i \to \infty, \forall \Phi \in C_0(\mathbf{R}^{nm}).$$

Since $C_0(\mathbf{R}^{nm})$ is separable by standard diagonalization arguments we can find a sequence $\phi_{i(k)}^k, k \in \mathbf{N}$, with the properties

$$\Phi(D\phi_{i(k)}^k) \overset{*}{\rightharpoonup} \langle \Phi; \nu\rangle \text{ in } L^\infty(\Omega),\ k \to \infty, \forall \Phi \in C_0(\mathbf{R}^{nm}).$$

This means that $D\phi_{i(k)}^k$ generate ν as a homogeneous Young measure.

In the case when ν_k are gradient L-Young measures we have also

$$L(D\phi_i^k) \rightharpoonup \langle L; \nu_k\rangle \text{ in } L^1 \text{ as } i \to \infty,\ k \in \mathbf{N}.$$

Because of the convergence $\langle L; \nu_k\rangle \to \langle L; \nu\rangle$, $k \to \infty$, the sequence $\phi_{i(k)}^k$ can be also chosen in such a way that

$$L(D\phi_{i(k)}^k) \rightharpoonup \langle L; \nu\rangle \text{ in } L^1(\Omega),\ k \to \infty.$$

Therefore ν is a gradient L-Young measure. **QED**

3 Proof of the main result

Proof of Theorem 1.2

If ν is a gradient L-Young measure centered at A then there exists a sequence $\phi_k \in l_A + \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)$ with the properties $L(D\phi_k) \rightharpoonup \langle L;\nu\rangle$ in L_1, $\Phi(D\phi_k) \overset{*}{\rightharpoonup} \langle \Phi;\nu\rangle$ in $L^\infty(\Omega)$ for all $\Phi \in C_0(\mathbf{R}^{nm})$. In this case

$$\int_\Omega \{L(D\phi_k) + \Phi(D\phi_k)\}dx \to \langle L+\Phi;\nu\rangle \operatorname{meas}\Omega, \forall \Phi \in C_0(R^{nm}),$$

that implies validity of (1.1).

To prove the converse we will first show that the set $G := \{\mathrm{Av}(D\phi)_\Omega : \phi \in l_A + \tilde{C}_0^\infty(\Omega;\mathbf{R}^m)\}$ is convex. Let $\nu^1 := \mathrm{Av}(D\phi_1)_\Omega$, $\nu^2 := \mathrm{Av}(D\phi_2)_\Omega$, $\lambda \in]0,1[$. Let Ω_1, Ω_2 be disjoint open subsets of Ω such that $\operatorname{meas}(\partial\Omega_1) = \operatorname{meas}(\partial\Omega_2) = 0$ and $\operatorname{meas}\Omega_1 = \lambda \operatorname{meas}\Omega$, $\operatorname{meas}\Omega_2 = (1-\lambda)\operatorname{meas}\Omega$. By Lemma 2.2 there exist functions $u^1 \in l_A + \tilde{C}_0^\infty(\Omega_1;\mathbf{R}^m)$, $u^2 \in l_A + \tilde{C}_0^\infty(\Omega_2;\mathbf{R}^m)$ such that $\mathrm{Av}(Du^1)_{\Omega_1} = \nu^1$, $\mathrm{Av}(Du^2)_{\Omega_2} = \nu^2$.

Let $u = u^1$ in Ω_1, $u = u^2$ in Ω_2. Then $\mathrm{Av}(Du)_\Omega \in G$ and $\mathrm{Av}(Du)_\Omega = \lambda\nu^1 + (1-\lambda)\nu^2$. This proves convexity of G.

The theorem will be proved if we will show that ν belongs to the closure of the set G in the following sense:

$$\inf_{\mu\in G}\{\rho(\mu,\nu) + |\langle L;\mu\rangle - \langle L;\nu\rangle|\} = 0. \tag{3.1}$$

In fact (3.1) implies existence of a sequence $\nu_k \in G$ such that $\rho(\nu_k,\nu) + |\langle L;\nu_k\rangle - \langle L;\nu\rangle| \to 0$, $k \to \infty$. Convergence of the first term to zero means that $\nu_k \overset{*}{\rightharpoonup} \nu$. Then, by Lemmata 2.2, 2.3 convergence of the second term to zero implies that ν is a gradient L-Young measure.

We will prove (3.1) by contradiction. Recall that

$$\rho(\mu,\nu) := \sum_{i=1}^\infty \frac{1}{2^i||\Phi_i||_C}\Big|\langle\Phi_i;\mu\rangle - \langle\Phi_i;\nu\rangle\Big|,$$

where the sequence $\{\Phi_i\}$ is dense in $C_0(\mathbf{R}^{nm})$.

If (3.1) does not hold, then for a sufficiently large l and an $\epsilon > 0$ we have

$$\inf_{\mu\in G}\{|\langle L;\mu\rangle - \langle L;\nu\rangle| + \sum_{i=1}^l \frac{1}{2^i||\Phi_i||_C}\Big|\langle\Phi_i;\mu\rangle - \langle\Phi_i;\nu\rangle\Big|\} > \epsilon.$$

Then, the subset of $\mathbf{R}^{l+1}$ given by the vectors

$$(\langle L;\mu\rangle, \frac{1}{2||\Phi_1||}\langle\Phi_1;\mu\rangle, \ldots, \frac{1}{2^l||\Phi_l||}\langle\Phi_l,\mu\rangle), \mu \in G,$$

is convex since G is convex, and the vector generated by ν does not belong to its closure. Hence, there exists a vector $c \in \mathbf{R}^{l+1}$ such that

$$\inf_{\mu\in G}\{c_0\langle L;\mu\rangle + \sum_{i=1}^{l} c_i\langle\Phi_i;\mu\rangle\} > c_0\langle L;\nu\rangle + \sum_{i=1}^{l} c_i\langle\Phi_i;\nu\rangle + \delta, \delta > 0.$$

Then

$$\inf_{\mu\in G}\langle\tilde{L};\mu\rangle > \langle\tilde{L};\nu\rangle + \delta, \text{ with } \tilde{L} = c_0 L + \sum_{i=1}^{l} c_i\Phi_i. \tag{3.2}$$

Note that the coefficient c_0 cannot be negative - otherwise the value at the left-hand side is $-\infty$. In the case $c_0 = 0$ we can replace $\tilde{L}$ by $\tilde{L}+\eta L$ and (3.2) still holds for $\eta > 0$ sufficiently small. Note that the integrand $\{\tilde{L} + \eta L\}/c_0$ is of the type $L+\Phi$, $\Phi \in C_0(\mathbf{R}^{nm})$, and due to (3.2) the inequality (1.1) fails for this integrand.

The above contradiction proves that (3.1) holds and that ν is a gradient L-Young measure. This completes the proof of the theorem. **QED**

4 Proof of Theorem 1.3

To prove Theorem 1.3 we will need the following two lemmata.

Lemma 4.1 *Assume that* $A \in \text{intco}\{v_1, \ldots, v_q\}$. *Then there is a function* $u_0 \in l_A + \tilde{C}_0^\infty(\Omega)$ *such that* $Du_0 \in \{v_1, \ldots, v_q\}$ *a.e. in* Ω.

Proof

Without loss of generality we can assume that $v_1, \ldots, v_q$ are extremum points of a compact convex subset of R^n.

To construct u_0 with desired properties consider the function

$$w_s(x) = \max_{1\le i\le q}\langle v_i - A, x\rangle - s, s > 0.$$

It easy to see that w_s is Lipschitz, $Dw_s(x) \in \{v_i - A : i = 1, \ldots, q\}$ a.e., and $w_s|_{\partial P_s} = 0$, where

$$P_s := \{x : \max_{1\le i\le q}\langle v_i - A, x\rangle \le s\}$$

is a compact set with Lipschitz boundary and nonempty interior.

Note that $P_s = sP_1$.

Since Vitaly covering arguments let us decompose Ω in disjoint sets of the form $y_i + s_iP_1$, $i \in \mathbf{N}$, and a set of zero measure, we can define u_0 as

$$l_A(x) + w_{s_i}(x - y_i) \text{ for } x \in y_i + s_iP_1, i \in \mathbf{N}.$$

Then $u_0 \in l_A + \tilde{C}_0^\infty(\Omega)$ and $Du_0 \in \{v_1, \dots, v_q\}$ a.e. in Ω.

QED

Lemma 4.2 *Let $v_i \in \mathbf{R}^n$, $c_i \geq 0$, $i = 1, \dots, q$, be such that $\sum_i c_i = 1$, $\sum_i c_i v_i = A$. Then for each $\epsilon > 0$ there exists a sequence of piece-wise affine functions $\phi_k \in l_A + \tilde{C}_0^\infty(\Omega)$ such that*

1) $D\phi_k(x) \in \{v_i : i = 1, \dots, q\} \cup B(A, \epsilon)$ for a.a. $x \in \Omega$;

2) $D\phi_k$ generates the measure $\sum_{i=1}^q c_i \delta_{v_i}$.

Proof

Without loss of generality we can assume that $c_i > 0$ for all i.

1.

Consider first the case when $A \notin \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$, but A has unique representation in the form of a convex combination of $v_1, \dots, v_q$. Then we have $A \in \operatorname{reint} \operatorname{co}\{v_1, \dots, v_q\}$. Let P be the largest subspace of $\mathbf{R}^n$ perpendicular to all vectors $v_i - A$, $i \in \{1, \dots, q\}$. Assume that $\dim P = m$ and $v_{q+1}, \dots, v_{q+m+1}$ are such vectors in P that $\operatorname{co}\{v_{q+1}, \dots, v_{q+m+1}\}$ has nonempty interior in P and 0 belongs to this interior.

For each $s > 0$, $\delta > 0$ consider the function

$$w_{s,\delta}(\cdot) = \max_{1 \leq i \leq q+m+1} \langle \tilde{v}_i - A, \cdot \rangle - s,$$

where

$$\tilde{v}_i = v_i \text{ for } i \in \{1, \dots, q\},$$

$$\tilde{v}_i = A + \delta v_i \text{ for } i \in \{q+1, \dots, q+m+1\}.$$

It is clear that for each $\delta > 0$ the inclusion $A \in \operatorname{int} \operatorname{co}\{\tilde{v}_1, \dots, \tilde{v}_{q+m+1}\}$ holds and $\tilde{v}_1, \dots, \tilde{v}_{q+m+1}$ are extremum points of a compact convex set.

Moreover, if $P_s := \{x : \max_{1\le i\le q+m+1}\langle v_i - A, x\rangle \le s\}$ then

$$\frac{\text{meas}\{x \in P_s : Dw_{s,\delta} \notin \{v_1 - A, \ldots, v_q - A\}\}}{\text{meas}\{x \in P_s : Dw_{s,\delta} \in \{v_1 - A, \ldots, v_q - A\}\}} \to 0$$

as $\delta \to 0$ uniformly with respect to s.

For $k \in \mathbf{N}$ consider a Vitaly covering of Ω by the supports $\Omega_i := x_i + P_{s_i}$ of the functions $\min\{0, w_{s_i,1/k}(\cdot - x_i)\}$ with $s_i \le 1/k$ and define $\phi_k(\cdot) = l_A(\cdot) + w_{s_i,1/k}(\cdot - x_i)$ in Ω_i $(i \in \mathbf{N})$, $\phi_k = l_A$ -otherwise. In this case

$$\text{meas}\{x \in \Omega : D\phi_k \notin \{v_1, \ldots, v_q\}\} \to 0, k \to \infty.$$

Therefore, if for a subsequence of ϕ_k (not relabeled)

$$c_i^k := \frac{\text{meas}\{x \in \Omega : D\phi_k = v_i\}}{\text{meas}\,\Omega} \to \tilde{c}_i, k \to \infty, i \in \{1, \ldots, q\},$$

then $\sum \tilde{c}_i = 1$, $\sum \tilde{c}_i v_i = A$, and because of uniqueness of the representation of A in the form of a convex combination of v_i $(i = 1, \ldots, q)$ we infer that $\tilde{c}_i = c_i$. Hence, $c_i^k \to c_i$ as $k \to \infty$ for the original sequences c_i^k, $i \in \{1, \ldots, q\}$.

Therefore the sequence $D\phi_k$ generates the measure $\sum_{i=1}^q c_i \delta_{v_i}$. It is also obvious that ϕ_k satisfies the requirement 1) for all sufficiently large $k \in \mathbf{N}$.

2.

The general case will be reduced to the one discussed above. We can assume without loss of generality that $A = 0$ and $v_i \ne A$, $c_i > 0$ for all $i \in \{1, \ldots, q\}$.

For $q = 2$ the result follows from the argument above since in this case 0 has unique representation in the form of a convex combination of v_1 and v_2.

Suppose the claim of the lemma is valid for $q = s$. We want to prove it for the case $q = s + 1$. If 0 has unique representation in the form of a convex combination of vectors $v_1, \ldots, v_{s+1}$, then the claim was already proved above. Otherwise we can find v_i and $\tilde{c}_j \ge 0$, $j \in \{1, \ldots, i-1, i+1, \ldots, s+1\}$, such that

$$v_i = \sum_{j\ne i, 1\le j\le s+1} \tilde{c}_j v_j, \quad \sum_{j\ne i, 1\le j\le s+1} \tilde{c}_j = 1.$$

In this case

$$0 = \sum_{j\ne i} c_j v_j + c_i \sum_{j\ne i} \tilde{c}_j v_j = \sum_{j\ne i} (c_j + c_i\tilde{c}_j) v_j = \sum_{j\ne i} \mu_j v_j,$$

where $\mu_j \geq c_j$, $j \in \{1, \ldots, i-1, i+1, \ldots, s+1\}$, and $\sum_{j \neq i} \mu_j = 1$.

We can find $\epsilon \in]0,1[$ such that $\epsilon \mu_j \leq c_j$ for all $j \neq i$ and at least for one $j_0 \neq i$ the identity $\epsilon \mu_{j_0} = c_{j_0}$ holds.

Then

$$0 = \epsilon \sum_{j \neq i} \mu_j v_j, \; 0 = \sum_{j \neq i} (c_j - \epsilon \mu_j) v_j + c_i v_i. \tag{4.1}$$

We can decompose Ω in two disjoint subsets Ω_1 and Ω_2 in such a way that

$$\text{meas}\, \Omega_1 = \epsilon\, \text{meas}\, \Omega, \text{meas}\, \Omega_2 = (1-\epsilon)\, \text{meas}\, \Omega.$$

Since each of the combinations in (4.1) contains at most s terms, by the induction assumption and Lemma 2.2 we can find sequences $u_k^1 \in l_A + \tilde{C}_0^\infty(\Omega_1)$, $u_k^2 \in l_A + \tilde{C}_0^\infty(\Omega_2)$ such that Du_k^1 generates the homogeneous Young measure $\nu_1 = \sum_{j \neq i} \mu_j \delta_{v_j}$, Du_k^2 generates the homogeneous Young measure $\nu_2 = \sum_{j \neq i} \frac{c_j - \epsilon \mu_j}{1-\epsilon} \delta_{v_j} + \frac{c_i}{1-\epsilon} \delta_{v_i}$, and

$$Du_k^1(x) \in \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s+1}\} \cup B(0,\epsilon) \text{ a.e. in } \Omega_1,$$

$$Du_k^2(x) \in \{v_1, \ldots, v_{j_0-1}, v_{j_0+1}, \ldots, v_{s+1}\} \cup B(0,\epsilon) \text{ a.e. in } \Omega_2.$$

Defining $u_k = u_k^1$ in Ω_1, $u_k = u_k^2$ in Ω_2 we obtain that $u_k \in l_A + \tilde{C}_0^\infty(\Omega)$, $Du_k(x) \in \{v_1, \ldots, v_{s+1}\} \cup B(0,\epsilon)$. Then, the claim of the lemma follows from Lemmata 2.2, 2.3 since $\text{Av}(Du_k) \stackrel{*}{\rightharpoonup} \nu$, where $\nu = \epsilon \nu_1 + (1-\epsilon)\nu_2 = \sum_{j=1}^{s+1} c_j \delta_{v_j}$.

This completes the proof of the lemma. **QED**

Proof of Theorem 1.3

Let ν be a probability measure with finite action on L. Let A be the center of mass of ν. Without loss of generality we can assume that $A = 0$.

We will construct convex combinations of Dirac masses ν_M ($M \in \mathrm{N}$) centered at 0 with the properties:

$$\nu_M = \sum c_i^M \delta_{v_i^M}; \langle L; \nu_M \rangle \to \langle L; \nu \rangle, \nu_M \stackrel{*}{\rightharpoonup} \nu, \text{ as } M \to \infty. \tag{4.2}$$

Before proving (4.2) we first note that it implies the theorem. Due to Lemma 2.3 to prove this claim it is enough to show that each ν_M is a gradient L-Young measure.

Fix $M \in \mathrm{N}$. Since $\langle L; \nu_M \rangle < \infty$ we infer that $v_i^M \in \{v \in R^n : L(v) < \infty\}$, $i \in \{1, \ldots, q\}$. Then we can find $\epsilon > 0$ and $\tilde{v}_i$, $i \in \{1, \ldots, p\}$, such that $L(\tilde{v}_i) < \infty$ for each $i \in \{1, \ldots, p\}$ and $B(0,\epsilon) \subset \text{intco}\{\tilde{v}_1, \ldots, \tilde{v}_p\}$.

By Lemma 4.2 we can find a sequence of functions $u_k \in l_0 + \tilde{C}_0^\infty(\Omega)$ such that $Du_k \in \{v_1^M, \dots, v_q^M\} \cup B(0,\epsilon)$ and Du_k generate Young measure $\sum c_i^M \delta_{v_i^M}$. Due to Lemma 4.1 in each subset of Ω, where Du_k is affine and $Du_k \not\in \{v_1^M, \dots, v_q^M\}$, we can modify u_k in such a way that the gradient of the modified function $\tilde{u}_k$ takes values in $\{\tilde{v}_1, \dots, \tilde{v}_p\}$ a.e. in the subset. Then $D\tilde{u}_k$ generates Young measure $\nu_M := \sum c_i^M \delta_{v_i^M}$ (since $\text{meas}\{x \in \Omega : D\tilde{u}_k(x) \neq Du_k(x)\} \to 0, k \to \infty$) and $L(D\tilde{u}_k) \rightharpoonup \langle L; \sum c_i^M \delta_{v_i^M} \rangle$ in L_1. Therefore, ν_M is a gradient L-Young measure.

To complete the proof of the theorem we have to construct measures ν_M satisfying (4.2). There exists a point $A' \in R^n$ and $\epsilon > 0$ such that $|L| < M_1$ in $B(A', 2\epsilon)$ and $\nu(B(A', \epsilon)) = c_0 > 0$. For each integer $M \geq M_1$ consider the set

$$U_M = \{v \in R^n : L(v) \leq M\}.$$

We can decompose U_M in sets U_M^i, $i = 1, \dots, l(M)$, $U_M^{i'}$, $i' = 1, \dots, l'(M)$, of diameters less than $1/M$ in such a way that oscillation of L in each of these sets does not exceed $1/M$, and $U_M^{i'} \subset B(A', \epsilon)$, $i' = 1, \dots, l'(M)$, $U_M^i \subset U_M \setminus B(A', \epsilon)$, $i = 1, \dots, l(M)$.

Let $c_{M,i} = \nu(U_M^i)$, $c_{M,i'} = \nu(U_M^{i'})$ and $c_M = \nu(R^n \setminus U_M)$. Note that

$$c_M + \sum_{i'=1}^{l'} c_{M,i'} + \sum_{i=1}^{l} c_{M,i} = 1.$$

Let $A_M^i \in U_M^i$, $i = 1, \dots, l(M)$, $A_M^{i'} \in U_M^{i'}$, $i' = 1, \dots, l'(M)$. Consider the probability measure

$$\mu_M := \sum_{i'} \frac{c_{M,i'}}{1 - c_M} \delta_{A_M^{i'}} + \sum_i \frac{c_{M,i}}{1 - c_M} \delta_{A_M^i}.$$

Let z_M be the center of mass of μ_M. It easy to check that because of superlinear growth of L the convergence $z_M \to 0$, $M \to \infty$, holds. Then the measure

$$\nu_M := \sum_{i'} \frac{c_{M,i'}}{1 - c_M} \delta_{(A_M^{i'} - z_M \frac{1-c_M}{c_0})} + \sum_i \frac{c_{M,i}}{1 - c_M} \delta_{A_M^i}.$$

is centered at 0. Moreover, by construction we have

$$\nu_M \stackrel{*}{\rightharpoonup} \nu, \langle L; \nu_M \rangle \to \langle L; \nu \rangle \text{ as } M \to \infty.$$

This way we establish validity of (4.2) and complete the proof. **QED**

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