REGRESSIVE RAMSEY NUMBERS ARE ACKERMANNIAN

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ABSTRACT. We give an elementary proof of the fact that regressive Ramsey numbers are Ackermannian. This fact was first proved b y Kanamori and McAloon with mathematical logic techniques.

Nous vivons encore sous le règne de la logique, voilà, bien entendu, à quoi je voulais en venir. Mais les procédés logiques, de nos jours, ne s'appliquent plus qu'à la résolution de problèmes d'intérêt secondaire. [1, 1924, p. 13] is

1. INTRODUCTION

Definition 1. 1. *let A be a set of natural numbers. A coloring $c : [A]^e \to \mathbb{N}$ of unordered e-tuples from A is* regressive *if $c(x) < \min x$ for all $x \in [A]^e$.*

2. *A subset $B \subseteq A$ is* min-homogeneous *for a coloring c of $[A]^e$ if for all $x \in [A]^e$ the color $c(x)$ depends only on* $\min x$.

Theorem 2 (Kanamori and McAloon). 1. *For every k and e there exists N such that for every regressive pair coloring on $\{1, 2, \dots, N\}$ there exists a min-homogeneous subset of size k.*

2. *The statement in (1) cannot be proved from the axioms of Peano Arithmetic (although it can be* phrased *in the language of PA)*
3. *Let $\nu(k)$ be the least N which satisfies 1 for $e = 2$. The function ν eventually dominates every primitive recursive function.*

Part (3) of Kanamori and McAloon's result [3] was proved with mathematical logic methods. We present below an elementary proof of 3.

2. THE LOWER BOUND

Define a sequence of (strictly increasing) integer functions $f_i, i \geq 1$ as follows:

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$$f_1(n) = n+1 \tag{1}$$
$$f_{i+1}(n) = f_i^{(\lfloor\sqrt{n}\rfloor)}(n) \tag{2}$$

Fix an integer $k > 2$. Define a sequence of semi-metrics $\langle d_i : i \in \mathbb{N}\rangle$ on $\{n : n \geq k^2\}$ by putting, for $k^2 \leq m \leq n$,

$$d_i(m,n) = |\{l \in \mathbb{N} : m \leq f_i^{(l)}(k^2) < n\}| \tag{3}$$

Let $i(m,n)$, for $k^2 \leq m < n$, be the greatest i for which $d_i(m,n)$ is positive, and $d(m,n) = d_{i(m,n)}(m,n)$.

Claim 3. *For all $n \geq m \geq k^2$, $d(m,n) \leq \sqrt{m}$.*

Proof. Trivial. □

Let us fix the following (standard) pairing function Pr on $\mathbb{N}^2$

$$\Pr(m,n) = \binom{m+n}{2} + n$$

Pr is a bijection between $[\mathbb{N}]^2$ and $\mathbb{N}$ and is monotone in each variable. Observe that if $m, n \leq l$ then $\Pr(m,n) < l^2$ for all $l > 3$.

Define a pair coloring c on $\{n : n \geq k^2\}$ as follows:

$$c(m,n) = \Pr(i(m,n), d(m,n)) \tag{4}$$

Claim 4. *For every $i \in \mathbb{N}$, every sequence $x_0 < x_1 < \cdots < x_i$ that satisfies $d_i(x_0, x_i) = 0$ is not min-homogeneous for c.*

Proof. The claim is proved by induction on i. If $i = 1$ then there are no $x_0 < x_1$ with $d_1(x,y) = 0$ at all. Suppose to the contrary that $i > 1$, that $x_0 < x_1 < \cdots < x_i$ is min-homogeneous with respect to c and that $d_i(x_0, x_i) = 0$. Necessarily, $i(x_0, x_j) = j < i$. By min-homogeneity, $i(x_0, x_1) = j$ as well, and $d_j(x_0, x_i) = d_j(x_0, x_1)$. Hence, $\{x_1, x, \ldots x_i\}$ is min-homogeneous with $d_j(x_1, x_i) = 0$ — contrary to the induction hypothesis. □

Claim 5. *The coloring c in (4) is regressive on the interval $[k^2, f_k(k^2))$.*

Proof. Clearly, $d_{k+1}(m,n) = 0$ for $k^2 \leq m < n < f_k(k^2)$ and therefore $i(m,n) \leq k < \sqrt{m}$. From Claim **??** we know that $d(m,n) \leq \sqrt{m}$. Thus, $c(m,n) = \Pr(i(m,n), d(m,n)) \leq \Pr(\sqrt{m}, \sqrt{m}) < m$, since $\sqrt{m} > 3$. □

We show that $f_k(k^2)$ grows eventually faster than every primitive recursive function by comparing the functions f_i with the usual approximations of Ackermann's function. It is well known that every primitive recursive function is dominated by some approximation of Ackermann's function (see, e.g. [2]).

Let $A_i(n)$ be defined as follows:

$$A_1(n) = n + 1 \tag{5}$$

$$A_{i+1}(n) = A_i^{(n)}(n) \tag{6}$$

The A_i-s are the usual approximations to Ackermann's function, which is defined by $Ack(n) = A_n(n)$.

Claim 6. 1. *$f_i(n) \geq 4n^2$ for $i, n \geq 4$.*
2. *$A_i(n) \leq f_{i+4}(4n^2) \leq f_{i+4}^{(2)}(n)$ for all i and $n \geq 4$.*
3. *$A_i(n) \leq f_{i+5}(n)$ for all i and $n \geq 4$.*

Proof. The first item is verified directly. The second inequality in the second item is by 1. The first inequality is proved by induction on i. Suppose $A_i(n) \leq f_{i+4}(4n^2)$. Since $A_i(n) \leq f_{i+4}^{(2)}(n)$, iterating n times yields $A_i^{(n)}(n) \leq f_{i+4}^{(2n)}(n)$, which is $\leq f_{i+4}^{(2n)}(4n^2) = f_{i+5}(4n^2)$. Thus $A_{i+1}(n) \leq f_{i+5}(4n^2)$.

The last item follows now trivially: $A_i(n) \leq f_{i+4}^{(2)}(n) \leq f_{i+5}(n)$ (as $n \geq 4$). □

Corollary 7. *The function $\nu(k)$ eventually dominates every primitive recursive func tion.*

3. Discussion

3.1. **Other Ramsey numbers.** Paris and Harrington [8] published in 1976 the first finite Ramsey-type statement that was shown to be independent over Peano Arithmetic. Soon after the discovery of the Paris-Harrington result, Erdős and Mills studied the Ramsey-Paris-Harrington numbers in [7]. Denoting by $R_c^e(k)$ the Ramsey-Paris-Harrington number for exponent e and c many colors, Erdős and Mills showed that $R_2^2(k)$ is double exponential in k and that $R_c^2(k)$ is Ackermannian as a function of k and c. In the same paper, several small Ramsey-Paris-Harrington numbers were computed. Later Mills tighten ed the double exponential upper bound for $R_2^2(k)$ in [5].

Canonical Ramsey numbers for pair colorings were treated in [4] and were also fond to be double exponential.

The second author showed that van der Waerden numbers are primitive recursive, refuting the conjecture that they were Ackermannian, in [9] (see also [6]).

We remark that an upper bound for regressive Ramsey numbers for pairs is $R_2^3(k)$ — the Ramsey-Paris-Harrington number for *triples.* Let N be large enough and suppose that c is regressive on $\{1, 2, \dots, N-1\}$. Color a triple $x < y < z$ red if $c(x, y) = c(x, z)$ and blue otherwise. Find a homogeneous set A of size at least k and so that $|A| > \min A+1$. The homogeneous color on A cannot be red for $k > 5$, and therefore A is min-homogeneous for c.

3.2. **Problems.** The following two problems about regressive Ramsey numbers remain open:

Problem 8. 1. *Find a concrete upper bound for regressive Ramsey numbers.*
2. *Compute small regressive Ramsey numbers*

REFERENCES

[1] André Breton. Manifeste du surréalisme. In *Manifestes du Surréalisme*, pages 11–66. Gallimard , 1972.
[2] Calude, Cristian. *Theories of computational complexity.* Annals of Discrete Mathematics, 35. Amsterdam etc.: North-Holl and. XII, 487 p., 1988.
[3] Akihiro Kanamori and Kenneth McAloon. On Gödel incompleteness and finite combinatorics. *Ann. Pure Appl. Logic*, 33(1):23–41, 1987.
[4] Hanno Leffman and Vojtěch Rödl. On canonical ramsey numbers for complete graphs versus paths. *Journal of Combinatorial theory, Series B*, 58:1–13, 1993 .
[5] George Mills. Ramsey–paris–harrington numbers for graphs. *Journal of Combinatorial theory, Series A*, 38:30–37, 198 5.
[6] Alon Nilli. Shelah's proof of the hales-jewett theorem. In Jaroslav Nešetřil and Vojtěch Rödl, ed itors, *Mathematics of Ramsey Theory*, volume 5 of *Algorithms and Combinatorics*, pages 151–152. Springer, Berlin, 1990.
[7] Paul Erdős and George Mills. Some bounds for the ramsey-paris-harrington numbers. *Journal of Combinatorial theory, Series A*, 30:53–70, 198 1.
[8] J. Paris and L. Harrington. A mathematical incompleteness in peano arithmetic. In J. Barwise, editor, *Handbook of Mathematical Logic.* North-Holland, 1977.
[9] Saharon Shelah. Primitive recursive bounds for van der Waerden numbers. *Journal of the American Mathematical Society*, 1:683–69 7, 1988.

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