# FORKING IN PREGEOMETRIES, PART I: THE BASICS

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ABSTRACT. The aim of this paper is to set the foundation to separate geometric model theory from model theory. Our thesis is that it is possible to lift results from geometric model theory to non first order logic (e.g. $L_{\omega_1,\omega}$). We introduce a relation between subsets of a pregeometry and show that it satisfies all the formal properties that forking satisfies in simple first order theories. This is important when one is trying to lift forking to nonelementary classes, in contexts where there exists pregeometries but not necessarily a well-behaved dependence relation (see for example [HySh]). We use these to reproduce S. Buechler's characterization of local modularity in general. These results are used by Lessmann to prove an abstract group configuration theorem in [Le2].

## 1. INTRODUCTION

At the center of stability theory is the notion of forking. Forking is a dependence relation discovered by S. Shelah. It satisfies the following properties in the first order stable case, see [Sh b].

(1) (Finite character) The type $p$ does not fork over $B$ if and only if every finite subtype $q \subseteq p$ does not fork over $B$.
(2) (Extension) Let $p$ be a type which does not fork over $B$. Let $C$ be given containing the domain of $p$. Then there exists $q \in S(C)$ extending $p$ such that $q$ does not fork over $B$;
(3) (Invariance) Let $f \in \mathrm{Aut}(\mathfrak{C})$ and $p$ be a type which does not fork over $B$. Then $f(p)$ does not fork over $f(B)$.
(4) (Existence) The type $p$ does not fork over its domain;
(5) (Existence of $\kappa(T)$) For every type $p$, there exists a set $B \subseteq \mathrm{dom}(p)$ such that $p$ does not fork $B$;
(6) (Symmetry) Let $p = \mathrm{tp}(\bar{a}/B\bar{c})$. Suppose that $p$ does not fork over $B$. Then $\mathrm{tp}(\bar{c}/B\bar{a})$ does not fork over $B$;
(7) (Transitivity) Let $B \subseteq C \subseteq A$. Let $p \in S(A)$. Then $p$ does not fork over $B$ if and only if $p$ does not fork over $C$ and $p \restriction C$ does not fork over $B$.

Already in the introduction of Chapter III of [Sh b], S. Shelah states what is important about the forking relation is that it satisfies properties (1)–(7). S. Shelah stated another property named by S. Buechler [Bu] the Pairs Lemma (see Proposition 16 for the statement) as one of the basic properties of forking, which he proved in [Sh b] using the

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Finite Equivalence Relation Theorem. Later Baldwin in his book [Bl] presented an axiomatic treatment of forking in stable theories. This allowed Baldwin to derive abstractly Shelah's Pairs Lemma from the other properties of forking. Following these ideas, it has now become common to characterize various stability conditions in terms of the axiomatic properties that forking satisfies.

One of the most difficult directions of pure model theory is the area called by Shelah classification for nonelementary classes. A major problem is to find a dependence relation which is as well-behaved as forking for first order theories. See for example [Gr 1], [Gr 2], [GrHa], [GrLe1], [GrLe2], [GrSh 1], [GrSh 2], [HaSh], [HySh], [Ki], [KlSh], [Le1], [MaSh], [Sh3], [Sh47], [Sh 87a], [Sh 87b], [Sh 88], [Sh tape], [Sh 299], [Sh 300], [Sh 394], [Sh 472], [Sh 576] and [Sh h]. The situation in nonelementary classes is very different from the first order case. In the first order case, the Extension property for forking comes for free; it holds for any theory and is a consequence of the Compactness Theorem. This is in striking contrast with the nonelementary cases; the Extension property is usually among the most problematic and does not hold over sets in general for any of the dependence relations introduced thus far.

A general dependence relation satisfying all the formal properties of forking has thus not been found yet for nonelementary classes. There are, however, several cases where pregeometries appear; i.e. sets with a closure operation satisfying the properties of linear dependence in a vector space. In the first order case, the pregeometries are the sets of realizations of a *regular* type, and the dependence is the one induced by forking and thus satisfies automatically many additional properties. In nonelementary classes the situation is different.

Let us describe several nonelementary examples. The first three examples have in common that there exists a rank, giving rise to a reasonable dependence relation. However the *Extension* property and the *Symmetry* property fail in general (they hold over sufficiently "rich" sets). The rank introduced for these classes are generalizations of what S. Shelah calls $R[\cdot, L, 2]$. Intuitively, a formula has rank $\alpha + 1$ if it can be partitioned in *two* pieces of rank $\alpha$ with some additional properties that are tailored to each context. It is noteworthy that extensions of Morley rank are inadequate, as partitioning a formula in countably many pieces makes sense only when the compactness theorem holds. In the last example, no rank is known, but pregeometries exist.

**Categorical sentences in $L_{\omega_1\omega}(Q)$:** S. Shelah started working on this context [Sh47] to answer a question of J.T.Baldwin: Can a sentence in $L(Q)$ have exactly one uncountable model? Shelah answers this question negatively using **V=L** (and later using different methods within ZFC) while developing very powerful concepts. One of the main tools is the introduction of a rank. This rank is bounded under the parallel to $\aleph_0$-stability. It gives rise to a dependence relation and pregeometries. Later, H. Kierstead [Ki] uses these pregeometries to obtain some results on the countable models of these sentences.

**Excellent Scott sentences:** In [Sh 87a] and [Sh 87b] S.Shelah introduces a simplification of the rank of [Sh47]. S. Shelah identifies the concept of *excellent Scott sentences* and proves (among many other things) the parallel to Morley's Theorem for them. Again, this rank induces a dependence relation on the subsets of the models. Later, R. Grossberg and B. Hart [GrHa] proved the existence of pregeometries

(regular types) for this dependence relation and used it to prove the main gap for excellent Scott sentences.

**Totally transcendental diagrams:** In [Le1] Lessmann introduced a rank for $\aleph_0$-stable diagrams. Finite diagrams were introduced by S. Shelah [Sh3] in 1970 (see also [GrLe1] for an exposition). They are classes of models omitting a prescribed set of types, with an additional condition. We call a finite diagram *totally transcendental* when the rank is bounded. The rank gives rise to a dependence relation on the subsets of the models and pregeometries exist often. This is used to give a proof of categoricity generalizing the Baldwin-Lachlan Theorem. In a work in preparation, [GrLe2], we prove the main gap for totally transcendental diagrams.

**Superstable diagrams:** In [HySh], Hyttinen and Shelah study stable finite diagrams ([Sh3] or [GrLe1]) under the additional assumption that $\kappa(D) = \aleph_0$. Such diagrams are called *superstable.* They introduce a relation between sets $A$, $B$ and an element $a$, written $a \downarrow_B A$. The main result is that the parallel of regular types exist. More precisely, for every pair of "sufficiently saturated" models $M \subseteq N$, $M \neq N$, there exists a type $p$ realized in $N - M$ such that the relation $a \downarrow_M C$ (standing for $a \notin \mathrm{cl}(C)$) induces a pregeometry among the realizations of $p$ in $N$.

Thus, pregeometries seem to appear naturally in nonelementary classes, while general well-behaved dependence relations are hard to find. The main goal of this paper is therefore to recover from *any* pregeometry a dependence relation over the subsets of the pregeometry that satisfies all the formal properties of forking. This is, of course, particularly useful when the pregeometry itself was *not* induced by forking.

A similar endeavor was attempted by John Baldwin in the early eighties. In [Bl1], J.Baldwin examined some pregeometries and several dependence relations in the first order case. From a pregeometry, he defines the relation $a \underset{B}{\downarrow} C$, by $a \in \mathrm{cl}(B \cup C) - \mathrm{cl}(B)$. He did not however introduce $A \underset{B}{\downarrow} C$, where $A$ is a *tuple* or a *set* as opposed to an element, which we do (see Definition 7). This is a crucial step; it is built-in in the model theory of first order, since forking is naturally defined for types of any arity. To make this more precise, fix $T$ a first order stable theory. Let us write

$$\bar{a} \overset{*}{\underset{B}{\downarrow}} C \quad \text{for} \quad \mathrm{tp}(\bar{a}/B \cup C) \text{ does not fork over } B.$$

Inside a regular type $p(x) \in S(B)$, the relation $a \in \mathrm{cl}(C)$ given by $a \overset{*}{\underset{B}{\downarrow}} C$ gives rise to a pregeometry. But, the relation $\bar{a} \overset{*}{\underset{B}{\downarrow}} C$ is defined in general whether or not $\bar{a}$ and $C$ consist of elements realizing $p$. Inside the pregeometry, the relation $\bar{a} \overset{*}{\underset{B}{\downarrow}} C$ holds (defined with forking) if and only if the relation $\bar{a} \underset{B}{\downarrow} C$ holds (defined formally from our definition using the closure operator of the pregeometry). This is a consequence of the Pairs Lemma, which holds for first order simple theories. When we start from an abstract pregeometry (or an abstract dependence relation), we do *not* have the formalism of types or the Pairs Lemma. Therefore the relation $\bar{a} \underset{B}{\downarrow} C$ has to be introduced for tuples, using the relation $a \underset{B}{\downarrow} C$ for elements. As a consequence, suppose we are given the corresponding notion of a regular type $p \in S(B)$ in a nonelementary context. Suppose there is some ambient dependence relation, written $A \overset{*}{\underset{B}{\downarrow}} C$ such that over realizations of $p$ the relation $a \in \mathrm{cl}(C)$, given by $a \overset{*}{\underset{B}{\not\downarrow}} C$, induces a pregeometry. Then, the truth value of the relation $\bar{a} \overset{*}{\underset{B}{\downarrow}} C$ (given from the ambient dependence relation) and $\bar{a} \underset{B}{\downarrow} C$ (defined from the closure operation in the pregeometry) may not coincide. They will coincide only if the Pairs Lemma holds for the dependence relation (and this fact is not known in general for nonelementary cases). Therefore, this abstract formalism allows us to introduce for nonelementary classes a (possibly) *better* dependence relation, inside the pregeometry.

As an illustration of the value of this general relation, we present S. Buechler's characterization of local modularity with parallel lines (see [Bu]) in this general context. This also has esthetic value as it allows one carry out this work in the general context of combinatorial geometry, without logic.

Finally, we add two more sections. One devoted to basic set-theoretic results and another to stable systems.

In the follow-up paper [Le2], Lessmann presents an abstract framework where, using the "forking relation" defined is this paper, he is able to derive a generalization of Zilber-Hrushovski group configuration theorem. We believe that this result has a lot of potential for the classification of nonelementary classes.

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## 2. Preliminaries

We recall a few standard and well-known facts about pregeometries. The notation is standard. We write $Ab$ for $A \cup \{b\}$.

**Definition 1.** We say that $(W, \mathrm{cl})$ is a *pregeometry* if $W$ is a set and $\mathrm{cl}\colon \mathcal{P}(W) \to \mathcal{P}(W)$ is a function satisfying the following four properties

1. (Monotonicity) For every set $X \in \mathcal{P}(W)$ we have $X \subseteq \mathrm{cl}(X)$;
2. (Finite Character) If $a \in \mathrm{cl}(X)$ then there is a finite set $Y \subseteq X$, such that $a \in \mathrm{cl}(Y)$;
3. (Transitivity) Let $X, Y \in \mathcal{P}(W)$. If $a \in \mathrm{cl}(X)$ and $X \subseteq \mathrm{cl}(Y)$ then $a \in \mathrm{cl}(Y)$;
4. (Exchange Property) For $X \in \mathcal{P}(W)$ and $a, b \in W$, if $a \in \mathrm{cl}(Xb)$ but $a \notin \mathrm{cl}(X)$, then $b \in \mathrm{cl}(Xa)$.

We always assume $\mathrm{cl}(\emptyset) \neq W$.

The next two basic properties are standard and easy.

**Fact 2.** If $(W, \mathrm{cl})$ is a pregeometry and $B \subseteq C \subseteq W$, then $\mathrm{cl}(B) \subseteq \mathrm{cl}(C)$.

**Fact 3.** If $(W, \mathrm{cl})$ is a pregeometry and $B \subseteq W$, then $\mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$.

**Definition 4.** Let $(W, \mathrm{cl})$ be a pregeometry.

(1) For $X \subseteq W$, we say that $X$ is *closed* if $X = \mathrm{cl}(X)$;
(2) $I \subseteq W$ is *independent* if for every $a \in I$, we have $a \notin \mathrm{cl}(I - \{a\})$;
(3) We say that $I \subseteq A$ *generates* $A$, if $\mathrm{cl}(I) = \mathrm{cl}(A)$;
(4) A *basis* for a set $A \subset W$ is an independent set $I$ generating $\mathrm{cl}(A)$;
(5) For $X \subseteq W$, the *dimension of* $X$, written $\dim(X)$, is the cardinality of a basis for $\mathrm{cl}(X)$.

**Fact 5.** Using the axioms of pregeometry, one can show that for every set, bases exist and that the dimension is well-defined see for example Appendix in [Gr]

**Definition 6.** Let $G = (W, \mathrm{cl})$ be a pregeometry.

(1) A bijection $f\colon W \to W$ is an *automorphism of* $G$ if for every $a \in W$ and $A \subseteq W$ we have
$$a \in \mathrm{cl}(A) \quad \text{if and only if} \quad f(a) \in \mathrm{cl}(f(A)).$$
We denote $Aut_A(G)$ the set of automorphisms of $G$ fixing $A$ pointwise.
(2) We say that $G$ is *homogeneous* if for every $a, b \in W$ and $A \subseteq W$, such that $a \notin \mathrm{cl}(A)$ and $b \notin \mathrm{cl}(A)$ there is an automorphism of $G$, fixing $A$ pointwise and taking $a$ to $b$.

## 3. Forking in Pregeometries

In this section, we introduce the main concept of the paper.

**Definition 7.** Let $(W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. We say that *$A$ depends on $C$ over $B$*, if there exist $a \in A$ and a finite $A' \subseteq A$ (possibly empty) such that
$$a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A').$$

If $A$ depends on $C$ over $B$, we write $A \not\downarrow_B C$;
If $A$ does not depend on $C$ over $B$, we write $A \downarrow_B C$.

**Remark 8.** An alternative definition with $A' = \emptyset$ does not permit a smooth extension to sets $A \downarrow_B C$ when $A$ is not a singleton.

**Remark 9.** $A \downarrow_B C$ if and only if $A \cup B \downarrow_B C \cup B$. Hence, we will often assume that $B \subseteq A \cap C$.

We now prove that the properties of forking in simple theories hold with this formalism, directly from the axioms of a pregeometry.

**Proposition 10** (Finite Character). *Let $(W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. Then*
$$A \downarrow_B C \quad \textit{if and only if} \quad A' \downarrow_B C',$$
*for every finite $A' \subseteq A$ and finite $C' \subseteq C$.*

*Proof.* If $A \not\downarrow_B C$, then there exist $a \in A$, and a finite $A' \subseteq A$ such that

$$a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A').$$

By Finite Character, there exist a finite $C' \subseteq C$ such that $a \in \mathrm{cl}(B \cup C' \cup A')$. Hence $A' \not\downarrow_B C'$, by definition.

For the converse, if there exist a finite $A' \subseteq A$ and a finite $C' \subseteq C$ such that $A' \not\downarrow_B C'$, then we can find $a \in A'$ and $A'' \subseteq A'$ such that

$$a \in \mathrm{cl}(B \cup C' \cup A'') - \mathrm{cl}(B \cup A'').$$

Since $C' \subseteq C$, we have $a \in \mathrm{cl}(B \cup C \cup A'')$, by Fact 2. Hence, $A \not\downarrow_B C$, by definition. □

**Proposition 11** (Continuity). *Let $(W, \mathrm{cl})$ be a pregeometry. Let $\langle C_i \mid i < \alpha \rangle$ be a continuous increasing sequence of sets in $W$, and $A, B \subseteq W$.*

(1) *If $A \downarrow_B C_i$ for every $i < \alpha$, then $A \downarrow_B \bigcup_{i<\alpha} C_i$.*
(2) *If $C_i \downarrow_B A$ for every $i < \alpha$, then $\bigcup_{i<\alpha} C_i \downarrow_B A$.*

*Proof.* By Finite Character. □

**Proposition 12** (Invariance). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$ and let $f \in Aut(G)$. Then*

$$A \downarrow_B C \quad \textit{if and only if} \quad f(A) \downarrow_{f(B)} f(C).$$

*Proof.* Note that since the inverse of an automorphism is an automorphism, it is enough to show one direction. Assume that $A \not\downarrow_B C$ and let $a \in A$ and $A' \subseteq A$ finite be such that

$$a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A').$$

Then $f(a) \in \mathrm{cl}(f(B \cup C \cup A')) - \mathrm{cl}(f(B \cup A'))$, by definition of automorphism. But since $f$ is a bijection

$$f(a) \in \mathrm{cl}(f(B) \cup f(C) \cup f(A')) - \mathrm{cl}(f(B) \cup f(A')).$$

Therefore, $f(A) \not\downarrow_{f(B)} f(C)$ by definition. □

**Proposition 13** (Monotonicity). *Let $(W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. Suppose $A \downarrow_B C$.*

(1) *If $A' \subseteq A$ and $C' \subseteq C$, then $A' \downarrow_B C'$;*
(2) *If $B' \subseteq C$, then $A \downarrow_{B \cup B'} C$.*

*Proof.* (1) Suppose that $A' \underset{B}{\not\perp} C'$. Let $a \in A'$ and $A^* \subseteq A'$ finite such that

$$a \in \mathrm{cl}(B \cup C' \cup A^*) - \mathrm{cl}(B \cup A^*).$$

Then, by Fact 2, we have $a \in \mathrm{cl}(B \cup C \cup A^*) - \mathrm{cl}(B \cup A^*)$. But $a \in A$ and $A^* \subseteq A$, so $A \underset{B}{\not\perp} C$.

(2) Suppose $A \underset{B \cup B'}{\not\perp} C$. Let $a \in A$ and $A' \subseteq A$ finite such that

$$a \in \mathrm{cl}(B \cup B' \cup C \cup A') - \mathrm{cl}(B \cup B' \cup A').$$

Since $B' \subseteq C$, we have $\mathrm{cl}(B \cup B' \cup C \cup A') = \mathrm{cl}(B \cup C \cup A')$. Also, $\mathrm{cl}(B \cup A') \subseteq \mathrm{cl}(B \cup B' \cup A')$. Hence $a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A')$. Therefore $A \underset{B}{\not\perp} C$. □

**Proposition 14** (Symmetry). *Let* $(W, \mathrm{cl})$ *be a pregeometry. Let* $A, B$ *and* $C$ *be subsets of* $W$. *Then*

$$A \underset{B}{\perp} C \quad \textit{if and only if} \quad C \underset{B}{\perp} A.$$

*Proof.* Suppose that $A \underset{B}{\not\perp} C$. Choose $a \in A$ and a finite $A' \subseteq A$ such that

(*) $$a \in \mathrm{cl}(B \cup C \cup A') - \mathrm{cl}(B \cup A').$$

By Finite Character and (*), there exist $c \in C$ and a finite (and possibly empty) $C' \subseteq C$ such that

(**) $$a \in \mathrm{cl}(B \cup C' \cup c \cup A') \quad \text{and} \quad a \notin \mathrm{cl}(B \cup C' \cup A').$$

Therefore, by the Exchange Property, we have

$$c \in \mathrm{cl}(B \cup C' \cup A' \cup a).$$

But $c \notin \mathrm{cl}(B \cup C' \cup A')$, (**). Hence,

$$c \in \mathrm{cl}(B \cup C' \cup A' \cup a) - \mathrm{cl}(B \cup C' \cup A').$$

Therefore, $C \underset{B}{\not\perp} A'$, for some finite subset $A'$ of $A$. Hence, $C \underset{B}{\not\perp} A$, by Finite Character. □

**Proposition 15** (Transitivity). *Let* $(W, \mathrm{cl})$ *be a pregeometry. Let* $A, B, C$ *and* $D$ *be subsets of* $W$ *such that* $B \subseteq C \subseteq D$. *Then,*

$$A \underset{C}{\perp} D \quad \textit{and} \quad A \underset{B}{\perp} C \quad \textit{if and only if} \quad A \underset{B}{\perp} D.$$

*Proof.* Suppose first that $A \underset{B}{\not\perp} D$. Choose $a \in A$ and a finite $A' \subseteq A$ such that

$$a \in \mathrm{cl}(D \cup A') - \mathrm{cl}(B \cup A').$$

Either $a \in \mathrm{cl}(C \cup A')$, and so

$$a \in \mathrm{cl}(C \cup A') - \mathrm{cl}(B \cup A'),$$

which implies that $A \underset{B}{\not\perp} C$. Or $a \notin \mathrm{cl}(C \cup A')$, and therefore

$$a \in \mathrm{cl}(D \cup A') - \mathrm{cl}(C \cup A'),$$

which implies that $A \underset{C}{\downarrow} D$.

The converse follows by Monotonicity since $B \subseteq C \subseteq D$. □

The following is proved in [Sh b] directly using the finite equivalence relation theorem. The proof that it follows from the other axioms of forking is due to J. Baldwin. We present it here for completeness.

**Proposition 16** (Pairs Lemma). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let $A, B, C$ and $D$ be subsets of $W$ such that $C \subseteq B \cap D$. Then*

$$A \cup B \underset{C}{\downarrow} D \quad \textit{if and only if} \quad A \underset{C \cup B}{\downarrow} D \cup B \quad \textit{and} \quad B \underset{C}{\downarrow} D.$$

*Proof.* Notice first, that by definition

(*) $$A \underset{C \cup B}{\downarrow} D \cup B \quad \text{if and only if} \quad A \underset{C \cup B}{\downarrow} D.$$

Therefore, by Symmetry and (*), it is equivalent to show that

$$D \underset{C}{\downarrow} A \cup B \quad \text{if and only if} \quad D \underset{C \cup B}{\downarrow} A \quad \text{and} \quad D \underset{C}{\downarrow} B,$$

which is true by Transitivity. □

**Remark 17.** Let $(W, \mathrm{cl})$ is a pregeometry. Let $A, B, C$ and $D$ be subsets of $W$. Then

$$AD \underset{B}{\downarrow} C \quad \text{if and only if} \quad A \underset{B}{\downarrow} CD.$$

*Proof.* Suppose $A \underset{B}{\downarrow} CD$. Then, by Monotonicity we have $A \underset{B}{\downarrow} D$. Therefore, by Symmetry, we have $D \underset{B}{\downarrow} D$. By Transitivity, we have $A \underset{BD}{\downarrow} CD$. Hence, $AD \underset{B}{\downarrow} C$ by Concatenation.

For the converse, suppose that $A \underset{B}{\downarrow} CD$. Then by Symmetry we must have $CD \underset{B}{\downarrow} A$. Hence, by the first paragraph, we know that $C \underset{B}{\downarrow} AD$, so by Symmetry, also $AD \underset{B}{\downarrow} C$. □

This finishes the list of usual properties of forking. We now prove a few propositions relating closure and $\downarrow$.

**Proposition 18** (Closed Set Theorem). *Let $(W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. Then*

$$A \underset{B}{\downarrow} C \quad \textit{if and only if} \quad A' \underset{B'}{\downarrow} C',$$

*provided that* $\mathrm{cl}(A \cup B) = \mathrm{cl}(A' \cup B')$, $\mathrm{cl}(B) = \mathrm{cl}(B')$ *and* $\mathrm{cl}(C \cup B) = \mathrm{cl}(C' \cup B')$.

*Proof.* It is clearly enough to prove one direction. Furthermore, by Symmetry, it is enough to show that $A \underset{B}{\perp} C$ implies $A \underset{B'}{\perp} C'$. Suppose that $A \underset{B'}{\not\perp} C'$. Let $a \in A$ and $A^* \subseteq A$ be such that

$$a \in \mathrm{cl}(B' \cup C' \cup A^*) - \mathrm{cl}(B' \cup A^*).$$

But, it follows from the assumption that $\mathrm{cl}(B' \cup C' \cup A^*) = \mathrm{cl}(B \cup C \cup A^*)$ and $\mathrm{cl}(B' \cup A^*) = \mathrm{cl}(B \cup A^*)$. Therefore

$$a \in \mathrm{cl}(B \cup C \cup A^*) - \mathrm{cl}(B \cup A^*),$$

which implies that $A \underset{B}{\not\perp} C$. □

**Remark 19.** In view of the previous result, when $A \underset{B}{\perp} C$, we can first choose a basis $B'$ of $B$, and choose $A' \subseteq A$ and $C' \subseteq C$, independent over $B$ (or equivalently $B'$), such that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A' \cup B)$ and $\mathrm{cl}(C \cup B) = \mathrm{cl}(C' \cup B)$, and thus $A' \underset{B}{\perp} C'$ and also $A' \underset{B'}{\perp} C'$.

**Proposition 20.** *Let* $(W, \mathrm{cl})$ *be a pregeometry. Let* $A, B$ *and* $C$ *be subsets of* $W$.

$$A \underset{B}{\perp} C \quad \textit{implies} \quad \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B).$$

*Proof.* Certainly $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B)$. Suppose that the reverse inclusion does not hold, and let $a \in \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B)$ such that $a \notin \mathrm{cl}(B)$. Then $a \in \mathrm{cl}(C \cup B) - \mathrm{cl}(B)$, so $\mathrm{cl}(A \cup B) \underset{B}{\not\perp} C$. But the previous proposition implies that $A \underset{B}{\not\perp} C$, which is a contradiction. □

**Remark 21.** In view of the definition and symmetry, when we look at $A \underset{B}{\perp} C$, we will generally assume that $B \subseteq A$ and $B \subseteq C$. Further, because of the closed set theorem, we may assume that $A, B$ and $C$ are closed, and finally, that $B = A \cap C$.

## 4. Buechler's Theorem

We list a few more definitions.

**Definition 22.** Let $(W, \mathrm{cl})$ be a pregeometry.

(1) $(W, \mathrm{cl})$ is called *modular* if for every closed subsets $S_1$ and $S_2$ of $W$ we have

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2);$$

(2) $(W, \mathrm{cl})$ is called *locally modular* if for every closed subsets $S_1$ and $S_2$ of $W$ we have

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2),$$

provided that $S_1 \cap S_2 \neq \emptyset$;

(3) $(W, \mathrm{cl})$ is called *projective* if for every $a, b \in W$ and $C \subseteq W$ such that

$$a \in \mathrm{cl}(C \cup \{b\}),$$

there exists $c \in C$ such that $a \in \mathrm{cl}(\{c, b\})$.

**Remark 23.** It is not too difficult to see that a pregeometry is projective if and only if it is modular.

**Definition 24.** Let $(W, \mathrm{cl})$ be a pregeometry.

(1) A closed set $L \subseteq W$ is a *line* if $\dim(L) = 2$;
(2) Two disjoint lines $L_1$ and $L_2$ are *parallel* if $\dim(L_1 \cup L_2) = 3$.

**Definition 25.** Let $G = (W, \mathrm{cl})$ be a pregeometry and $A \subseteq W$. Define the *localization of $G$ at $A$*, written $G_A = (W_A, \mathrm{cl}_A)$, by

$$W_A = W - A \quad \text{and} \quad \mathrm{cl}_A(X) = \mathrm{cl}(X \cup A) - A, \text{ for } X \subseteq W_A.$$

**Remark 26.** It is easy to see that if $G$ is a pregeometry, then $G_A$ is a pregeometry. In $G_A$, we denote the dimension of $X$ by $\dim(X/A)$.

**Remark 27.** If $G = (W, \mathrm{cl})$ is locally modular, then $G_A$ is modular for any finite subset $A$ of $W - \mathrm{cl}(\emptyset)$.

**Proposition 28.** *Let $(W, \mathrm{cl})$ be a pregeometry. Let $S_1, S_2$ be finite dimensional closed sets satisfying $S_0 = S_1 \cap S_2$. Then,*

$$S_1 \underset{S_0}{\downarrow} S_2 \quad \textit{if and only if} \quad \dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2).$$

*Proof.* Suppose first that $S_1 \underset{S_0}{\downarrow} S_2$. Let $I$ be a basis for $S_0$, and let $I_i \supseteq I$ be a basis for $S_i$ for $i = 1, 2$. Clearly, $\mathrm{cl}(S_1 \cup S_2) = \mathrm{cl}(I_1 \cup I_2)$. We claim, in addition, that $I_1 \cup I_2$ is independent. Otherwise there is $a \in \mathrm{cl}(I_1 \cup I_2 - \{a\})$. Without loss of generality, we may assume that $a \in I_1$. Now, since $I_1$ is independent, $a \notin \mathrm{cl}(I_1 - \{a\})$, thus

$$a \in \mathrm{cl}(I_1 \cup I_2 - \{a\}) - \mathrm{cl}(I_i - \{a\}), \quad \text{for } i = 1, 2.$$

We may also assume that $a \notin I$. To see this, assume that $a \in I$. Choose $I_i' \subseteq I_i - I$, minimal with respect to inclusion, such that $a \in \mathrm{cl}(I_1' \cup I_2' \cup I - \{a\})$, $I_i' \neq \emptyset$, for $i = 1, 2$. By the Exchange Property, there is $b \notin I$, such that

$$b \in \mathrm{cl}(I_1' \cup I_2' \cup I \cup \{b\}) \subseteq \mathrm{cl}(I_1 \cup I_2 - \{b\}).$$

But, if $a \notin I$, then $\mathrm{cl}(I_1 - \{a\}) = \mathrm{cl}(I \cup I_1 - \{a\})$ so

$$a \in \mathrm{cl}(I_2 \cup (I_2 - \{a\})) - \mathrm{cl}(I \cup (I_2 - \{a\})),$$

which means that $S_1 \underset{S_0}{\not\downarrow} S_2$, a contradiction. Hence $I_1 \cup I_2$ is independent. Therefore $\dim(S_1 \cup S_2) = |I_1 \cup I_2|$. But $|I_1 \cup I_2| + |I| = |I_1| + |I_2|$, so

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2).$$

For the converse, suppose $S_1 \underset{S_0}{\not\downarrow} S_2$. Let $a \in S_1$ and $A_1 \subseteq S_1$ such that

$$(*) \qquad a \in \mathrm{cl}(S_2 \cup A_1) - \mathrm{cl}(S_0 \cup A_1).$$

Choose $a$ such that $A_1$ has minimal cardinality. This implies that $A_1 \cup \{a\}$ is independent over $S_0$, and $A_1$ is independent over $S_2$. Thus, we can pick a basis $I_0$ for $S_0$, and extend

$I_0 \cup A_1 \cup \{a\}$ to a basis $I_1$ of $S_1$. Now choose $I_2'$ disjoint from $I_0$, such that $I_0 \cup I_2'$ is a basis of $S_2$. But, $I_0 \cup A_1 \cup \{a\} \cup I_2'$ is not independent by (*). Hence

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) < \dim(S_1) + \dim(S_2),$$

which finishes the proof. □

In the previous section, we showed that in any pregeometry, there is a relation that satisfies all the properties that forking satisfies in the context of simple theories. This allows us to show a theorem of Buechler [Bu], originally proved for stable theories, when the pregeometry comes from forking.

**Theorem 29** (Buechler). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Then $G$ is locally modular if and only if $G_A$ has no parallel lines for every finite $A \subseteq W$, such that $A \not\subseteq \mathrm{cl}(\emptyset)$.*

*Proof.* Suppose first that there is a finite $A \subseteq W$, such that $A \not\subseteq \mathrm{cl}(\emptyset)$ and $G_A$ contain parallel lines. Thus, let $L_1$ and $L_2$ be disjoint lines in $G_A$ such that $\dim(L_1 \cup L_2/A) = 3$. Let $L_i' = \mathrm{cl}(L_i \cup A)$ for $i = 1, 2$. Then $A \subseteq L_1' \cap L_2'$, so $L_1' \cap L_2' \not\subseteq \mathrm{cl}(\emptyset)$, $L_i'$ is closed for $i = 1, 2$, and

$$\dim(L_1' \cup L_2') + \dim(L_1' \cap L_2') \neq \dim(L_1') + \dim(L_2').$$

This shows that $G$ is not locally modular.

For the converse, suppose that $G$ is not locally modular. Then there are closed $S_1$ and $S_2$ subsets of $W$ such that $S_1 \cap S_2 \not\subseteq \mathrm{cl}(\emptyset)$ and

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) \neq \dim(S_1) + \dim(S_2).$$

We may assume that $S_1$ and $S_2$ are finite dimensional. Let $S_0 = S_1 \cap S_2$. By Proposition 28, this implies that $S_1 \underset{S_0}{\not\downarrow} S_2$.

Let $\mathcal{D}$ be the set of pairs of integers $\langle d_1, d_2 \rangle$ such that there are closed sets $S_1$ and $S_2$ such that

- $S_0 = S_1 \cap S_2$ and $S_0 \not\subseteq \mathrm{cl}(\emptyset)$;
- $d_1 = \dim(S_1/S_0)$ and $d_2 = \dim(S_2/S_0)$;
- $S_1 \underset{S_0}{\not\downarrow} S_2$.

By assumption $\mathcal{D} \neq \emptyset$. Choose $\langle d_1, d_2 \rangle$ minimal with respect to the lexicographic order. We claim that $\langle d_1, d_2 \rangle = \langle 2, 2 \rangle$. Note that this is enough to prove the theorem since $\mathrm{cl}_{S_0}(S_1 - S_0)$ and $\mathrm{cl}_{S_0}(S_2 - S_0)$ are parallel lines in $G_{S_0}$.

Certainly, $d_1 > 1$. Otherwise, $\dim(S_1/S_0) = 1$ and since $S_1 \underset{S_0}{\not\downarrow} S_1$ there must exist $a \in S_1 - S_0$ such that $a \in \mathrm{cl}(S_2) - \mathrm{cl}(S_0)$. Since $S_2$ and $S_0$ are closed, we have $a \in S_1 \cap S_2 - S_0$, a contradiction, since $S_1 \cap S_2 = S_0$.

We now show that $d_1 < 3$. Suppose $d_1 = \dim(S_1/S_0) \geq 3$. We will show that this contradicts the minimality of $d_1$. We first show that

(*) $$S_1 \cap \mathrm{cl}(S_2 a) = \mathrm{cl}(S_0 a), \quad \text{for any } a \in S_1 - S_0.$$

First, notice that $S_0a \subseteq S_1$ and $S_0a \subseteq \mathrm{cl}(S_2a)$, so

$$S_1 \cap \mathrm{cl}(S_2a) \supseteq \mathrm{cl}(S_0a), \quad \text{for any } a \in S_1 - S_0.$$

Hence, if (*) does not hold, it is because for some $a \in S_1 - S_0$, there exists

$$b \in (S_1 \cap \mathrm{cl}(S_2a)) - \mathrm{cl}(S_0a).$$

By definition, this implies that $\{ a, b \} \underset{S_0}{\not\downarrow} S_2$.

Let $S_1' = \mathrm{cl}(S_0ab)$. Then $S_1' \cap S_2 = S_0$ and $S_0 \not\subseteq \mathrm{cl}(\emptyset)$. Furthermore $S_1' \underset{S_0}{\not\downarrow} S_2$. But $\dim(S_2/S_0) = d_2$ and $\dim(S_1'/S_0) = 2 < 3 \leq d_1$, which contradicts the minimality of $d_1$. Therefore, (*) holds.

Now, since $S_1 \underset{S_0}{\not\downarrow} S_2$, there exist $a \in S_1$ and a finite $A \subseteq S_1$ such that

$$a \in \mathrm{cl}(S_2 \cup A) - \mathrm{cl}(S_0 \cup A). \tag{**}$$

But $A \not\subseteq S_0$. Otherwise, by (**) we have $a \in \mathrm{cl}(S_2) - \mathrm{cl}(S_0)$. This shows that $a \in S_2 - S_1$ since $S_2$ and $S_0$ are closed. But $a \in S_1$, so $a \in (S_1 \cap S_2) - S_0 = \emptyset$, which is impossible. Hence, there is $b \in A - S_0$. Then, since $Ab = A$, we have

$$a \in \mathrm{cl}(S_2 \cup A) - \mathrm{cl}(S_0b \cup A).$$

Hence $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2$.

Now consider $S_2' := \mathrm{cl}(S_2b)$. Then, $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2$ implies that $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2'$. By (*) we have $S_1 \cap S_2' = \mathrm{cl}(S_0b)$. Finally, $\dim(S_1/(S_0b)) < \dim(S_1/S_0) = d_1$ and $d_2 = \dim(S_2/S_0) = \dim(S_2'/S_0b)$. This contradicts the minimality of $d_1$. We prove similarly that $d_2 = 2$, which finishes the proof. □

## 5. Some "set theory"

In this section, we gather several observations with a set-theoretic flavor. The next theorem is a generalization of a lemma from J. Baumgartner, M. Foreman and O. Spinas [BFS]. Although the proof is easy, it does not follow from the analog theorem involving models as we do not have control over the cardinality of the closures. The value of this theorem is that it makes it possible to attach a club as an invariant of the pregeometry.

**Theorem 30.** *Let $G = (W, \mathrm{cl})$ be a pregeometry. Suppose $\dim(W) = \lambda$ is regular and uncountable. Let $I = \{ a_i \mid i < \lambda \}$ and $J = \{ b_i \mid i < \lambda \}$ be bases of $W$. Then*

$$C = \{ i < \lambda \ : \ \mathrm{cl}(\{ a_j \mid j < i \}) = \mathrm{cl}(\{ b_j \mid j < i \}) \}$$

*is a closed and unbounded subset of $\lambda$.*

*Proof.* We first show that $C$ is closed. Let $\delta = \sup(\delta \cap C)$. Then, for any $i < \delta$ there is $i_1 \in C$ such that $i < i_1 < \delta$. Hence, by definition of $C$

$$\mathrm{cl}(\{ a_j \mid j < i_1 \}) = \mathrm{cl}(\{ b_j \mid j < i_1 \}). \tag{*}$$

Lemma 4 and (*) implies that $a_i \in \mathrm{cl}(\{ b_j \mid j < \delta \})$. Hence,

$$\{ a_j \mid j < \delta \} \subseteq \mathrm{cl}(\{ b_j \mid j < \delta \}),$$

and therefore

$$\mathrm{cl}(\{ a_j \mid j < \delta \}) \subseteq \mathrm{cl}(\{ b_j \mid j < \delta \}),$$

by Fact 2 again. The other inclusion is similar and so

$$\mathrm{cl}(\{ a_j \mid j < \delta \}) \supseteq \mathrm{cl}(\{ b_j \mid j < \delta \}).$$

This shows that $\delta \in C$, by definition of $C$.

We now show that $C$ is unbounded in $\lambda$. Let $i < \lambda$ be given. We construct $i_n < \lambda$ for $n \in \omega$ increasing with $i_0 = i$ such that

(1) $\mathrm{cl}(\{ a_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i_{n+1} \})$ if $n$ is even;
(2) $\mathrm{cl}(\{ b_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ a_j \mid j < i_{n+1} \})$ if $n$ is odd.

This is enough: Let $i(*) = \sup\{ i_n \mid n \in \omega \}$. Then $i(*) < \lambda$ since $\lambda$ is regular uncountable. Further $\mathrm{cl}(\{ a_j \mid j < i(*) \}) = \mathrm{cl}(\{ b_j \mid j < i(*) \})$, since if $i < i(*)$, then there is $i_n$ with $n$ even such that $i < i_n$, so

$$a_i \in \mathrm{cl}(\{ a_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i_{n+1} \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i(*) \}),$$

hence

$$\mathrm{cl}(\{ a_j \mid j < i(*) \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i(*) \}).$$

The other inclusion is proved similarly. Thus $i < i(*) \in C$, which shows that $C$ is unbounded.

This is possible: Given $i < \lambda$, we let $i_0 = i$. Assume that $i_n < \lambda$ has been constructed. Suppose $n$ is even. For each $j < i_n$, we have that $a_j \in W = \mathrm{cl}(\{ b_j \mid j < \lambda \})$ since $J$ is a basis. By Finite Character, there is a finite $S_j \subseteq \lambda$ such that $a_j \in \mathrm{cl}(\{ b_k \mid k \in S_j \})$. Let $k_j = \sup S_j < \lambda$, so $a_j \in \mathrm{cl}(\{ b_l \mid l \leq k_j \})$, and by increasing $k_j$ if necessary, we may assume that $k_j \geq i_n$. Set $i_{n+1} = \sup\{ k_j + 1 \mid j < i_n \}$. Then $i_{n+1} < \lambda$ since $\lambda$ is regular and satisfies our requirement. The case when $n$ is odd is handled similarly. □

**Proposition 31** (Downward Theorem). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. Suppose $A \underset{B}{\perp} C$ and $A'$ is a subset of $A$, of cardinality at most $\lambda$, for $\lambda$ an infinite cardinal. Then there is $B' \subseteq B$ of cardinality at most $\lambda$ such that $A' \underset{B'}{\perp} C$.*

*Proof.* Let $A' \subseteq A$ of cardinality $\lambda$ be given. Let $\{ \langle a_i, A_i \rangle \mid i < \lambda \}$ be an enumeration of all the pairs such that $a_i \in A'$ and $A_i \subseteq A'$ is finite. Such an enumeration is possible since $\lambda$ is infinite. Since $A \underset{C}{\perp} B$, necessarily

(*) $$a_i \notin \mathrm{cl}(B \cup C \cup A_i) - \mathrm{cl}(B \cup A_i), \quad \text{for every } i < \lambda.$$

Hence, either $a_i \notin \mathrm{cl}(B \cup C \cup A_i)$, or $a_i \in \mathrm{cl}(B \cup A_i)$. If the latter holds, by Finite Character, we can find a finite $B_i \subseteq B$ such that $a_i \in \mathrm{cl}(B_i \cup A_i)$. We let $B_i = \emptyset$, if $a_i \notin \mathrm{cl}(B \cup A_i)$. Let $B' = \bigcup B_i$. Then $B' \subseteq B$, and $|B'| \leq \lambda$.

We claim that $A' \underset{B'}{\perp} C$. Otherwise, there exist $a \in A'$ and a finite $A^* \subseteq A'$, such that

(**) $$a \in \text{cl}(B' \cup C \cup A^*) - \text{cl}(B' \cup A^*).$$

Choose $i < \lambda$ such that $a = a_i$ and $A^* = A_i$. Thus, $a_i \in \text{cl}(B' \cup C \cup A_i)$, and so by Fact 2 we have $a_i \in \text{cl}(B \cup C \cup A_i)$. Therefore, by (*) we have that $a_i \in \text{cl}(B \cup A_i)$. Hence $a_i \in \text{cl}(B_i \cup A_i)$ by construction. But $B_i \subseteq B'$, and so $a_i \in \text{cl}(B' \cup A_i)$ by Fact 2. This contradicts (**) since $A^* = A_i$. □

**Corollary 32.** *Let $G = (W, \text{cl})$ be a pregeometry. Let $A, B$ and $C$ be subsets of $W$. Suppose that $A, B$ and $C$ have cardinality at least $\lambda$ for some $\lambda$ infinite. If $A \underset{C}{\perp} B$, then we can find $A' \subseteq A$, $B' \subseteq B$ and $C' \subseteq C$ of cardinality $\lambda$, such that $A' \underset{C'}{\perp} B'$.*

*Proof.* By the previous theorem using monotonicity. □

**Proposition 33** (Ultraproducts of Pregeometries). *Let $I$ be a set and $\mathfrak{D}$ an $\aleph_1$-complete ultrafilter on $I$. Suppose that $(W_i, \text{cl}_i)$ is a pregeometry for each $i \in I$. Consider $W = \Pi_{i \in I} W_i$ and for $a \in W$ and $B \subseteq W$, define*

$$a \in \text{cl}(B) \quad \textit{if} \quad \{\, i \in I \mid a(i) \in \text{cl}_i(B(i)) \,\} \in \mathfrak{D}.$$

*Then $(W, \text{cl})$ is a pregeometry.*

*Proof.* We only show Finite Character, since all the other axioms of a pregeometry are routine. Suppose $a \in \text{cl}(B)$. Then $J = \{\, i \in I \mid a(i) \in \text{cl}_i(B(i)) \,\} \in \mathfrak{D}$, and by Finite Character of $\text{cl}_i$, for each $i \in J$, there is a finite $B'(i) \subseteq B(i)$, such that $a(i) \in \text{cl}_i(B'(i))$. Let $J_n = \{\, i \in J \mid B'(i)$ has $n$ elements$\}$. Then

$$\{\, i \in J \mid a(i) \in \text{cl}_i(B'(i)) \,\} = \bigcup_{n<\omega} J_n.$$

Hence, by $\aleph_1$-completeness, there exist $n < \omega$ such that $J_n \in \mathfrak{D}$. We now write $B'(i) = \{\, b^i_1, \dots, b^i_n \,\}$ for $i \in J_n$. Let $A = \{\, f_1, \dots, f_n \,\} \subseteq B$ be given by $f_k(i) = b^i_k$ when $i \in J_n$ and $f_k(i) \in B(i)$ arbitrary when $i \notin J_n$. Then

$$\{\, i \in I \mid a(i) \in \text{cl}_i(A(i)) \,\} \supseteq J_n \in \mathfrak{D},$$

by construction. Hence $\{\, i \in I \mid a(i) \in \text{cl}_i(A(i)) \,\} \in \mathfrak{D}$. Thus, $a \in \text{cl}(A)$ and $A$ is a finite subset of $B$, which is what we needed. □

## 6. Stable systems

We now introduce stable systems, a notion originally developed in model theory. They are used for example in [Sh 87a], [Sh 87b] and later in the proof of the main gap [Sh b]. See also [Ma].

**Definition 34.** Let $G = (W, \text{cl})$ be a pregeometry.

(1) We call $S = \langle A_s \mid s \in I \rangle$ a *system*, if $A_s \subseteq W$, $I$ is a subset of $\bigcup I$ closed under subsets and $s \subseteq t$ implies $A_s \subseteq A_t$. We denote by $s^-$ the immediate predecessor of $s$ in $I$ if one exists;

(2) We call $S = \langle A_s \mid s \in I \rangle$ a *stable system*, if $S$ is a system which satisfies in addition

$$A_s \underset{A_{s^-}}{\perp} \bigcup \{A_t \mid t \not\supseteq s, t \in I\}, \quad \text{for every } s, t \in I.$$

**Proposition 35** (Generalized Symmetry Lemma). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let $S = \langle A_s \mid s \in I \rangle$ be a system. Suppose there is an enumeration $I = \langle s(i) \mid i < \alpha \rangle$ such that*

(1) *$s(i) \subseteq s(j)$ implies $i \leq j$, for every $i, j < \alpha$;*
(2) $A_{s(i)} \underset{A_{s(i)^-}}{\perp} \bigcup \{A_{s(j)} \mid j < i\}$.

*Then $S$ is a stable system.*

*Proof.* By Finite Character, we may assume that $I$ is finite. We prove this by induction on $|I|$. The base case is obvious. Suppose it is true for $|I| = n < \omega$. Suppose $I = \langle s(i) \mid i \leq n \rangle$ is an enumeration satisfying (1) and (2). Assume for a contradiction that $S$ is not a stable system. By induction hypothesis, we have either

(*) $$A_{s(n)} \underset{A_{s(n)^-}}{\not\perp} \bigcup \{A_{s(j)} \mid s(j) \not\subseteq s(n)\},$$

or there exists $i < n$ with $s(i) \not\subseteq s(n)$ such that

(**) $$A_{s(i)} \underset{A_{s(i)^-}}{\not\perp} \bigcup \{A_{s(j)} \mid s(j) \not\subseteq s(i), s(j) \neq s(n)\} \cup A_{s(n)}.$$

By assumption, we know that

(†) $$A_{s(n)} \underset{A_{s(n)^-}}{\perp} \bigcup \{A_{s(j)} \mid j < n\}.$$

By (1), we have that

$$\bigcup \{A_{s(j)} \mid s(j) \not\subseteq s(n)\} \subseteq \bigcup \{A_{s(j)} \mid j < n\}.$$

Hence (*) is impossible, by Monotonicity and (†).

Now if $s(i) \subseteq s(n)$, then $s(i)^- \subseteq s(n)^-$. Hence, $A_{s(i)^-} \subseteq A_{s(n)^-}$ since $S$ is a system. By Monotonicity used twice, (†) implies that

$$A_{s(n)} \underset{A_{s(i)^-}}{\perp} \bigcup \{A_{s(j)} \mid s(j) \not\subseteq s(i), s(j) \neq s(n)\} \cup A_{s(i)}.$$

But this and Remark 17 contradicts (**). Hence $s(i) \not\subseteq s(n)$. □

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