FORKING IN PREGEOMETRIES, PART II: ABSTRACT GROUP CONFIGURATION

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ABSTRACT. The aim of this paper is to make progress towards a geometric model theory for non first order theories. The main difficulty is to work in an environment where the compactness theorem fails. This paper continues the work started in [GrLe1]. The main result is an axiomatic approach to the Hrushovski-Zilber group configuration theorem.

1. INTRODUCTION

A central result in Geometric Stability Theory is the presence in very general circumstances of a definable group among the definable (maybe infinitely definable) sets of a model. This is referred to by W. Hodges [Ho] as the Zilber Group Configuration Theorem, and by others as the Hrushovski Group Configuration Theorem. We will call it the Hrushovski-Zilber Group Configuration Theorem. It has an ancient flavor; it is in a line of work which dates back to Veblen and Young around 1910. The general template is the emergence of algebraic structures from certain geometric configurations.

The Hrushovski-Zilber Group configuration Theorem for the first order, countable $\aleph_1$-categorical case is due to Boris Zilber [Zi]. It builds on the methods of Baldwin-Lachlan [BlLa]). It was extended to stable theories by Ehud Hrushovski [Hr1] (see also the exposition of Elizabeth Bouscaren [Bo]). This generalization was done using S. Shelah's notions of forking, regular types and p-simple technology.

These methods have since developed into a field of its own. See for example the recent books of Steve Buechler [Bu1] and Anand Pillay [Pi]. They have been used to answer classical logical questions, for example B. Zilber's solution to the finitely axiomatization problem [Zi]; to general classification theory questions, for example E. Hrushovski's proof that unidimensional stable theories are superstable [Hr2], S. Buechler's work on Vaught's Conjecture [Bu2], and have found several applications outside of model theory [CH], [HP1], [HP2], [EvHr1], [EvHr2], [Hr3].

In parallel, much work was done in classification for classes of models that are not first order [Gr 1], [Gr 2], [GrHa], [GrLe1], [GrLe2], [GrSh 1], [GrSh 2], [HaSh], [HySh], [Ki], [KlSh], [Le], [MaSh], [Sh3], [Sh47], [Sh 87a], [Sh 87b], [Sh 88], [Sh tape], [Sh 299], [Sh 300], [Sh 394], [Sh 472], [Sh 576] and [Sh h]. This field was called classification for

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nonelementary classes by S. Shelah. Often, the techniques are combinatorial and set-theoretic, using basic definability properties. In the current state of the theory there are no geometric considerations.

Our aim here is to separate the model-theoretic aspects from the combinatorial geometry in the Hrushovski-Zilber Group Configuration Theorem to enable us to transfer this tool to non first order contexts. This was started in [GrLe1].

The setting of the Hrushovski-Zilber Group Configuration Theorem is the following. We have a pregeometry where the closure operation comes from forking. Technically speaking, the pregeometry is the set of realization of a stationary type p with the additional property that the closure operation given by

$$a \in \mathrm{cl}(B) \text{ if and only if } \mathrm{tp}(a/B \cup \mathrm{dom}(p)) \text{ forks over } \mathrm{dom}(p).$$

Here are several of the key ingredients in the first order case that are used. (1) The notion of types (2) The fact that the pregeometry comes from forking guarantees that the ambient dependence relation is well-behaved. (3) (T stable) Every type is definable. (4) Work in T^{eq}, which allows one to use the Canonical Basis Theorem. All these results rely on the Compactness Theorem.

There are several known pregeometries in nonelementary cases for applications:

Categorical sentences in $L_{\omega_1\omega}(Q)$: S. Shelah in [Sh47] introduced a rank which is bounded under the parallel to $\aleph_0$-stability (following from $\aleph_1$-categoricity). It gives rise to a dependence relation and pregeometries. Later, H. Kierstead [Ki] uses these pregeometries to obtain some information on the countable models of these sentences.

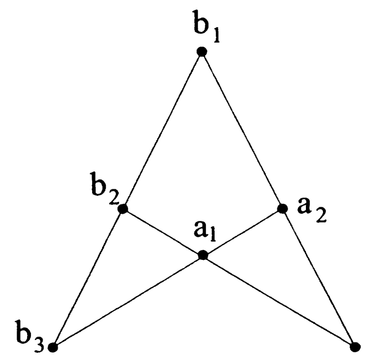
Excellent Scott Sentences: In [Sh 87a] and [Sh 87b] S.Shelah introduces a simplification of the rank of [Sh47]. This rank induces a dependence relation on the subsets of the models. S. Shelah also defines the concept of *excellent Scott sentences*. Later, R. Grossberg and B. Hart [GrHa] continued the classification of excellent Scott sentences. They proved the existence of pregeometries and used them to prove the main gap.

Totally transcendental diagrams: In [Le] we introduced a rank for $\aleph_0$-stable diagrams, introduced by S. Shelah [Sh3] in 1970. They are classes of models omitting a prescribed set of types, with an additional condition. We call a finite diagram *totally transcendental* when the rank is bounded. The rank gives rise to a dependence relation on the subsets of the models. We can also show that pregeometries exist.

Superstable diagrams: In [HySh], Hyttinen and Shelah study stable finite diagrams ([Sh3]) under the assumption that $\kappa(D) = \aleph_0$. Such diagrams are called *superstable*. They introduce a relation between sets A, B and an element a, written $a \downarrow_B A$. The main result is that pregeometries exist with respect to this dependence relation.

Let us examine how the above contexts allow us to circumvent the difficulties posed by the absence of the Compactness Theorem. In each of them, we have (1) a good notion of types. (2) In spite of the fact that the dependence relation is not necessarily as well-behaved as forking, there exists pregeometries. By work started in [GrLe1], this implies that we can define another dependence relation which satisfies all the formal properties of forking for first order theories. (3) In many of them, there is a notion of stationary types and those are definable. (4) There are several ways (as yet unpublished, some due to myself, some to Saharon Shelah) of introducing substitutes to T^{eq} and get the Canonical Basis Theorem.

The aim of this paper is to take into account the technology available (or being developed in nonelementary classes) to find some natural axioms (behind which the logical framework is hidden) under which group configurations may yield a group. Let us make this more precise. The Hrushovski-Zilber Group Configuration Theorem states in essence that if in a *definable* pregeometry we have the following dependence configuration (called group configuration), then there exists a *definable* group.



The way to read this diagram is as follows. Any two points are independent from each other, and any three points are dependent if and only if they are on the same line.

There are two steps in the Hrushovski-Zilber Group Configuration Theorems.

Step 1: Starting from the group configuration, where the dependence relation is forking, to obtain a similar group configuration, where in addition, some points are *uniquely determined* by others. This is often called the unique definability condition.

Step 2: From this special configuration, one derives a definable group.

Both steps rely on the general properties of forking and the canonical basis theorem for stable theories.

Step 1 seems decidedly model-theoretic and there is little hope for general conditions for the existence of an abstract theorem generalizing it. However, Step 2 is amenable.

There are two aspects of definability: By *syntactic definability*, we mean some model theoretic notion; we live in an ambient model M satisfying some axioms (not necessarily first order) and have a notion of formula. A set A is said to be syntactically definable over B if there exists a set of formulas p over B such that $a \in A$ if and only if a realizes all the formulas in p. Now given an automorphism group Γ, there is also a notion of *semantic definability*. We say that a set A is semantically definable over B if for every $f \in \Gamma$ fixing B pointwise, f fixes A setwise.

Now, syntactic definability implies semantic definability in case the automorphism group is (a subgroup) of the automorphism group of the model M. The converse is more delicate.

Here is the framework of the paper. We work inside a pregeometry (W, cl), given with an automorphism group Γ. We require that the pregeometry be homogeneous with respect to this automorphism group, which in our context means that the automorphism group is rich. We then have a notion of semantic definable sets (henceforth just called definable). We also consider a subcollection $\mathfrak{D}$ of definable sets (which in the applications are going to be the syntactically definable sets). We assume that $\mathfrak{D}$ satisfies an axiom parallel to (i) the definability of (stationary) types and another axiom parallel to (ii) the canonical basis theorem. If we assume in addition that in the unique definability condition of Step 2, the definable sets are in $\mathfrak{D}$, then this implies the existence of a group, which is equal to a (potentially) infinite intersection of sets in $\mathfrak{D}$.

If one is interested in applications to model theory for nonelementary classes and in particular issues of definability, we will be given a natural notion of syntactical definable sets and this theorem will give a definable group in this language (provided this notion satisfies the condition of $\mathfrak{D}$). All the first order notions for definability used so far belong to this set and the axioms hold in the well-known first order cases.

We can also look at this without a notion of syntactically definable sets. This allows us to ignore $\mathfrak{D}$, that is to assume that $\mathfrak{D}$ is the set of all semantically definable sets. Then, we do not need an axiom on definability of stationary types and just consider the canonical basis theorem for semantically definable sets. This gives a very smooth theorem in the context of combinatorial geometry.

The presentation owes much to [Ho], [Bo] and [EvHr1]. In fact, the setting of [EvHr1] is a particular case of our setting: Let $K \subseteq L$ be alebraically closed fields. The pregeometry (W, cl) is given by $W = L - K$ and $a \in \mathrm{cl}(C)$ if and only if a is in the algebraic closure (in L) of the field generated by $K \cup C$. The automorphism group Γ is $\mathrm{aut}(L/K)$. All the axioms are satisfied. Using the fact that they work in algebraically closed fields, they managed to obtain additional information on the definable groups.

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2. The Context

Let (W, cl) be a pregeometry and Γ be a group of automorphisms of (W, cl).

We always assume $\mathrm{cl}(\emptyset) \neq W$, in fact we will make the following assumption:

Hypothesis 1 (Nontriviality Assumption). We assume that (W, cl) is infinite dimensional.

Notation 2.
(1) We denote Γ_X the group of automorphisms of (W, cl) fixing X pointwise.
(2) Given a sequence A of elements of W. We denote by $\Gamma_X(A)$ the orbit of A under Γ_X, namely
$$\Gamma_X(A) = \{f(A) \mid f \in \Gamma_X\}.$$

For a sequence $A = \langle a_i \mid i < \alpha \rangle$, we write $f(A)$ for $\langle f(a_i) \mid i < \alpha \rangle$.

In [GrLe1], we introduced the following relation between subsets of a pregeometry. For convenience and readability, we use the usual notation $\mathop{\perp}$ (introduced by Makkai in [Ma]). The justification for this use can be seen in the theorem below.

Definition 3. Let (W, cl) be a pregeometry. Let A, B and C be subsets of W. We say that *A depends on C over B*, if there exist $a \in A$ and a finite $A' \subseteq A$ (possibly empty) such that

$$a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A').$$

If A depends on C over B, we write $A \not\mathop{\perp}_B C$;
If A does not depend on C over B, we write $A \mathop{\perp}_B C$.

In [GrLe1], we proved that this dependence relation satisfies the familiar axioms of forking, as introduced by Shelah (see, for example [Sh3]). As a result, we have given them their usual name.

Fact 4 (Forking Relations).

(1) (Definition) $A \mathop{\perp}_B C$ if and only if $A \mathop{\perp}_B B \cup C$;
(2) (Existence) $A \mathop{\perp}_C C$.
(3) (Finite Character) $A \mathop{\perp}_C B$ if and only if $A' \mathop{\perp}_C B'$ for every finite $A' \subseteq A$ and finite $B' \subseteq B$;
(4) (Invariance) If $f \in \Gamma$, then $A \mathop{\perp}_B C$ if and only if $f(A) \mathop{\perp}_{f(B)} f(C)$;
(5) (Monotonicity) Let $B \subseteq B_1 \subseteq C' \subseteq C$. Then $A \mathop{\perp}_B C$ implies $A \mathop{\perp}_{B_1} C'$;
(6) (Symmetry) $A \mathop{\perp}_B C$ if and only if $C \mathop{\perp}_B A$;
(7) (Transitivity) If $B \subseteq C \subseteq D$, then $A \mathop{\perp}_B D$ if and only if $A \mathop{\perp}_B C$ and $A \mathop{\perp}_C D$;
(8) ($\kappa(T) = \aleph_0$) For every $\bar{a}$ and C there exists $B \subseteq C$, $|B| < \aleph_0$, i.e. finite, such that $\bar{a} \mathop{\perp}_B C$;
(9) (Closed Set) $A \mathop{\perp}_B C$ if and only if $\mathrm{cl}(A) \mathop{\perp}_{\mathrm{cl}(B)} \mathrm{cl}(C)$.

Remark 5. Definition, Existence, Finite Character, Invariance, Monotonicity, $\kappa(T) = \aleph_0$ and Closed Set are obvious. The difficulty is to obtain (6) and (7).

To provide intuition, we give another property, which can with Finite Character, be used as a definition.

Proposition 6. *Let S_1, S_2 be finite dimensional closed sets satisfying $S_0 = S_1 \cap S_2$. Then,*

$$S_1 \mathop{\perp}_{S_0} S_2 \quad \textit{if and only if} \quad \dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2).$$

The first axiom corresponds to the extension property of forking as well as some saturation.

Axiom 7 (Extension). Let $\bar{a}$ be given and X be finite dimensional. Then, there exists $\bar{a}' \in \Gamma(\bar{a})$ such that $\bar{a}' \perp X$.

The next axioms correspond the uniqueness of the nonforking extension. I call it Homogeneity because a pregeometry satisfying H1 is called homogeneous. The axioms H2 and H3 have a similar flavor and in first order model theoretic cases follow from the same facts: stationarity and saturation.

Axiom 8 (Homogeneity).

H1 If $a, b \notin \mathrm{cl}(X)$ then there is $f \in \Gamma_{\mathrm{cl}(X)}$ such that $f(a) = b$;

H2 If $\bar{a}_1 \in \Gamma_X(\bar{a}_2)$, $\bar{b}_1 \in \Gamma_X(\bar{b}_2)$ and $\bar{a}_i \underset{X}{\perp} \bar{b}_i$ for $i = 1, 2$, then there is $g \in \Gamma_X$ such that $g(\bar{a}_1) = \bar{a}_2$ and $g(\bar{b}_1) = \bar{b}_2$;

H3 If $\bar{a} \underset{X}{\perp} \bar{b}$, $\bar{a}' \underset{X}{\perp} \bar{b}$ and $\bar{a} \in \Gamma_X(\bar{a}')$, then $\bar{a} \in \Gamma_{\mathrm{cl}(X\bar{b})}(\bar{a}')$.

Fact 9. If $\dim(X) < \dim(W)$ and $|\Gamma_X(a)| < \aleph_0$, then $a \in \mathrm{cl}(X)$.

Proof. Since W is infinite dimensional, there exists an infinite set $\{a_n \mid n < \omega\} \subseteq W - \mathrm{cl}(X)$. By Homogeneity, if $a \notin \mathrm{cl}(X)$, then $\{a_n \mid n < \omega\} \subseteq \Gamma_{\mathrm{cl}(X)}(a) \subseteq \Gamma_X(a)$, a contradiction. □

The next definition is a substitute for the logical notions of algebraic or definable closure.

Definition 10. (1) We say that a is in the *definable closure* of X, written $a \in \mathrm{dcl}(X)$, if $|\Gamma_X(a)| = 1$, i.e. $\Gamma_X(a) = \{a\}$;
(2) We say that a is in the *algebraic closure* of X, written $a \in \mathrm{acl}(X)$, if $|\Gamma_X(a)| < \aleph_0$;

Remark 11. For small dimensional sets $X \subseteq W$ and elements $a \in W$, Fact 9 implies that if $a \in \mathrm{acl}(X)$ or $a \in \mathrm{dcl}(X)$, then $a \in \mathrm{cl}(X)$.

In the rest of this section, we introduce the notions that can be used to bypass the general $\mathfrak{C}^{eq}$ technology, in particular Shelah's Canonical Basis Theorem.

Definition 12.

(1) We say that a set $A \subseteq W^n$ is *definable over* $X \subseteq W$, if every $f \in \Gamma_X$ fixes A setwise;
(2) We say that $X \subseteq W$ is the *support* of a set $A \subseteq W^n$ if for every $f \in \Gamma$, f fixes A setwise if and only if f fixes X pointwise.

Fact 13.

(1) Any automorphism f fixes X pointwise if and only if f fixes $\mathrm{dcl}(X)$ pointwise, so by definition of support, we have $X = \mathrm{dcl}(X)$.

(2) The support of A is unique if it exists. Let X and Y be supports of A, Let $f \in \Gamma$ fixing X pointwise. Then f fixes A setwise since X is a support and so f fixes Y pointwise since Y is a support also. Thus $\Gamma_X(Y) = Y$ so $\mathrm{dcl}(Y) = Y \subseteq \mathrm{dcl}(X)$. We are done by symmetry.

Remark 14. By the previous fact, if A has support X, we define $\dim(A) := \dim(X)$ and $A \underset{B}{\perp} C$ if and only if $\mathrm{supp}(A) \underset{\mathrm{supp}(B)}{\perp} \mathrm{supp}(C)$. All these notions are well-defined and satisfy all the facts we have already proved. There will be no ambiguity since we will not deal with $A \subseteq W$.

We consider a collection $\mathfrak{D}$ of definable sets (without) parameters. We require that $\mathfrak{D}$ be closed under union and intersection, projections, product and permutation. We do *not* require closure under complementation. For clarity, we use the usual first order notation with formulas. For example, by $\phi(\bar{x}) \in \mathfrak{D}$ we mean a definable subset of $W^{\ell(\bar{x})}$. We write $\models \phi[a, \bar{b}]$ to say that $(a, \bar{b})$ is in the definable set $\phi(\bar{x})$.

We require that if $a \in \mathrm{dcl}(\bar{b})$, then there is $\phi(x, \bar{y}) \in \mathfrak{D}$ such that $\models \phi[a, \bar{b}]$ and for every a' such that $\models \phi[a', \bar{b}]$, we have $a = a'$. We also require that the sets of if $\mathfrak{D}$ are compatible with Γ, i.e. if $\models \phi[a, \bar{b}]$, then also $\models \phi[f(a), f(\bar{b})]$ for $f \in \Gamma$.

Now on to the last axioms.

Axiom 15 (Definability of Stationary types). Let $\bar{a}, \bar{b} \in W$ and $R \in \mathfrak{D}$ be a relation on the orbits of $\bar{a}$ and $\bar{b}$. Then there is $d_R \in \mathfrak{D}$ such that for all $\bar{a}' \in \Gamma(\bar{a})$ we have $\bar{a}' \in d_R \in \mathfrak{D}$ if and only if for every $\bar{b}' \in \Gamma(\bar{b})$ if $\bar{a}' \perp \bar{b}'$, then $(\bar{a}', \bar{b}') \in R$.

Axiom 16 (Canonical Basis). If $E(\bar{x}, \bar{y}) \in \mathfrak{D}$ is an equivalence relation over orbits of W, then each equivalence class $\bar{b}/E$ has a support.

3. The group configuration

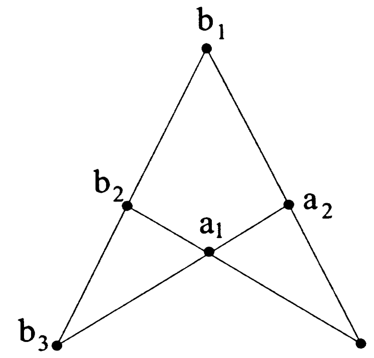
In this section, we show that if a pregeometry, its automorphism group and the collection of definable sets satisfy our list of axioms, then Hrushovski and Zilber nice group configuration gives rise to a definable group.

Hypothesis 17. There exist b_i, a_i for $i = 1, 2, 3$, sequences of dimension 1, such that

(1) All sequences are pairwise independent;
(2) $\dim(b_1 b_2 b_3) = 2,\ \dim(b_i a_j a_k) = 2$, for all $i \neq j \neq k$, and

$$\dim(b_1 b_2 b_3 a_1 a_2 a_3) = 3;$$

(3) $a_2 \in \mathrm{dcl}(b_1 a_3)$, $a_1 \in \mathrm{dcl}(b_2 a_3)$, and $a_3 \in \mathrm{dcl}(b_1 a_2) \cap \mathrm{dcl}(b_2 a_1)$.



Given sets of sequences A, B, we will denote $A + B$, the set

$$A + B = \{(\bar{a}, \bar{b}) \mid \bar{a} \in A, \bar{b} \in B, \text{ and } \bar{a} \perp \bar{b}\}.$$

Given $(b'_1, b'_2) \in \Gamma(b_1) + \Gamma(b_1)$ and $a \in \Gamma(a_2) - \mathrm{cl}(b'_1 b'_2)$, we define $h_{(b'_1, b'_2)}(a)$ as follows. Choose $f \in \Gamma$ such that $f(b_1) = b'_1$, $f(b_2) = b'_2$ and $f(a_2) = a$. To do this, choose first $\sigma \in \Gamma$ such that $\sigma(b_1) = b'_1$, $\sigma(b_2) = b'_2$. Clearly σ exists by Axiom H2, since $b'_i \in \Gamma(b_i)$ for $i = 1, 2$. Then, choose $\tau \in \Gamma$ such $\tau(\sigma(a_2)) = a$ and $\tau \restriction b_1 b_2 = id$. This is possible by Axiom H1 since by assumption on the configuration $a_2 \notin \mathrm{cl}(b_1 b_2)$, so $\sigma(a_2) \notin \mathrm{cl}(b'_1 b'_2)$ and $a \notin \mathrm{cl}(b'_1 b'_2)$ by choice of a.

We now make a few observations. First, $f(a_3)$ is uniquely determined since $a_3 \in \mathrm{dcl}(b_1 a_2)$. Indeed, suppose $g(b_1) = b'_1$ and $g(a_2) = a$. Then $g^{-1} f \in \Gamma_{b_1 a_2}$ so $g^{-1} f(a_3) = a_3$. Therefore $g(a_3) = f(a_3)$. Second, notice that $f(a_1)$ is uniquely determined, since $a_1 \in \mathrm{dcl}(b_2 a_3)$. Indeed, suppose $g(b_2) = b'_2$ and $g(a_3) = f(a_3)$. Then $g^{-1} f \in \Gamma_{b_2 a_3}$, so $g^{-1} f(a_1) = a_1$, and $g(a_1) = f(a_1)$.

We define $h_{(b'_1, b'_2)}(a) = f(a_1)$. In view of the previous considerations, this is well-defined and furthermore $f(a_1) \in \Gamma(a_2)$. Notice also that $\langle \bar{b}'_1, \bar{b}'_2, a, a' \rangle \in \mathfrak{D}$, for all $h_{(b'_1, b'_2)}(a) = a'$, using projections and intersection.

We wish to extend the action of $\Gamma(b_1) + \Gamma(b_1)$ on all elements of $\Gamma(a_2)$. To do this, we define the following relation on $\Gamma(b_1) + \Gamma(b_1)$:

$$(b'_1, b'_2) \sim (b''_1, b''_2) \quad \text{if} \quad h_{(b'_1, b'_2)}(a) = h_{(b''_1, b''_2)}(a), \quad \text{for all } a \in \Gamma(a_2) - \mathrm{cl}(b'_1 b'_2 b''_1 b''_2).$$

Claim. $\sim$ is an equivalence relation on $\Gamma(b_1) + \Gamma(b_1)$.

Proof. Reflexivity and Symmetry are obvious. To see that Transitivity holds, we first show that we can replace "for all" by "there exists" in the definition of a. Indeed, suppose that $a, a' \in \Gamma(a_2) - \mathrm{cl}(b'_1 b'_2 b''_1 b''_2)$ and that $h_{(b'_1, b'_2)}(a) = h_{(b''_1, b''_2)}(a)$. By Axiom H1, there exists $\sigma \in \Gamma_{\mathrm{cl}(b'_1 b'_2 b''_1 b''_2)}$ such that $\sigma(a) = a'$. Notice that $h_{(b'_1, b'_2)}(\sigma(a)) = \sigma(h_{(b'_1, b'_2)}(a)$ and similarly, $h_{(b''_1, b''_2)}(\sigma(a)) = \sigma(h_{(b''_1, b''_2)}(a)$, and hence $h_{(b'_1, b'_2)}(a') = h_{(b''_1, b''_2)}(a')$. Transitivity now follows easily. □

We denote by $[b'_1, b'_2]$ the equivalence class of (b'_1, b'_2) under $\sim$. It now follows from Axiom 15 that $\sim \in \mathfrak{D}$. Hence, each $[b'_1, b'_2] \in \mathfrak{D}$ and by Axiom 16 must have a support. Clearly, $\mathrm{supp}[b'_1, b'_2]\, \mathrm{cl}(b'_1, b'_2) \subseteq W$.

Let $H = \{[b_1', b_2'] \mid (b_1', b_2') \in \Gamma(b_1) + \Gamma(b_1)\}$.

Notice that Γ acts transitively on the elements of H in the following sense: if $[b_1, b_2], [c_1, c_2]$ are elements of H, there is $f \in \Gamma$ such that $f([b_1, b_2]) = [c_1, c_2]$. To see this, recall that $c_i \in \Gamma(b_i)$ for $i = 1, 2$ and that by definition of H we have that each sequence is independent and $b_1 \mathop{\smile\!\!\!\!\!\!|}\, b_2$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}\, c_2$. Hence, by Axiom H2 there exists $f \in \Gamma$ such that $f(b_i) = c_i$ for $i = 1, 2$. Then $f([b_1, b_2]) = [f(b_1), f(b_1)] = [c_1, c_2]$ as required.

Notice also that by Axiom 16, every element $\alpha \in H$ has a support X_α, so that we can extend forking and dimensions on elements of H. Elements of H are called *germs* and they each act on $\Gamma(a_2)$. We will want to compose germs, but we will want to make sure that the composition is also an element of H. For this, some more work is needed. We can express H by an infinite intersection of elements of $\mathfrak{D}$ by Axiom 15.

Lemma 18. $[b_1, b_2] \subseteq \mathrm{cl}(b_3)$ *and therefore* $[b_1, b_2] \mathop{\smile\!\!\!\!\!\!|}\, b_i$ *for* $i = 1, 2$.

Proof. First, observe that

(*) $$[b_1, b_2](a_2) = a_1 \in \mathrm{cl}(a_2 b_3),$$

by definition and the configuration. We want to show that

$$X := \mathrm{supp}([b_1, b_2]) \subseteq \mathrm{cl}(b_3).$$

By definition of support, it is enough to show that for all $f \in \Gamma_{\mathrm{cl}(b_3)}$, we have

$$f([b_1, b_2]) = [b_1, b_2],$$

i.e. $[b_1, b_2]$ is fixed setwise by f.

For this, fix $f \in \Gamma_{\mathrm{cl}(b_3)}$ and let $a \in \Gamma(a_2) - \mathrm{cl}(X f(X) b_1 b_2 b_3)$.

We claim that $[b_1, b_2](a) \in \mathrm{cl}(a b_3)$. To see this, it is enough to find an automorphism $\sigma \in \Gamma_{\mathrm{cl}(b_1 b_2 b_3)}$ such that $\sigma(a_2) = a$ and then applying σ to (*). But the existence of σ follows from H1 if we can show that $a_2 \notin \mathrm{cl}(b_1 b_2 b_3)$. This follows from the configuration. Suppose $a_2 \in \mathrm{cl}(b_1 b_2 b_3)$. Then $a_1 \in \mathrm{cl}(b_1 b_2 b_3)$, since $a_1 \in \mathrm{cl}(a_2 b_3)$ and also $a_3 \in \mathrm{cl}(b_1 b_2 b_3)$ since $a_3 \in \mathrm{cl}(a_1 b_2)$. This is a contradiction since

$$\dim(a_1 a_2 a_3 b_1 b_2 b_3) = 3 \neq 2 = \dim(b_1 b_2 b_3).$$

Now choose $g \in \Gamma_{\mathrm{cl}(b_3 a)}$ such that $g \upharpoonright X = f \upharpoonright X$.

Then, we have the following equalities:

$$\begin{aligned} [b_1, b_2](a) &= g([b_1, b_2](a)) && (\text{since } [b_1, b_2](a) \in \mathrm{cl}(b_3 a)) \\ &= g([b_1, b_2])(g(a)) && (g \text{ is an automorphism}) \\ &= g([b_1, b_2])(a) && (g(a) = a) \\ &= f([b_1, b_2])(a) && (f \upharpoonright X = g \upharpoonright X) \end{aligned}$$

Thus, by definition of the germs, $[b_1, b_2] = f([b_1, b_2])$. This finishes the proof. □

The elements of H act on $\Gamma(a_2)$. It makes sense to compose them. Let $\alpha, \beta \in H$. We write $\alpha * \beta$ for an element $\gamma \in H$, such that for all $a \in \Gamma(a_2)$ such that $a \mathop{\smile\!\!\!\!\!\!|}\, \alpha\beta\gamma$, we have $\gamma(a) = \alpha(\beta(a))$. Such γ's do not necessarily exist. We will show that, in fact, $\alpha * \beta$ exists if $\alpha \perp \beta$.

Suppose $[c_1, c_2]$ and $[c_2, c_3]$ are in H. Then in this case, it is easy to see that $[c_1, c_2] * [c_2, c_3] = [c_1, c_3]$. We will show that, in fact, this is the typical situation when $\alpha \perp \beta$. This is done by the following lemmas.

Lemma 19. *If c_1, c_2 and $c_3 \in \Gamma(b_1)$ are such that $\dim(c_1c_2c_3) = 3$, then $[c_1c_2] \perp [c_2, c_3]$*

Proof. First, let $X = \mathrm{supp}([c_1, c_2])$ and $Y = \mathrm{supp}([c_2, c_3])$. Suppose $[c_1c_2] \not\perp [c_2, c_3]$. Then, by definition, $X \not\perp Y$, and so $X \subseteq Y$ since they are both closed, and furthermore $X = Y$ since they have dimension 1. By the dimension, $c_1 \perp c_2c_3$, so since $Y \subseteq \mathrm{cl}(c_2c_3)$, we must have $c_1 \perp c_2Y$. But, since $X = Y$, we now have also $c_1 \perp c_2X$. Since $c_2 \perp X$, we thus have $\dim(c_1c_2X) = 3$. But $\mathrm{cl}(c_1c_2X) = \mathrm{cl}(c_1c_2)$, so that's impossible. □

Lemma 20. *Let $\alpha, \beta \in H$. If $\alpha \perp \beta$, then there exist c_1, c_2 and c_3 such that $\alpha = [c_1, c_2]$, $\beta = [c_2, c_3]$ and $\dim(c_1, c_2, c_3) = 3$.*

Proof. Notice that Γ acts transitively over $H + H$, via the supports: let $\alpha_1 \perp \beta_1$ and $\alpha_2 \perp \beta_2$. Denote by X_{α_i}, (respectively X_{β_i}) the supports of α_i (respectively β_i). Then, by definition $X_{\alpha_i} \perp X_{\beta_i}$, for $i = 1, 2$ and further, $X_{\alpha_1} \in \Gamma(X_{\alpha_2})$, and $X_{\beta_1} \in \Gamma(X_{\beta_2})$ by a homogeneous axiom. The result follows by Stationarity. Now, by the previous lemma, if c_1, c_2 and $c_3 \in \Gamma(a)$ are such that $\dim(c_1c_2c_3) = 3$, then $[c_1c_2] \perp [c_2, c_3]$. Thus, by transitivity, we can find $f \in \Gamma$ such that $f([c_1, c_2]) = \alpha$ and $f([c_2, c_3]) = \beta$. Thus, $\alpha = [f(c_1), f(c_2)]$ and $\beta = [f(c_2), f(c_3)]$. Clearly, $\dim(f(c_1), f(c_2), f(c_3)) = 3$. We are done. □

Lemma 21. *If $\alpha, \beta \in H$ with $\alpha \perp \beta$, then $\alpha * \beta$ is a well-defined element of H. Moreover, $\alpha * \beta \perp \alpha$ and $\alpha * \beta \perp \beta$.*

Proof. Choose c_1, c_2 and c_3 such that $\alpha = [c_1, c_2]$ and $\beta = [c_2, c_3]$, with $\dim(c_1, c_2, c_3) = 3$. Check that $\gamma = [c_1, c_3]$. The rest is now immediate. □

Define an equivalence relation on $H + H$,

$$(\alpha_1, \beta_1) \approx (\alpha_2, \beta_2) \quad \text{if} \quad \alpha_1 * \beta_1(e) = \alpha_2 * \beta_2(e),$$

for every $e \in \Gamma(j)$ such that $\alpha_1 * \beta_1(e)$ and $\alpha_2 * \beta_2(e)$ are both defined. Let G be the set of equivalence classes. Let us call $[\alpha, \beta]$ the equivalence class of (α, β) under $\approx$. We define

$$[\alpha_1, \alpha_2] * [\beta_1, \beta_2] = [\gamma, \delta],$$

where $\alpha_1 * \alpha_2 * \beta_1 * \beta_2 \approx \gamma * \delta$, and $(\gamma, \delta) \in H + H$. By considerations similar to H, G can be expressed by an infinite intersection of sets in $\mathfrak{D}$, and also its product by Axiom 16.

The next claim shows that G is closed under composition.

Claim. $(G, *)$ is closed under composition.

Proof. Let $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2] \in G$ be given. Then $\alpha_1 \perp \alpha_2$ and $\alpha_1 * \alpha_2 \perp \alpha_i$, for $i = 1, 2$ by a previous lemma. Similarly, $\beta_1 \perp \beta_2$ and $\beta_1 * \beta_2 \perp \beta_i$, for $i = 1, 2$.

We distinguish two cases. Let $\alpha := \alpha_1 * \alpha_2$. If $\alpha \perp \beta_1 * \beta_2$, then both $[\alpha, (\beta_1 * \beta_2)]$ and $[\beta_1, \beta_2] \in G$, and obviously $\alpha_1 * \alpha_2 * \beta_1 * \beta_2 \approx \alpha * (\beta_1 * \beta_2)$.

If $\alpha \not\perp \beta_1 * \beta_2$, then $\alpha \in \mathrm{cl}(\beta_1 * \beta_2)$ and so since $\beta_1 * \beta_2 \perp \beta_1$, also $\alpha \perp \beta_1$. Thus, $\alpha \perp \beta_1$ and $\beta_1 \perp \beta_2$.

First, choose $\delta \in H$ such that $\delta \perp \alpha\beta_1\beta_2$. In particular, $\beta_1 \perp \delta$. Now choose $\delta_1 \in H$ such $\delta_1 \perp \delta$. Then $\delta_1 * \delta$ is well-defined, and $\delta_1 * \delta \perp \delta$. Since Γ acts transitively on $H + H$, we can find $g \in \Gamma$ such that $g(\delta) = \delta$ and $g(\delta * \delta_1) = \beta_1$. Thus, $\beta_1 = g(\delta_1) * \delta$. Call $g(\delta_1) = \delta' \in H$. Then $\alpha * \beta_1 * \beta_2 = (\alpha * \delta) * (\delta' * \beta_2)$. We are done in we can show that $\alpha \perp \delta$ and $\delta' \perp \beta_2$. Certainly $\alpha \perp \delta$ by choice of δ. Now if $\delta' \not\perp \beta_2$, then $\beta_2 \in \mathrm{cl}(\delta')$. But $\beta_1 \in \mathrm{cl}(\delta, \delta')$ so $\delta' \in \mathrm{cl}(\beta_1 \delta)$. Hence, $\dim(\delta, \beta_1, \beta_2) = 2$, contradicting the choice of δ.

This finishes the proof. □

Lemma 22. *$(G, *)$ is a group.*

Proof. G is nonempty. Since H is closed under inverse, it is easy to see that the inverse of $[\alpha, \beta]$ is $[\beta^{-1}, \alpha^{-1}]$, so G is closed under inverse. The previous claim shows that G is closed under composition. Finally, $(G, *)$ acts on $\Gamma(a_2)$ as described. □

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