# CHARACTERIZATION OF HOMOGENEOUS SCALAR VARIATIONAL PROBLEMS SOLVABLE FOR ALL BOUNDARY DATA

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# Characterization of homogeneous scalar variational problems solvable for all boundary data*

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## Abstract

In this paper we introduce a condition on lower semicontinuous integrands $L: R^n \to R$ both necessary and sufficient for all problems of the form

$$J(u) = \int_\Omega L(Du)dx \to \min, u \in W^{1,1}(\Omega), u\Big|_{\partial\Omega} = f$$

to have a solution, provided $\Omega$ and $f$ are sufficiently regular or certain conditions on growth of $L$ at infinity are assumed.

## 1 Introduction

In this paper we deal with minimization problems

$$J(u) \to \min, u\Big|_{\partial\Omega} = f, u \in W^{1,1}(\Omega) \tag{1.1}$$

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for integral functionals of the form

$$J(u) = \int_\Omega L(Du(x))dx,$$

where $\Omega$ is an open bounded subset of $R^n$ with *Lipschitz boundary* and $L : R^n \to R$ is a lower semicontinuous function.

A function $u \in W^{1,1}(\Omega)$ is called an *admissible* function for the problem (1.1) if $u\big|_{\partial\Omega} = f$ and the negative part of the function $L(Du)$ is integrable. In this case $J(u)$ is well defined but equals $+\infty$ if the positive part of $L(Du)$ is not summable.

We accept the following notations: for a subset $A$ of $R^n$ the sets int$A$, reint$A$, co$A$, and extr$A$ are respectively the interior, the relative interior, the convex hull, and the set of extremum points of $A$. $B(a,\epsilon)$ denotes the ball of radius $\epsilon$ centered at the point $a \in R^n$; $l_a$ is a linear function with gradient equal to $a$ everywhere. $\partial L(F)$ denotes the subgradient of $L$ at a point $F$:

$$\partial L(F) := \{l \in R^n : L(v) - L(F) - \langle l, v - F\rangle \geq 0,\ \forall v \in R^n\}.$$

$L^{**}$ is the convexification of $L$: the epigraph of $L^{**}$ is convexification of the epigraph of $L$, that is

$$L^{**}(v_0) := \{\sum c_i L(v_i) : c_i \geq 0, v_i \in R^n, \sum c_i = 1, \sum c_i v_i = v_0\}.$$

Note that $L^{**}$ exists as a function from $R^n$ to $R$ if and only if $L$ exceeds an affine function everywhere (see Lemma 2.1).

Weak and strong convergences of sequences are denoted by $\rightharpoonup$ and $\to$ respectively.

We will frequently utilize the following version of Vitaly covering theorem (see [S,p.109]).

A family $F$ of closed subsets of $R^n$ is said to be a Vitaly cover of a bounded set $A$ if for any $x \in A$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x, \epsilon_k)$ with $\epsilon_k \to 0$, and a sequence $C_k \in F$ such that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and (meas $C_k/$ meas $B(x, \epsilon_k)) > r(x)$ for all $k \in N$.

The version of Vitaly covering theorem from [S,p.109] says that each Vitaly cover of $A$ contains at most countable subfamily of disjoint sets $C_k$ such that meas $(A \setminus \cup_k C_k) = 0$.

Problems (1.1) were studied recently in the framework of Existence Theory in Elasticity: when dealing with homogeneous materials undergoing antiplane shear deformations

$$(x, y, z) \in R^3 \to (x, y, z + u(x, y)) \in R^3$$

the problem of minimization of the free energy is of the form (1.1). While the existence results are well-known for problems (1.1) with convex integrands (see e.g. [ET],[Da],[Mo]), the situation is poorly understood in the case of nonconvex problems. Note that active research in the area of nonconvex variational problems started since the work [B], where the first existence results for realistic problems in Elasticity were established in the general case (without restrictions on the class of admissible deformations).

Some recent efforts were devoted to the question of solvability of problems (1.1) under restrictions on integrands motivated by physical reasons (see [BP], [GT], [R], [SH]). In these papers solvability problem was treated for particular boundary data. Moreover, the papers [C1], [C2], [F] indicated conditions on integrands both necessary and sufficient for the problem (1.1) with linear boundary conditions $f = l_F$, $F \in R^n$ to have a solution.

The answer is given by the following theorem

**Theorem 1.1** *The problem (1.1) with linear boundary conditions $f = l_F$ has a solution if and only if either $\partial L(F) \neq \emptyset$ or there exist $v_1, \ldots, v_q \in R^n$ ($q \in N$) such that $F \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$ and $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$.*

Here we state a slightly more general result since Theorem 1.1 was proved in [C1], [C2] for lower semicontinuous integrands with superlinear growth at infinity and, independently, in [F] for continuous integrands bounded from below. We also utilize different terminology in order to formulate the result in terms of $L$ only.

However, a crucial ingredient of the proofs (more precisely of their sufficient parts) is utilization of special functions proposed in [C1], [C2], [F].

If $v_1, \ldots, v_q$ are extremum points of a convex compact subset of $R^n$, and $F \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$, then the function

$$w_s(x) = \max_{1 \leq i \leq q} \langle v_i - F, x \rangle - s \tag{1.2}$$

is Lipschitz, $Dw_s(x) \in \{v_i - F : i = 1, \dots, q\}$ a.e., and $w_s|_{\partial P_s} = 0$, where

$$P_s = \{x : \max_{1 \le i \le q} \langle v_i - F, x \rangle \le s\}$$

is a compact set with Lipschitz boundary and nonempty interior.

Note that $P_s = sP_1$.

Since Vitaly covering arguments lets us decompose $\Omega$ into disjoint sets of the form $y_i + s_i P_1$ and a set of nonzero measure, we can define $u_0$ as

$$\langle F, x \rangle + w_{s_i}(x - y_i) \text{ for } x \in y_i + s_i P_1.$$

Then $u_0\Big|_{\partial\Omega} = l_F$ on $\partial\Omega$, $u_0 \in W^{1,\infty}(\Omega)$. In order to prove that $u_0$ is a solution of the problem (1.1) note that if $l \in \cap_{i=1}^q \partial L(v_i)$ then for any admissible function $u \in W^{1,1}(\Omega)$ we have

$$J(u) - J(u_0) = \int_\Omega \{L(Du) - L(Du_0) - \langle l, Du - Du_0 \rangle\} dx =$$

$$\int_\Omega \{L(Du) - L(v_1) - \langle l, Du - v_1 \rangle\} dx - \int_\Omega \{L(Du_0) - L(v_1) - \langle l, Du_0 - v_1 \rangle\} dx.$$

It is easy to see that all functions $L(v_i) + \langle l, v - v_i \rangle$, $i = 1, \dots, q$, coincide. Then $L(v) - L(v_1) - \langle l, v - v_1 \rangle \ge 0$ everywhere with the equality in the case $v \in \{v_1, \dots, v_q\}$. Hence the first term is nonnegative while the second one equals zero. Hence $J(u) - J(u_0) \ge 0$.

This proves that the condition $\cap_{i=1}^q \partial L(v_i) \ne \emptyset$ with $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$ implies solvability of the problem. If $\partial L(F) \ne \emptyset$ then the function $l_F$ is a solution. Therefore, each of these two conditions implies solvability of the problem. These arguments prove the "sufficient" part of Theorem 1.1.

The converse will be proved in §2.

Before explaining what kind of influence these simple arguments had on further developments of solvability theory, let us state the results of this paper.

**Theorem 1.2** *Let $L : R^n \to R$ be a lower semicontinuous function such that $L \ge \theta$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$.*

*Then all problems of the form (1.1) with $\partial\Omega \in C^2$ of positive curvature and $f \in C^2(\partial\Omega)$ are solvable if and only if for each $F \in R^n$ either $\partial L(F) \ne \emptyset$ or there exist $v_1, \dots, v_q \in R^n$ such that $\cap_{i=1}^q \partial L(v_i) \ne \emptyset$ and $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$. Moreover, all solutions are Lipschitz continuous functions.*

**Remark** We are not aware whether it is possible to weaken the conditions on $\Omega$ and $f$ in this theorem essentially (see §4 for a discussion).

In the case when the growth of $L$ provides a certain a priori regularity of minimizers, an analogous result holds for arbitrary $\Omega$ and $f$, for which at least one admissible function $u \in W^{1,1}(\Omega)$ with $J(u) < \infty$ exists. However, in this case we can not state Lipschitz regularity of solutions.

**Theorem 1.3** *Let $L : R^n \to R$ be a lower semicontinuous function such that $L \geq \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$.*

*Then each problem of the form (1.1), for which at least one admissible function $u \in W^{1,1}(\Omega)$ with $J(u) < \infty$ exists, has a solution if and only if for each $F \in R^n$ either $\partial L(F) \neq \emptyset$ or there exist $v_1, \ldots, v_q \in R^n$ such that $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$, and $F \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$.*

**Remark** Note that the only role of the growth condition $L \geq \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$, is to provide differentiability of solutions to the relaxed problem in the classical sense almost everywhere. Hence, the result of the theorem holds under any other conditions on lower semicontinuous integrands $L : R^n \to R$ with superlinear growth, which imply this property of solutions to the relaxed problem.

Note that regularity in minimization problems was studied typically in the context of continuity of solutions and their derivatives (everywhere or on an open set of full measure). However, here we need an intermediate property - differentiability in the classical sense almost everywhere. It seems that not too much is known in this direction. Indeed, results on continuity of solutions are not sufficient in our situation. Simultaneously, partial regularity of derivatives, which is more than enough for our purposes, was treated usually for elliptic integrands, cf. e.g. [G].

The proofs of theorems 1.2, 1.3 are further refinements of the above discussed arguments, which were also developed recently in a deeper way in the context of Theory of Differential Inclusions.

Note that the sufficient part of Theorem 1.1 is equivalent to a particular differential inclusion $Du(x) \in \{v_1, \ldots, v_q\}$ a.e. in $\Omega$, $u = l_F$ on $\partial\Omega$.

When anyone deals with nonlinear boundary conditions, more complicated differential inclusions should be considered. The typical one is: $Du(x) \in \operatorname{extr} U$ for a.e. $x \in \Omega$, $u = f$ on $\partial\Omega$, where $f \in W^{1,\infty}(\Omega)$ and $Df(x) \in U$

for a.e. $x \in \Omega$ (here $U$ is a compact convex subset of $R^n$ with nonempty interior).

It was observed in [DP] that the same functions $w_s$ with $F = Df(x_0)$ (see (1.2)) can be utilized to perturb $f$ by $\phi_s := w_s(\cdot - x_0) - f(x_0) - \langle Df(x_0), \cdot - x_0\rangle$ in such a way that $D\min\{f, \phi_s\} \in \text{extr}U$ for each $x$ in an open subset $\tilde{\Omega}$ of $\Omega$ such that $x_0 + P_{s/2} \subset \tilde{\Omega} \subset x_0 + P_{2s}$ and $\phi_s = f$ on $\partial\tilde{\Omega}$. Refining arguments from [DP] it is easy to see that for each $x_0 \in \Omega$, where $Df(x_0)$ exists in the classical sense and $Df(x_0) \in \text{int}U$, such a perturbation exists for all $s > 0$ sufficiently small (see Lemma 3.2). Applying Vitaly covering arguments we solve the inclusion.

In the context of variational problems this means that if $U$ is an $n$-dimensional proper face of $L^{**}$ (note that $L = L^{**}$ on the set of extremal points of $U$, cf. Lemma 3.1), then we can perturb a solution $u_0$ of the problem

$$\int_\Omega L^{**}(Du)dx \to \min, u\Big|_{\partial\Omega} = f$$

on the set that includes almost all points of the set in which $Du_0 \in \text{int}U$, in such a way that $Du_0 \in \text{extr}U$ a.e. in this set. Since gradient of the perturbed function lies in $U$ on the set of perturbation, this function is also a solution. Consequentially, there exists a subset of full measure of the set $\{x : Du_0(x) \in \text{int}U\}$, which can be complemented by a subset of the set $\{x \in \Omega : Du_0(x) \in \partial U\}$ and a set of zero measure to become an open set.

These arguments were utilized in [Z] in order to prove that problems (1.1) with integrands $L$, which coincide with $L^{**}$ everywhere with exception of a finite collection of distinct $n$-dimensional proper faces, have a solution. Being more precise, the perturbation arguments can not be applied if the function $u_0$ is only in $W^{1,1}(\Omega)$, since low regularity of $u_0$ does not let us assert that $u_0 = \phi_s$ in the boundary of an open subset of $\Omega$. The author of [Z] was not careful and utilized a mistaken fact that for each function $u \in W^{1,1}(\Omega)$ and each $\delta > 0$ there exists an open subset $\Omega_\delta$ of $\Omega$ such that meas $(\Omega \setminus \Omega_\delta) < \delta$ and $||u_0||_{W^{1,\infty}(\Omega_\delta)} < \infty$. Indeed, it is easy to construct a function $u \in W^{1,1}(\Omega)$ such that $\text{ess}\sup|Du|$ is unbounded in any nontrivial open subset of $\Omega$. However, we also noticed that the perturbation arguments still work at points of differentiability of $u_0$. This remark simplifies basic arguments from [DP] and let us apply them to the case $u_0 \in W^{1,p}(\Omega)$ with $p > n$, since in this case $u_0$ is differentiable almost everywhere in the classical sense (cf. [EG,p.234]). The final version of the paper [Z] will contain these corrections.

It turns out that further refinements of these arguments can be utilized in order to prove Theorem 1.3, which is a characterization result. It is helpful here to utilize simple direct arguments constructing a sequence of solutions to the relaxed problem, which converges strongly to a solution of the original problem, instead of Baire category arguments and other techniques from [DP], [Z] (see also the papers mentioned therein) traditional for Theory of Differential Inclusions.

The proof of Theorem 1.2 needs more subtle arguments since growth of $L$ does not guarantee almost everywhere differentiability (in the classical sense) of functions, which give finite values to the integral functional. It seems to be an open question whether this property holds for solutions to the relaxed problems.

In the case of Theorem 1.2 we first prove solvability of the relaxed problem in the class of Lipschitz continuous functions, following arguments introduced first in the context of solvability theory for the Plateau problem (see [Gi]). Then, careful construction of special perturbations of this solution gives a solution to the original problem in the class of Lipschitz functions. Next, we utilize a nonsmooth analogue of the Euler-Lagrange equation to prove that such solutions are automatically solutions of the boundary value problem (1.1).

We prove Theorem 1.1 and Theorem 1.3 in §2 and §3 respectively. In §4 we recall some facts on solvability of problems of the form (1.1) in the class of Lipschitz continuous functions, provided certain regularity on $\partial\Omega$ and $f$ is assumed and $L$ is convex. Here we also prove a nonsmooth analog of the Euler-Lagrange equation. Theorem 1.2 is proved in §5.

## 2 Proof of Theorem 1.1 and some auxiliary propositions

Recall first some basic facts about convex functions. By Caratheodory theorem, for each subset $A$ of $R^n$ we have

$$\text{co}A = \{\sum_{i=1}^{n+1} c_i v_i : c_i \geq 0, v_i \in A, \sum_{i=1}^{n+1} c_i = 1\}.$$

Since the dimension of the epigraph of any lower semicontinuous function $L : R^n \to R$ does not exceed $n+1$, for each $v_0 \in R^n$ we have

$$L^{**}(v_0) := \inf\{\sum_{i=1}^{q} c_i L(v_i) : q \in N, c_i \geq 0, v_i \in R^n, \sum_{i=1}^{q} c_i = 1, \sum_{i=1}^{q} c_i v_i = v_0\} =$$

$$\inf\{\sum_{i=1}^{n+2} c_i L(v_i) : c_i \geq 0, v_i \in R^n, \sum_{i=1}^{n+2} c_i = 1, \sum_{i=1}^{n+2} c_i v_i = v_0\}.$$

It is also well-known that if $L : R^n \to R \cup \{\infty\}$ is a lower semicontinuous convex function, which is bounded in a neighborhood of $v_0$, then $L$ is Lipschitz in a smaller neighborhood of $v_0$. Moreover $\partial L(v_0) \neq \emptyset$.

Recall also a version of the Hahn-Banach theorem. If $U$ is a closed convex subset of $R^n$ and $v_0 \notin \text{int}U$ then there exists $l \in R^n$ such that

$$\langle l, v_0 \rangle \geq \langle l, v \rangle, \forall v \in U.$$

All these facts can be found in any textbook containing chapters on Convex Analysis, see e.g. [ET].

Before proving Theorem 1.1 we state and prove two auxiliary propositions which will be utilized frequently later on.

**Lemma 2.1** *Let $L : R^n \to R$ be a lower semicontinuous function. Then the following assertions are equivalent:*

*1)*

$$L^{**}(\cdot) := \inf\{\sum_{i=1}^{q} c_i L(v_i) : q \in N, c_i \geq 0, \sum_{i=1}^{q} c_i = 1, \sum_{i=1}^{q} c_i v_i = \cdot\}$$

*is a convex continuous function;*

*2) there exists $l \in R^n$ and $c \in R$ such that*

$$L(v) \geq \langle l, v \rangle + c, \forall v \in R^n;$$

*3) there exists a point $F \in R^n$ such that*

$$\inf\{\sum_{i=1}^{q} c_i L(v_i) : q \in N, c_i \geq 0, v_i \in R^n, \sum_{i=1}^{q} c_i = 1, \sum_{i=1}^{q} c_i v_i = F\} > -\infty.$$

**Proof**

If $L^{**} : R^n \to R$ is a convex continuous function then $\partial L^{**}(0) \neq 0$, and, as a consequence, for $l \in \partial L^{**}(0)$ we have

$$L(v) - L^{**}(0) \geq L^{**}(v) - L^{**}(0) \geq \langle l, v \rangle.$$

Hence, 1) implies 2). The implication 2) $\Rightarrow$ 3) is obvious.

Let us prove the last assertion of the lemma. Without loss of generality we can assume that $F = 0$. Consider auxiliary functions $L_k : R^n \to R \cup \{\infty\}$ defined as follows: $L_k = L$ in $\bar{B}(0,k)$, $L_k = \infty$ - otherwise. Let also $L_k^{**}$ be convexification of $L_k$.

Then, for each $k$ the function $L_k^{**}\big|_{\bar{B}(0,k)}$ is a lower semicontinuous convex function, which is locally continuous in $B(0,k)$. Note also that $L_k^{**}(v)$ is a nonincreasing sequence for each $v \in R^n$. Moreover the sequence $L_k^{**}(0)$ is bounded from below. Hence, if $l_k \in \partial L_k^{**}(0)$ then $\sup_k |l_k| < \infty$ and for each limit point $l_0$ of $l_k$ ($l_0 = \lim_{j\to\infty} l_{k_j}$) and each $v \in R^n$ we have

$$\lim_{k\to\infty} \{L_k^{**}(v) - L_k^{**}(0)\} - \langle l_0, v \rangle = \lim_{j\to\infty} \{L_{k_j}^{**}(v) - L_{k_j}^{**}(0) - \langle l_{k_j}, v \rangle\} \geq 0.$$

Since $L_k^{**}(0)$ is bounded from below, the function $L^{**} := \lim_{k\to\infty} L_k^{**}$ majorizes $L^{**}(0) + \langle l_0, v \rangle$ everywhere. Note that $L^{**}$ is convex as a pointwise limit of nonincreasing sequence of convex functions. Since it is also locally bounded, it is continuous.

The proof of Lemma 2.1 is complete.

**Corollary 2.2** *Lower semicontinuous function $L : R^n \to R$ has nonempty subgradient at $F \in R^n$ if and only if $\sum_{i=1}^q c_i L(v_i) \geq L(F)$ for any $q \in N$, $v_i \in R^n$, $c_i \geq 0$ $(i = 1, \ldots, q)$ such that $\sum_{i=1}^q c_i = 1$, $\sum_{i=1}^q c_i v_i = F$.*

**Proof**

Let $l \in \partial L(F)$. Then for all $c_i \geq 0$, $v_i \in R^n$ such that $\sum_{i=1}^q c_i = 1$, $\sum_{i=1}^q c_i v_i = F$, $q \in N$, we obtain

$$\sum_{i=1}^q c_i L(v_i) - L(F) = \sum_{i=1}^q c_i L(v_i) - L(F) - \sum_{i=1}^q c_i \langle l, v_i - F \rangle =$$

$$\sum_{i=1}^q c_i \{L(v_i) - L(F) - \langle l, v_i - F \rangle\} \geq 0.$$

To prove the converse note that by Lemma 2.1 $L^{**}$ is a convex continuous function. Since $L(F) = L^{**}(F)$, $\partial L^{**}(F) \neq \emptyset$, and $L \geq L^{**}$ everywhere, we infer that $\partial L(F) \neq 0$.

The proof is complete.

**Lemma 2.3** *Let $L : R^n \to R$ be a lower semicontinuous function. Let $v_1, \ldots, v_q$ be such points in $R^n$ that $\sum_{i=1}^q c_i v_i = F$ for some $c_i \geq 0$ with $\sum_{i=1}^q c_i = 1$.*

*Then, there exists a bounded in $W^{1,\infty}(\Omega)$ sequence $u_k$ such that $u_k\big|_{\partial\Omega} = l_F$, and $J(u_k) \to \sum_{i=1}^q c_i L(v_i)\,\mathrm{meas}\,\Omega$.*

This lemma is a version of the well-known relaxation theorem (see [ET,Ch.10]). The main difference is that here we have a lower than usual regularity of integrands.

**Proof**

Without loss of generality we can assume that $c_i > 0$ for all $i$.

Consider first the case when $F$ has unique representation as a convex combination of $\{v_1, \ldots, v_q\}$. In this case $v_1, \ldots, v_q$ are extremum points of a compact convex set.

In the case $F \in \mathrm{int\,co}\{v_1, \ldots, v_q\}$ the claim was proved in Introduction since there we proved existence of a function $u_0$ such that $Du_0 \in \{v_1, \ldots, v_q\}$ $u_0\big|_{\partial\Omega} = l_F$. Indeed, in this case we have

$$\int_\Omega Du(x)dx = \sum_{i=1}^q (\bar{c}_i\,\mathrm{meas}\,\Omega)v_i = F\,\mathrm{meas}\,\Omega, \sum_{i=1}^q \bar{c}_i = 1, \bar{c}_i \geq 0.$$

Since the representation of $F$ in the form of a convex combination of $v_1, \ldots, v_q$ is unique, we obtain that $\bar{c}_i = c_i$ for each $i$. Hence, defining $u_k$ as $u$ for all $k \in N$ we obtain that $J(u_k) = \sum_{i=1}^q c_i L(v_i)\,\mathrm{meas}\,\Omega$.

Consider now the case when $F \notin \mathrm{int\,co}\{v_1, \ldots, v_q\}$. In any case $F \in \mathrm{reint\,co}\{v_1, \ldots, v_q\}$. Let $P$ be the largest subspace of $R^n$ perpendicular to all vectors $v_i - F$, $i \in \{1, \ldots, q\}$. Assume that $\dim P = m$ and $v_{q+1}, \ldots, v_{q+m+1}$ are such points in $P$ that $\mathrm{co}\{v_{q+1}, \ldots, v_{m+1+q}\}$ has nonempty interior in $P$ and 0 belongs to this interior.

For each $\delta > 0$ consider the function

$$w_{s,\delta}(\cdot) = \max_{1 \leq i \leq q+m+1} \langle \tilde{v}_i - F, \cdot \rangle - s,$$

where

$$\tilde{v}_i = v_i \text{ for } i \in \{1, \ldots, q\},$$

$$\tilde{v}_i = F + \delta v_i \text{ for } i \in \{q+1, \ldots, m+q+1\}.$$

It is clear that for each $\delta > 0$ the inclusion $F \in \operatorname{int} \operatorname{co}\{\tilde{v}_1, \ldots, \tilde{v}_{m+1+q}\}$ holds and $\tilde{v}_1, \ldots, \tilde{v}_{q+m+1}$ are extremum points of a compact convex set.

Moreover,

$$\frac{\operatorname{meas}\{x \in P_s : Dw_{s,\delta} \notin \{v_1 - F, \ldots, v_q - F\}\}}{\operatorname{meas}\{x \in P_s : Dw_{s,\delta} \in \{v_1 - F, \ldots, v_q - F\}\}} \to 0$$

as $\delta \to 0$ uniformly with respect to $s$.

For a $k \in N$ consider Vitaly covering of $\Omega$ by the supports $\Omega_i := x_i + P_{s_i}$ of the functions $\min\{0, w_{s_i,1/k}(\cdot - x_i)\}$ and define $\tilde{u}_k = l_F + w_{s_i,1/k}(\cdot - x_i)$ in $\Omega_i$ $(i \in N)$, $\tilde{u}_k = l_F$ -otherwise. In this case

$$\operatorname{meas}\{x \in \Omega : D\tilde{u}_k \notin \{v_1, \ldots, v_q\}\} \to 0$$

as $k \to \infty$. Therefore, if for a subsequence of $\tilde{u}_k$ (not relabeled)

$$c_i^k := \frac{\operatorname{meas}\{x \in \Omega : D\tilde{u}_k = v_i\}}{\operatorname{meas} \Omega} \to \tilde{c}_i, i \in \{1, \ldots, q\},$$

then $\sum \tilde{c}_i = 1$, $\sum \tilde{c}_i v_i = F$, and because of uniqueness of the representation of $F$ in the form of a convex combination of $v_i$ $(i = 1, \ldots, q)$ we infer that $\tilde{c}_i = c_i$. Hence, $c_i^k \to c_i$ as $k \to \infty$ for the original sequence $c_i^k$.

But we can not assert yet that $J(\tilde{u}_k) \to \sum_{i=1}^q c_i L(v_i)$ since $L$ can be unbounded in the set $\{F + \delta v_{q+1}, \ldots, F + \delta v_{q+m+1}; \delta \in [0,1]\}$. In order to overcome this difficulty notice that for all $\delta > 0$ sufficiently small the vectors $\tilde{v}_i := \delta v_i + F$ $(i = q+1, \ldots, m+q+1)$ lie in the interior of the set $\operatorname{co}\{v_1, \ldots, v_{q+m+1}\}$. Hence, for all $k$ sufficiently large the function $\tilde{u}_k$ can be redefined by the above described procedure in each set $\{x \in \Omega : D\tilde{u}_k = \tilde{v}_i\}$, $i \in \{q+1, \ldots, m+q+1\}$, (we denote the new function as $u_k$) in such a way that $Du_k \in \{v_1, \ldots, v_{q+m+1}\}$ a.e. on this set and $u_k = \tilde{u}_k$ on the boundary of this set. Since $u_k = \tilde{u}_k$ a.e. on the set $\{x \in \Omega : D\tilde{u}_k(x) \in \{v_1, \ldots, v_q\}\}$ and $|L(Du_k)| \leq c < \infty$ we infer that $J(u_k) \to \sum_{i=1}^q c_i L(v_i)$.

The general case can be reduced to the one discussed above. We can assume without loss of generality that $v_i \neq F$, $c_i > 0$ for all $i \in \{1, \ldots, q\}$.

For $q = 2$ we can assert that there exists a sequence of piecewise affine functions $u_k$ such that $u_k\big|_{\partial\Omega} = l_F$, $\text{meas}\,\{x \in \Omega : Du_k = v_i\} \to c_i\,\text{meas}\,\Omega$ $(i = 1, 2)$, and $J(u_k) \to \sum c_i L(v_i)\,\text{meas}\,\Omega$, since $F$ has unique representation in the form of a convex combination of $v_1, v_2$.

Let this claim be valid for $q = s$. To prove it for $q = s+1$, consider vectors $\tilde{v}_1, \dots, \tilde{v}_s$ such that $\tilde{v}_i = v_i$ for $i \le s-1$, $\tilde{v}_s = (c_s v_s + c_{s+1}v_{s+1})/(c_s + c_{s+1})$. Then $F = \sum_{i=1}^{s} \tilde{c}_i \tilde{v}_i$, where $\tilde{c}_i = c_i$ for $i \le s-1$ and $\tilde{c}_s = c_s + c_{s+1}$. By the induction assumption there exists a sequence of piece-wise affine functions $u_k$ such that $u_k\big|_{\partial\Omega} = l_F$, $\text{meas}\,\{x \in \Omega : Du_k(x) = \tilde{v}_i\} \to \tilde{c}_i\,\text{meas}\,\Omega$ $(i = 1, \dots, s)$, and $J(u_k) \to \sum \tilde{c}_i L(\tilde{v}_i)\,\text{meas}\,\Omega$. For a $k \in N$ let $\Omega_k := \text{int}\{x \in \Omega : Du_k(x) = v_s\}$. We can find a sequence $u_j^k$ such that $u_j^k = u_k$ in $\partial\Omega_k$, $||u_j^k||_{W^{1,\infty}(\Omega_k)} \le c < \infty$, and

$$\text{meas}\,\{x \in \Omega_k : u_j^k \ne v_i\} \to \frac{c_i}{\tilde{c}_s}\,\text{meas}\,\Omega_k\ (i = s, s+1), j \to \infty,$$

$$J(u_j^k; \Omega_k) \to \sum_{i=s}^{s+1} \frac{c_i}{\tilde{c}_s} L(v_i)\,\text{meas}\,\Omega_k = \sum_{i=s}^{s+1} c_i L(v_i)\,\text{meas}\,\Omega, j \to \infty.$$

Then, for a subsequence $w_k := u_{j(k)}^k$ $(k \to \infty)$ we get the convergence $J(w_k) \to \sum_{i=1}^{q} c_i L(v_i)\,\text{meas}\,\Omega$.

The proof of the theorem is complete.

**Proof of Theorem 1.1**

Sufficiency of the condition

*either* $\partial L(F) \ne \emptyset$ *or there exist* $v_1, \dots, v_q \in R^n$ *such that* $F \in \text{int}\,\text{co}\{v_1, \dots, v_q\}$ *and* $\cap_{i=1}^{q} \partial L(v_i) = \emptyset$

for solvability of the problem

$$\int_\Omega L(Du)dx \to \min, u\big|_{\partial\Omega} = l_F, u \in W^{1,1}(\Omega)$$

has been proved in Introduction.

In order to prove its necessity, first note that

$$\inf\{\sum_{i=1}^{q} c_i L(v_i) : q \in N, c_i \ge 0, v_i \in R^n, \sum_{i=1}^{q} c_i = 1, \sum_{i=1}^{q} c_i v_i = F\} > -\infty. \quad (2.1)$$

Indeed, for each $c_i, v_i$ $(i = 1, \ldots, q)$ from (2.1) by Lemma 2.3 we get

$$\inf\{J(u) : u\Big|_{\partial\Omega} = l_F, u \in W^{1,\infty}\} \leq \sum_{i=1}^{q} c_i L(v_i). \tag{2.2}$$

Since solvability of the problem (1.1) implies the inequality

$$\inf\{J(u) : u\Big|_{\partial\Omega} = l_F, u \in W^{1,\infty}\} > -\infty,$$

we infer that (2.1) holds.

By Lemma 2.1 we infer that $L^{**}$ is a convex continuous function. Moreover, if $u_0$ is a solution of the problem (1.1) then (2.1), (2.2) imply that $J(u_0) \leq L^{**}(F) \operatorname{meas} \Omega$.

Let $l \in \partial L^{**}(F)$. For each admissible $u$ we have

$$J(u) - L^{**}(F) \operatorname{meas} \Omega = \int_\Omega \{L(Du) - L^{**}(F) - \langle l, Du - F\rangle\} dx \geq 0.$$

Hence, $J(u_0) = L^{**}(F) \operatorname{meas} \Omega$.

Let

$$P_l = \{v \in R^n : L(v) - L^{**}(F) - \langle l, v - F\rangle = 0\}.$$

Since $J(u_0) = L^{**}(F) \operatorname{meas} \Omega$ and $\int_\Omega \langle l, Du_0 - F\rangle dx = 0$ we infer that $Du_0(x) \in P_l$ for a.a. $x \in \Omega$.

It is obvious that $P_l$ is a closed set. Moreover, we claim that $F \in \operatorname{int} \operatorname{co} P_l$ if $L(F) \neq L^{**}(F)$. Otherwise by the Hahn-Banach theorem there exists an $a \in R^n$ such that $\langle F, a\rangle \geq \langle v, a\rangle$ for any $v \in \operatorname{co} P_l$. Then $\langle F, a\rangle \geq \langle Du_0, a\rangle$ a.e. on $\Omega$. Since $\int_\Omega \langle Du_0, a\rangle dx = \langle F, a\rangle \operatorname{meas} \Omega$ we infer that $Du_0 \in \{v \in R^n : \langle v - F, a\rangle = 0\}$ a.e. on $\Omega$. As a consequence,

$$\frac{\partial(u_0 - l_F)}{\partial a} = \langle Du_0 - F, a\rangle = 0$$

a.e. on $\Omega$. Since $u_0 = l_F$ on $\partial\Omega$ we infer that $u_0 = l_F$ a.e. on $\Omega$. Hence $F \in P_l$, and, as a consequence, $L(F) = L^{**}(F)$. This is a contradiction.

We have proved that either $F \in \operatorname{int} \operatorname{co} P_l$ or $L(F) = L^{**}(F)$. In the first case there exist $v_1, \ldots, v_q \in P_l$ such that $F \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$. It is obvious that in this case $l \in \partial L(v_i)$ for any $i \in \{1, \ldots, q\}$. Hence $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$. This completes the proof of Theorem 1.1.

## 3 Proof of Theorem 1.3

We will need one more lemma with respect to properties of convexifications.

**Lemma 3.1** *Let $L : R^n \to R$ be a lower semicontinuous function such that $L(v) \geq \theta(v)$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$. Let $F \in R^n$ and $l \in \partial L^{**}(F)$. Let also*

$$P_l := \{v \in R^n : L^{**}(v) - L^{**}(F) - \langle l, v - F\rangle = 0\}.$$

*Then $L = L^{**}$ in the set of extremum points of $P_l$.*

**Proof**

Assume that $v_0 \in \text{extr}P_l$. By Caratheodory theorem there exists $c_i^k \geq 0$, $v_i^k \in R^n$ $(i = 1, \dots, n+2)$ such that $\sum_{i=1}^{n+2} c_i^k = 1$, $\sum_{i=1}^{n+2} c_i^k v_i^k = v_0$ and $\sum_{i=1}^{n+2} c_i^k L(v_i^k) \to L^{**}(v_0)$ as $k \to \infty$.

We can assume also that $c_i^k \to c_i$ and either $v_i^k \to v_i$ or $|v_i^k| \to \infty$ as $k \to \infty$. Since $c_i^k|v_i^k| \to 0$ in the case $|v_i^k| \to \infty$ (recall that $\theta(v)/|v| \to \infty$ as $|v| \to \infty$), we obtain that for all $i \in \{1, \dots, n+2\}$ such that $c_i > 0$ the convergence $v_i^k \to v_i$ holds and $\sum c_i v_i = v_0$. Because of lower semicontinuity of $L$ we have $\sum c_i L(v_i) = L^{**}(v_0)$. Since $L(v) - L^{**}(v_0) - \langle l, v - v_0\rangle \geq 0$ everywhere, we infer that $L(v_i) - L^{**}(v_0) - \langle l, v_i - v_0\rangle = 0$ for each $v_i$. Then $v_i \in P_l$ for each $i$. Because $v_0 \in \text{extr}P_l$ we obtain that $v_i = v_0$ for all $i$ under consideration. Hence $L(v_0) = L^{**}(v_0)$.

The proof is complete.

**Lemma 3.2** *Let $u_0 \in C(B(x_0, r))$ be differentiable at $x_0$ in the classical sense. Let $U$ be a convex compact subset in $R^n$, and let $v_1, \dots, v_q \in \text{extr}U$ be such that*

$$Du_0(x_0) \in \text{int}\,\text{co}\{v_1, \dots, v_q\}.$$

*Then, for all $s > 0$ sufficiently small the function*

$$\phi_s(\cdot) := w_s(\cdot - x_0) + u_0(x_0) + \langle Du_0(x_0), \cdot - x_0\rangle,$$

*where $w_s(x) := \max_{1\leq i\leq q}\langle v_i - Du_0(x_0), x\rangle - s$, has the properties:*

$$\phi_s < u_0, x \in x_0 + P_{s/2};\ \phi_s > u_0, x \in x_0 + \partial P_{2s},\ \textit{where}$$

$$P_s = \{x \in R^n : \max_{1\leq i\leq q}\langle v_i - Du_0(x_0), x\rangle \leq s\}.$$

**Proof** is straightforward. We have

$$u_0(x) - \phi_s(x) = u_0(x) - \langle Du_0(x_0), x - x_0\rangle - u_0(x_0) - w_s(x - x_0) =$$

$$o(|x - x_0|) - w_0(x - x_0) + s.$$

Since $|w_0(\cdot - x_0)| \leq s/2$ inside $x_0 + P_{s/2}$ we obtain that $u_0 - \phi_s > 0$ inside $x_0 + P_{s/2}$ if $s > 0$ is sufficiently small.

Since $w_0(x - x_0) = 2s$ for $x \in x_0 + P_{2s}$, we infer that $u_0 - \phi_s < 0$ in $\partial P_{2s}$ if $s > 0$ is sufficiently small.

The proof is complete.

**Proof of Theorem 1.3**

By Theorem 1.1 solvability of all problems (1.1) with linear boundary data implies that

*for each $F \in R^n$ either $\partial L(F) \neq \emptyset$ or there exist $v_1, \dots, v_q \in R^n$ such that $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ and $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$*

We need to prove that this condition suffices for solvability of all problems (1.1) with boundary data $f$ admitting at least one function $u \in W^{1,1}(\Omega)$ such that $J(u) < \infty$.

First note that the function $L^{**} : R^n \to R$ is a continuous convex function satisfying the growth condition $L^{**} \geq \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$.

Let $\Omega$ and $f$ be of the described above type. Let $u_0$ be a solution of the problem

$$\int_\Omega L^{**}(Du(x))dx \to \min, u\Big|_{\partial\Omega} = f, u \in W^{1,1}(\Omega). \tag{3.1}$$

We will construct a solution $\tilde{u}$ of the problem (3.1), for which the inclusion $D\tilde{u}(x) \in \{v : L(v) = L^{**}(v)\}$ holds a.e. in $\Omega$, as a limit of a sequence of perturbations of $u_0$, each of which is also a solution of the problem (3.1). Note that $\tilde{u}$ is automatically a solution to the original problem (1.1).

Let $\tilde{\Omega}$ be the set of those points $x \in \Omega$, where $u_0$ is differentiable in the classical sense and $L(Du_0(x)) \neq L^{**}(Du_0(x))$. Note that $u$ is differentiable in the classical sense almost everywhere in $\Omega$ since $u \in W^{1,p}(\Omega)$ with $p > n$ (cf. [EG, p.234]).

Let $x_0 \in \tilde{\Omega}$. There exist $v_1, \dots, v_q$, which are extremum points of a compact convex set, and $l \in R^n$ such that $Du_0(x_0) \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$, $l \in \cap_i \partial L(v_i)$. Note that $L^{**}(\cdot) = L(v_1) + \langle l, \cdot - v_1\rangle$ in $co\{v_1, \dots, v_q\}$.

By Lemma 3.2 for all sufficiently small $s > 0$ the function

$$\phi_s := \langle Du_0(x_0), \cdot - x_0\rangle + u_0(x_0) + w_s(\cdot - x_0), \text{ where}$$

,

$$w_s(x) = \max_{1\le i\le q}\langle v_i - Du_0(x_0), x\rangle - s,$$

satisfies the inequalities:

$$\phi_s < u_0, x \in x_0 + P_{s/2}, \ \phi_s > u_0, x \in x_0 + \partial P_{2s} \tag{3.2}$$

with $P_s := \{x \in R^n : \max_{1\le i\le q}\langle v_i - Du_0(x_0), x\rangle \le s\}$.

Hence, the function $u_1$, which is equal to $u_0$ outside the set $x_0 + P_{2s}$ and to $\min\{\phi_s, u_0\}$ inside this set, is well defined and is an element of $W^{1,p}(\Omega)$.

Since problems (3.1) with integrands $L^{**}$ and $L^{**} + \langle l, \cdot\rangle + c$ have the same solutions, we can assume without loss of generality that $L^{**} = 0$ in $U := \text{co}\{v_1, \ldots, v_q\}$ and $L^{**} \ge 0$ otherwise.

At the same time, if $\Omega' := \{x \in \Omega : u_1 \ne u_0\}$ then $Du_1 \in \text{extr}U$ a.e. in $\bar{\Omega}'$ and, as a consequence, we have $L(Du_1) = 0$ a.e. in $\bar{\Omega}'$. Hence $J(u_1) = J(u_0)$.

Since sets of the form $\bar{\Omega}'$ form the Vitaly cover of $\tilde{\Omega}$ (see (3.2)), by the Vitaly covering theorem we can decompose $\tilde{\Omega}$ on disjoint closed sets $\bar{\Omega}_j, j = 1, 2, \ldots$, and a set of zero measure such that for each $j \in N$ there exists a function $\psi_j \in W_0^{1,\infty}(\Omega_j)$ such that $Du_0 + D\psi_j \in \{v : L(v) = L^{**}(v)\}$ a.e. in $\bar{\Omega}_j$, and

$$\int_{\bar{\Omega}_j} L(Du_0)dx = \int_{\bar{\Omega}_j} L(Du_0 + D\psi_j)dx.$$

Define $u_i$ as $u_0 + \psi_j$ in $\bar{\Omega}_j$, $j \le i$, and as $u_0$ otherwise. Then $u_i$ is a sequence of solutions of the problem (3.1). Note that this sequence converges strongly in $W^{1,1}(\Omega)$. Indeed, in view of the growth conditions on $L$ we have

$$||Du_k - Du_l||_{L_1} = ||Du_k - Du_l||_{L_1(\cup_{k\le j\le l}\bar{\Omega}_j)} \le$$

$$2\int_{\cup_{k\le j\le l}\bar{\Omega}_j} (c_1 L^{**}(Du_0) + c_2)dx \to 0, \text{ as } k, l \to \infty.$$

Therefore, the function $\tilde{u}$, which is the limit of $u_i$ in $W^{1,1}(\Omega)$, is also a solution of the problem (3.1). Simultaneously $\text{meas}\{x \in \Omega : L(Du_i) \ne L^{**}(Du_i)\} \to 0$ and, as a consequence, $D\tilde{u} \in \{v : L(v) = L^{**}(v)\}$ a.e. in $\Omega$. Hence, $\tilde{u}$ is a solution of the original problem (1.1).

The proof of the theorem is complete.

# 4 Some auxiliary facts related to solvability of boundary value minimization problems with convex integrands and validity of the Euler-Lagrange equation for their solutions

In this section we recall some standard facts about solvability of problems (1.1) with convex integrands. These facts were established in the context of solvability theory for the Plateau problem (see [Gi]). We also prove a version of the Euler-Lagrange equation, which is valid for all Lipschitz minimizers of problems (1.1) with convex integrands.

Recall that boundary data $f$ is said to satisfy *boundary slope condition* if there exists $M > 0$ such that for each point $x_0 \in \partial\Omega$ we can find $l_1, l_2 \in R^n$ such that $|l_1|, |l_2| \leq M$ and $\langle l_1, x - x_0\rangle + f(x_0) \leq f(x) \leq \langle l_2, x - x_0\rangle + f(x_0)$, $\forall x \in \partial\Omega$.

For the proof of the following theorem see, e.g., [Gi].

**Theorem 4.1** *Let $L : R^n \to R$ be a convex continuous function. Let boundary data $f$ satisfy the boundary slope condition with $M > 0$. Then there exists a solution $u_0$ of the problem $J(u) \to \min$, $u\big|_{\partial\Omega} = f$ in the class of Lipschitz functions. Moreover, $u_0$ can be chosen satisfying the inequality $||Du_0||_{L^\infty} \leq M$.*

**Remark**

Let $\Omega$ be a convex domain with $\partial\Omega \in C^2$ of positive curvature and let $f \in C^2(\partial\Omega)$. Then $f$ satisfies boundary slope condition with certain $M > 0$.

Solutions of the minimization problems always satisfy a nonsmooth version of the Euler-Lagrange equation.

**Theorem 4.2** *Let $L : R^n \to R$ be a continuous convex function. Let $u_0 \in W^{1,\infty}(\Omega)$ be a local minimizer of the functional $J$: $J(u_0) \leq J(u_0 + \phi)$ for all $\phi \in C_0^1(\Omega)$ with $||\phi||_{C^1} \leq \epsilon$, $\epsilon > 0$.*

*Then there exists a function $l \in L^\infty(\Omega)$ such that $l(x) \in \partial L(Du_0(x))$ a.e. in $\Omega$ and*

$$\int_\Omega \langle l(x), D\phi(x)\rangle dx = 0, \forall \phi \in C_0^1(\Omega).$$

**Proof**

Let $M > ||u_0||_{W^{1,\infty}(\Omega)}$. Define $L^M$ to be equal to $L$ for $|v| \leq M+1$ and to $\infty$ for $|v| > M+1$.

Let

$$K := \sup\{|l| : l \in \partial L^M(v), |v| \leq M\} + \sup\{|L^M(v)| : |v| \leq M\}.$$

Consider convexification $G^{**}$ of the function $G := \min\{L^M, K|v| + K(M+1)\}$.

Because of lower semicontinuity of $G$, by Lemma 2.1 we infer continuity and convexity of $G^{**}$. Note that $G^{**} = L$ for $|v| < M$. Indeed, for these $v$ we have $|L(v)| \leq K \leq K|v| + K(M+1)$. Then $G = L$ for $|v| < M$. Moreover, for each $v_0 \in B(0, M)$ and each $l \in \partial L^M(v_0)$ we have:

$$L(v_0) + \langle l, v - v_0 \rangle \leq L^M(v),$$

$$|L(v_0) + \langle l, v - v_0 \rangle| \leq K + KM + K|v|.$$

Hence $l \in \partial G(v_0)$ and, as a consequence, $\partial G(v_0) = \partial L^M(v_0) \neq \emptyset$. Then, by Corollary 2.2 we get $L = G = G^{**}$ in $B(0, M)$.

Since $G^{**} = L$ in $B(0, M)$, the function $u_0$ is a local minimizer for the integral functional with the integrand $F$, where $F(x, v) := G^{**}(v) + |v - Du_0(x)|^2$. In this case $u_0$ is automatically a solution of the minimization problem. To prove this, note that for each nontrivial $\phi \in C_0^1(\Omega)$ the function

$$I(\epsilon) := \int_\Omega \{F(x, Du_0(x) + \epsilon D\phi(x)) - F(x, Du_0(x))\} dx$$

is a convex function of $\epsilon$ and $I(0) = 0$. Moreover, for $\epsilon > 0$ sufficiently small we have $I(\epsilon) \geq I(0) = 0$, since $u_0$ is a local minimizer. Because of strict convexity of $I$ we infer that $I(\epsilon) > 0$ everywhere. Since $\phi \in C_0^1(\Omega)$ is arbitrary we obtain that $u_0$ is a unique global minimizer.

The proof reduces to finding a function $l_M \in L^\infty(\Omega; R^n)$ such that $l_M(x) \in \partial_v F(x, Du_0(x))$ for a.e. $x \in \Omega$ and

$$\int_\Omega \langle l_M(x), D\phi(x) \rangle dx = 0, \forall \phi \in C_0^1(\Omega). \tag{4.1}$$

Indeed, since for a.e. $x \in \Omega$ the identity $\partial_v F(x, Du_0(x)) = \partial L^M(Du_0(x))$ holds, we obtain that $l_M(x) \in \partial L^M(Du_0(x))$ a.e. in $\Omega$. Note that for each $v \in R^n$ the identity $\partial L(v) = \cap_M \partial L^M(v)$ holds. Note also that, since

$\cup_{x\in\Omega}\partial L^M(Du_0(x))$ is a nonincreasing sequence of bounded sets, we infer that all functions $l_M$ are equabounded in $L^\infty$. Then, by Banach-Mazur theorem (see, e.g., [ET;Ch.1,Sect.1]) there exists a sequence $\tilde{l}_k := \sum_{i=M_k+1}^{M_{k+1}} c_i l_i$ with $M_k \to \infty$, $c_i \geq 0$ such that $\sum_{i=M_k+1}^{M_{k+1}} c_i = 1$ and $\tilde{l}_k \to l_0$ in $L_1$. Since $\tilde{l}_k(x) \in \partial L^{M_k}(Du_0(x))$ for a.a. $x \in \Omega$ we obtain that $l_0(x) \in \cap_M \partial L^M(Du_0(x)) = \partial L(Du_0(x))$ a.e. in $\Omega$. It is also clear that (4.1) holds with $l_0$ instead of $l_M$. This proves the claim of Theorem 4.2.

In order to prove (4.1) notice that in the case $F(x,\cdot) \in C^1$ for a.e. $x \in \Omega$ the identity (4.1) holds with $l_M(x) = F_v(x, Du_0(x))$. The general case can be reduced to this one by approximation arguments.

Consider functions $F^\epsilon : \Omega \times R^n \to R$ such that for each $x_0 \in \Omega$, $v_0 \in R^n$

$$F^\epsilon(x_0, v_0) = \int_{R^n} F(x_0, v) * \rho_\epsilon(v - v_0) dv,$$

where $\rho \geq 0$ is a usual mollifying kernel, i.e. $\rho$ is smooth with the support in the unit ball, $\int_{R^n} \rho = 1$, and $\rho_\epsilon = \epsilon^{-n}\rho(x/\epsilon)$.

It is easy to see that $F^\epsilon$ is convex in $v$ and $F^\epsilon(x,\cdot) \in C^\infty$ for a.e. $x \in \Omega$. Moreover,

$$A_1|v|^2 + B_1 \leq F^\epsilon(x,v) \leq A_2|v|^2 + B_2, \epsilon \in ]0,1], A_2 \geq A_1 > 0,$$

and for a.e. $x \in \Omega$ the family $F^\epsilon(x,\cdot)$ converges to $F(x,\cdot)$ uniformly in each compact set.

Since each problem $J^\epsilon \to \min$, $u\big|_{\partial\Omega} = f$, $u \in W^{1,2}$ has a solution $u^\epsilon$ we infer that $u^\epsilon, \epsilon \in ]0,1]$, form a relatively compact set in the weak topology of $W^{1,2}$. Then, because of lower semicontinuity of convex functionals with respect to weak convergence in $W^{1,2}$ we infer that

$$\liminf_{\epsilon\to 0} J^\epsilon(u^\epsilon) \geq J(\tilde{u})$$

for each limit function $\tilde{u}$ of $u^\epsilon$ ($u^{\epsilon_k} \to \tilde{u}$ in $L_2$ for some $\epsilon_k \to 0$), see e.g. [Syc1].

Since $u_0$ is the unique solution of the original problem we infer that $u^\epsilon \to u_0$ in $L_2$. Then $u^\epsilon \rightharpoonup u_0$ in $W^{1,2}$, where $\rightharpoonup$ denotes the weak convergence. For strictly convex functionals convergences $u^\epsilon \rightharpoonup u_0$ in $W^{1,2}$, $J^\epsilon(u^\epsilon) \to J(u_0)$ imply strong convergence of $u^\epsilon$ to $u_0$ in $W^{1,2}$ (see [Syc1] for a simple proof, and [Syc2] for the characterization of this property of integral functionals in terms of integrands).

For each $\epsilon > 0$ we have

$$\int_\Omega \langle F_v^\epsilon(x, Du^\epsilon(x)), D\phi \rangle dx = 0, \forall \phi \in C_0^1(\Omega), \qquad (4.2)$$

where $F^\epsilon(x, Du^\epsilon)$ ($\epsilon \in ]0,1]$) form a relatively compact set in the weak topology of $L^1(\Omega)$.

Locally uniform convergence of $F^{\epsilon_k}(x,\cdot)$ to $F(x,\cdot)$ for an $x \in \Omega$ implies that for each sequence $v_k \in R^n$ and each sequence $l_k \in F_v^{\epsilon_k}(x, v_k)$ such that $\epsilon_k \to 0$, $v_k \to v_0$, $l_k \to \tilde{l}$ the inclusion $\tilde{l} \in \partial_v F(x, v_0)$ holds.

Without loss of generality we can assume that $F_v^{\epsilon_k}(\cdot, Du^{\epsilon_k}(\cdot))$ converge to $l \in L^1$ weakly in $L^1$. Since for a.e. $x \in \Omega$ all limit points of the sequence $F_v^{\epsilon_k}(x, Du^{\epsilon_k}(x))$ belong to $\partial_v F(x, Du_0(x))$ (recall that $Du^{\epsilon_k} \to Du_0$ a.e. in $\Omega$) and $\partial_v F(x, Du_0)$ is a compact convex set, we infer that $l(x) \in \partial_v F(x, Du_0(x))$ for a.e. $x \in \Omega$. Being the weak limit of $F_v^{\epsilon_k}(\cdot, Du^{\epsilon_k}(\cdot))$, the function $l(\cdot)$ satisfies (4.2) automatically.

The proof is complete.

# 5 Proof of Theorem 1.2

In this section we give proof to the last result of this paper - Theorem 1.2.

**Proof of Theorem 1.2.** Due to Theorem 1.1 solvability of all problems (1.1) with linear boundary conditions and a fixed $\Omega$ implies that

*for any $F \in R^n$ either $\partial L(F) \neq \emptyset$ or there exist $v_1, \ldots, v_q \in R^n$ such that $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ and $F \in \operatorname{int}\operatorname{co}\{v_1, \ldots, v_q\}$*

To prove the converse, fix $\Omega$ with $\partial\Omega \in C^2$ of positive curvature and $f \in C^2(\partial\Omega)$.

By Lemma 3.2 $L^{**} : R^n \to R$ is a continuous convex function. It is clear also that $L^{**} \geq \theta$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$.

By Theorem 4.1 and the remark to it there is a solution $u_0 \in W^{1,\infty}(\Omega)$ of the problem

$$J^{**}(u) \to \min, u\Big|_{\partial\Omega} = f \qquad (5.1)$$

in the class $u \in W^{1,\infty}(\Omega)$.

We can also prove that $u_0$ is a solution of the problem (5.1) in $W^{1,1}(\Omega)$. Indeed, by Theorem 4.2 there exists $l \in L^\infty(\Omega)$ such that $l(x) \in \partial L(Du_0(x))$

for a.e. $x \in \Omega$ and

$$\int_\Omega \langle l(x), D\phi(x)\rangle dx = 0, \forall \phi \in W_0^{1,1}(\Omega).$$

If $u\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega} = f$ then we obtain

$$J^{**}(u) - J^{**}(u_0) = \int_\Omega \{L^{**}(Du) - L^{**}(Du_0) - \langle l(x), Du - Du_0\rangle\} dx.$$

Since the expression in the brackets is nonnegative in $\Omega$ a.e. we obtain that $u_0$ is a solution in $W^{1,1}$. Note also that in the case $\operatorname{ess\,sup}|Du|$ is sufficiently large the expression in the brackets is positive in a set of positive measure since $L^{**}$ has superlinear growth at infinity, and, as consequence, $J^{**}(u) - J^{**}(u_0) > 0$. Therefore, all solutions to the problem (5.1) in $W^{1,1}(\Omega)$ are bounded in $W^{1,\infty}(\Omega)$.

Let $M := ||Du_0||_{L^\infty(\Omega)}$. By Rademacher's theorem (cf.[EG,p.81]) $u_0$ has classical derivative a.e. in $\Omega$. Let $\tilde{\Omega}$ be the set, where $Du_0$ exists in the classical sense and $L(Du_0(x_0)) \neq L^{**}(Du_0(x_0))$. There exists $M_1$ such that for each point $x \in \tilde{\Omega}$ there exists $v_1, \dots, v_q \in B(0, M_1)$ such that $Du_0(x) \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$ and $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ (as a consequence, $L^{**}$ is affine on $\operatorname{co}\{v_1, \dots, v_q\}$). Indeed, because of superlinear growth of $L^{**}$ at infinity, the union of those compact convex sets intersecting $B(0, M)$, on each of which $L^{**}$ is affine, is a bounded set.

Therefore, for any $x_0 \in \tilde{\Omega}$ we can isolate extremum points $v_i, i \in \{1, \dots, q\}$, of a compact convex set such that $v_1, \dots, v_q \in B(0, M_1)$, $Du_0(x_0) \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$, and $\cap_i \partial L(v_i) \neq \emptyset$. Let $w_s$ be functions from (1.2) with $F = Du_0(x_0)$. By Lemma 3.2 we have that for all $s > 0$ sufficiently small the function

$$\phi_s := u_0(x_0) + \langle Du_0(x_0), \cdot - x_0\rangle + w_s(\cdot - x_0)$$

has properties:

$$\phi_s < u_0, x \in x_0 + P_{s/2}, \ \phi > u_0, x \in x_0 + \partial P_{2s},$$

where $P_s = \{x \in R^n : \max_{1 \le i \le q} \langle v_i - Du_0(x_0), x\rangle \le s\}$.

Hence, we can define a perturbation $u_1$ of $u_0$ as follows:

$$u_1 = u_0, x \in (\Omega \setminus \{x_0 + P_{2s}\}), \ \ u_1 = \min\{\phi_s, u_0\} - \ \text{otherwise}.$$