A PRIMER OF SIMPLE THEORIES

by

Rami Grossberg

José Iovino

and

Olivier Lessmann

Department of Mathematical Sciences

Carnegie Mellon University

Pittsburgh, PA 15213

Research Report No. 98-210
February, 1998

A PRIMER OF SIMPLE THEORIES

RAMI GROSSBERG, JOSÉ IOVINO, AND OLIVIER LESSMANN

ABSTRACT. We present a self-contained exposition of the basic aspects of simple theories while developing the fundamentals of forking calculus. We expound most of the deeper aspects of S. Shelah's 1980 paper *Simple unstable theories*. The concept of weak dividing has been replaced with that of forking. The exposition is from a contemporary perspective and takes into account contributions due to E. Hrushovski, B. Kim, and A. Pillay.

INTRODUCTION

The question of how many models a complete theory can have has been at the heart of some of the most fundamental developments in the history of model theory. The most basic question that one may ask in this direction is whether a given first order theory has only one model up to isomorphism. Erwin Engeler, Cesław Ryll-Nardzewski, and Lars Svenonius (all three independently) published in 1959 a complete characterization of the countable theories that have a unique countable model (see Theorem 2.3.13 of [CK]). The next landmark development occurred in 1962, when Michael Morley proved in his Ph.D. thesis that if a countable theory has a unique model in some uncountable cardinality, then it has a unique model in every uncountable cardinality. (See [Mo].) This answered positively a question of Jerzy Łős [Lo] for countable theories. The conjecture of Łős in full generality (including uncountable theories) was proved by Saharon Shelah in 1970. (See [Sh31].)

The problem of counting the number of uncountable models of a first order theory led Shelah to develop an awesome body of mathematics which he called *classification theory*. A fundamental distinction that emerges in this context is that between two classes of theories: *stable* and *unstable* theories. For a cardinal λ, a theory T is called *stable in* λ if whenever M is a model of T of cardinality λ, the number of complete types over M is also λ. A theory is called *stable* if it is stable in some cardinal.

The *stability spectrum* of T is the class of cardinals λ such that T is stable in λ. In his ground-breaking paper [Sh3], Shelah gave the first description to the stability spectrum of T. He characterized the class of cardinalities $\lambda \geq 2^{|T|}$ such that T is stable in λ. For the combinatorial analysis of models involved, he devised an intricate tool which he called *strong splitting*. Later, in order to describe the full stability spectrum (*i.e.*, include the cardinals $\lambda < 2^{|T|}$ such that T is stable in λ), he refined the concept of strong splitting, and developed the fundamental notion of *forking*.

Between the early 1970's and 1978, Shelah concentrated his efforts in model theory to the completion of his monumental treatise [Sha]. The complete description of the stability spectrum of T is given in Section III-5. Shelah, however, realized quickly that the range of applicability of the concept of forking extends well beyond the realm of the spectrum problem.

Intuitively, if p is a type over a set A, an extension $q \supseteq p$ is called *nonforking* if q imposes no more dependency relations between its realizations and the elements of A than

Date: January 30, 1998.

those already present in p. This yields a general concept of *independence* in model theory, of which the concepts of linear independence in linear algebra and algebraic independence in field theory are particular examples.

Shelah's original presentation of the basics of forking appeared to be complicated and required time for the reader to digest. This fact, combined with the rather unique exposition style of the author, made [Sha] difficult to read, even by experts.

In 1977, Daniel Lascar and Bruno Poizat published [LaPo] an alternative approach to forking which appeared more understandable than Shelah's. They replaced the original "combinatorial" definition with one closely related to Shelah's notion of *semidefinability* in Chapter VII of [Sha]. The approach of Lascar and Poizat had a remarkable impact on the dissemination of the concept of forking in the logical community. Several influential publications, such as the books of Anand Pillay [Pi] and Daniel Lascar [La3] and the papers of Victor Harnik and Leo Harrington [HH] and Michael Makkai [Ma], adopted the French approach and avoided Shelah's definition of forking. Both of these approaches were presented in John Baldwin's book [Ba].

Parallel to these events, Shelah isolated an important class of theories which extends that of stable theories, the class of *simple theories*. This concept originated in the study of yet another combinatorial property of theories, namely,

> $(\lambda, \kappa) \in \mathrm{SP}(T)$: Every model of T of cardinality λ has a κ-saturated extension of cardinality λ.

For stable T, the class of pairs λ, κ such that $(\lambda, \kappa) \in \mathrm{SP}(T)$ had been completely identified in Chapter VIII-4 of [Sha] (using the stability spectrum theorem and some combinatorial set theory).

Shelah wondered whether there is a natural class of theories extending the class of stable theories where a characterization of the class of pairs λ, κ such that $(\lambda, \kappa) \in \mathrm{SP}(T)$ holds is still possible. In order to prove a consistency result in this direction, he introduced in [Sh93] the class of *simple theories*, and showed that a large part of the apparatus of forking from stability theory could be developed in this weaker framework.

Some of the complexity of the paper is due to the fact that he did not realize that, for simple theories, the notion of forking is equivalent to the simpler notion of *dividing*. (He was aware in 1979, however, that these two concepts are equivalent when the underlying theory is stable.)

It should be remarked that Shelah's main goal in [Sh93] was not to extend the apparatus of forking from stable to simple theories, but rather to prove the aforementioned consistency result (Theorem 4.10 in [Sh93]). In fact, after the proof of the theorem, he adds

> This theorem shows in some sense the distinction between simple and not simple theories is significant.

In the early 1990's, Ehud Hrushovski noticed that the fact that the first order theory of an ultraproduct of finite fields is simple (and unstable) has far reaching consequences. (See [Hr].) Hrushovski's spectacular applications to Diophantine geometry, as well as his collaboration with Anand Pillay [HP1], [HP2] and Zoe Chatzidakis [CH] attracted much attention to the general theory of simple theories.

Anand Pillay consequently prompted his Ph.D. student Byunghan Kim to study in the general context of simple theories a property that he and Hrushovski (see [HP1]) isolated and called the *Independence Property*.

In a clever two-page argument, Kim proved that, for simple theories, the notions of forking and dividing are equivalent. See [Ki]. From this equivalence, Kim and Pillay derived easily the properties of Symmetry and Transitivity of forking for simple theories.

For this work, Kim shared with Itay Neeman the 1997 Sacks Prize, which honors the best dissertation in logic of the year.

The purpose of this paper is to present a self contained introduction to the basic properties of simple theories and forking, including the recent elegant contributions due to Kim, some results of [KP1], as well as two recent unpublished simplifications due to Shelah. The presentation should be accessible to a reader who has had a basic course in model theory, for example, little more than the first three chapters of [CK] will suffice. We also assume that the reader is accustomed to using the concept of *monster model*.

The first author would like to seize this opportunity to announce that much more is included in [Gr].

At the end of the paper we have included historical notes.

The notation is standard. Throughout the paper, T denotes a complete first order theory. without finite models. The language of T is denoted $L(T)$. The monster model is denoted by $\mathfrak{C}$. If A is a set and $\bar{a}$ is a (finite) tuple, $A\bar{a}$ denotes the union of A with the terms of $\bar{a}$.

The paper is organized as follows:

- **Section 1:** We introduce dividing, forking and Morley sequences, and present the main properties that hold when there is no assumption on the underlying theory: Finite Character, Extension, Invariance, and Monotonicity.
- **Section 2:** We introduce the main rank and define simple theories. We also present several equivalent characterizations of simple theories. Among other things, it is shown that the rank is bounded if and only if it is finite. We also study Shelah's original rank, which includes a fourth parameter.
- **Section 3:** We continue the treatment of forking, but now under the assumption that the underlying theory is simple. We prove the property of Existence and the characterization that he theory T is simple if and only if $\kappa(T) \leq |T|$. We then prove Symmetry, Transitivity, and the Independence Theorem.
- **Section 4:** We introduce a distance between sequences of indiscernibles which yields a notion of equivalence among them. The main result is that in any theory, a Morley sequence over a set A has a bounded number of nonequivalent A-automorphic images.
- **Section 5:** We show that for simple theories, it is consistent to have a "nice" description of the class SP(T), namely, There is a model of set theory where there are cardinalities $\lambda > \kappa$ such that $\lambda^{<\kappa} > \lambda$ and $\lambda^{|T|} = \lambda$. (Hence, it is not possible to use cardinal arithmetic to show that $(\lambda, \kappa) \in$ SP(T).) It is shown that for some λ and κ as above. $(\lambda, \kappa) \in$ SP(T). The model theoretic content of this section is the fact that the set of nonforking extensions of a type (up to logical equivalence) forms a semi-upper lattice, satisfying the $(2^{|T|})^+$-chain condition. This partial order is embedded into a natural complete boolean algebra. We then use a set-theoretic property of boolean algebras, namely, satisfying a chain condition, to construct κ-saturated extensions of cardinality λ.
- **Appendix A:** We present an improvement of Theorem 1.11. The theorem is a is a revision of a Theorem of Morley within the more modern setting of Hanf numbers (following Barwise and Kunen). The result has been included here because we could not find the precise statement needed in the literature.
- **Appendix B:** Here we have included several historical remarks, as well as a list of credits.

In the last week of 1997 we sent a final draft of this paper to John Baldwin and Saharon Shelah. We are grateful for several comments we received and incorporated in the text.

1. Forking

In this section we make no particular assumptions on the theory T (besides completeness).

Recall that a sequence I in $\mathfrak{C}$ is indiscernible over a set A if any two finite subsequences of I of the same length have the same type over A. We always assume that the index set does not have a last element.

For $k < \omega$, we will say that a set of formulas $q(\bar{x})$ is *k-contradictory* if every subset of q of k elements is inconsistent. Note that if $\langle \bar{a}_i \mid i < \omega \rangle$ is an indiscernible sequence and the set $\{ \varphi(\bar{x}, \bar{a}_i) \mid i < \omega \}$ is inconsistent, then it is k-contradictory for some $k < \omega$.

We begin by introducing the fundamental notion of *dividing*.

Definition 1.1.

(1) A formula $\varphi(\bar{x}, \bar{b})$ *divides over* A if there exist $\{ \bar{b}_i \mid i < \omega \}$ and $k < \omega$ such that
   (a) $\text{tp}(\bar{b}_i/A) = \text{tp}(\bar{b}/A)$ for every $i < \omega$;
   (b) The set $\{ \varphi(\bar{x}, \bar{b}_i) \mid i < \omega \}$ is k-contradictory.

(2) A type p *divides over* A if there exists a formula $\varphi(\bar{x}, \bar{b})$ such that $p \vdash \varphi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{b})$ divides over A.

Lemma 1.2. *A formula $\varphi(\bar{x}, \bar{b})$ divides over A if and only if there exist $k < \omega$ and an sequence $\langle \bar{b}_i \mid i < \omega \rangle$ indiscernible over A such that $\bar{b}_0 = \bar{b}$ and $\{ \varphi(\bar{x}, \bar{b}_i) \mid i < \omega \}$ is k-contradictory.*

Proof. Necessity is clear. We prove sufficiency. Assume that $\varphi(\bar{x}, \bar{b})$ divides over A and take $k < \omega$ and $I = \{ \bar{b}_i \mid i < \omega \}$ such that $\text{tp}(\bar{b}_i/A) = \text{tp}(\bar{b}/A)$ for every $i < \omega$ and $\{ \varphi(\bar{x}, \bar{b}_i) \mid i < \omega \}$ is k-contradictory. Expand the language with names for the elements of A and let $\{ \bar{c}_n \mid n < \omega \}$ be constants not in the language of T. Consider the union of the following sets of sentences:

- T;
- $\neg\exists\bar{x}\, [\varphi(\bar{x}, \bar{c}_{i_0}) \wedge \cdots \wedge \varphi(\bar{x}, \bar{c}_{i_{k-1}})]$, whenever $i_0 < \cdots < i_{k-1} < \omega$;
- $\psi(\bar{c}_0, \ldots, \bar{c}_n, \bar{d}) \leftrightarrow \psi(\bar{c}_{i_0}, \ldots, \bar{c}_{i_n}, \bar{d})$, whenever $i_0 < \cdots < i_n < \omega$, $\bar{d} \in A$, and ψ is in the language of T;
- $\psi(\bar{c}_0, \bar{d})$, for every $\psi(\bar{x}, \bar{d}) \in \text{tp}(\bar{b}_0/A)$.

An application of Ramsey's Theorem shows that our set of sentences is consistent.

Let N be a model for it and let $\bar{d}_n$ be the interpretation of $\bar{c}_n$ in the model N. Then, $\langle \bar{d}_n \mid n < \omega \rangle$ is a sequence indiscernible over A and $\{ \varphi(\bar{x}, \bar{b}_n) \mid n < \omega \}$ is k-contradictory. Furthermore, there exists an A-automorphism f such that $f(\bar{d}_0) = \bar{b}$. Therefore, the sequence $\langle f(\bar{d}_n) \mid n < \omega \rangle$ satisfies the requirements of the lemma. ⊣

The next lemma is crucial to analyze forking and dividing. It will be used in the proof of the Concatenation Lemma 1.13, in the characterization of forking through the rank (Theorem 3.11), and the Independence Theorem (Theorem 3.12).

Lemma 1.3. *The following conditions are equivalent.*

(1) $\text{tp}(\bar{a}/A\bar{b})$ *does not divide over A;*

(2) For every infinite sequence I indiscernible over A with $\bar{b} \in I$ there exists $\bar{a}'$ realizing $\text{tp}(\bar{a}/A\bar{b})$ *such that I is indiscernible over $A\bar{a}'$;*

(3) For every infinite sequence I indiscernible over A with $\bar{b} \in I$ there exists an A-automorphism f such that $f(I)$ is indiscernible over $A\bar{a}$.

Proof. The equivalence between (2) and (3) is a consequence of the homogeneity of $\mathfrak{C}$.

(2) $\Rightarrow$ (1): By contradiction, suppose $\text{tp}(\bar{a}/A\bar{b})$ divides over A and take $\varphi(\bar{x}, \bar{c}, \bar{b}) \in \text{tp}(\bar{a}/A\bar{b})$ with $\bar{c} \in A$ such that $\varphi(\bar{x}, \bar{c}, \bar{b})$ divides over A. Lemma 1.2 provides a sequence $I = \langle \bar{b}_i \mid i < \omega \rangle$ indiscernible over A such that $\bar{b}_0 = \bar{b}$ and $\{\varphi(\bar{x}, \bar{c}, \bar{b}_i) \mid i < \omega\}$ is k-contradictory. By (2) there exists $\bar{a}'$ realizing $\text{tp}(\bar{a}/A\bar{b})$ such that I is indiscernible over $A\bar{a}'$. But then $\models \varphi[\bar{a}', \bar{c}, \bar{b}_0]$, and $\models \varphi[\bar{a}', \bar{c}, \bar{b}_i]$ for every $i < \omega$ by indiscernibility. This contradicts the fact that $\{\varphi(\bar{x}, \bar{c}, \bar{b}_i) \mid i < \omega\}$ is k-contradictory.

(1) $\Rightarrow$ (2): Suppose that (2) fails, and choose an indiscernible sequence $I = \langle \bar{b}_i \mid i < \omega \rangle$ and $\bar{b} \in I$ which witness this failure, *i.e.*,

(*) I is not indiscernible over $A\bar{a}'$, for every $\bar{a}'$ realizing $\text{tp}(\bar{a}/A\bar{b})$.

Denote $p(\bar{x}, \bar{b}) := \text{tp}(\bar{a}/A\bar{b})$ and let $q := \bigcup_{\bar{b}_i \in I} p(\bar{x}, \bar{b}_i)$. We claim that q is consistent.

If q is inconsistent, there exist a finite $I^* \subseteq I$ and a formula $\varphi(\bar{x}, \bar{c}, \bar{b}) \in \text{tp}(\bar{a}/A\bar{b})$ such that $\{\varphi(\bar{x}, \bar{c}, \bar{b}_i) \mid \bar{b}_i \in I^*\}$ is inconsistent. By the indiscernibility of I over A, $\{\varphi(\bar{x}, \bar{c}, \bar{b}_i) \mid \bar{b}_i \in I\}$ is $|I^*|$-inconsistent, so $\text{tp}(\bar{a}/A\bar{b})$ divides over A. But this is a contradiction.

Now let $\Gamma(\bar{x})$ be the union of the following formulas:

- $q(\bar{x})$;
- $\psi(\bar{x}, \bar{b}_0, \dots, \bar{b}_{n-1}, \bar{d}) \leftrightarrow \psi(\bar{x}, \bar{b}_{i_0}, \dots, \bar{b}_{i_{n-1}}, \bar{d})$, whenever $i_0 < \dots < i_{n-1} < \omega$, $n < \omega$, $\psi \in L$ and $\bar{d} \in A$.

We show that $\Gamma(\bar{x})$ is consistent, which contradicts (*). The proof is by induction on the cardinality of the finite subsets of $\Gamma(\bar{x})$. For the induction step, it is sufficient to show that for any $\bar{d} \in A$, $i_0 < \dots < i_n$ and $\varphi(\bar{x}, \bar{b}_0, \bar{b}_1, \dots, \bar{b}_{i_{n-1}}, \bar{d}) \in q(\bar{x})$ we have

(**)
$$\models \exists \bar{x} \left[\varphi(\bar{x}, \bar{b}_0, \bar{b}_1, \dots, \bar{b}_{i_{n-1}}, \bar{d}) \wedge \left[\psi(\bar{x}, \bar{b}_0, \dots, \bar{b}_{n-1}, \bar{d}) \leftrightarrow \psi(\bar{x}, \bar{b}_{i_0}, \dots, \bar{b}_{i_{n-1}}, \bar{d})\right] \right].$$

Fix $\bar{a}'$ realizing q. By Ramsey's Theorem there is an infinite subsequence I' of I which is φ-indiscernible over $A\bar{a}'$. Take $\bar{b}'_0, \dots \bar{b}'_{n-1}, \bar{b}'_{i_0}, \dots \bar{b}'_{i_{n-1}} \in I'$. Then,

$$\models \varphi[\bar{a}', \bar{b}'_0, \bar{b}'_1, \dots, \bar{b}'_{i_{n-1}}, \bar{d}] \wedge \left[\psi[\bar{a}', \bar{b}'_0, \dots, \bar{b}'_{n-1}, \bar{d}] \leftrightarrow \psi[\bar{a}', \bar{b}'_{i_0}, \dots, \bar{b}'_{i_{n-1}}, \bar{d}] \right].$$

Therefore,

$$\models \exists \bar{x} \left[\varphi(\bar{x}, \bar{b}'_0, \bar{b}'_1, \dots, \bar{b}'_{i_{n-1}}, \bar{d}) \wedge \left[\psi(\bar{x}, \bar{b}'_0, \dots, \bar{b}'_{n-1}, \bar{d}) \leftrightarrow \psi(\bar{x}, \bar{b}'_{i_0}, \dots, \bar{b}'_{i_{n-1}}, \bar{d})\right] \right],$$

which implies (**) by the indiscernibility of I over A. ⊣

Lemma 1.4. *If* $\text{tp}(\bar{a}_0/A\bar{b})$ *does not divide over* A *and* $\text{tp}(\bar{a}_1/A\bar{b}\bar{a}_0)$ *does not divide over* $A\bar{a}_0$, *then* $\text{tp}(\bar{a}_0\bar{a}_1/A\bar{b})$ *does not divide over* A.

Proof. Let I be a sequence indiscernible over A such that $\bar{b} \in I$. By Lemma 1.3, showing that $\text{tp}(\bar{a}_0\bar{a}_1/A\bar{b})$ does not divide over A is equivalent to finding $\bar{c}_0\bar{c}_1$ realizing $\text{tp}(\bar{a}_0\bar{a}_1/A\bar{b})$ such that I is indiscernible over $A\bar{c}_0\bar{c}_1$. By Lemma 1.3, since $\text{tp}(\bar{a}_0/A\bar{b})$ does not divide over A, we can find $\bar{c}_0$ realizing $\text{tp}(\bar{a}_0/A\bar{b})$ such that I is indiscernible over $A\bar{c}_0$. Take an $A\bar{b}$-automorphism f such that $f(\bar{a}_0) = \bar{c}_0$. Since $\text{tp}(\bar{a}_1/A\bar{b}\bar{a}_0)$ does not divide over $A\bar{a}_0$, the type $\text{tp}(f(\bar{a}_1)/A\bar{b}\bar{c}_0)$ does not divide over $A\bar{c}_0$. Since I is indiscernible over $A\bar{c}_0$, By Lemma 1.3 we can choose $\bar{c}_1$ realizing $\text{tp}(f(\bar{a}_1)/A\bar{b}\bar{c}_0)$ such that I is indiscernible over $A\bar{c}_0\bar{c}_1$. We have $\text{tp}(\bar{c}_0\bar{c}_1/A\bar{b}) = \text{tp}(f(\bar{a}_0)f(\bar{a}_1)/A\bar{b}) = \text{tp}(\bar{a}_0\bar{a}_1/A\bar{b})$, so we are done. ⊣

We now introduce one of the main concepts of this paper.

Definition 1.5.

(1) A formula $\varphi(\bar{x}, \bar{b})$ *forks over* A if there exist $n < \omega$ and $\{ \varphi_i(\bar{x}, \bar{b}^i) \mid i < n \}$, such that
 (a) $\varphi_i(\bar{x}, \bar{b}^i)$ divides over A for every $i < n$;
 (b) $\varphi(\bar{x}, \bar{b}) \vdash \bigvee_{i<n} \varphi_i(\bar{x}, \bar{b}^i)$.
(2) A type p *forks over* A if there exists a formula $\varphi(\bar{x}, \bar{b})$ such that $p \vdash \varphi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{b})$ forks over A.

Remark 1.6.

(1) If $\varphi(\bar{x}, \bar{b})$ divides over A, then $\varphi(\bar{x}, \bar{b})$ forks over A;
(2) If $\varphi(\bar{x}, \bar{b}) \vdash \bigvee_{i<n} \varphi_i(\bar{x}, \bar{b}^i)$, and $\varphi_i(\bar{x}, \bar{b}^i)$ forks over A for every $i < n$, then $\varphi(\bar{x}, \bar{b})$ forks over A;
(3) (Finite Character of Forking) If $p \in S(B)$ forks over A, then there exists $\varphi(\bar{x}, \bar{b}) \in p$ such that $\varphi(\bar{x}, \bar{b})$ forks over A.

The following is also a fundamental property of forking. The proof is immediate.

Remark 1.7 (Invariance). Let p be a type and A a set. The following conditions are equivalent:

(1) The type p does not divide (fork) over A;
(2) For every A-automorphism f, the type $f(p)$ does not divide (respectively, fork) over A;
(3) There exists an A-automorphism f such that the type $f(p)$ does not divide (respectively, fork) over A.

The reader can observe that the advantage of forking over dividing is exactly that argument below can be carried out.

Theorem 1.8 (Extension). *If p does not fork over A and* $\text{dom}(p) \subseteq B$, *then there exists $q \in S(B)$ extending p such that q does not fork over A.*

Proof. Consider

$$\Gamma := \{ \neg\psi(\bar{x}, \bar{b}) \mid \bar{b} \in B, \psi(\bar{x}, \bar{b}) \text{ forks over } A \}.$$

Let us show that $p \cup \Gamma$ is consistent. If $p \cup \Gamma$ is inconsistent, then there exists $\{ \neg\psi_i(\bar{x}, \bar{b}_i) \mid i < n \} \subseteq \Gamma$ such that $p \cup \{ \neg\psi_i(\bar{x}, \bar{b}_i) \mid i < n \}$ is inconsistent. But then, $p \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and every $\psi_i(\bar{x}, \bar{b}_i)$ forks over A. Hence, p forks over A, which is a contradiction.

Choose a complete extension $q \in S(B)$ of $p \cup \Gamma$. If q forks over A, there exists $\psi(\bar{x}, \bar{b}) \in q$ such that $\psi(\bar{x}, \bar{b})$ forks over A. By the definition of Γ, we have $\neg\psi(\bar{x}, \bar{b}) \in \Gamma \subseteq q$, which is a contradiction. Hence, q does not fork over A. ⊣

Definition 1.9. Let $A \subseteq B$ and p be a type over B. Then, $\langle \bar{a}_n \mid n < \omega \rangle$ is a *Morley sequence for p over A* if

(1) For every $n < \omega$, $\bar{a}_n$ realizes p;
(2) For every $n < \omega$, the type $\text{tp}(\bar{a}_n / B \cup \{ \bar{a}_m \mid m < n \})$ does not fork over A;
(3) $\langle \bar{a}_n \mid n < \omega \rangle$ is indiscernible over B.

Remark 1.10.

(1) Any subsequence of a Morley sequence is a Morley sequence (for the same type).
(2) An A-automorphic image of a Morley sequence for p is a Morley sequence for the image of p under the automorphism.

(3) If $\langle \bar{a}_n \bar{b}_n \mid n < \omega \rangle$ is a Morley sequence for $\text{tp}(\bar{a}_0 \bar{b}_0 / B)$ over A, then $\langle \bar{a}_n \mid n < \omega \rangle$ is a Morley sequence for $\text{tp}(\bar{a}_0 / B)$ over A and $\langle \bar{b}_n \mid n < \omega \rangle$ is a Morley sequence for $\text{tp}(\bar{b}_0 / B)$ over A.

The following theorem is used to produce Morley sequences much in the same way that Ramsey's Theorem is used to produce indiscernibles. A better upper bound for the length the initial sequence is given in Appendix A.

Theorem 1.11. *For every sequence $\langle \bar{a}_i \mid i < \beth_{(2^{|T|})^+} \rangle$ there exists an indiscernible sequence $\langle \bar{b}_n \mid n < \omega \rangle$ with the following property: for every $n < \omega$ there exist $i_0 < \cdots < i_{n-1}$ such that*

$$\text{tp}(\bar{b}_0, \ldots, \bar{b}_{n-1}/\emptyset) = \text{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/\emptyset).$$

Proof. Fix $\langle \bar{a}_i \mid i < \beth_{(2^{|T|})^+} \rangle$ in order to find an indiscernible sequence as in the statement of the theorem. Consider

$$\Gamma_n = \{ p \in S(\emptyset) \mid \text{There exist } i_1 < \cdots < i_n < (2^{|T|})^+ \text{ such that } \models p(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \}.$$

Let $\{ \bar{c}_n \mid n < \omega \}$ be constants not in the language of T. We will find a sequence of types

$$(*) \qquad \langle p_n(\bar{x}_1, \ldots, \bar{x}_n) \mid n < \omega \rangle, \qquad p_n \in \Gamma_n$$

such that the union of the following sets of sentences is consistent:

- T;
- $p_n(\bar{c}_{i_1}, \ldots, \bar{c}_{i_n})$, whenever $i_1 < \cdots < i_n < \omega$.

This will clearly prove the proposition.

To prove the consistency of above sentences, we will construct, by induction on n, a sequence of cofinal subsets of $(2^{|T|})^+$,

$$\langle F_n \mid n < \omega \rangle,$$

a family of subsets of $\beth_{(2^{|T|})^+}$,

$$\{ X_{\xi,n} \mid \xi \in F_n,\ n < \omega \},$$

and a sequence of types (*) such that the following conditions hold:

(1) $F_{n+1} \subseteq F_n$;
(2) $|X_{\xi,n}| > \beth_\lambda(2^{|T|})$, where ξ is the λth element of F_n;
(3) $\models p_n(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n})$, whenever $i_1 < \cdots < i_n$ are in $X_{\xi,n}$.

We let $F_0 = (2^{|T|})^+$, and $X_{\xi,0} = \beth_{(2^{|T|})}$ for every $\xi \in F_0$. Then, (1)–(3) are obvious. Assume, then, that F_n and the $X_{\xi,n}$'s have been defined, and let us define F_{n+1} and the $X_{\xi,n+1}$'s.

Let α be the order type of F_n, and let $g \colon \alpha \to F_n$ be the unique order isomorphism. Define

$$G_n = \{ g(\lambda + n) \mid \lambda < \alpha \}.$$

Then G_n is also cofinal in $(2^{|T|})^+$. Furthermore, if $\xi = g(\lambda + n)$,

$$(**) \qquad |X_{\xi,n}| > \beth_{\lambda+n}(2^{|T|}).$$

The map

$$(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \mapsto \text{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/\emptyset)$$

is a partition of $[X_{\xi,n}]^n$ into $2^{|T|}$-many classes. Hence, by (**) and the Erdős-Rado theorem (see, for example, [CK] Theorem 7.2.1), there exist $X_{\xi,n+1} \subseteq X_{\xi,n}$ with

$$|X_{\xi,n+1}| > \beth_\lambda(2^{|T|})$$

and a type $p_{\xi,n+1} \in \Gamma_{n+1}$ such that (3) holds with $p_{\xi,n+1}$ in place of p_{n+1}. Now the pigeonhole principle allows us to find a cofinal $F_{n+1} \subseteq G_n$ and a type p_{n+1} such that $p_{\xi,n+1} = p_{n+1}$ for every $\xi \in F_{n+1}$. Renumbering the $X_{\xi,n+1}$'s with respect to the ordering of F_{n+1} preserves (2). This concludes the construction. ⊣

Theorem 1.12. *Let $A \subseteq B$. Suppose that p is over B and does not fork over A. Then there exists a Morley sequence for p over A. Moreover, if $p = \text{tp}(\bar{b}/B)$, the Morley sequence can be chosen with $\bar{b}$ as its first element.*

Proof. Let us first expand the language with constants for the elements of B and call T^* the resulting expansion T. Now we use Theorem 1.8 to construct by induction a sequence $\langle \bar{a}_i \mid i < \beth_{(2^{|T^*|})^+} \rangle$ such that $\bar{a}_i \models p$ and $\text{tp}(\bar{a}_i/B \cup \{\bar{a}_j \mid j < i\})$ does not fork over A.

By Theorem 1.11, there exists a sequence $I = \langle \bar{b}_n \mid n < \omega \rangle$ indiscernible over B (since $L(T^*)$ has names for the elements of B) such that for every $n < \omega$ there are $i_0 < \cdots < i_{n-1} < \beth_{(2^{|T^*|})^+}$ satisfying

$$\text{tp}(\bar{b}_0, \ldots, \bar{b}_{n-1}/B) = \text{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/B).$$

We claim that I is a Morley sequence for p over A. First, I is a indiscernible over B, and every $\bar{b}_n$ realizes p. Now suppose, by contradiction, that $\text{tp}(\bar{b}_n/B \cup \{\bar{b}_m \mid m < n\})$ forks over A for some $n < \omega$. Then there is $\varphi(\bar{x}, \bar{c}, \bar{b}_0, \ldots, \bar{b}_{n-1}) \in \text{tp}(\bar{b}_n/B \cup \{\bar{b}_m \mid m < n\})$, such that $\varphi(\bar{x}, \bar{c}, \bar{b}_0, \ldots, \bar{b}_{n-1})$ forks over A. Choose $i_0 < \cdots < i_n$ such that $\text{tp}(\bar{b}_0, \ldots, \bar{b}_n/B) = \text{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/B)$. Then, $\varphi(\bar{x}, \bar{c}, \bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}) \in \text{tp}(\bar{a}_{i_n}/B \cup \{\bar{a}_j \mid j < i_n\})$, and $\varphi(\bar{x}, \bar{c}, \bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}})$ forks over A. Thus, $\text{tp}(\bar{a}_{i_n}/B \cup \{\bar{a}_j \mid j < i_n\})$ forks over A, which is a contradiction.

The last sentence follows by taking a B-automorphism mapping $\bar{b}_0$ to $\bar{b}$. ⊣

The following lemma will be used in the characterization of dividing in terms of Morley sequences (Theorem 3.7, which in turn will be the main tool to prove the equivalence between forking and dividing in simple theories) and in the proof of the main result of Section 4 (Theorem 4.11).

Lemma 1.13 (Concatenation). *Let $\langle \bar{b}_n \mid n < \omega \rangle$ be a Morley sequence for $\text{tp}(\bar{b}_0/A)$. Suppose that*

$$(*) \qquad \langle \bar{a}_0^k{}^\frown \bar{a}_1^k{}^\frown \ldots {}^\frown \bar{a}_j^k \mid k < \omega \rangle$$

is indiscernible over A and every term of (*) realizes $\text{tp}(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_j/A)$. Then there exists a sequence $\langle \bar{c}_n \mid n < \omega \rangle$ indiscernible over A with $\ell(\bar{c}_n) = \ell(\bar{b}_0)$ such that for every $k < \omega$, the following equalities hold:

$$\begin{aligned}\text{tp}(\bar{a}_0^k, \bar{a}_1^k, \ldots, \bar{a}_j^k, \bar{c}_0, \bar{c}_1, \ldots /A) &\\ = \text{tp}(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_j, \bar{b}_{j+1}, \bar{b}_{j+2}, \ldots /A) &\\ = \text{tp}(\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_j, \bar{c}_{j+1}, \bar{c}_{j+2}, \ldots /A).&\end{aligned}$$

Proof. Let $I = \langle \bar{b}_n \mid n < \omega \rangle$. For each $m < \omega$ we construct $\bar{c}_m$ by induction on m such that:

(1) For every $k < \omega$,

$$\mathrm{tp}(\bar{a}_0^k, \bar{a}_1^k, \dots, \bar{a}_j^k, \bar{c}_0, \bar{c}_1, \dots, \bar{c}_{m-1}/A) = \mathrm{tp}(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_j, \bar{b}_{j+1}, \dots, \bar{b}_{j+m}/A);$$

(2) For every $k < \omega$, the sequence

$$(\dagger) \qquad \langle \bar{a}_0^{k\frown} \dots {}^\frown\bar{a}_j^{k\frown}\bar{c}_0{}^\frown \dots {}^\frown\bar{c}_{m-1} \mid k < \omega \rangle \text{ is indiscernible over } A$$

and every term of (†) realizes $\mathrm{tp}(\bar{b}_0, \dots, \bar{b}_j, \bar{b}_{j+1}, \dots, \bar{b}_{j+m-1}/A)$.

Condition (1) and the indiscernibility of I imply the conclusion of the lemma. Condition (2) helps in carrying out the induction.

When $m = 0$, conditions (1) and (2) hold by assumption. Having already defined $\bar{c}_n$ for $n < m$, consider

$$p(\bar{x}, \bar{b}_0, \dots, \bar{b}_{j+m}) := \mathrm{tp}(\bar{b}_{j+m+1}/A \cup \{\bar{b}_0, \dots, \bar{b}_{j+m}\}).$$

Since I is a Morley sequence, the type $p(\bar{x}, \bar{b}_0, \dots, \bar{b}_{j+m})$ does not fork over A, and hence does not divide over A. By the induction hypothesis (1), we have in particular that

$$\mathrm{tp}(\bar{a}_0^0, \dots, \bar{a}_j^0, \bar{c}_0, \dots, \bar{c}_{m-1}/A) = \mathrm{tp}(\bar{b}_0, \dots, \bar{b}_{j+m}/A).$$

Therefore, the type $p(\bar{x}, \bar{a}_0^0, \dots, \bar{a}_j^0, \bar{c}_0, \dots, \bar{c}_{m-1})$ does not divide over A. By Lemma 1.3 applied to the induction hypothesis (2) (with $\bar{b} = \bar{a}_0^{0\frown} \dots {}^\frown\bar{a}_j^{0\frown}\bar{c}_0{}^\frown \dots {}^\frown\bar{c}_{m-1}$ and $\bar{a}' = \bar{c}_m$), we find $\bar{c}_m$ such that

$$\bar{c}_m \models p(\bar{x}, \bar{a}_0^0, \dots, \bar{a}_j^0, \bar{c}_0, \dots, \bar{c}_{m-1})$$

and

$$\langle \bar{a}_0^{k\frown} \dots {}^\frown\bar{a}_j^{k\frown}\bar{c}_0{}^\frown \dots {}^\frown\bar{c}_{m-1} \mid k < \omega \rangle \text{ is indiscernible over } A \cup \bar{c}_m.$$

Then, $\langle \bar{a}_0^{k\frown} \dots {}^\frown\bar{a}_j^{k\frown}\bar{c}_0{}^\frown \dots {}^\frown\bar{c}_m \mid k < \omega \rangle$ is indiscernible over A, and (2) is satisfied.

To see that (1) holds, notice that by the induction hypothesis (2) we must have

$$\begin{aligned} \mathrm{tp}(\bar{a}_0^k, \dots, \bar{a}_j^k, \bar{c}_0, \dots, \bar{c}_m/A) &= \mathrm{tp}(\bar{a}_0^0, \dots, \bar{a}_j^0, \bar{c}_0, \dots, \bar{c}_m/A), && \text{for every } k < \omega \\ &= \mathrm{tp}(\bar{b}_0, \dots, \bar{b}_{j+m+1}/A) && (\text{by the choice of } \bar{c}_m). \end{aligned}$$

⊣

2. Ranks and Simple Theories

The concept of simplicity is defined in terms of a rank.

Definition 2.1. Let $p(\bar{x})$ be a set of formulas, possibly with parameters. Let Δ be a set of formulas and let $k < \omega$. We define the *rank* $D[p, \Delta, k]$. The rank $D[p, \Delta, k]$ is either an ordinal, or -1, or ∞. The relation $D[p, \Delta, k] \geq \alpha$, is defined by induction on α.

(1) $D[p, \Delta, k] \geq 0$ if p is consistent;
(2) $D[p, \Delta, k] \geq \delta$ when δ is a limit if $D[p, \Delta, k] \geq \beta$ for every $\beta < \delta$;
(3) $D[p, \Delta, k] \geq \alpha + 1$ if for every finite $r \subseteq p$, there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ and a set $\{\bar{a}_i \mid i < \omega\}$ with $\ell(\bar{a}_i) = \ell(\bar{y})$ such that:
  (a) $D[r \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha$ for every $i < \omega$;
  (b) The set $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k-contradictory.

We write

$D[p, \Delta, k] = -1$ if p is not consistent;
$D[p, \Delta, k] = \alpha$ when $D[p, \Delta, k] \geq \alpha$ but $D[p, \Delta, k] \not\geq \alpha + 1$;
$D[p, \Delta, k] = \infty$ when $D[p, \Delta, k] \geq \alpha$ for every ordinal α.

The next propositions establish some of the most basic properties of the rank. The proofs are rather easy exercises. However,we include them here for completeness.

Proposition 2.2.

(1) (Monotonicity) *If* $p_1 \vdash p_2$, $\Delta_1 \subseteq \Delta_2$, *and* $k_1 \leq k_2$, *then* $D[p_1, \Delta_1, k_1] \leq D[p_2, \Delta_2, k_2]$;
(2) (Finite Character) *For every type* p *there exists a finite* $r \subseteq p$ *such that* $D[p, \Delta, k] = D[r, \Delta, k]$
(3) (Invariance) *If* f *is an automorphism, then* $D[p, \Delta, k] = D[f(p), \Delta, k]$.

Proof. (1) We prove by induction on the ordinal α, that

$$D[p_1, \Delta_1, k_1] \geq \alpha \quad \text{implies} \quad D[p_2, \Delta_2, k_2] \geq \alpha.$$

If $D[p_1, \Delta_1, k_1] \geq 0$, then p_1 is consistent, so p_2 is consistent since $p_1 \vdash p_2$, and hence $D[p_2, \Delta_2, k_2] \geq 0$.

When α is a limit ordinal, the implication is immediate from the induction hypothesis.

Suppose $D[p_1, \Delta_1, k_1] \geq \alpha + 1$ and let $r_2 \subseteq p_2$ be finite. Since $p_1 \vdash p_2$, there is a finite $r_1 \subseteq p_1$ such that $r_1 \vdash r_2$. By the definition of the rank, there exist $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\{\bar{a}_i \mid i < \omega\}$ with $\ell(\bar{a}_i) = \ell(\bar{y})$ such that $D[r_1 \cup \varphi(\bar{x}, \bar{a}_i)\, \Delta_1, k_1] \geq \alpha$ for every $i < \omega$ and $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k_1-contradictory. Now, $r_1 \cup \varphi(\bar{x}, \bar{a}_i) \vdash r_2 \cup \varphi(\bar{x}, \bar{a}_i)$, so $D[r_2 \cup \varphi(\bar{x}, \bar{a}_i), \Delta_2, k_2] \geq \alpha$ for every $i < \omega$ by induction hypothesis. But $\varphi(\bar{x}, \bar{y}) \in \Delta_2$ (since $\Delta_1 \subseteq \Delta_2$), and $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k_2-contradictory (since $k_2 \geq k_1$). Hence, $D[p_2, \Delta_2, k_2] \geq \alpha + 1$, by definition of the rank.

(2) If $D[p, \Delta, k] = -1$, then p is inconsistent and by the compactness theorem there is an inconsistent finite $r \subseteq p$. Then, $D[r, \Delta, k] = -1$.

If $D[p, \Delta, k] = \infty$, then for every finite $r \subseteq p$, $D[r, \Delta, k] = \infty$ by Monotonicity.

If $D[p, \Delta, k] = \alpha$, then $D[p, \Delta, k] \geq \alpha$ and $D[p, \Delta, k] \not\geq \alpha + 1$, so there exists a finite $r \subseteq p$ with $D[r, \Delta, k] \geq \alpha$ such that there are no $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $\{\bar{a}_i \mid i < \omega\}$ such that $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k-contradictory and $D[r \cup \varphi(\bar{x}, \bar{a}_i)\, \Delta, k] \geq \alpha$ for every $i < \omega$. But this demonstrates that $D[r, \Delta, k] \not\geq \alpha + 1$. Thus, $D[r, \Delta, k] = \alpha$.

(3) Immediate. ⊣

Lemma 2.3 (Ultrametric Property). *For every* p, Δ, k, $n < \omega$ *and formulas* $\{\psi_l(\bar{x}, \bar{b}_l) \mid l < n\}$, *we have*

$$D[p \cup \bigvee_{l<n} \psi_l(\bar{x}, \bar{b}_l), \Delta, k] = \max_{l<n} D[p \cup \psi_l(\bar{x}, \bar{b}_l), \Delta, k].$$

Proof. By Monotonicity, for every $l < n$ we have

$$D[p \cup \psi_l(\bar{x}, \bar{b}_l), \Delta, k] \leq D[p \cup \bigvee_{l<n} \psi_l(\bar{x}, \bar{b}_l), \Delta, k].$$

Hence,

$$\max_{l<n} D[p \cup \psi_l(\bar{x}, \bar{b}_l), \Delta, k] \leq D[p \cup \bigvee_{l<n} \psi_l(\bar{x}, \bar{b}_l), \Delta, k].$$

To prove the reverse inequality, we show by induction that for every ordinal α and every type p,

$$D[p \cup \bigvee_{l<n} \psi_l(\bar{x}, \bar{b}_l), \Delta, k] \geq \alpha \quad \text{implies} \quad \max_{l<n} D[p \cup \psi_l(\bar{x}, \bar{b}_l), \Delta, k] \geq \alpha.$$

When $\alpha = 0$ or α is a limit ordinal, the implication is easy. For the successor stage suppose, by contradiction, that

(*)

$$D[\,p \cup \bigvee_{l<n} \psi_l(\bar{x}, \bar{b}_l)\,, \Delta, k] \geq \alpha + 1, \quad \text{but} \quad \max_{l<n} D[\,p \cup \psi_l(\bar{x}, \bar{b}_l)\,, \Delta, k] \not\geq \alpha + 1.$$

Then, for every $l < n$ we have $D[\,p \cup \psi_l(\bar{x}, \bar{b}_l)\,, \Delta, k] \leq \alpha$. Choose a finite $r_l \subseteq p$ such that

$$D[\,r_l \cup \psi_l(\bar{x}, \bar{b}_l)\,, \Delta, k] \leq \alpha$$

and let $r := \bigcup_{l<n} r_l$. Then $r \subseteq p$ is finite, so, by (*) and the definition of the rank, there exist $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $\{\,\bar{a}_i \mid i < \omega\,\}$ with $\ell(\bar{a}_i) = \ell(\bar{y})$ such that $\{\,\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\,\}$ is k-contradictory and for every $i < \omega$

$$D[\,r \cup \bigvee_{l<n} \psi(\bar{x}, \bar{b}_l) \cup \varphi(\bar{x}, \bar{a}_i)\,, \Delta, k] \geq \alpha.$$

By induction hypothesis, for every $i < \omega$

$$\max_{l<n} D[\,r \cup \psi_l(\bar{x}, \bar{b}_l) \cup \varphi(\bar{x}, \bar{a}_i)\, \Delta, k] \geq \alpha,$$

so there exists $l_i < n$ such that

$$D[\,r \cup \psi_{l_i}(\bar{x}, \bar{b}_{l_i}) \cup \varphi(\bar{x}, \bar{a}_i)\,, \Delta, k] \geq \alpha.$$

By the pigeonhole principle, we may assume that $l_i = l^* < n$ is fixed and

$$D[\,r \cup \psi_{l^*}(\bar{x}, \bar{b}_{l^*}) \cup \varphi(\bar{x}, \bar{a}_i)\,, \Delta, k] \geq \alpha,$$

for every $i < \omega$. By definition of the rank,

$$D[\,r \cup \psi_{l^*}(\bar{x}, \bar{b}_{l^*})\,, \Delta, k] \geq \alpha + 1,$$

and by Monotonicity,

$$D[\,r_{l^*} \cup \psi_{l^*}(\bar{x}, \bar{b}_{l^*})\,, \Delta, k] \geq \alpha + 1.$$

But this contradicts the choice of r_{l^*}. Therefore,

$$\max_{l<n} D[\,p \cup \psi_l(\bar{x}, \bar{b}_l)\,, \Delta, k] \geq \alpha + 1.$$

⊣

The proof of the following lemma is similar to that of Theorem 1.8.

Lemma 2.4. *Let p be a type, Δ and Φ be sets of formulas and $k < \omega$. Suppose that $D[\,p, \Delta, k] < \infty$. Then, for every set B there exists a type $q \in S_\Phi(B)$ such that*

$$D[\,p, \Delta, k] = D[\,p \cup q, \Delta, k].$$

Proof. We may assume that Φ is closed under conjunction. Suppose that $D[p, \Delta, k] = \alpha$. Consider

$$\Gamma := \{\,\neg\psi(\bar{x}, \bar{b}) \mid \bar{b} \in B, \psi(\bar{x}, \bar{y}) \in \Phi,\ D[p \cup \psi(\bar{x}, \bar{b}), \Delta, k] < \alpha\}.$$

Let us show that $p\cup\Gamma$ is consistent. If $p\cup\Gamma$ were inconsistent, there would be $\{\,\neg\psi_i(\bar{x}, \bar{b}_i) \mid i < n\,\} \subseteq \Gamma$ such that $p \cup \{\,\neg\psi_i(\bar{x}, \bar{b}_i) \mid i < n\,\}$ is inconsistent. But then, $p \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$. By Monotonicity and Lemma 2.3, we have

$$\alpha = D[p, \Delta, k] \leq D[p \cup \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i), \Delta, k] = \max_{i<n} D[\,p \cup \psi_i(\bar{x}, \bar{b}_i)\,, \Delta, k] < \alpha,$$

which is, of course, a contradiction.

Choose $q \in S_\Phi(B)$ extending Γ. If $D[p \cup q, \Delta, k] < \alpha$, then by Finite Character, there exists $\psi(\bar{x}, \bar{b}) \in q$ such that $D[p \cup \psi(\bar{x}, \bar{b}), \Delta, k] < \alpha$. Hence, by definition of Γ, we must have $\neg\psi(\bar{x}, \bar{b}) \in \Gamma \subseteq q$, which is a contradiction. ⊣

Definition 2.5. A first order theory T is *simple*, if $D[\bar{x} = \bar{x}, \Delta, k] < \infty$ for every finite Δ and every $k < \omega$.

As we will see in Theorem 2.8, any stable theory is simple. However, there are important examples of simple unstable theories. Historically, the motivating example was the theory of the random graph.

There are several equivalent formulations of model theoretical stability. One of them is via the following rank. Recall that the types p and q are *explicitly contradictory* if there exists $\varphi(\bar{x}, \bar{b})$ such that $\varphi(\bar{x}, \bar{b}) \in p$ and $\neg\varphi(\bar{x}, \bar{b}) \in q$ (or vice versa).

Definition 2.6. Let $p(\bar{x})$ be a set of formulas, possibly with parameters. Let Δ be a set of formulas. We define the *rank* $R[p, \Delta, \aleph_0]$. The rank $R[p, \Delta, \aleph_0]$ is either an ordinal, or -1, or ∞. The relation $R[p, \Delta, \aleph_0] \geq \alpha$, is defined by induction on α.

(1) $R[p, \Delta, \aleph_0] \geq 0$ if p is consistent;
(2) $R[p, \Delta, \aleph_0] \geq \delta$ when δ is a limit if $R[p, \Delta, \aleph_0] \geq \beta$ for every $\beta < \delta$;
(3) $R[p, \Delta, \aleph_0] \geq \alpha + 1$ if for every finite $r \subseteq p$ there exists a set of Δ-types $\{ q_i \mid i < \omega \}$ such that:
  (a) $R[r \cup q_i, \Delta, \aleph_0] \geq \alpha$ for every $i < \omega$;
  (b) The types q_i and q_j are explicitly contradictory if $i \neq j < \omega$.

We write

$R[p, \Delta, \aleph_0] = -1$ if p is not consistent;
$R[p, \Delta, \aleph_0] = \alpha$ when $R[p, \Delta, \aleph_0] \geq \alpha$ but $R[p, \Delta, \aleph_0] \not\geq \alpha + 1$;
$R[p, \Delta, \aleph_0] = \infty$ when $R[p, \Delta, \aleph_0] \geq \alpha$ for every ordinal α.

The following fact follows from Theorems 2.2 and 2.13 of Chapter II in [Sha].

Fact 2.7. *T is stable if and only if $R[\bar{x} = \bar{x}, \Delta, \aleph_0] < \omega$ for every finite set of formulas Δ.*

Theorem 2.8. *Let T be a (complete) first order theory. If T is stable, then T is simple.*

Proof. Using Fact 2.7 it suffices to show that for every finite Δ and every $k < \omega$,

$$D[\bar{x} = \bar{x}, \Delta, k] \leq R[\bar{x} = \bar{x}, \Delta, \aleph_0].$$

We shall show by induction on the ordinal α that for every type p, every finite set of formulas Δ, and every $k < \omega$

$$D[p, \Delta, k] \geq \alpha \quad \text{implies} \quad R[p, \Delta, \aleph_0] \geq \alpha.$$

When $\alpha = 0$ or α is a limit ordinal, the implication is obvious. Suppose $D[p, \Delta, k] \geq \alpha + 1$. Let $r \subseteq p$ be finite subtype. Then there exist a formula $\varphi \in \Delta$ and $\{ \bar{a}_i \mid i < \omega \}$ such that the set $\{ \varphi(\bar{x}, \bar{a}_i) \mid i < \omega \}$ is k-contradictory, and $D[r \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha$ for every $i < \omega$. Let $A := \bigcup \{ \bar{a}_i \mid i < \omega \}$. By Lemma 2.4, for every $i < \omega$ there exists $q_i \in S_\Delta(A)$ such that

$$D[r \cup \varphi(\bar{x}, \bar{a}_i) \cup q_i, \Delta, k] \geq \alpha, \qquad \text{for every } i < \omega.$$

Therefore, by the induction hypothesis,

$$R[r \cup \varphi(\bar{x}, \bar{a}_i) \cup q_i, \Delta, \aleph_0] \geq \alpha, \qquad \text{for every } i < \omega.$$

Now, since $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k-contradictory and $\varphi(\bar{x}, \bar{a}_i) \in q_i$, any k-element subset of $\{q_i \mid i < \omega\}$ contains two types that are explicitly contradictory. Hence, there exists an infinite $S \subseteq \omega$ such that for any $i \neq j$ in S the Δ-types q_i and q_j are explicitly contradictory. By definition, we must have $R[r, \Delta, \aleph_0] \geq \alpha + 1$, which finishes the induction. ⊣

We now provide various charaterizations of simplicity.

Lemma 2.9. *Let $\varphi(\bar{x}, \bar{y})$ be a formula, $k < \omega$ and α be an ordinal. The following conditions are equivalent:*

(1) $D[\bar{x} = \bar{x}, \varphi, k] \geq \alpha$;
(2) There exists $\{\bar{a}_\eta \mid \eta \in {}^{\alpha>}\omega\}$ such that
  (a) For every $\eta \in {}^{\alpha}\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction\beta}) \mid \beta < \alpha\}$ is consistent;
  (b) For every $\eta \in {}^{\alpha>}\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta^\frown n}) \mid n < \omega\}$ is k-contradictory.

Proof. By induction on α. ⊣

The preceding lemma is generalized in Proposition 2.14

Definition 2.10. A theory T had the *tree property* if there exist a formula $\varphi(\bar{x}, \bar{y})$, an integer $k < \omega$, and $\{\bar{a}_\eta \mid \eta \in {}^{\omega>}\omega\}$ such that

(1) For every $\eta \in {}^{\omega}\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction l}) \mid l < \omega\}$ is consistent;
(2) For every $\eta \in {}^{\omega>}\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta^\frown n}) \mid n < \omega\}$ is k-contradictory.

Corollary 2.11. *The following conditions are equivalent:*

(1) T is simple;
(2) $D[\bar{x} = \bar{x}, \Delta, k] < \omega$, for every finite Δ and $k < \omega$;
(3) T does not have the tree property.

Proof. By the compactness theorem and the previous lemma (and coding finite sets of formulas by a single formula) ⊣

Shelah's original definition of the rank in [Sh93] is more general than that in Definition 2.1. We now compare both definitions and derive a few simple facts which we will need in Section 3. The reader interested mainly in the basics of forking may skip the rest of this section, and return to it only as needed.

Definition 2.12. Let $p(\bar{x})$ be a set of formulas, possibly with parameters. Let Δ be a set of formulas, $k < \omega$ and λ a cardinality (not necessarily infinite). We define the *rank* $D[p, \Delta, k, \lambda]$. The rank $D[p, \Delta, k, \lambda]$ is either an ordinal, or -1, or ∞. The relation $D[p, \Delta, k, \lambda] \geq \alpha$, is defined by induction on α.

(1) $D[p, \Delta, k, \lambda] \geq 0$ if p is consistent;
(2) $D[p, \Delta, k, \lambda] \geq \delta$ when δ is a limit if $D[p, \Delta, k, \lambda] \geq \beta$ for every $\beta < \delta$;
(3) $D[p, \Delta, k, \lambda] \geq \alpha + 1$ if for every finite $r \subseteq p$ and every $\mu < \lambda$ there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ and a set $\{\bar{a}_i \mid i < \mu\}$ with $\ell(\bar{a}_i) = \ell(\bar{y})$ such that:
  (a) $D[r \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k, \lambda] \geq \alpha$ for every $i < \mu$;
  (b) The set $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \mu\}$ is k-contradictory.

As usual, we write

$D[p, \Delta, k, \lambda] = -1$ if p is not consistent;
$D[p, \Delta, k, \lambda] = \alpha$ when $D[p, \Delta, k, \lambda] \geq \alpha$ but $D[p, \Delta, k, \lambda] \not\geq \alpha + 1$;
$D[p, \Delta, k, \lambda] = \infty$ when $D[p, \Delta, k, \lambda] \geq \alpha$ for every ordinal α.

Remark 2.13. Clearly, the function $D[p, \Delta, k, \cdot]$ is nonincreasing and $D[p, \Delta, k] = D[p, \Delta, k, \aleph_1]$. The statements Proposition 2.2 are also true for this new definition. The Ultrametric Property (Lemma 2.3) holds for $D[p, \Delta, k, \lambda]$, when λ is uncountable.

Proposition 2.14. *Let p be finite, φ be a formula, $k < \omega$, $n < \omega$ and $\lambda = \mu^+$ (or $\mu + 1$ when μ is finite). Then $D[p, \varphi, k, \lambda] \geq \alpha$ if and only if the union of the following sets of formulas is consistent:*

- $\{ p(\bar{x}_\eta) \mid \eta \in {}^\alpha\mu \}$;
- $\{ \neg\exists\bar{x} \bigwedge_{i \in w} \varphi(\bar{x}; \bar{y}_{\eta^\frown i}) \mid w \subseteq \mu,\ |w| = k,\ \eta \in {}^{\alpha>}\mu \}$;
- $\{ \varphi(\bar{x}_\eta; \bar{y}_{\eta\restriction(l+1)}) \mid \eta \in {}^n\mu, l < \alpha \}$.

Proof. Immediate from Definition 2.12. ⊣

Corollary 2.15. *Let p be a type, Δ be a finite set of formulas and $k < \omega$. The following conditions are equivalent:*

(1) $D[p, \Delta, k] \geq n$;
(2) $D[p, \Delta, k, m] \geq n$ *for every* $m < \omega$.

Proof. When p is finite, the statement follows from from Proposition 2.14. The general case follows immediately by compactness. ⊣

Corollary 2.16. *Let p be a type, Δ be a finite set of formulas, and $k < \omega$.Then,*

$$D[p, \Delta, k, \lambda] = D[p, \Delta, k], \quad \textit{for every}\ \lambda > \aleph_0.$$

Proof. We may assume that $\Delta = \{\varphi\}$. Clearly $D[p, \Delta, k, \lambda] \leq D[p, \Delta, k]$. The reverse inequality follows by Finite Character and Proposition 2.14. ⊣

Remark 2.17. Corollary 2.16 implies that if λ, Δ, and α are finite, then for every $\psi(\bar{x}; \bar{y})$ the set

$$\{ \bar{b} \mid D[\{\psi(\bar{x}; \bar{b})\}, \Delta, k, \lambda] = \alpha \}$$

is first order definable in $\mathfrak{C}$.

Corollary 2.18. *In Definition 2.1 we can add at the successor stage the condition*

(c) $\langle \bar{a}_i \mid i < \omega \rangle$ *is indiscernible over* $\mathrm{dom}(p)$.

Proof. We wish to prove that if $D[p, \Delta, k] \geq \alpha + 1$, then for every finite $r \subseteq p$ there exist $\varphi \in \Delta$ and a sequence $\langle \bar{a}_i \mid i < \omega \rangle$ indiscernible over $\mathrm{dom}(p)$ such that $\{ \varphi(\bar{x}, \bar{a}_i) \mid i < \omega \}$ is k-contradictory and

(1) $$D[r \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha, \quad \text{for every } i < \omega.$$

Let $A := \mathrm{dom}(p)$ and fix $k < \omega$. Let $\chi := (2^{|A|+|T|})^+$ and $\lambda := (\beth_\chi)^+$. By Corollary 2.16, we have

$$D[p, \Delta, k, \lambda] = D[p, \Delta, k] \geq \alpha + 1.$$

Hence, for every finite $r \subseteq p$ there exist a formula $\varphi \in \Delta$ and a set $\{ \bar{b}_i \mid i < \beth_\chi \}$ such that $\{ \varphi(\bar{x}, \bar{b}_i) \mid i < \beth_\chi \}$ is k-contradictory and

(*) $$D[r \cup \varphi(\bar{x}, \bar{b}_i), \Delta, k, \lambda] \geq \alpha, \quad \text{for every } i < \beth_\chi.$$

Theorem 1.11 provides a sequence $\langle \bar{a}_i \mid i < \omega \rangle$ indiscernible over A such that for every $n < \omega$ there exist $i_1 < \cdots < i_n < \beth_\chi$ satisfying

(**) $$\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_{n-1}/A) = \mathrm{tp}(\bar{b}_{j_1}, \ldots, \bar{b}_{j_n}/A).$$

Clearly, $(**)$ guarantees that $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k-contradictory. Now, $(*)$, $(**)$, and Invariance imply

$$D[r \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha, \quad \text{for every } i < \omega.$$

$\dashv$

3. Forking in Simple Theories

In this section we study the connections between forking and the rank D, and establish the main properties of forking which hold when the theory T is simple.

Lemma 3.1. *Let T be simple. Suppose that p is a type over A and that $\varphi(\bar{x}, \bar{a})$ forks over A. Then there are Δ_0 finite and $k_0 < \omega$ such that*

$$D[\, p \cup \varphi(\bar{x}, \bar{a})\, , \Delta, k] < D[\, p, \Delta, k],$$

for every $\Delta \supseteq \Delta_0$ finite and every $k \geq k_0$.

Proof. Suppose first that $\varphi(\bar{x}, \bar{a})$ divides over A. By lemma 1.2 there exist a set $\{\bar{a}_i \mid i < \omega\}$ and $k < \omega$ such that $\bar{a}_0 = \bar{a}$, $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{a}_i/A)$ for every $i < \omega$, and $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is k-contradictory. Let $\Delta_0 := \{\varphi(\bar{x}, \bar{y})\}$ and $k_0 := k$. Suppose that there exist a finite $\Delta \supseteq \Delta_0$ finite and $l \geq k_0$ such that

$$D[\, p \cup \varphi(\bar{x}, \bar{a})\, , \Delta, l] \nless D[\, p, \Delta, l].$$

By Monotonicity,

$$D[\, p \cup \varphi(\bar{x}, \bar{a})\, , \Delta, l] = D[\, p, \Delta, l].$$

Since T is simple, there is an ordinal α such that $D[\, p, \Delta, l] = \alpha$. By Finite Character, there is a finite $r \subseteq p$ such that $D[r, \Delta, l] = D[\, p, \Delta, l] = \alpha$. Since $p \vdash r \cup \varphi \vdash r$, we have

(*) $$D[r \cup \varphi(\bar{x}, \bar{a})\, , \Delta, l] = \alpha.$$

Since $\mathrm{tp}(\bar{a}_i/A) = \mathrm{tp}(\bar{a}/A)$, there exists an A-automorphism f such that $f_i(\bar{a}) = \bar{a}_i$. By Invariance of the rank,

(**) $$D[\, f_i(r) \cup \varphi(\bar{x}, f_i(\bar{a}))\, , \Delta, l] = \alpha, \qquad \text{for every } i < \omega.$$

Since f_i fixes A pointwise and $\mathrm{dom}(r) \subseteq A$, from (**) we obtain

$$D[r \cup \varphi(\bar{x}, \bar{a}_i)\, , \Delta, l] = \alpha, \qquad \text{for every } i < \omega,$$

but $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $\{\varphi(\bar{x}, \bar{a}_i) \mid i < \omega\}$ is l-contradictory (since $l \geq k$). Hence, $D[r, \Delta, l] \geq \alpha + 1$, which is a contradiction.

The lemma is therefore true if $\varphi(\bar{x}, \bar{a})$ divides over A. If $\varphi(\bar{x}, \bar{a})$ forks over A, there exist $n < \omega$ and $\varphi_i(\bar{x}, \bar{a}^i)$ for $i < n$, such that $\varphi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \varphi_i(\bar{x}, \bar{a}^i)$ and $\varphi_i(\bar{x}, \bar{a}^i)$ divides over A, for every $i < n$. By the preceding argument, for every $i < n$ there exist a finite Δ^i and $k^i < \omega$ such that

(†) $$D[\, p \cup \varphi_i(\bar{x}, \bar{a}^i)\, , \Delta, l] < D[\, p, \Delta, l], \qquad \text{for every } \Delta \supseteq \Delta^i \text{ finite and } k^i \leq l < \omega.$$

Let $\Delta_0 := \bigcup_{i<n} \Delta^i$, $k_0 := \max_{i<n} k^i$. We prove that these Δ_0 and k_0 satisfy the conclusion of the lemma.

Suppose $\Delta \supseteq \Delta_0$ is finite and $k_0 \leq l < \omega$. We have

$$\begin{aligned} D[p \cup \varphi(\bar{x}, \bar{a}), \Delta, l] &\leq D[p \cup \bigvee_{i<n} \varphi_i(\bar{x}, \bar{a}^i), \Delta, l] && (\text{since } \varphi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \varphi_i(\bar{x}, \bar{a}^i)) \\ &= \max_{i<n} D[p \cup \varphi_i(\bar{x}, \bar{a}^i), \Delta, l] && (\text{by Lemma 2.3}) \\ &< D[p, \Delta, l] && (\text{by } (\dagger)), \end{aligned}$$

which is what we sought to prove. ⊣

Theorem 3.2 (Existence). *Suppose that T is simple. Then no type forks over its domain.*

Proof. Suppose that p is over A and forks over A. Then there exists a formula $\varphi(\bar{x}, \bar{a})$ such that $p \vdash \varphi(\bar{x}, \bar{a})$, and $\varphi(\bar{x}, \bar{a})$ forks over A. But since $p \vdash p \cup \varphi \vdash p$,

$$D[p, \Delta, k] = D[p \cup \varphi(\bar{x}, \bar{a}), \Delta, k], \qquad \text{for every } \Delta \text{ and } k < \omega,$$

which contradicts Lemma 3.1. ⊣

Corollary 3.3. *Let T be simple. Then for every p over A there exists a Morley sequence for p over A.*

Proof. By Theorem 3.2 the type p does not fork over A, so a Morley sequence for p over A exists by Theorem 1.12. ⊣

Theorem 3.4. *Suppose that T is simple. If p is over A, there exists $B \subseteq A$ with $|B| \leq |T|$ such that p does not fork over B.*

Proof. Since T is simple, $D[p, \Delta, k] < \infty$ for every Δ finite and $k < \omega$. Fix Δ finite, $k < \omega$, and a finite type $q_{\Delta,k} \subseteq p$, such that

$$(*) \qquad D[p, \Delta, k] = D[q_{\Delta,k}, \Delta, k].$$

Let

$$q := \bigcup \{ q_{\Delta,k} \mid \Delta \subseteq L(T),\ \Delta \text{ finite},\ k < \omega \} \subseteq p \qquad \text{and} \qquad B := \operatorname{dom}(q).$$

Then $|B| \leq |T|$, and since $p \vdash q \vdash q_{\Delta,k}$, by (*) we have

$$(**) \qquad D[p, \Delta, k] = D[q, \Delta, k], \qquad \text{for every finite } \Delta \text{ and } k < \omega.$$

We will show that p does not fork over B.

Suppose p forks over B. Then there exists $\varphi(\bar{x}, \bar{a})$ such that $p \vdash \varphi(\bar{x}, \bar{a})$, and $\varphi(\bar{x}, \bar{a})$ forks over B. Since $p \vdash q \cup \varphi(\bar{x}, \bar{a}) \vdash q$,

$$D[p, \Delta, k] \leq D[q \cup \varphi(\bar{x}, \bar{a}), \Delta, k] \leq D[q, \Delta, k], \qquad \text{for every finite } \Delta \text{ and } k < \omega.$$

Therefore, by (**),

$$D[q \cup \varphi(\bar{x}, \bar{a}), \Delta, k] = D[q, \Delta, k], \qquad \text{for every finite } \Delta \text{ and } k < \omega.$$

This contradicts Lemma 3.1 since q is over B. ⊣

Definition 3.5. We call $\kappa(T)$ the least cardinal κ such that every type does not fork over a subset of its domain of cardinality κ.

Theorem 3.4 says that $\kappa(T) \leq |T|^+$ if T is simple. We now show that a rather strong converse of Theorem 3.4 also holds. If $\kappa(T) < \infty$, then $\kappa(T) \leq |T|^+$ and T is simple. Therefore, T is simple if and only if $\kappa(T) \leq |T|^+$. This equivalence is sometimes used as the definition of simplicity. A complete first order theory T is called *supersimple* if $\kappa(T) = \aleph_0$. See [Ki1].

Theorem 3.6. *Suppose that T is not simple. Then, for every any cardinal κ there exists a type p such that p forks over all subsets of cardinality κ of its domain.*

Proof. Let κ be given and let $\mu = (2^{\kappa+|T|})^+$. Since T is not simple, T has the tree property by Corollary 2.11. Let φ and $k < \omega$ witness this. By compactness, we can find $\{\bar{a}_\eta \mid \eta \in {}^{\kappa^+>}\mu\}$ such that

(1) For every $\eta \in {}^{\kappa^+}\mu$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction\beta}) \mid \beta < \mu\}$ is consistent;

(2) For every $\eta \in {}^{\kappa^+>}\mu$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\frown i}) \mid i < \mu\}$ is k-contradictory.

By the pigeonhole principle and compactness, there exist $\{\bar{b}_\eta \mid \eta \in {}^{\kappa^+>}\mu\}$ such that

(3) For every $\eta \in {}^{\kappa^+}\omega$, the set $\{\varphi(\bar{x}, \bar{b}_{\eta\restriction\beta}) \mid \beta < \kappa^+\}$ is consistent;

(4) For every $\eta \in {}^{\kappa^+>}\omega$, the set $\{\varphi(\bar{x}, \bar{b}_{\eta\frown n}) \mid n < \omega\}$ is k-contradictory;

(5) For every $\eta \in {}^{\kappa^+>}\omega$ and every $n < \omega$,

$$\mathrm{tp}(\bar{b}_{\eta\frown 0}/\bigcup\{\bar{b}_\nu \mid \nu \leq \eta\}) = \mathrm{tp}(\bar{b}_{\eta\frown n}/\bigcup\{\bar{b}_\nu \mid \nu \leq \eta\}).$$

Let $\eta \in {}^{\kappa^+}\omega$ and consider the set $p := \{\varphi(\bar{x}, \bar{b}_{\eta\restriction\beta}) \mid \beta < \kappa^+\}$ (which is a type by (1)). For every subset A of $\mathrm{dom}(p)$ of cardinality at most κ there is $\alpha < \kappa^+$ such that $p \restriction A \subseteq \{\varphi(\bar{x}, \bar{b}_{\eta\restriction\beta}) \mid \beta \leq \alpha\}$. By (4) and (5), p divides, and hence forks over A. ⊣

Theorem 3.7. *Let T be simple. The following conditions are equivalent:*

(1) The formula $\varphi(\bar{x}, \bar{b})$ divides over A;

(2) For every Morley sequence $\langle \bar{b}_n \mid n < \omega\rangle$ for $\mathrm{tp}(\bar{b}/A)$, the set $\{\varphi(\bar{x}, \bar{b}_n) \mid n < \omega\}$ is inconsistent;

(3) For some Morley sequence $\langle \bar{b}_n \mid n < \omega\rangle$ for $\mathrm{tp}(\bar{b}/A)$, the set $\{\varphi(\bar{x}, \bar{b}_n) \mid n < \omega\}$ is inconsistent.

Proof. (3) $\Rightarrow$ (1) is obvious and (2) $\Rightarrow$ (3) is given by Corollary 3.3.

(1) $\Rightarrow$ (2): Let $\langle \bar{b}_n \mid n < \omega\rangle$ be a Morley sequence for $\mathrm{tp}(\bar{b}/A)$. Assume for the sake of contradiction that

(*) $$\{\varphi(\bar{x}, \bar{b}_n) \mid n < \omega\} \text{ is consistent.}$$

Since $\varphi(\bar{x}, \bar{b})$ divides over A, there exist a sequence $\langle \bar{a}_n \mid n < \omega\rangle$ indiscernible over A and $k < \omega$ such that $\bar{a}_0 = \bar{b}$ and

(**) $$\{\varphi(\bar{x}, \bar{a}_n) \mid n < \omega\} \text{ is } k\text{-contradictory.}$$

By the Concatenation Lemma (Lemma 1.13), there exists $\langle \bar{c}_n \mid n < \omega\rangle$ such that for every $n < \omega$ the following equalities hold:

(***) $$\mathrm{tp}(\bar{a}_n, \bar{c}_0, \bar{c}_1, \ldots/A) = \mathrm{tp}(\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots/A) = \mathrm{tp}(\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots/A).$$

Let $p := \{\varphi(\bar{x}, \bar{c}_n) \mid n < \omega\}$. The second equality of (***) and (*) imply that p is consistent. Hence, since T is simple, there is an ordinal α such that $D[p, \varphi(\bar{x}, \bar{y}), k] = \alpha$. By the first equality of (***), for every $n < \omega$ there exists an A-automorphism f_n such that $f_n(\bar{a}_n) = \bar{c}_0$ and $f_n(\bar{c}_{m+1}) = \bar{c}_m$ for every $m < \omega$. By Invariance of the rank,

$$D[\varphi(\bar{x}, \bar{a}_n) \cup p, \varphi(\bar{x}, \bar{y}), k] = D[f_n(\varphi(\bar{x}, \bar{a}_n) \cup p), \varphi(\bar{x}, \bar{y}), k] = D[p, \varphi(\bar{x}, \bar{y}), k] = \alpha,$$

for every $n < \omega$. Now (**) and the definition of the rank imply that $D[p, \varphi(\bar{x}, \bar{y}), k] \geq \alpha + 1$, which is a contradiction. ⊣

The proof of the next result was kindly communicated to us by Shelah. It is considerably simpler than Kim's original proof in [Ki]

Theorem 3.8. *Let T be simple. The following conditions are equivalent:*

(1) $\varphi(\bar{x}, \bar{b})$ divides over A;

(2) $\varphi(\bar{x}, \bar{b})$ forks over A.

Proof. (1) $\Rightarrow$ (2) is obvious. We prove (2) $\Rightarrow$ (1): Since $\varphi(\bar{x}, \bar{b})$ forks over A, there exist $m < \omega$ and $\psi_i(\bar{x}, \bar{a}^i)$ such that $\psi_i(\bar{x}, \bar{a}^i)$ divides over A for $i < m$ and

$$(*) \qquad \varphi(\bar{x}, \bar{b}) \vdash \bigvee_{i<m} \psi_i(\bar{x}, \bar{a}^i).$$

Let $\bar{a} := \langle \bar{a}^0, \dots, \bar{a}^{m-1} \rangle$. By Corollary 3.3 we can fix a Morley sequence for $\mathrm{tp}(\bar{a}\bar{b}/A)$,

$$\langle \bar{a}_n\bar{b}_n \mid n < \omega \rangle, \quad \text{with} \quad \bar{a}_0 = \bar{a}, \bar{b}_0 = \bar{b}.$$

Let us write $\bar{a}_n = \langle \bar{a}_n^0, \dots, \bar{a}_n^{m-1} \rangle$ for every $n < \omega$.

Since $\langle \bar{b}_n \mid n < \omega \rangle$ is indiscernible over A, to show that $\varphi(\bar{x}, \bar{b})$ divides over A, it suffices to show that the set $\{ \varphi(\bar{x}, \bar{b}_n) \mid n < \omega \}$ is inconsistent. Assume that it is consistent and let $\bar{c}$ realize it. By the definition of Morley sequence,

$$\mathrm{tp}(\bar{a}\bar{b}/A) = \mathrm{tp}(\bar{a}_n\bar{b}_n/A), \quad \text{for every } n < \omega.$$

Hence, using an A-automorphism and (*), we conclude that

$$(**) \qquad \varphi(\bar{x}, \bar{b}_n) \vdash \bigvee_{i<m} \psi_i(\bar{x}, \bar{a}_n^i), \quad \text{for every } n < \omega.$$

By the choice of $\bar{c}$, we have $\models \varphi[\bar{c}, \bar{b}_n]$ for every $n < \omega$. Therefore, (**) implies that for every $n < \omega$ there exists $i(n) < m$ such that $\models \psi_{i(n)}[\bar{c}, \bar{a}_n^{i(n)}]$. By the pigeonhole principle, there exist an infinite $S \subseteq \omega$ and a fixed $k < m$ such that

$$(\dagger) \qquad \models \psi_k[\bar{c}, \bar{a}_n^k], \quad \text{for every } n \in S.$$

But $\langle \bar{a}_n^k \mid n \in S \rangle$ is a Morley sequence for $\mathrm{tp}(\bar{a}_0^k/A)$. Furthermore, (†) shows that the set $\{ \psi_k(\bar{x}, \bar{a}_n^k) \mid n \in S \}$ is consistent. Thus, $\psi_k(\bar{x}, \bar{a}^k)$ does not divide over A by Theorem 3.7. This contradicts the choice of $\psi_k(\bar{x}, \bar{a}^k)$. ⊣

Theorem 3.9 (Symmetry). *Let T be simple. Then $\mathrm{tp}(\bar{a}/A\bar{b})$ forks over A if and only if $\mathrm{tp}(\bar{b}/A\bar{a})$ forks over A.*

Proof. It is, of course, sufficient to prove one direction. Suppose $\mathrm{tp}(\bar{a}/A\bar{b})$ forks over A and take $\varphi(\bar{x}, \bar{c}, \bar{b}) \in \mathrm{tp}(\bar{a}/A\bar{b})$ such that $\varphi(\bar{x}, \bar{c}, \bar{b})$ forks over A. By Theorem 3.8, the formula $\varphi(\bar{x}, \bar{c}, \bar{b})$ divides over A. If $\mathrm{tp}(\bar{b}/A\bar{a})$ does not fork over A, we can choose a Morley sequence $I = \langle \bar{b}_n \mid n < \omega \rangle$ for $\mathrm{tp}(\bar{b}/A\bar{a})$ over A such that $\bar{b}_0 = \bar{b}$. We have $\models \varphi[\bar{a}, \bar{c}, \bar{b}]$, so, by the indiscernibility of I over $A\bar{a}$, we also have $\models \varphi[\bar{a}, \bar{c}, \bar{b}_n]$ for every $n < \omega$. Thus, $\{\varphi(\bar{x}, \bar{c}, \bar{b}_n) \mid n < \omega\}$ is consistent (as it is realized by $\bar{a}$). But $\langle \bar{c}\bar{b}_n \mid n < \omega \rangle$ is a Morley sequence for $\mathrm{tp}(\bar{c}\bar{b}/A)$, and hence $\{\varphi(\bar{x}, \bar{c}, \bar{b}_n) \mid n < \omega\}$ is inconsistent by Theorem 3.7. This contradiction shows that $\mathrm{tp}(\bar{b}/A\bar{a})$ must fork over A. ⊣

Theorem 3.10 (Transitivity). *Let T be simple and $A \subseteq B \subseteq C$. If $\mathrm{tp}(\bar{a}/C)$ does not fork over B and $\mathrm{tp}(\bar{a}/B)$ does not fork over A, then $\mathrm{tp}(\bar{a}/C)$ does not fork over A.*

Proof. It suffices to show that $\mathrm{tp}(\bar{a}/A\bar{c})$ does not fork over A for any $\bar{c} \in C$. By Symmetry, this is equivalent to showing that $\mathrm{tp}(\bar{c}/A\bar{a})$ does not fork over A for any $\bar{c} \in C$. Fix $\bar{c} \in C$ and suppose $\varphi(\bar{x}, \bar{d}, \bar{a}) \in \mathrm{tp}(\bar{c}/A\bar{a})$ forks over A. By Theorem 3.8,

$$(*) \qquad \varphi(\bar{x}, \bar{d}, \bar{a}) \text{ divides over } A.$$

Since $\text{tp}(\bar{a}/C)$ does not fork over B, the type $\text{tp}(\bar{a}/A\bar{c})$ does not fork over B, so by Symmetry and the fact that dividing implies forking,

(**) $\text{tp}(\bar{c}/A\bar{a})$ does not divide over B.

Now, since $\text{tp}(\bar{a}/B)$ does not fork over A, by Corollary 3.3 we can choose a Morley sequence $\langle \bar{a}_n \mid n < \omega \rangle$ for $\text{tp}(\bar{a}/B)$ over A. Let $I := \langle \bar{d}\bar{a}_n \mid n < \omega \rangle$, Then I is a Morley sequence for $\text{tp}(\bar{d}\bar{a}/B)$. By the definition of dividing and (**), the set $\{ \varphi(\bar{x}, \bar{d}, \bar{a}_n) \mid n < \omega \}$ must be consistent. Since I is a Morley sequence for $\text{tp}(\bar{d}\bar{a}/A)$, the formula $\varphi(\bar{x}, \bar{d}, \bar{a})$ does not divide over A by Theorem 3.7. This contradicts (*). Hence, $\text{tp}(\bar{a}/C)$ cannot fork over A. ⊣

We can now show the converse of Lemma 3.1.

Theorem 3.11. *Let T be simple. Let p be a type over B and $A \subseteq B$. The following conditions are equivalent:*

(1) p does not fork over A;

(2) $D[p, \Delta, k] = D[p \restriction A, \Delta, k]$, for every $k < \omega$ and Δ finite.

Proof. (2) ⇒ (1) is Lemma 3.1.

(1) ⇒ (2). By Finite Character, we may assume that $B = A \cup \bar{b}$ for some tuple $\bar{b}$. We show by induction on α that

$$D[p \restriction A, \Delta, k] \geq \alpha \quad \text{implies} \quad D[p, \Delta, k] \geq \alpha.$$

When $\alpha = 0$ or α is a limit ordinal, the implication is easy. Suppose that

$$D[p \restriction A, \Delta, k] \geq \alpha + 1.$$

By Corollary 2.18, we can find a formula $\varphi \in \Delta$ and a sequence $\langle \bar{a}_i \mid i < \omega \rangle$ indiscernible over A such that $\{ \varphi(\bar{x}, \bar{a}_i) \mid i < \omega \}$ is k-contradictory and

(*) $D[(p \restriction A) \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha$, for every $i < \omega$.

By the definition of the rank, it suffices to show that the sequence $\langle \bar{a}_i \mid i < \omega \rangle$ can be chosen so that

(**) $D[p \cup \varphi(\bar{x}, \bar{a}_i), \Delta, k] \geq \alpha$, for every $i < \omega$.

Using Lemma 2.4, we find $\bar{c}_i$ for $i < \omega$ realizing $(p \restriction A) \cup \varphi(\bar{x}, \bar{a}_i)$ such that

$$D[\text{tp}(\bar{c}_i/A \cup \bar{a}_i), \Delta, k] \geq \alpha.$$

Now, using Theorem 1.11 (as in the proof of Corollary 2.18), we may assume that the sequence $\langle \bar{c}_i\hat{\ }\bar{a}_i \mid i < \omega \rangle$ is indiscernible over A. By taking an A-automorphism, we may also assume that $\bar{c}_0$ realizes p. Since, by hypothesis, $\text{tp}(\bar{c}_0/A\bar{b})$ does not fork over A, the type $\text{tp}(\bar{b}/A\bar{c}_0)$ does not fork over A by Symmetry. Now, using Extension and an $A\bar{c}_0$-automorphism, we may further assume that $\text{tp}(\bar{b}/A\bar{c}_0\bar{a}_0)$ does not fork over A. Since forking and dividing are equivalent for simple theories, we can use Lemma 1.3 and an $A\bar{c}_0\bar{a}_0$-automorphism to assume that

(†) $\langle \bar{c}_i\hat{\ }\bar{a}_i \mid i < \omega \rangle$ is indiscernible over B.

Using now Extension and an $A\bar{c}_0\bar{a}_0$-automorphism, we may further assume that

(‡) $\text{tp}(\bar{b}/A \cup \{ \bar{c}_i, \bar{a}_i \mid i < \omega \})$ does not fork over A.

From (†) and the fact that $\bar{c}$ realizes p, we conclude that $\bar{c}_i$ realizes p for every $i < \omega$. By (‡) and Symmetry, the type $\text{tp}(\bar{c}_i/B\bar{a}_i)$ does not fork over A for any $i < \omega$. Therefore, by Monotonicity, $p \cup \varphi(\bar{x}, \bar{a}_i)$ does not fork over A, for any $i < \omega$. Thus, by induction hypothesis, $D[p, \Delta, k] \geq \alpha + 1$. ⊣

When the theory is stable, the following theorem follows from Shelah's the fact that types over models are stationary. The theorem is due to Kim and Pillay and generalizes a result of Hrushovski and Pillay in [HP1]).

The Independence Theorem will play a crucial role in the proof of the chain condition (Theorem 5.8).

Theorem 3.12 (The Independence Theorem). *Let T be simple and M be a model of T. Let A and B be sets such that $\mathrm{tp}(A/MB)$ does not fork over M. Let $p \in S(M)$. Let q be a nonforking extension of p over MA and r be nonforking extension of p over MB. Then, $q \cup r$ is a nonforking extension of p over MAB.*

Proof. Since p is complete, we know by Robinson's Consistency Lemma that $q \cup r$ is consistent. By Finite Character, it is sufficient to prove that if $\bar{a}, \bar{b}$ are such that $\mathrm{tp}(\bar{a}/M\bar{b})$ does not fork over M and $q(\bar{x}, \bar{y}), r(\bar{x}, \bar{z})$ are types with parameters from M such that $q(\bar{x}, \bar{a}), r(\bar{x}, \bar{b})$ do not fork over, then the union $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{b})$ does not fork over M. By Robinson's Consistency Lemma, we may assume that $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{b})$ is a type. Suppose, by contradiction, that $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{b})$ forks over M.

By Extension, we can find a realization $\bar{c}$ of $r(\bar{x}, \bar{b})$ such that $\mathrm{tp}(\bar{c}/M\bar{b})$ does not fork over M. By Existence, the type $r(\bar{c}, \bar{z}) \cup \mathrm{tp}(\bar{b}/M)$ does not fork over $M\bar{c}$, so by Extension we can find a realization $\bar{b}'$ of it such that

(*) $\mathrm{tp}(\bar{b}'/M\bar{a}\bar{c})$ does not fork over $M\bar{c}$.

Now, $\mathrm{tp}(\bar{c}/M\bar{b}) = \mathrm{tp}(\bar{c}/M\bar{b}')$, so, by Symmetry,

(**) $\mathrm{tp}(\bar{b}'/M\bar{c})$ does not fork over M.

By $(*)$, $(**)$ and Transitivity, $\mathrm{tp}(\bar{b}'/M\bar{a}\bar{c})$ does not for over M, so $\mathrm{tp}(\bar{c}/M\bar{a}\bar{b}')$ does not for over M by Symmetry. By the choice of $\bar{c}$ and $\bar{b}'$, we conclude that $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{b}')$ does not fork over M.

We will eventually contradict the following claim.

Claim. There does not exist a sequence $\langle \bar{a}_i{}^\frown\bar{d}_i \mid i < \omega \rangle$ indiscernible over M such that

- $q(\bar{x}, \bar{a}_0) \cup r(\bar{x}, \bar{d}_0)$ does not fork over M, and
- $q(\bar{x}, \bar{a}_0) \cup r(\bar{x}, \bar{d}_1)$ forks over M.

Proof of the claim. Suppose that there is $\langle \bar{a}_i{}^\frown\bar{d}_i \mid i < \omega \rangle$ as above. By Lemma 1.3 and Extension, there exists $\bar{c}$ realizing $q(\bar{x}, \bar{a}_0) \cup r(\bar{x}, \bar{d}_0)$ such that $\langle \bar{a}_i{}^\frown\bar{d}_i \mid i < \omega \rangle$ is indiscernible over $M\bar{c}$ and $\mathrm{tp}(\bar{c}/M \cup \{ \bar{a}_i{}^\frown\bar{d}_i \mid i < \omega \})$ does not fork over M. By Monotonicity, $q(\bar{x}, \bar{a}_0) \cup r(\bar{x}, \bar{d}_1)$ does not fork over M, a contradiction. ⊣

Since M is a model, there exists an ultrafilter $\mathcal{D}$ over M such that $\mathrm{tp}(\bar{b}/M) = \mathrm{tp}(\bar{b}'/M) = \mathrm{Av}(\mathcal{D}, M)$. Define $\langle \bar{b}_i \mid i < \omega \rangle$ such that $\bar{b}_0 = \bar{b}$ and $\bar{b}_i$ realizes $\mathrm{Av}(\mathcal{D}, M \cup \{ \bar{b}_j \mid j < i \})$. Then $\langle \bar{b}_i \mid i < \omega \rangle$ is indiscernible over M. By Lemma 1.3, since forking is the same as dividing, we may assume, by using an $M\bar{b}$-automorphism, that $\langle \bar{b}_i \mid i < \omega \rangle$ is indiscernible over $M\bar{a}$. Similarly, we define $\langle \bar{b}'_i \mid i < \omega \rangle$ such that $\bar{b}'_0 = \bar{b}'$ and $\bar{b}'_i$ realizes $\mathrm{Av}(\mathcal{D}, M \cup \{ \bar{b}'_j \mid j < i \})$. Again, we may assume, that $\langle \bar{b}'_i \mid i < \omega \rangle$ is indiscernible over $M\bar{a}$. Now, let us construct a third indiscernible sequence $\langle \bar{c}_i \mid i < \omega \rangle$, but now such that $\bar{c}_i$ realizes

$$\mathrm{Av}(\mathcal{D}, M \cup \{ \bar{b}_i \mid i < \omega \} \cup \{ \bar{b}'_i \mid i < \omega \} \cup \{ \bar{c}_j \mid j < i \}).$$

Then, both $\langle \bar{b}_i \mid i < \omega \rangle + \langle \bar{c}_i \mid i < \omega \rangle$ and $\langle \bar{b}'_i \mid i < \omega \rangle + \langle \bar{c}_i \mid i < \omega \rangle$ are indiscernible over M. Furthermore, by taking a longer sequence if necessary, we may assume that

$$\mathrm{tp}(\bar{a}\bar{c}_i/M) = \mathrm{tp}(\bar{a}\bar{c}_j/M), \quad \text{for every } i, j < \omega.$$

Suppose that for some (and hence all) $i < \omega$ the type $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c}_i)$ does not fork over M. We will find $\langle \bar{a}_i \mid i < \omega \rangle$ and $\langle b^i_m \mid m < \omega \rangle$ for $i < \omega$ such that for every $m < \omega$

$$\mathrm{tp}(\bar{a}_i \bar{b}^i_m / M) = \mathrm{tp}(\bar{a}\bar{c}_0 / M),$$
$$\mathrm{tp}(\bar{a}_{i+1} \bar{b}^i_m / M) = \mathrm{tp}(\bar{a}\bar{b}'_0 / M).$$

By Ramsey's Theorem, we can assume that $\langle \bar{a}_i {}^\frown \bar{b}^i_0 \mid i < \omega \rangle$ is indiscernible over M, and the claim is contradicted.

We construct $\bar{a}_i$ and $\langle \bar{b}^i_m \mid m < \omega \rangle$ by induction on i. Let f be an M-automorphism such that $f(\bar{c}_m) = \bar{b}'_m$ for $m < \omega$. We let $\bar{a}_0 = \bar{a}$, $\bar{b}^0_m = \bar{c}_m$ for every $m < \omega$, and define

$$\bar{a}_{i+1} = f(\bar{a}_i),$$
$$\bar{b}^{i+1}_m = f(\bar{b}^i_m), \qquad \text{for } m < \omega$$

(so, in particular, $\bar{b}^1_m = \bar{b}'_m$).

In case $q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c}_i)$ forks over M, for some $i < \omega$, we use the $\bar{b}_i$s, rather than the $\bar{b}'_i$s, to derive a similar contradiction.

$\dashv$

4. Indiscernibles Based on a Set

This section can be skipped without loss of continuity.

Corollary 4.13 assumes that T is simple. The rest of the section is valid for all first order theories.

We start by defining a distance between indiscernible sequences. If I and J are sequences of indiscernibles, we denote by $I + J$ the sequence that results from placing every element of J above every element of I.

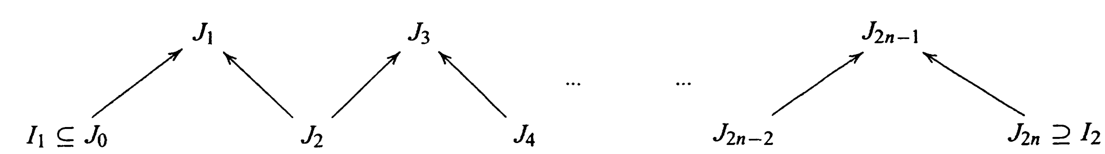
Definition 4.1. Let I_1 and I_2 be two infinite sequences indiscernible over the set A. We define the *distance between* I_1 *and* I_2 *over* A, denoted $d_A(I_1, I_2)$. We say that $d_A(I_1, I_2) \leq n$ for $n < \omega$, if there exists a sequence $\langle J_k \mid k \leq 2n \rangle$ satisfying the following conditions:

(1) every J_k is an infinite sequence, indiscernible over A;
(2) $I_1 \subseteq J_0$ and $I_2 \subseteq J_{2n}$;
(3) For every $k < n$ the sequence $J_{2k} + J_{2k+1}$ is indiscernible over A;
(4) For every $k < n$ the sequence $J_{2k+2} + J_{2k+1}$ is indiscernible over A.

We write:

$d_A(I_1, I_2) = \infty$ if there is no $n < \omega$ such that $d_A(I_1, I_2) \leq n$;
$d_A(I_1, I_2) = n$ if $n < \omega$ is smallest such that $d_A(I_1, I_2) \leq n$.

The following graph illustrates the case $d_A(I_1, I_2) \leq n$. An arrow from I to J indicates that $I + J$ is indiscernible over A.



Definition 4.2. Let I_1 and I_2 be indiscernible over the set A. We say that I_1 *is equivalent to* I_2 *over* A, denoted $I_1 \approx_A I_2$, if $d_A(I_1, I_2) < \infty$.

Remark 4.3. When the theory is stable, indiscernible sequences are indiscernible sets, so the relation $\approx_\emptyset$ coincides with the equivalence relation defined in [Sha], Chapter III.

We first state some easy lemmas.

Lemma 4.4. *Let A be a given set.*

(1) d_A is a metric on the class of sequences indiscernible over A;

(2) $\approx_A$ is an equivalence relation on the class of sequences indiscernible over A.

Proof. (2) is immediate from (1) and (1) is proved easily from the definitions. ⊣

Recall that a sequence is said to be (Δ, m)-indiscernible over a set A, if any two increasing subsequences of length m have the same Δ-type over A. When $\Delta = L(T)$, we simply say m-indiscernible over A. The following Proposition is central to derive the Corollary 4.6, which states that being at distance k is a first order property.

Proposition 4.5. *Let A be a set and $\ell, k < \omega$. Suppose $\ell > 0$. Let*

$$I_l := \langle \bar{a}_n^l \mid n < \omega \rangle$$

be indiscernible over A, for $l = 1, 2$, such that with $\ell(\bar{a}_0^1) = \ell(\bar{a}_0^2) = \ell$. The following conditions are equivalent:

(1) $d_A(I_1, I_2) \leq k$;

(2) For every $m < \omega$, for every finite $\Delta \subseteq L(T)$, every $\bar{a} \in A$, and every choice of $\bar{a}_0^l < \cdots < \bar{a}_m^l$ in I_l ($l = 1, 2$), there exists a set $\{ \bar{b}_i^j \mid j \leq 2k,\ i \leq m \}$ such that

(a) For every $i \leq m$ we have $\bar{b}_i^0 = \bar{a}_i^1$ and $\bar{b}_i^{2k} = \bar{a}_i^2$;

(b) For every $j < k$, both of the sequences

$$\langle \bar{b}_i^{2j} \mid i \leq m \rangle + \langle \bar{b}_i^{2j+1} \mid i \leq m \rangle, \qquad \langle \bar{b}_i^{2j+2} \mid i \leq m \rangle + \langle \bar{b}_i^{2j+1} \mid i \leq m \rangle$$

are (Δ, m)-indiscernible over $\bar{a}$.

Proof. Immediate, by compactness. ⊣

Corollary 4.6 (Type Definability of d_A). *Let A be a set, $k, \ell < \omega$ and suppose $\ell > 0$. Then there exists a type*

$$r_A^{k,\ell} := r_A^{k,\ell}(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1, \dots /A)$$

with parameters from A, such that for any two sequences $I_l = \langle \bar{a}_n^l \mid n < \omega \rangle$ indiscernible over A such that $\ell(\bar{a}_n^l) = \ell$ for $l = 1, 2$,

$$d_A(I_1, I_2) \leq k \quad \text{if and only if} \quad \bar{a}_0^1{}^\frown\bar{a}_0^2{}^\frown\bar{a}_1^1{}^\frown\bar{a}_1^2{}^\frown \dots \qquad \text{realizes } r_A^{k,\ell}.$$

Proof. We will use Proposition 4.5. Let $\{ \bar{x}_i \mid i < \omega \}$ be new variables for the sequence I_1, and $\{ \bar{y}_i \mid i < \omega \}$ for I_2. For every finite $\Delta \subseteq L(T)$, every finite $\bar{a} \in A$, and every $m < \omega$, let $\{ \bar{z}_i^j \mid i \leq m, j \leq 2k \}$ be new variables (to play the role of $\bar{b}_i^j$ in the previous proposition). Let the formula $\psi_{\Delta,m,\bar{a}}(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_m, \bar{y}_m)$ be the existential closure with respect to the variables $\{ \bar{z}_i^j \mid i \leq m, j \leq 2k \}$ of the conjunction of the following formulas (written informally for readability):

- $\bar{x}_i = \bar{z}_i^0$ for every $i \leq m$;
- $\bar{y}_i = \bar{z}_i^{2k}$ for every $i \leq m$;
- $\langle \bar{z}_i^{2j} \mid i \leq m \rangle + \langle \bar{z}_i^{2j+1} \mid i \leq m \rangle$ is (Δ, m)-indiscernible over $\bar{a}$ for every $j < k$;
- $\langle \bar{z}_i^{2j+2} \mid i \leq m \rangle + \langle \bar{z}_i^{2j+1} \mid i \leq m \rangle$ is (Δ, m)-indiscernible over $\bar{a}$ for every $j < k$.

This is possible since Δ and $\bar{a} \in A$ and m are finite. Then, let

$$r_A^{k,\ell} := \{ \psi_{\Delta,m,\bar{a}} \mid \Delta \subseteq L(T),\ \Delta \text{ is finite}, m < \omega, \bar{a} \in A \}.$$

Proposition 4.5 shows that $r_A^{k,\ell}$ is as required. ⊣

Definition 4.7. Let I be a sequence indiscernible over the set A and let $\Phi \subseteq \mathrm{Aut}_A(\mathfrak{C})$.

(1) The *orbit* of I with respect to Φ is

$$\Gamma^{\Phi}_A(I) := \{\, F(I)/\approx_A \mid F \in \Phi \,\}.$$

When $\Phi = \mathrm{Aut}_A(\mathfrak{C})$, we simply write $\Gamma_A(I)$.

(2) We say that *I is based on A* if $|\Gamma_A(I)| < \|\mathfrak{C}\|$. [1]

Theorem 4.8. *Let I be a sequence indiscernible over the set A. Then I is based on A if and only if $|\Gamma_A(I)| \leq 2^{|T|+|A|}$.*

We first prove a combinatorial lemma. The method used in the proof is often used by model theorists to prove the Erdős-Rado Theorem.

Lemma 4.9 (End Homogeneity Lemma). *Let λ be an infinite cardinal, and let $n < \omega$. Suppose that*

$$G\colon [(2^{\lambda})^+]^{n+1} \to 2^{\lambda}.$$

Then there exist a set $S \subseteq (2^{\lambda})^+$ of cardinality λ^+ and an ordinal $\alpha^* < (2^{\lambda})^+$ such that

$$G(i_1, \ldots, i_n, i_{n+1}) = G(i_1, \ldots, i_n, \alpha^*), \qquad \textit{whenever } i_1 < \cdots < i_n < i_{n+1} \in S.$$

Proof. Let G be given as in the statement of the lemma above and form the following model:

$$M = \langle\, (2^{\lambda})^+, \in, G, \alpha \,\rangle_{\alpha \in 2^{\lambda}}.$$

Using the Downward-Löwenheim-Skolem Theorem, construct an increasing continuous sequence of models $\langle\, M_i \mid i < \lambda^+ \,\rangle$ such that

(1) $M_i \prec M$;
(2) $\|M_i\| = 2^{\lambda}$;
(3) M_{i+1} realizes all the types over subsets of M_i of cardinality λ which are already realized in M.

Let us see that this implies the conclusion of the lemma. Let $M^* := \bigcup_{i<\lambda} M_i$. By (3), we can choose an ordinal $\alpha^* \in M - M^*$ such that $\alpha^* > \sup M^*$. Condition (2) allows us to find $\alpha_i \in M_{i+1} - M_i$ such that

$$\mathrm{tp}(\alpha_i/\{\, \alpha_j \mid j < i \,\}) = \mathrm{tp}(\alpha^*/\{\, \alpha_j \mid j < i \,\}), \qquad \text{for every } i < \lambda^+.$$

It is easy to see that $S = \{\, \alpha_i \mid i < \lambda^+ \,\}$ and α^* are as required. ⊣

Proof of Theorem 4.8. Necessity is trivial. To prove sufficiency, let $\lambda = |T| + |A|$ and suppose that there exist $I = \langle\, \bar{a}_n \mid n < \omega \,\rangle$ and A as in the statement of the theorem such that

$$\big|\, \{\, F(I)/\approx_A \mid F \in \mathrm{Aut}_A(\mathfrak{C}) \,\} \,\big| \geq (2^{\lambda})^+.$$

For each $i < (2^{\lambda})^+$, fix an A-automorphism F_i such that $F_i(I) \not\approx_A F_j(I)$ if $i \neq j$. Let $\bar{a}^i_n = F_i(\bar{a}_n)$. By Corollary 4.6, for every $k < \omega$ and $i \neq j$ we can find

$$\varphi^k_{i,j}(\bar{x}_0, \bar{y}_0, \ldots, \bar{x}_{n^k(i,j)}, \bar{y}_{n^k(i,j)}; \bar{b}^k_{i,j}) \in r^{k,\ell(\bar{a}_0)}_A$$

such that

$$\models \neg\varphi^k_{i,j}[\bar{a}^i_0, \bar{a}^j_0, \ldots, \bar{a}^i_{n^k(i,j)}, \bar{a}^j_{n^k(i,j)}; \bar{b}^k_{i,j}], \qquad \text{for } i \neq j.$$

---

[1] This is equivalent to saying that the class $\Gamma_A(I)$ is a set.

The End Homogeneity Lemma provides a set $S \subseteq \lambda^+$ of cardinality λ^+ and $j^* < (2^\lambda)^+$ such that if we let $\varphi_i^k = \varphi_{i,j^*}^k$ and $n^k(i) = n^k(i, j^*)$, we have

$$\models \neg\varphi_i^k[\bar{a}_0^i, \bar{a}_0^j, \dots, \bar{a}_{n^k(i)}^i, \bar{a}_{n^k(i)}^j; \bar{b}_i^k], \qquad \text{for every } i \in S \text{ and } k \in \omega.$$

Without loss of generality, we may assume $S = \lambda^+$. By the pigeonhole principle, we construct a sequence $\langle S_k \mid k < \omega \rangle$ and $i_k < (2^\lambda)^+$ such that if we let $\varphi^k = \varphi_{i_k}^k$ and $n^k = n^k(i)$, we have

(1) $S_{k+1} \subseteq S_k$ and $S_0 = S$;
(2) $|S_k| = \lambda^+$;
(3) $\models \neg\varphi^k[\bar{a}_0^i, \bar{a}_0^j, \dots, \bar{a}_{n^k}^i, \bar{a}_{n^k}^j; \bar{b}^k]$ for every $i \neq j \in S_k$.

Let us now expand the language of T with names for the elements of A. In addition, for every $n < \omega$ and every $\alpha < \|\mathfrak{C}\|$ let $\bar{c}_n^\alpha$ be new constants. Let T^* be the union of the following sentences, written informally for readability:

· T;
· For every α the sequence $I_\alpha = \langle \bar{c}_n^\alpha \mid n < \omega \rangle$ is indiscernible over A;
· $\neg\varphi^k(\bar{c}_0^\alpha, \bar{c}_0^\beta, \bar{c}_1^\alpha, \bar{c}_1^\beta, \dots; \bar{b}^k)$, whenever $\alpha < \beta$ and $k < \omega$.

Then T^* is consistent by (1)–(3). The conclusion of the theorem follows since the third item guarantees that $d_A(I_\alpha, I_\beta) = \infty$ for $\alpha < \beta$. ⊣

Proposition 4.10. *Let I be a sequence of indiscernibles over a set A and let k be an integer greater than* 1. $\Gamma \subseteq \mathrm{Aut}_A(\mathfrak{C})$ *be such that*

$$d_A(f(I), g(I)) \geq k, \quad \textit{for } f(I) \neq g(I) \textit{ in } \Gamma.$$

Then $|\Gamma| < \|\mathfrak{C}\|$ *implies* $|\Gamma| \leq 2^{|T|+|A|}$.

Proof. Similar to the proof of Theorem 4.8, using a single application of the Erdős-Rado theorem. ⊣

Theorem 4.11. *Let $A \subseteq B$ and $p \in S(B)$. Let I be a Morley sequence for p over A. Then, for any $\Phi \subseteq \mathrm{Aut}_A(\mathfrak{C})$ of cardinality $(2^{|A|+|T|})^+$ there exist $f \neq g$ in Φ such that $d_A(f(I), g(I)) \leq 1$.*

Proof. Let $p(\bar{x}) \in S(B)$ not fork over A. Let $\ell = \ell(\bar{x})$ and let $I = \langle \bar{b}_k \mid k < \omega \rangle$ be a Morley sequence for p over A.Suppose, by contradiction, that there exists $\Phi \subseteq \mathrm{Aut}_A(\mathfrak{C})$ with $|\Phi| > 2^{|A|+|T|}$ such that

$$d_A(f(I), g(I)) \geq 2, \qquad \text{for every } f, g \in \Phi \text{ with } f \neq g.$$

Let $\lambda := 2^{|A|+|T|}$. By Proposition 4.10 for every $\alpha < \beth_{\lambda^+ + 2}$ we find an A-automorphism f_α such that

$$(*) \qquad d(f_\alpha(I), f_\beta(I)) \geq 2, \qquad \text{for } \alpha < \beta < \beth_{\lambda^+ + 2}.$$

For convenience, let us write $I_\alpha = f_\alpha(I) = \langle \bar{b}_k^\alpha \mid k < \omega \rangle$. By Corollary 4.6, there exists a first order formula

$$\varphi_{\alpha,\beta}(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_{k(\alpha,\beta)}, \bar{y}_{k(\alpha,\beta)}; \bar{b}_{\alpha,\beta}) \in r_A^{2,\ell}$$

witnessing (*), *i.e.*,

$$\models \neg\varphi_{\alpha,\beta}[\bar{b}_0^\alpha, \bar{b}_0^\beta, \dots, \bar{b}_{k(\alpha,\beta)}^\alpha, \bar{b}_{k(\alpha,\beta)}^\beta; \bar{b}_{\alpha,\beta}], \qquad \text{for } \alpha < \beta.$$

By an application of the Erdős-Rado to the function

$$(\alpha, \beta) \mapsto \varphi_{\alpha,\beta}(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_{k(\alpha,\beta)}, \bar{y}_{k(\alpha,\beta)}; \bar{b}_{\alpha,\beta}),$$

we may fix a formula $\varphi(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_k, \bar{y}_k; \bar{b})$ over A such that for every $\alpha < \beta < \beth_{\lambda^+}$

$$\varphi(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_k, \bar{y}_k; \bar{b}) = \varphi_{\alpha,\beta}(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_{k(\alpha,\beta)}, \bar{y}_{k(\alpha,\beta)}; \bar{b}_{\alpha,\beta}).$$

Claim. For every $n < \omega$ there exists a sequence $J_n = \langle \bar{a}^n_k \mid k < \omega \rangle$ indiscernible over A such that

(1) $\mathrm{tp}(\bar{a}^n_0, \bar{a}^n_1, \dots / A) = \mathrm{tp}(\bar{b}_0, \bar{b}_1, \dots / A)$ for every $n < \omega$;
(2) The sequence $\langle \bar{a}^{n\frown}_0 \dots {}^\frown\bar{a}^n_k \mid n < \omega \rangle$ is indiscernible over A, for every $k < \omega$;
(3) For every $m < n$ and every sufficiently large l there exists $\alpha < \beta$ such that

$$\mathrm{tp}(\bar{a}^m_0\bar{a}^n_0, \bar{a}^m_1\bar{a}^n_1, \dots, \bar{a}^m_l\bar{a}^n_l / A) = \mathrm{tp}(\bar{b}^\alpha_0\bar{b}^\beta_0, \bar{b}^\alpha_1\bar{b}^\beta_1, \dots, \bar{b}^\alpha_l\bar{b}^\beta_l / A).$$

Proof of the claim. We construct a sequence $\langle p_m \mid m < \omega \rangle$ such that p_m is a consistent set of formulas with free variables from $\{ \bar{x}^n_k \mid n, k \leq m \}$ and parameters from A, and

(1) $p_m \subseteq p_{m+1}$;
(2) For every $\gamma < \lambda^+$ there exists an increasing sequence

$$\bar{\alpha}^\gamma := \langle \alpha_i(\gamma) \mid i < \beth_\gamma \rangle$$

of ordinals less than $\beth_{\lambda^+}$ satisfying

(a) For every $m < \omega$ the sequence $\langle \bar{a}^{\alpha_i(\gamma)\frown}_0 \dots {}^\frown\bar{a}^{\alpha_i(\gamma)}_m \mid i < \beth_{\lambda^+} \rangle$ is $m+1$-indiscernible over the set A;
(b) For every $m < \omega$ and $i_0 < \dots < i_m < \beth_{\lambda^+}$

$$p_m = \mathrm{tp}(\{ \bar{a}^{\alpha_{i_n}(\gamma)}_k \mid n, k \leq m \} / A).$$

The construction is by induction on m. For $m = 0$, we let

$$p_0 := \mathrm{tp}(\bar{a}_0 / A) \quad \text{and} \quad \bar{\alpha}^\gamma := \langle i \mid i < \beth_\gamma \rangle.$$

Now suppose $m \geq 0$ and that (a) and (b) hold for m. Given $\gamma < \lambda^+$, define

$$\gamma_0 := \gamma + m^2 \cdot \ell(\bar{a}_0) + 1 + m + 2.$$

By induction hypothesis, choose $\bar{\alpha}^{\gamma_0}$ satisfying (a) and (b) at stage m. The function

$$(\beta_0, \dots, \beta_{m+1}) \mapsto \mathrm{tp}(\{ \bar{a}^{\beta_n}_k \mid n, k \leq m+1 \} / A)$$

is a coloring from $[\bar{\alpha}^{\gamma_0}]^{m+2}$ into $S(A)$. Since $|\bar{\alpha}^{\gamma_0}| = \beth_{\gamma_0}$, by the Erdős-Rado Theorem there exist a type $p_{m+1} \in S(A)$ and a sequence $S \subseteq \bar{\alpha}^{\gamma_0}$ of cardinality $\beth_{\gamma+m^2\cdot l(\bar{a}_0)+1}$, satisfying (b) at stage $m+1$. To see that (a) holds, consider the coloring on $[S]^{m+2}$ given by

$$(\beta_0, \dots, \beta_{m+1}) \mapsto \mathrm{tp}(\bar{a}^{\beta_0}_0 \dots \bar{a}^{\beta_0}_m,\ \bar{a}^{\beta_1}_0 \dots \bar{a}^{\beta_1}_m, \dots,\ \bar{a}^{\beta_m}_0 \dots \bar{a}^{\beta_m}_m / A).$$

By the Erdős-Rado Theorem, there exists an increasing subsequence of S,

$$\bar{\alpha}^\gamma := \langle \alpha_i(\gamma) \mid i < \beth_\gamma \rangle,$$

monochromatic with respect to the coloring. Thus, (a) holds at stage $m+1$ and (b) still holds for this subsequence.

This suffices to prove the claim. The consistency of every p_m and (1) imply that $\bigcup_{m<\omega} p_m$ is a consistent type in the variables $\{ \bar{x}^n_k \mid k, n < \omega \}$ over the set A. Pick $\{ \bar{b}^n_k \mid k, n < \omega \}$ realizing $\bigcup_{m<\omega} p_m$. Then, (1) of the claim follows from (b) at stage m; (2) follows from the fact that (a) holds for every $m < \omega$, and (3) is implied by (b). ⊣

Condition (3) of the claim with $l = k$ and the choice of $\varphi(\bar{x}_0, \bar{y}_0, \dots, \bar{x}_k, \bar{y}_k; \bar{b})$ yields $d_A(J_0, J_1) \geq 2$.

We shall derive a contradiction by showing that

$$(\dagger) \qquad d_A(J_0, J_1) \leq 1.$$

In order to prove (†), let $\{ \bar{c}_n \mid n < \omega \}$ be a set of constants not in the language, and let Γ be the union of the following set of sentences, written informally for readability:

- T;
- $\langle \bar{a}_0^0, \bar{a}_1^0, \bar{a}_2^0, \dots, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots \rangle$ is indiscernible over A;
- $\langle \bar{a}_0^1, \bar{a}_1^1, \bar{a}_2^1, \dots, \bar{c}_0, \bar{c}_1, \bar{c}_2, \dots \rangle$ is indiscernible over A.

We use the claim and the Concatenation Lemma (Lemma 1.13) to prove the consistency of Γ. By (1) of the claim, J_0 is a Morley sequence for $\mathrm{tp}(\bar{a}_0^0/A)$, since I is a Morley sequence. By (1) and (2) of the Claim and the Concatenation Lemma, for every $n < \omega$ there exists an A-automorphism g_n such that

$$\langle \bar{a}_0^k, \dots, \bar{a}_n^k, g_n(\bar{b}_0), g_n(\bar{b}_1), \dots \rangle \text{ is indiscernible over } A, \quad \text{for every } k < \omega.$$

Therefore,

$$\langle \bar{a}_0^0, \dots, \bar{a}_n^0, g_n(\bar{b}_0), g_n(\bar{b}_1), \dots \rangle \text{ is indiscernible over } A$$

and also

$$\langle \bar{a}_0^1, \dots, \bar{a}_n^1, g_n(\bar{b}_0), g_n(\bar{b}_1), \dots \rangle \text{ is indiscernible over } A.$$

Thus, Γ is finitely consistent. The proof is now complete. ⊣

The next corollary is immediate.

Corollary 4.12. *If I is a Morley sequence for $p \in S(B)$ over A, then I is based on A.*

Corollary 4.13. *Suppose that T is simple. Then for every set A and every type $p \in S(A)$ there exists a sequence $\langle \bar{a}_n \mid n < \omega \rangle$ indiscernible over A such that every $\bar{a}_n$ realizes p and $\langle \bar{a}_n \mid n < \omega \rangle$ is based on A.*

Proof. By Corollary 3.3 and the preceding theorem. ⊣

5. Shelah's Boolean Algebra

The argument we present in this section differs from [Sh93] only in that we use a different partial order. The partial order considered in [Sh93] is defined through weak dividing; here we use forking. The proof of the chain condition was communicated to us by Shelah in a recent correspondence. It uses the Hrushovski-Kim-Pillay Independence Theorem (Theorem 3.12). We are grateful to Shelah for allowing us to include it here.

Recall that a pair of cardinals (λ, κ) is in $\mathrm{SP}(T)$ if every model of cardinality λ has a κ-saturated elementary extension of the same cardinality. (See the introduction.)

Definition 5.1. Let $p(\bar{x})$ be a type over C and A be a set containing C. We define

$$W(p, A) := \{ \varphi(\bar{x}; \bar{a}) \mid \text{The formula } \varphi(\bar{x}; \bar{a}) \text{ does not fork over } p,\ \bar{a} \in A \}.$$

We identify two formulas $\varphi(\bar{x}; \bar{b})$ and $\psi(\bar{x}; \bar{c})$ when $\varphi(\mathfrak{C}; \bar{b}) = \psi(\mathfrak{C}; \bar{c})$.

The set $W(p, A)$ is partially ordered by

$$\varphi(\bar{x}; \bar{b}) \leq \psi(\bar{x}; \bar{c}) \quad \text{if and only if} \quad \psi(\bar{x}; \bar{c}) \vdash \varphi(\bar{x}; \bar{b}).$$

Proposition 5.2. *Suppose $C \subseteq A$ and $p \in S(A)$. Then the partially ordered set*

$$\langle W(p, A), \leq \rangle$$

is a distributive semi-lower lattice.

Proof. It suffices to show that $W(p, A)$ is closed under finite disjunctions, but this is an obvious property of forking. ⊣

We will say that two formulas $\varphi, \psi \in W(p, A)$ are *incompatible* if there is no $\rho \in W(p, A)$ such that $\rho \geq \varphi$ and $\rho \geq \psi$.

Proposition 5.3. *The formulas $\varphi(\bar{x}; \bar{b}), \psi(\bar{x}; \bar{c}) \in W(p, A)$ are compatible if and only if $\varphi(\bar{x}; \bar{b}) \wedge \psi(\bar{x}; \bar{c}) \in W(p, A)$.*

Proof. Sufficiency is trivial. To prove the converse, let $\rho(\bar{x}; \bar{d})$ be such that $\rho \geq \varphi$ and $\rho \geq \psi$. By definition, $\varphi(\bar{x}; \bar{b}) \vdash \rho(\bar{x}; \bar{d})$ and $\psi(\bar{x}; \bar{c}) \vdash \rho(\bar{x}; \bar{d})$. Therefore, $[\varphi(\bar{x}; \bar{b}) \wedge \psi(\bar{x}; \bar{c})] \vdash \rho(\bar{x}; \bar{d})$. Since $\rho(\bar{x}; \bar{d})$ does not fork over p, also $\varphi(\bar{x}; \bar{b}) \wedge \psi(\bar{x}; \bar{c})$ does not fork over p. ⊣

We will use some facts about the completion of partially ordered sets to boolean algebras. In particular, we need to recall the following notion. A partially ordered set $\langle P, \leq \rangle$ is *separative* if for every $p, q \in P$ such that $p \not\leq q$ there exists $r \in P$ with $r \not\leq q$ and $q \not\leq r$.

Fact 5.4. *If $\langle P, \leq \rangle$ is separative, there exists a unique complete boolean algebra $B - \{0\}$ containing P such that*

(1) The order of B extends that of P;
(2) P is dense in B.

For the proof, see Lemma 17.2 in Thomas Jech's book [Je]

Notice that since the $W(p, A)$ is in general not closed under propositional conjunctions, the partially ordered set $W(p, A)$ is not separative. However, the following fact (Lemma 17.3 of [Je]) allows us to circumvent this difficulty.

Fact 5.5. *Let $\langle P, \leq_P \rangle$ be an arbitrary partially ordered set. Then there exist a unique separative partially ordered set $\langle Q, \leq_Q \rangle$ and a function $h : P \to Q$ such that*

(1) If $p \leq_P q$, then $h(p) \leq_Q h(q)$;
(2) p and q are compatible in P if and only if $h(p)$ and $h(q)$ are compatible in Q.

From Facts 5.4 and 5.5 one gets:

Corollary 5.6. *Let $C \subseteq A$ be sets and $p \in S(C)$. Then there exist a unique complete boolean algebra $B_{p,A}$ and a function e_p: $W(p, A) \to B_{p,A}$ such that*

(1) If $\varphi(\bar{x}; \bar{b}) \vdash \psi(\bar{x}; \bar{c})$, then $e_p(\psi) \leq_{B_{p,A}} e_p(\varphi)$;
(2) The formulas $\varphi(\bar{x}; \bar{b})$ and $\psi(\bar{x}; \bar{c})$ are compatible in $W(p, A)$ if and only if in $B_{p,A}$

$$e_p(\varphi(\bar{x}; \bar{b})) \cdot e_p(\psi(\bar{x}; \bar{c})) \neq 0;$$

(3) The image of $W(p, A)$ under e_p is dense in $B_{p,A}$.

In [Sh 80], Shelah introduced a generalization of Martin's Axiom that consistently holds above the continuum. He denotes that principle by $(\mathrm{Ax}_0\mu)$. The tradeoff is that the countable chain condition is replaced by stronger requirements, namely,

- The forcing notion is complete, and
- The forcing conditions are compatible essentially on a club.

For a more complete description see [Sh 80]. The proof is a variant of the traditional finite support iteration used to show that the consistency of ZFC+GCH implies the consistency of ZFC+$\neg$CH + MA.

In [Sh93] Shelah claims that if the ground model satisfies GCH, then for a class of regular cardinalities $\mathcal{R}$ (such that for every $\mu \in \mathcal{R}$ the next element of $\mathcal{R}$ is much larger than μ) there exists a generic extension preserving cardinals and cofinalities such that

(1) $(\mathrm{Ax}_0\mu)$ holds;
(2) $2^\mu >> \mu^+$ and $\mu^{<\mu} = \mu$, for every cardinal $\mu \in \mathcal{R}$.

The construction is by class forcing.

Recall that a boolean algebra is said to have the *μ-chain condition* if the size of every antichain is less than μ.

Rather than stating $(\mathrm{Ax}_0\mu)$ specifically, we will quote as a fact the only consequence of it that we will use. The enthusiastic reader can find a complete proof of the following in Lemma 4.13 of [Sh93].

Fact 5.7. *Suppose* $(\mathrm{Ax}_0\mu)$ *and* $\mu^{<\mu} = \mu$ *holds. Let B be a boolean algebra of cardinality less than 2^μ satisfying the μ-chain condition. Then $B - \{0\}$ is the union of μ ultrafilters.*

Corollary 5.6 will be used together with Fact 5.7 to find κ-saturated elementary extensions of models of a simple theory.

In order to apply Fact 5.7 we need to show that for $C \subseteq A$ and $p \in S(C)$, the boolean algebra $B_{p,A}$ has the μ^+ chain condition. Notice that, below, μ is independent of A and depends only on $|C| + |T|$.

Theorem 5.8. *Let T be simple. For every $C \subseteq A$ and every $p \in S(C)$ the partially ordered set $W(p, A)$ has the $(2^{|T|+|C|})^+$-chain condition.*

Proof. Let $\lambda = (2^{|T|+|C|})^+$ and fix $\{\varphi_i(\bar{x}; \bar{a}_i) \mid i < \lambda\} \subseteq W(p, A)$. We will show that $\{\varphi_i(\bar{x}; \bar{a}_i) \mid i < \lambda\}$ is not an antichain by finding $i < j < \lambda$ such that

$$p \cup \{\varphi_i(\bar{x}; \bar{a}_i), \varphi_j(\bar{x}; \bar{a}_j)\} \text{ does not fork over } C.$$

To this end, choose $\langle M_i \mid i < \lambda\rangle$ an increasing, continuous chain of models such that:

(1) $C \subseteq M_0$;
(2) $\bar{a}_i \in M_{i+1}$, for $i < \lambda$;
(3) $\|M_i\| = 2^{|T|+|C|}$, for $i < \lambda$;

Consider the following stationary subset of λ.

$$S := \{\delta < \lambda \mid \mathrm{cf}\,\delta = |T|^+\}.$$

Define a function $f: S \to \lambda$ by

$$f(\delta) := \min\{j \mid \mathrm{tp}(\bar{a}_\delta/M_\delta) \text{ does not fork over } M_j\}.$$

Since T is simple, for every $\delta \in S$ there exists $B \subseteq M_\delta$ of cardinality at most $|T|$ such that $\mathrm{tp}(\bar{a}_\delta/M_\delta)$ does not fork over B. Since $\mathrm{cf}\,\delta = |T|^+$, there is $j < \delta$ such that $B \subseteq M_j$. This shows that $f(\delta) < \delta$ for every $\delta \in S$. Hence, by Fodor's Lemma ([Je], Theorem 1.7.22), there exists a stationary $S^* \subseteq S$ and a fixed $j < \lambda$ such that $\mathrm{tp}(\bar{a}_\delta/M_\delta)$ does not fork over M_j, for every $\delta \in S^*$. Without loss of generality, we may assume that $S^* = \lambda$ and $j = 0$, *i.e.*,

$$\mathrm{tp}(\bar{a}_i/M_i) \text{ does not fork over } M_0, \quad \text{for every } i < \lambda.$$

By simplicity (see Theorem 3.4), for every $i < \lambda$ there exists $N_i \prec M_0$ of cardinality $|C| + |T|$ such that N_i contains C and $\mathrm{tp}(\bar{a}_i/M_i)$ does not fork over N_i. But, there are at most $2^{|C|+|T|}$ subsets of M_0 of cardinality $|C| + |T|$. Hence, by the pigeonhole principle, there exists a subset $S^* \subseteq \lambda$ of cardinality λ and a model $N \prec M_0$ of cardinality $|C| + |T|$ such that $N_i = N$ for every $i \in S^*$. Without loss of generality, we may assume that $S^* = \lambda$, *i.e.*,

(*) $\qquad \operatorname{tp}(\bar{a}_i/M_i)$ does not fork over N, for every $i < \lambda$.

Now, $p \cup \varphi_i(\bar{x}; \bar{a}_i)$ does not fork over C by definition. Hence, by Extension, we can find $q_i \in S(N\bar{a}_i)$ extending $p \cup \varphi_i(\bar{x}; \bar{a}_i)$ such that

(**) $\qquad q_i$ does not fork over C, for every $i < \lambda$.

But, $|S(N)| \leq 2^{|C|+|T|}$, so, by the pigeonhole principle again, there exists a subset $S^* \subseteq \lambda$ of cardinality λ and a type $q \in S(N)$ such that $q_i \upharpoonright N = q$ for every $i \in S^*$. Without loss of generality, we may assume that $S^* = \lambda$, *i.e.*,

$$q_i \upharpoonright N = q, \quad \text{for every } i < \lambda.$$

Thus, by the choice of q_i,

(***) $\qquad q \cup \varphi_i(\bar{x}; \bar{a}_i)$ is a nonforking extension of $q \in S(N)$, for every $i < \lambda$.

Now, fix $i < j < \lambda$. Recall that $\bar{a}_i \in M_j$ (by (2)) and $N \prec N_j$. Therefore, by (*) and Monotonicity, we conclude that

(†) $\qquad \operatorname{tp}(\bar{a}_j/N\bar{a}_i)$ does not fork over N.

But now, the Independence Theorem (Theorem 3.12) applied to (***) and (†) shows that

(‡) $\qquad q \cup \{\varphi_i(\bar{x}; \bar{a}_i), \varphi_j(\bar{x}; \bar{a}_j)\}$ does not fork over N.

By (**), (‡) and Transitivity,

$$q \cup \{\varphi_i(\bar{x}; \bar{a}_i), \varphi_j(\bar{x}; \bar{a}_j)\} \text{ does not fork over } C.$$

Thus, $p \cup \{\varphi_i(\bar{x}; \bar{a}_i), \varphi_j(\bar{x}; \bar{a}_j)\}$ does not fork over C by Monotonicity. ⊣

Corollary 5.9. *Let T be simple. Let $C \subseteq A$ and $p \in S(C)$ the boolean algebra $B_{p,A}$ has the $(2^{|T|+|C|})^+$-chain condition.*

Proof. By Theorem 5.8 and Corollary 5.6. ⊣

The following facts are fairly well-known (see, for example, [Sha]) and show that, for T simple, the problem of characterizing the pairs (λ, κ) such that $(\lambda, \kappa) \in \mathrm{SP}(T)$ is interesting only when T is unstable and $\kappa < \lambda$ and $\lambda^{<\kappa} > \lambda \geq |D(T)|$.

Fact 5.10.

(1) Let $\lambda \geq 2^{|T|}$. If $\lambda^{<\kappa} = \lambda$, then $(\lambda, \kappa) \in \mathrm{SP}(T)$.
(2) Let $\lambda \geq 2^{|T|}$. If $\lambda^{<\lambda} > \lambda$, then $(\lambda, \lambda) \in \mathrm{SP}(T)$ if and only if T is stable in λ.

We now proceed to the proof of the main theorem. We will first prove three simple propositions. The reader may want to skip to Theorem 5.15 below before reading the proofs of Propositions 5.11, 5.13, and 5.14.

Proposition 5.11. *Let $\lambda > \kappa$. Then $(\lambda, \kappa) \in \mathrm{SP}(T)$ if and only if the following property holds.*

> (♦): *For every set A of cardinality λ there exists $S \subseteq S(A)$ of cardinality λ such that every type over a subset of A of cardinality less than κ has an extension in S.*

Proof. Sufficiency is clear. To prove necessity, let M be a model of cardinality λ and construct an increasing and continuous sequence of models $\langle M_i \mid i < \kappa^+ \rangle$, such that

(1) $M_0 = M$;
(2) $\|M_i\| = \lambda$, for every $i < \kappa^+$;
(3) M_{i+1} realizes every type over subsets of M_i of cardinality less than κ.

We let $N = \bigcup_{i<\kappa^+} M_i$. Then N is a κ-saturated extension of M of cardinality λ. ⊣

Remark 5.12. Proposition 5.11 can be regarded as a statement about the boolean algebras $B_{p,A}$ as follows. Let $p \in S(C)$ and suppose $q \in S(A)$ extends p and does not fork over C. Then, $q \subseteq W(p, A) \subseteq B_{p,A}$, and q has the finite intersection property. Conversely, if $F \subseteq B_{p,A}$ is an ultrafilter, then the set of formulas

$$q_F = \{ \varphi(x, \bar{a}) \in W(p, A) \mid \text{There existse } e \in F \text{ with } \varphi(x, \bar{a}) \leq e \}$$

is a complete type extending p which does not fork over C.

Proposition 5.13. *Suppose that T is simple, and let $\lambda > \kappa \geq \kappa(T)$, such that $\lambda^{<\kappa(T)} = \lambda \geq |D(T)|$. Then $(\lambda, \kappa) \in \mathrm{SP}(T)$ if the following property holds.*

> (♦♦): *For every set A of cardinality λ and every complete type p over a subset of A of cardinality less than $\kappa(T)$ the boolean algebra $B_{p,A}$ contains a family $\mathfrak{D}_{p,A}$ of cardinality λ of ultrafilters of $B_{p,A}$, such that every subset of $B_{p,A}$ of cardinality less than κ with the finite intersection property can be extended to an ultrafilter in $\mathfrak{D}_{p,A}$.*

Proof. We show that Condition (♦♦) implies Condition (♦) of Proposition 5.11. Since $\lambda^{<\kappa(T)} = \lambda$, there are only λ subsets of A of cardinality $\kappa(T)$. Since $\lambda \geq |D(T)|$, there are at most λ complete types over every of these subsets. Therefore there are only λ boolean algebras of the form$B_{p,A}$. Thus,

$$(\dagger) \qquad \Big| \bigcup_{p,A} \mathfrak{D}_{p,A} \Big| = \lambda$$

Now let $S \subseteq S(A)$ be the set of types of the form

$$q_F = \{ \varphi(x, \bar{a}) \in W(p, A) \mid \text{There exists } e \in F \text{ with } \varphi(x, \bar{a}) \leq e \},$$

where $F \in \bigcup_{p,A} \mathfrak{D}_{p,A}$. We claim that S satisfies Condition (♦). By (†), S has cardinality λ. Now, let q be a type over a subset of A of cardinality less than κ. Then, q does not fork over a subset C of A of cardinality less than $\kappa(T)$. Let $p = q \restriction C$. Thus, q is a subset of $W(p, A)$ of cardinality less than κ with the finite intersection property, so by (♦♦), there exists an ultrafilter F in $\mathcal{D}_{p,A}$ extending q. This implies that q_F extends q, as required. ⊣

Proposition 5.14. *Suppose that $\mu = \mu^{<\kappa}$. Let B be a boolean algebra. Assume that there exists a family $\mathfrak{F}$ of cardinality μ of ultrafilters of B such that $B - \{0\} = \bigcup \mathfrak{F}$. Then, there exists a family of ultrafilter $\mathfrak{D}$ of B of cardinality μ satisfying the following property.*

> (♦♦♦): *Every subset of B of cardinality less than κ with the finite intersection property can be extended to an ultrafilter in $\mathfrak{D}$.*

Proof. Let $\mathfrak{F}$ be as in the hypothesis of the proposition. Write $\mathfrak{F} = \{ F_i \mid i < \mu \}$. To construct $\mathfrak{D}$, let us consider for every $\chi < \kappa$ the family I_χ of finite subsets of χ, and for $\alpha < \kappa$ let $I_\chi(\alpha)$ be the set $\{ t \in I_\chi \mid \alpha \in t \}$. Notice that $\{ I_\chi(\alpha) \mid \alpha < \kappa \}$ is closed under finite intersections, so we can pick an ultrafilter $E_\chi \subseteq \mathbf{P}(I_\chi)$ extending $\{ I_\chi(\alpha) \mid \alpha < \kappa \}$.

Given $\chi < \kappa$ and a function $f : I_\chi \to \mu$, define $D_f \subseteq B$ as follows:

$$a \in D_f \quad \text{if and only if} \quad \{ t \in I_\chi \mid a \in F_{f(t)} \} \in E_\chi.$$

We prove that D_f is an ultrafilter of B.

Given $a \in B$, let $S_a = \{ t \in I_\chi \mid a \in F_{f(t)} \}$. We show that D_f is upwardly closed. Suppose that $a \in D_f$ and $b \geq a$ is in B. Since $F_{f(t)}$ is a filter, $a \in F_{f(t)}$ and $a \leq b$, imply $b \in F_{f(t)}$. Thus, $S_a \subseteq S_b$. But E_χ is a filter, so $S_a \in E_\chi$, implies $S_b \in E_\chi$. Therefore, $b \in D_f$.

D_f is also closed under $\wedge$. Suppose $a, b \in D_f$. Then $S_a, S_b \in E_\chi$, so $S_a \cap S_b \in E_\chi$ since E_χ is a filter. But, $a, b \in F_{f(t)}$, implies $a \wedge b \in F_{f(t)}$ since $F_{f(t)}$ is a filter. Hence, $S_a \cap S_b \subseteq S_{a\wedge b}$ and $S_{a\wedge b} \in E_\chi$, since E_χ is a filter. Thus, $a \wedge b \in D_f$.

Now we show that D_f is maximal: Suppose $a \in B - D_f$. By the definition of S_a and since E_χ is an ultrafilter, we have $S_a \notin E_\chi$. Hence, $\{ t \in I_\chi \mid a \notin F_{f(t)} \} \in E_\chi$. Since $F_{f(t)}$ is an ultrafilter, we must have $S_{1-a} \in E_\chi$, that is, $1 - a \in D_f$.

Thus, D_f is an ultrafilter. Define

$$\mathfrak{D} := \{ D_f \mid \chi < \kappa,\ f : I_\chi \to \mu \}.$$

Notice that $|\mathfrak{D}| \leq \kappa \cdot \sum_{\chi<\kappa} \mu^\chi \leq \mu^{<\kappa} = \mu$. It remains to show that $\mathfrak{D}$ satisfies (♦♦♦).

Suppose that $D := \{ a_i \mid i < \chi < \kappa \} \subseteq B$ has the finite intersection property. For every $t \in I_\chi$ let $a_t := \bigwedge_{j\in t} a_j$. Then $a_t \neq 0$. Let $f : I_\chi \to \mu$ be defined by

$$f(t) := \min\{ i < \mu \mid a_t \in F_i \}.$$

(Our assumption on on $\mathfrak{F}$ guarantees that f is well-defined.) We now check that $D \subseteq D_f$. Take $a_i \in D$. Then $a_i \geq a_t$, for $t \in I_\chi(i)$. But $a_t \in F_{f(t)}$ by the definition of f, so $a_i \in F_{f(t)}$ since $F_{f(t)}$ is a filter. We have shown that

$$I_\chi(i) \subseteq \{ t \in I_\chi \mid a_i \in F_{f(t)} \}.$$

But $I_\chi(i) \in E_\chi$ and E_χ is a filter, so

$$\{ t \in I_\chi \mid a_i \in F_{f(t)} \} \in E_\chi.$$

Hence, $a_i \in D_f$. ⊣

We can now prove the theorem.

Theorem 5.15. *Let T be simple. Let $\lambda = \lambda^{|T|} > \kappa > |D(T)|$ and suppose that there exists $\mu > 2^{|T|}$ such that* $(\mathrm{Ax}_0\mu)$ *holds and $\mu = \mu^{<\mu} \leq \lambda < 2^\mu$. Then $(\lambda, \kappa) \in \mathrm{SP}(T)$.*

Proof. By Theorem 3.4 $\kappa(T) \leq |T|^+$, so the assumption on λ, guarantees that $\lambda^{<\kappa(T)} = \lambda$. Therefore, by Proposition 5.13 it suffices to show that given A of cardinality λ, $C \subseteq A$ of cardinality $\kappa(T)$ and $p \in S(C)$ there is a family of ultrafilters satisfying Condition (♦♦) of Proposition 5.13. By Proposition 5.14, it suffices to show that for any boolean algebra $B_{p,A}$ and any subalgebra $B'_{p,A} \leq B_{p,A}$ of cardinality λ containing $W(p, A)$ there exists a family of ultrafilters $\mathfrak{F}$ of $B'_{p,A}$ of cardinality μ such that $B'_{p,A} - \{0\} = \bigcup \mathfrak{F}$.

Since T is simple, the boolean algebra $B_{p,A}$ satisfies the $(2^{|C|+|T|})^+$-chain condition by Corollary 5.6. Since $|C| \leq |T|$, $|B'_{p,A}| = \lambda < 2^\mu$ and $\mu > 2^{|T|}$, the algebra $B'_{p,A}$ satisfies the μ-chain condition. But, since $|W(p, A)| = \lambda$, Fact 5.7 implies the existence of a family of ultrafilters $\mathfrak{F}$ as desired. ⊣

Appendix A. A Better Bound for Theorem 1.11

Theorem 1.11 had a crucial role in producing Morley sequences.In this Appendix we improve the bound on the length of the sequence in the hypothesis of the theorem.

Given a first order complete theory T and a set Γ of (not necessarily complete) types over the empty set, we let

$$\mathrm{EC}(T, \Gamma) = \{ M \models T \mid M \text{ omits every type in } \Gamma \}.$$

For a cardinal λ, the ordinal $\delta(\lambda)$ is defined as the least ordinal δ such that for any T and Γ and M, if

$|T| \leq \lambda$,
$T \vdash (P, <)$ is a linear order,
$M \in \mathrm{EC}(T, \Gamma)$, and
$(P^M, <^M)$ has order type at least δ,

then there exists $N \in \mathrm{EC}(T, \Gamma)$, such that $(P^N, <^N)$ is not well-ordered.

The following lemma is well-known. We provide a proof at the end of this Appendix for the sake of completeness.

Theorem A.1. *Let P be a unary predicate and $<$ a binary predicate in $L(T)$ such that*

$$T \vdash \text{“} < \text{ is a linear order on } P\text{”}.$$

Suppose $M \in \mathrm{EC}(T, \Gamma)$ with $P^M = \{a_i : i < (2^{|T|})^+\}$ is such that

$$M \models a_i < a_j \quad \text{if and only if} \quad i < j < (2^{|T|})^+.$$

Then there exists $N \in \mathrm{EC}(T, \Gamma)$ such that P^N is not well-ordered by $<^N$.

Corollary A.2. $\delta(|T|) < (2^{|T|})^+$.

We could not find a proof of the following theorem in the literature, so we have included a complete proof here.

Theorem A.3. *Let T be any theory. For every $\langle a_i \mid i < \beth_{\delta(|T|)} \rangle$ there exists an indiscernible sequence $\langle b_n \mid n < \omega \rangle$ with the following property: for every $n < \omega$ there are $i_0 < \cdots < i_{n-1}$ satisfying*

$$\mathrm{tp}(b_0, \ldots, b_{n-1}/\emptyset) = \mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/\emptyset).$$

Proof. Let $I = \langle a_i \mid i < \beth_{\delta(|T|)} \rangle$. We define the following functions:

- For every $n < \omega$, functions $f_n : [\beth_{\delta(|T|)}]^n \to D(T)$ given by $f_n(i_0, \ldots, i_{n-1}) = \mathrm{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/\emptyset)$;
- A bijection $g : \beth_{\delta(|T|)} \to I$, defined by $g(i) = \bar{a}_i$;
- A bijection $h : D(T) \to \kappa$, where $\kappa = |D(T)|$;

Let χ be a regular cardinal large enough so that $H(\chi)$ contains $L(T)$, $D(T)$, I, $\beth_{\delta(|T|)}$ and the functions f_n, g and h as subsets. Assume in addition that $H(\chi)$ “knows” the Erdő-Rado Theorem; more precisely:

$$H(\chi) \models \beth_n(\beth_\alpha) \to (\beth_\alpha^+)^{n+1}_{\beth_\alpha} \quad \forall \alpha \in \delta(|T|) \, \forall n \in \omega.$$

Next, choose new predicates J, D, λ, μ; new constants κ and φ for every $\varphi \in L(T)$; and new function symbols f_n for every $n \in \omega$, g,h, and b. Now form the following expansion of $H(\chi)$.

$$V = \langle H(\chi), \in, J, D, , \lambda, \mu, \kappa, f_n, g, h, b, \varphi \rangle_{\varphi \in L(T), n \in \omega},$$

where $J^V = I$, $D^V = D(T)$, $\lambda^V = \beth_{\delta(|T|)}$, $\mu^V = (2^{|T|})^+$, $f_n^V = f_n$, $g^V = g$, $h^V = h$, $b(\alpha)^V = \beth_\alpha$, $\kappa^V = \kappa$, and $\varphi^V = \varphi$ for $\varphi \in L(T)$.

Let $T^* = \mathrm{Th}(V)$. Then T^* contains the following sentences (written informally for readability):

- $|J| = \lambda$;
- $|D| = \kappa < \mu$;
- $\forall \alpha \in \mu \; b(\alpha + n + 1) \to (b(\alpha))^{n+1}_\kappa$, for every $n \in \omega$;

We define a set of types Γ in the language of T^* as follows:

(*) $$\Gamma := \{p \in S(\emptyset) \mid \text{ no } \bar{a} \in I \text{ realizes } p\}.$$

Then $V \in \mathrm{EC}(T^*, \Gamma)$ and $\mu^V = \delta(|T|)$, so by CorollaryA.2, there is $V' \in \mathrm{EC}(T^*, \Gamma)$ such that $\mu^{V'}$ is not well-ordered. Let $\{\alpha_n \mid n < \omega\} \subseteq \mu^{V'}$ witness this. We may assume that $V' \models \alpha_n > \alpha_{n+1} + n + 1$.

We now construct sets $X_n \subseteq \lambda^{V'}$ for $n < \omega$ such that

(1) $V' \models |X_n| \geq b(\alpha_n)$ for every $n < \omega$;

(2) $V' \models f_n(g(i_0), \ldots, g(i_{n-1})) = f_n(g(j_0), \ldots, g(j_{n-1}))$, for $i_0 < \cdots < i_{n-1} \in X_n$ and $j_0 < \cdots < j_{n-1} \in X_n$.

The construction is by induction on n. Let $X_0 = \lambda^{V'}$ and clearly $V' \models |X_0| \geq b(\alpha_0)$ since $V' \models b(\alpha_0) \in \lambda$.

Having constructed X_n, notice the following.

(i) $V' \models |X_n| \geq b(\alpha_n)$;

(ii) $V' \models c\colon [X_n]^{n+1} \to D$, with $c(i_0, \ldots, i_n) := f_{n+1}(g(i_0), \ldots, g(i_n))$;

(iii) $V' \models |D| = \kappa$;

(iv) $V' \models b(\alpha_n) \to (b(\alpha_{n+1}))^{n+1}_{\kappa}$.

Then (4) applied to (1) and (2) implies that there is $X_{n+1} \subseteq X_n$ such that $V' \models |X_{n+1}| \geq b(\alpha_{n+1})$ monochromatic with respect to c, so X_{n+1} is as required.

Since every $\varphi \in L$ has a name in $L(T^*)$, there are $p_n \in D(T)$ for $n \in \omega$ such that $\mathrm{tp}(g(i_0), \ldots, g(i_n)/\emptyset) = p_n$ for $\bar{a}_{i_0} = g(i_0), \ldots, \bar{a}_{i_{n-1}} = g(i_{n-1}) \in J^{V'}$, and $i_0 < \cdots < i_{n-1} \in X_n$.

Now let $\{c_n \mid n < \omega\}$ constants not in $L(T^*)$ and let T_1 be the union of the following set of sentences:

- T^*;
- $p_n(c_0, \ldots, c_{n-1})$, for every $n < m$;
- $g(c_n) < g(c_m)$, for every $n < m$;
- $\varphi(c_0, \ldots, c_{n-1}) \leftrightarrow \varphi(c_{i_0}, \ldots, c_{i_{n-1}})$, whenever $\varphi \in L(T)$, $i_0 < \ldots i_{n-1}$, and $n < \omega$.

Then T_1 is consistent: for every finite subset of T_1 use $g(X_n)$ to realize the c_k's.

Let $N_1 \models T_1$ and $b_n = c_n^{N_1}$ for every $n \in \omega$. Certainly $\{b_n \mid n < \omega\}$ is indiscernible (in $L(T)$). Now we show that for every $n < \omega$ there exist $i_0 < \cdots < i_{n-1} < \beth_{(2^{|T|})^+}$ such that

$$\mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/\emptyset) = p_n.$$

But by construction, there are $j_0 < \cdots < j_{n-1}$ such that

$$q_n = \mathrm{tp}_{L(T^*)}(g(j_0), \ldots, g(j_{n-1})/\emptyset) = p_n.$$

Hence, $\{x_i \in J\} \in q_n$ for every $i < n$, and since $\{J, g, \in\} \subseteq L(T^*)$, we have

$$\{g(x_0) < g(x_1), \ldots, g(x_{n-2}) < g(x_{n-1})\} \subseteq q_n$$

But since $V' \in \mathrm{EC}(T^*, \Gamma)$, we must have $q_n \notin \Gamma$. Thus, by (*), there are $a_{i_0}, \ldots a_{i_{n-1}}$ in I realizing q_n. ⊣

Now to the proof of Theorem A.1:

Proof. Let T^* be an expansion of T of the same cardinality with Skolem functions. We construct sequences of sets $\langle S_n \mid n < \omega \rangle$, types

$$p_n(x_0, \ldots, x_{n-1}) \in S^n_{L(T^*)}(\emptyset) \quad \text{for } n > 0,$$

and ordinals

$$\alpha_i(n,0),\dots,\alpha_i(n,n-1)\in S_n,\qquad \text{for } i\in S_n$$

such that

(1) $S_0 = (2^{|T|})^+$ and $S_{n+1} \subseteq S_n$;
(2) $|S_n| = (2^{|T|})^+$;
(3) $(2^{|T|})^+ > \alpha_i(n,0) > \dots > \alpha_i(n,n-1) > i$ for every $i \in S_n$;
(4) $p_n \subseteq p_{n+1}$;
(5) For every $n < m$, $i \in S_n$, and $j \in S_m$,

$$p_n = \mathrm{tp}(a_{\alpha_i(n,0)},\dots,a_{\alpha_i(n,n-1)}/\emptyset, M) = \mathrm{tp}(a_{\alpha_j(m,0)},\dots,a_{\alpha_j(m,n-1)}/\emptyset, M).$$

The construction is by induction on n. For $n = 0$, we only need to define S_0 and we let $S_0 = (2^{|T|})^+$.

Having constructed S_n for every $i \in S_n$, fix $j \in S_n$ with $j > i$ (which exists by (2)). Define

$$\begin{aligned}\alpha_i(n+1,0) &:= \alpha_j(n,0)\\ &\vdots\\ \alpha_i(n+1,n-1) &:= \alpha_j(n,n-1)\\ \alpha_i(n+1,n) &:= j.\end{aligned}$$

We also let

$$p^i_{n+1} := \mathrm{tp}(a_{\alpha_i(n+1,0)},\dots,a_{\alpha_i(n+1,n)}/\emptyset, M).$$

Then $p^i_{n+1} \in S^{n+1}(\emptyset)$. Since $|S_n| = (2^{|T|})^+ > |S^{n+1}_{L(T^*)}(\emptyset)|$, there exist $S_{n+1} \subseteq S_n$ of cardinality $(2^{|T|})^+$ and $p_{n+1} \in S^{n+1}(\emptyset)$ such that $p^i_{n+1} = p_{n+1}$ for every $i \in S_{n+1}$. It is easy to see that all the requirements are satisfied.

Now let $\{c_n \mid n < \omega\}$ be new constants and let T_1 be the union of the following set of sentences:

- T^*;
- $c_n > c_{n+1}$, for every $n < \omega$;
- $p_n(c_0,\dots,c_{n-1})$, for every $n < \omega$.

Then T_1 is consistent: If Γ is a finite subset of T and n maximal appearing in Γ, then $\Gamma \subseteq T^* \cup \{c_i > c_{i+1} \mid i < n\} \cup p_{n+1}(c_0,\dots,c_n)$ is satisfied by interpreting c_i as $\alpha_{(n,i)}$.

Let $N_1 \models T_1$ and $b_n = c_n^{N_1}$. Let N be the closure of $\{b_n \mid n < \omega\}) \restriction L(T)$. under the Skolem functions. Then $N \models T$ and P^N is not well-ordered. We claim that $N \in \mathrm{EC}(T,\Gamma)$. Otherwise there exist $\bar{d} \in N$ and $p \in \Gamma$ such that $\bar{d} \models p$. and $\bar{\tau} \in L(T^*)$ such that $\bar{d} = \bar{\tau}(b_0,\dots,b_{n-1})$. Hence,

$$p = \mathrm{tp}(\bar{a}/\emptyset) = \mathrm{tp}(\bar{\tau}(b_0,\dots,b_{n-1})/\emptyset) \subseteq p_n.$$

By construction of p_n and S_n, there are $\langle a_{i_0},\dots,a_{i_{n-1}}\rangle \in P^M$ realizing p_n, so $\bar{\tau}(a_{i_0},\dots,a_{i_{n-1}})$ realizes p, which is a contradiction, since $\bar{\tau}(a_{i_0},\dots,a_{i_{n-1}}) \in M \in \mathrm{EC}(T,\Gamma)$.

⊣

Appendix B. Historical Notes

Introduction: Although Baldwin's book [Ba] was published in 1988, early versions of it circulated since 1980 and had significant influence on several publications with earlier publication dates.

We have learned from Baldwin that a preliminary version of [Sh93] was titled "Treeless unstable theories".

Section 1: The concept of Definition 1.9 was introduced in [Sha] and [Sh93], where it is called *indiscernible based on* (A, B). However, since the name "Morley sequence" has become standard since the 1970's (see[Sa] and [Po]), we have departed from Shelah at this point.

The proof of the Concatenation Lemma (Lemma 1.13) appears on page 198 of [Sh93].

We do not know of any complete proof of Theorem 1.11 in print. It is stated as Lemma 6.3 in [Sh93] and used heavily in the proof of Claim 6.4, the precursor of Kim's proof. The scant proof of Theorem 1.11 offered in [Sh93] contains the line "by the method Morley proved his omitting types theorem".

Section 2: Our $D[p, \Delta, k]$ is Shelah's $D^m[p, \Delta, \aleph_1, k]$ in [Sh93]. All the material in this section is due to Shelah and is taken from [Sh93] and Chapter III of [Sha], where the facts are often stated without proof. Occasionally, we have chosen more modern language.

Section 3: The equivalence between forking and dividing for stable theories was discovered by Baldwin and Shelah in 1979 while discussing a preliminary version of [Sh93]. It appears in print for the first time in [Pi], and later in [Ba]. The fact that Shelah did not include it in [Sha] leads us to speculate that he was not aware of it in 1978.

The characterization of dividing through Morley sequences (Theorem 3.7) and the equivalence between forking and dividing (Theorem 3.8) in simple theories is due to Kim [Ki]. The proofs Symmetry and Transitivity as consequences of this equivalence are from [Ki], where the credit is given to Pillay.

The proof of Theorem 3.8 included here is new, and was shown to us by Shelah.

The Symmetry property for strongly minimal sets was discovered by William Marsh [Mar] and used by Baldwin and Lachlan in [BL]. Lascar discovered the Symmetry property for superstable theories [La1, La2]. Independently, Shelah generalized it to stable theories. The implication (1) $\Rightarrow$ (2) of Theorem 3.11 is from [KP1]. This theorem is an analog of Shelah's characterization of non-forking extension via local rank for stable theories (see Theorem III.4.1 of [Sha]). The implication (2) $\Rightarrow$ (1) of Theorem 3.11 is due to Shelah.

The Independence Theorem 3.12 for simple theories is due to Kim and Pillay in [KP1] and is a generalization of a result of Hrushovski and Pillay about S_1-structures, namely, (i)$\Rightarrow$(ii) in Lemma 5.22 of [HP1].

Section 4: All the material in this section is contained [Sh93].

Section 5: The boolean algebra introduced here is a variation of that in [Sh93]; we have replaced weak dividing with forking. The proof that this boolean algebra satisfies the chain condition was communicated to us by Shelah.

REFERENCES

[Ba] John T. Baldwin. **Fundamentals of Stability Theory**. Springer-Verlag, Berlin, 1988.

[BL] John T. Baldwin and Alistair H. Lachlan. On strongly minimal sets. *J. Symbolic Logic*, **36**, 79–96, 1971.

[CK] C. C. Chang and H. Jerome Keisler. **Model Theory**. North-Holland Publishing .Co., Amsterdam, 1990.

[CH] Zoe Chatzidakis and Ehud Hrushovski. Algebraically closed fields with an automorphism. *Trans. of AMS.*

[Gr] Rami Grossberg. **A course in model theory**. Book in preparation.

[HH] Victor Harnik and Leo Harrington. Fundamentals of forking. *Ann. Pure Appl. Logic*, **26**, 245–286, 1984.

[Hr] Ehud Hrushovski. Pseudofinite fields and related structures. Manuscript.

[HP1] Ehud Hrushovski and Anand Pillay, Groups definable in local fields and pseudo-finite fields, *Israel J. of Mathematics*, **85**, 203–262, 1994.

[HP2] Ehud Hrushovski and Anand Pillay, Definable groups of algebraic groups over finite fields. *Crelle's Journal*, **462**, 69–91, 1995.

[Je] Thomas Jech. **Set Theory**. Academic Press, New York, 1978.

[Ki] Byunghan Kim. Forking in simple unstable theories. *J. of London Math. Society*, To appear.

[Ki1] Byunghan Kim. Supersimple. *

[KP1] Byunghan Kim and Anand Pillay. Simple theories *Annals of Pure and Applied Logic*, To appear.

[KP2] Byunghan Kim and Anand Pillay. Simple survey. To appear.

[La1] Daniel Lascar. Types définissables et produit de types. *C. R. Acad. Sci. Paris Sir. A-B*, **272**, A1233–A1236, 1971.

[La2] Daniel Lascar. Ranks and definability in superstable theories. *Israel J. Math.*, **23**, 53–87, 1976.

[La3] Daniel Lascar. **Stability in Model Theory**. John Wiley & Sons, Inc., New York, 1987.

[LaPo] Daniel Lascar and Bruno Poizat. Fundamentals of forking. *J. Symbolic Logic*, 1977.

[Lo] Jerzy Łös. On the categoricity in power of elementary deductive systems and related problems. *Colloq. Math.* , **3**, (1954) 58–62.

[Ma] Michael Makkai. A survey of basic stability theory, with particular emphasis on orthogonality and regular types. *Israel J. Math.*, **49**, 181–238, 1984.

[Mar] William Marsh. *On ω_1-categorical and not ω-categorical theories*. PhD thesis, Dartmouth College, 1966.

[Mo] Michael Morley. Categoricity in power. *Trans. Amer. Math. Soc.*, **114**, 514–538, 1965.

[Pi] Anand Pillay. **An Introduction to Stability Theory**. Clarendon Press, Oxford University Press, New York, 1983.

[Po] Bruno Poizat. Suites d'indiscernables dans les théories stables. Technical report, Univ. Pierre et Marie Curie, Paris, 1978.

[Sa] Gerald Sacks. **Saturated Model Theory**. W. A. Benjamin, Inc., Reading, 1972.

[Sh3] Saharon Shelah. Finite diagrams stable in power. *Ann. Math. Logic*, **2**, 69–118, 1970/1971.

[Sh31] Saharon Shelah. Categoricity of uncountable theories. In L.A. Henkin et al., editors, **Proceedings of the Tarski Symposium**, pages 187–203, Providence, R.I, 1974.

[Sha] Saharon Shelah. **Classification Theory and the Number of Nonisomorphic Models**. North-Holland Publishing Co., Amsterdam, 1978.

[Sh 80] Saharon Shelah. A weak generalization of MA to higher cardinals. *Israel Journal of Mathematics*, **30**:297–306, 1978.

[Sh93] Saharon Shelah. Simple unstable theories. *Ann. Math. Logic*, **19**, 177–203, 1980.

E-mail address, Rami Grossberg: `rami@cmu.edu`

E-mail address, José N. Iovino: `iovino@cmu.edu`

E-mail address, Olivier Lessmann: `lessmann@cmu.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213